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NOTE ON MY PAPER "A SIMPLE PROOF FOR VON NEUMANN'S MINIMAX THEOREM"

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At present the proofs for the minimax principles avoiding Brower's fixed point theorem are essentially of two types. One method is based on the application of some separating theorems (cf. Parthasaraty [9] and Balakrishnan [10]). Recently a different treatment was developed in [2]—[7] which can be called the method of level sets, and which turns out to be more farreaching in several aspects than the previous one (as it is pointedout in the work of L. L. Stachó [5]).

The question naturally arises: what is the deeper connection between the two approaches?

One of the aims of this paper is to point out how the finite intersection property of the level sets follows from the Hahn—Banach theorem. Theorem 1 below is essentially a consequence of [5, Proposition 3] because convex (resp. concave) functions are automatically continuous in the interior of a straight line segment in a vector space. However our proof is more direct and bypasses the technical difficulty arising from the [5, Proposition 3] should be applied to $(X \setminus \text{ext}\, X) \times (Y \setminus \text{ext}\, Y)$ instead of $X \times Y$ immediately.

1. Let E and F be arbitrary vector spaces, and let $X \subset E$ and $Y \subset F$ be arbitrary convex sets. Let $f(x, y)$ be a real function on $X \times Y$ which is concave in x for any fixed $y \in Y$ and convex in y for any fixed $x \in X$. Denote $H_y^{(c)} = \{x \in X \colon f(x, y) \geqq c\}$ $(y \in Y)$ and $\mathscr{C}$ the set of real numbers c for which $H_y^{(c)} \neq \emptyset$ whenever $y \in Y$. Denote $c^* = \sup \mathscr{C}$.

THEOREM 1. *For any $c < c^*$ the system of sets $\{H_y^{(c)}, y \in Y\}$ has the finite intersection property.*

PROOF. First we show that for any $y_1, y_2 \in Y$ and $c < c^*$ we have $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} \neq \emptyset$. If $c^* = -\infty$ then the statement of Theorem 1 is trivial, hence we may assume $c^* > -\infty$. Suppose $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} = \emptyset$ for some $y_1, y_2 \in Y$ and $c < c^*$. Then there exists $\lambda \in [0, 1]$ such that $H_{y_\lambda}^{(c)} = \{x \colon f(x, y) = c\}$ for $y_\lambda \overset{\text{def}}{=} \lambda y_1 + (1-\lambda) y_2$. Indeed, suppose the contrary, i.e. for every $\lambda \in [0, 1]$ there exists $x \in X$ with $f(x, y_\lambda) > c$. Then, because of the convexity of f in y we have: for every $\lambda \in [0, 1]$ there exists $x \in X$ such that

$$\lambda f(x, y_1) + (1-\lambda) f(x, y_2) > c$$

i.e.

$$((f(x, y_1) - c,\ f(x, y_2) - c),\ (\lambda, 1-\lambda)) > 0.$$

Then, according to a classical separating theorem for disjoint convex sets in R^2 we obtain: there exist $x_1, x_2, \dots, x_n \in X$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{R}_+$ with $\Sigma\lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i (f(x_i, y_1) - c) \geqq 0, \quad \sum_{i=1}^{n} \lambda_i (f(x_i, y_2) - c) \geqq 0.$$

Since f is concave in x we obtain

$$f(x_\lambda, y_1) \geqq c, \quad f(x_\lambda, y_2) \geqq c \quad (x_\lambda \overset{\text{def}}{=} \sum \lambda_i x_i).$$

This means that $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} \neq \emptyset$ which contradicts our indirect assumption. So we have proved the existence of $y_\lambda \in Y$ with the property $H_{y_\lambda}^{(c)} = \{x \in X : f(x, y_\lambda) = c\}$. Hence $c \geqq c^*$ follows and this contradicts $c < c^*$. We proved that $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} \neq \emptyset$ if $y_1, y_2 \in Y$ and $c < c^*$. Suppose $\bigcap_{i=1}^{n} H_{y_i}^{(c)} \neq \emptyset$ if $n \leqq N$, $y_i \in Y$ and $c < c^*$. Let $y_1, y_2, \dots, y_{N+1} \in Y$ and $c < c^*$ be arbitrary. We prove that $\bigcap_{1}^{N+1} H_{y_i}^{(c)} \neq \emptyset$. We know that $\tilde{X} \overset{\text{def}}{=} \bigcap_{1}^{N-1} H_{y_i}^{(c)} \neq \emptyset$ and the set $\tilde{X}$ is convex. We can repeat the proof above for $\tilde{f} = f|_{\tilde{X} \times Y}$ and $\tilde{H}_{y_N}^{(c)} = H_{y_N}^{(c)} \cap \tilde{X}$, $\tilde{H}_{y_{N+1}}^{(c)} = H_{y_{N+1}}^{(c)} \cap \tilde{X}$, and Theorem 1 follows by induction on N.

