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Bicircular projections on some Banach spaces

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Abstract

In this paper we show that the bicircular projections are precisely the Hermitian projections and prove some immediate consequences of this result.

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1. Introduction

Recently, Stacho and Zalar [13] introduced a new class of operators for Banach spaces called bicircular projections. They proved some nice results for these projections on the Banach spaces $B(H)$, $S(H)$, and $A(H)$. These spaces consist, respectively, of all bounded operators, all bounded symmetric, and all bounded antisymmetric linear operators on a complex Hilbert space H. In a later paper [14], they gave an isomorphic characterization of Hilbert space using bicircular projections on JB^* triples. In [13], it was remarked that a natural problem is to describe the bicircular projections of a given Banach space, cf. the reviews of both articles [13] and [14]. Our aim in this note is to address this question by showing the relationship between bicircular projections and norm hermitian projections. This relationship leads to results in Banach spaces which do not have the algebraic structure required in [13]. Let X be a Banach space over $\mathbb{C}$ and P a bounded linear projection on X and $\overline{P}$ its complementary projection . The projection P is called a bicircular projection provided $e^{i\alpha}P + e^{i\beta}\overline{P}$ is an isometry for all real values of α and β.

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As observed in [13], this condition is equivalent to the requirement that $P + e^{i\theta}\overline{P}$ is an isometry for every real value of θ. This fact is extremely important in our study of these projections.

2. Bicircular projections on certain ideals of operators

We begin by recalling the definition of a Hermitian operator on a complex Banach space. There are several equivalent definitions of Hermitian operators, cf. [5, p. 109], but the following is most useful form for this study.

Definition 2.1. Let X be a complex Banach space and $T \in B(X)$. The operator T is said to be Hermitian if $e^{i\theta T}$ is an isometry for every $\theta \in \mathbb{R}$.

Definition 2.2. Let X be a complex Banach space and P be a projection. P is a bicircular projection if $e^{i\alpha}P + e^{i\beta}(I - P)$ is an isometry for all real values of α and β.

From the fact that $P - (I - P)$ is an isometry, it follows immediately that every bicircular projection P is bounded. The following lemma clarifies the relationship between Hermitian projections and bicircular projections.

Lemma 2.1. *A projection on a complex Banach space is a bicircular projection if and only if it is a Hermitian projection.*

Proof. Let P be a projection on X. Then

$$e^{i\theta P} = \sum_{n=0}^{\infty} \frac{(i\theta P)^n}{n!} = e^{i\theta}(P + e^{-i\theta}(I - P)),$$

P is bicircular if and only if for any $x \in X$ we have

$$\|e^{i\theta P}x\| = \|e^{i\theta}(P + e^{-i\theta}\overline{P})x\| = \|x\|$$

or equivalently P is a Hermitian projection. $\square$

We now give some results which follow immediately from the proceeding lemma. We first recall some terminology and background on ideals of operators, see [12]. Let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space, and $\mathbf{B}(\mathscr{H})$ be the bounded operators on $\mathscr{H}$. A *symmetric norm ideal* $(\mathscr{J}, \nu)$ consists of a proper two sided ideal $\mathscr{J}$ in $\mathbf{B}(\mathscr{H})$ together with a norm ν on $\mathscr{J}$ which satisfies the conditions

- $\nu(A) = \|A\|$for rank 1 operators and
- $\nu(UAV) = \nu(A)$ for every A $\in \mathscr{J}$ and every pair of unitary operators U and V.

It is also assumed that the set of finite rank operators is dense in $\mathscr{J}$. The canonical example of such an ideal $\mathscr{J}$ is the Schatten class $C_p(\mathscr{H})$. Sourour in [12] proved the following theorem concerning Hermitian operators on such ideals of operators. It extends an earlier result in [6].

Theorem 2.1. *Let $(\mathscr{J}, \nu)$ be a minimal norm ideal in $\mathbf{B}(\mathscr{H})$ other than $\mathscr{C}_2$, and let τ be a linear transformation on $\mathscr{J}$. Then τ is a bounded Hermitian operator if and only if there are bounded self-adjoint operators A and B on $\mathscr{H}$ such that $\tau(T) = AT + TB$, for every $T \in \mathscr{J}$.*

From this result Sourour observes the following.

Corollary 2.1. *Let τ be a bounded operator on $\mathscr{J}$. Then τ and τ^2 are Hermitian if and only if τ is either a left multiplication or a right multiplication by a self-adjoint operator on $\mathscr{H}$.*

The following corollary is immediate and is similar in flavor to Theorem 2.2 of [13].

Corollary 2.2. *Let P be a bicircular projection on $\mathscr{J}$. Then there is orthogonal projection Q on $\mathscr{H}$ such that $P(T) = QT$ or $P(T) = TQ$ for every $T \in \mathscr{J}$.*

3. Bicircular projections on some other Banach spaces

We have observed that every bicircular projection on a complex Banach space must be Hermitian. As a consequence, on $L^p(\mu)$ $1 \leqslant p < \infty$, $p \neq 2$, and more generally reflexive Orlicz spaces $L^{\phi}(\mu)$ the only bicircular projections are multiplications by characteristic functions of measurable sets see [10]. If Ω is a compact Hausdorff space with a closed and open subset set G, then multiplication by χ_G is a Hermitian projection (hence bicircular projection). The space $C[0, 1]$ admits only trivial bicircular projections because the Hermitian operators on $C[0, 1]$ are multiplications by real valued continuous functions.

Some important Banach spaces admit only trivial Hermitian operators (real multiples of the identity). For example, each of the following spaces: Bergman spaces $L^p_a(\Delta)\, p \neq 2$, Hardy spaces $H^p(\Delta)$ $(p \neq 2)$, $AC[0, 1]$, $C^1[0, 1]$, $Lip([0, 1])$ admit only trivial Hermitian operators and consequently possess only trivial bicircular projections, cf. [3,2,8]. There are properties which an abstract Banach space may have that imply existence of only trivial bicircular projections. For example, Doust and Ricker [7] say a Banach space X has property $\mathscr{P}$ if every projection $P \neq I$ with dimension of the range greater than 1 satisfies $\|P\| > 1$. Since $\|P\| \leqslant 1$ for every bicircular projection P, the next result is immediate.

Corollary 3.1. *Banach spaces with property $\mathscr{P}$ admit only* 0, I *as a bicircular projections.*

There are Banach spaces which admit nontrivial bicircular projections. The Dirichlet space $\mathscr{D}^p(\Delta)$ is the Banach space of functions holomorphic in the disk Δ for which $f' \in L^p(\Delta, \mathrm{d}A)$ with $\mathrm{d}A$ denoting normalized area measure. The norm is given by

$$\|f\| = \left(|f(0)|^p + \int_\Delta |f'(z)|^p \,\mathrm{d}A\right)^{1/p}.$$

The space $\mathscr{S}^p(\Delta)$ is the space of functions holomorphic in the disk with derivative belong to the Hardy space $H^p(\Delta)$. The norm is given by

$$\|f\| = (|f(0)|^p + \|f'\|^p)^{1/p}.$$

In [8] the authors prove the following.

Theorem 3.1. *Let A be Hermitian operator on $\mathscr{S}^p(\Delta)$ or $\mathscr{D}^p(\Delta)$. Then there exists real constants a and b such that*

$$Af(z) = af(0) + bf(z)$$

for every f in the space.

Corollary 3.2. *Let P be a bicircular projection* (*different from* 0 *or* I) *on* $\mathscr{S}^p(\Delta)$ *or* $\mathscr{D}^p(\Delta)$. *Then P is of the following form*:

- $Pf(z) = f(0)$ *or*
- $Pf(z) = f(z) - f(0)$.

Banach spaces with certain types of Schauder bases also admit nontrivial bicircular projections. Let E be a Banach space of sequences $x = (x(k))_{k=1}^{\infty}$ so that the standard unit vectors $\{e_n\}_{n=1}^{\infty}$ defined by $e_n(k) = \delta_{n,k}$ is a Schauder basis of E. This sequence is said to be a 1-symmetric basis of E provided that every operator of the form

$$Vx = (\lambda(k)x(\pi(k)))_{k=1}^{\infty}$$

is an isometry of E whenever the scalars $\lambda(k)$ all have modulus one and π is a permutation of the positive integers. Arazy [1] has proven the following

Theorem 3.2. *Let E be a complex sequence space different from l_2. Then a bounded operator T on E is Hermitian if and only $T = M_a$ with $a = (a(k))_{k=1}^{\infty}$ a bounded sequence of real numbers, where*

$$M_a x = ax = (a(k)x(k))_{k=1}^{\infty}.$$

It now follows immediately that

Corollary 3.3. *Every bicircular projection on a complex symmetric sequence space E is a projection onto a subspace spanned by a subset of the canonical basis elements e_n's.*

For vector valued function spaces, many results concerning Hermitian operators are known. It is easy to determine the Hermitian projections in these cases. For example, from Theorem 4.2 in [11] we have the following corollary.

Corollary 3.4. *An operator P is a bicircular projection on the Bochner space $L^p(\Omega, X)$ for $1 \leqslant p < \infty$, $p \neq 2$ if and only if there is strongly measurable map $Q : \Omega \to Pr(X)$, the bounded Hermitian projections on X such that $(Pf)(x) = Q(x)f(x)$ for every $f \in L^p(\Omega, X)$.*

We end this note with an observation related to the characterization of Hilbert spaces using bicircular projections in [14]. An element x of a complex Banach space X is said to be Hermitian [9] if there is an Hermitian ($\equiv$ bicircular) projection P_x whose range is the span of x. Let $h(X)$ denote the set of Hermitian ($\equiv$ bicircular) elements of X. Results in [4] or Corollary 4.4 in [9] implies that X is a Hilbert space if and only if $h(X) = X$.

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