THEOREM 2. *Let X, Y be arbitrary sets and let f: $X \times Y \to \mathbf{R}$ be an arbitrary function. Then*

$$(B \overset{\text{def}}{=}) \sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y) (\overset{\text{def}}{=} J)$$

if and only if for every $c < c^*$ $(H^{(c)} \overset{\text{def}}{=}) \bigcap_{y \in Y} H_y^{(c)} \neq \emptyset$. *In this case* $B = J = c^*$.

PROOF. It is easy to see that $B \leqq J \leqq c^*$. Indeed $B \leqq J$ is trivial. If $c > c^*$ then there exists $y \in Y$ such that $f(x, y) < c$ for every $x \in X$, i.e. $J \leqq c$. This proves the inequality $J \leqq c^*$. Now let $c < c^*$ and $x_c \in H^{(c)}$. Then $f(x_c, y) \geqq c$ for every $y \in Y$, i.e. $B \geqq c$. Hence $B = J = c^*$ follows.

Now suppose $B = J$. Let $c < c^*$ and $\varepsilon > 0$ such that $c + \varepsilon < c^*$. This means that for every $y \in Y$ there exists $x \in X$ such that $f(x, y) \geqq c + \varepsilon$, i.e. $J \geqq c + \varepsilon$. This means (using $B = J$) that there exists $x \in X$ such that $f(x, y) \geqq c$ for every $y \in Y$, i.e. $\bigcap_{y \in Y} H_y^{(c)} \neq \emptyset$.

REMARK. It is easy to see that for an arbitrary function f we have $J = c^*$. $J \leqq c^*$ is proved above. Now we prove $J \geqq c^*$. Let $c < c^*$. Then for every $y \in Y$ there exists $x \in X$ such that $f(x, y) \geqq c$, hence $J \geqq c$ and the desired inequality $J > c^*$ follows.

From Theorems 1 and 2 we obtain immediately

THEOREM 3. *Suppose the conditions of Theorem* 1 *are fulfilled and E is a topological vector space, $X \subset E$ is a compact subset, further $f(x, y)$ is upper semi continuous in x for every fixed $y \in Y$. Then we have*

$$\inf_{y \in Y} \max_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

References

[1] J. von Neumann, Zur Theorie der Gesellschaftsspiele, *Math. Ann.*, **100** (1928), 295—320.
[2] I. Joó, Remarks to a theorem of B. Š. Kasin on convergence systems, *Proc. of Conf. on Constructive Function Theory*, (Sofia, 1977) pp. 345—351.
[3] I. Joó and A. Sövegjártó, A fixed point theorem, *Ann. Univ. Sci. Budapest, Sect. Math.*, **24** (1981), 9—11.
[4] I. Joó, A simple proof for von Neumann's minimax theorem, *Acta Sci. Math. (Szeged)*, **42** (1980), 91—94.
[5] L. L. Stachó, Minimax theorems beyond topological vector spaces, *Acta. Sci. Math. (Szeged)*, **42** (1980), 157—164.
[6] V. Komornik, Minimax theorems for upper semicontinuous function, *Acta Math. Acad. Sci. Hungar.*, **40** (1982).
[7] I. Joó and L. L. Stachó, A note on Ky Fan's minimax theorem, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 401—407.
[8] K. Fan, Minimax theorems, *Proc. Acad. Sci. USA*, **39** (1953), 42—48.
[9] T. Parthasarathy and T. E. Raghavan, *Some Topics in two Person games* (New York, 1971).
[10] A. V. Balakrishnan, *Appliced Functional Analysis* (New York, 1976).

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