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Linear Algebra and its Applications 429 (2008) 2025–2029
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Generalized bicircular projections via the operator equation $\alpha X^2AY^2 + \beta XAY + A = 0$ ☆

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Received 26 April 2008; accepted 27 May 2008
Available online 10 July 2008
Submitted by C.-K. Li

Abstract

The operator equation $\alpha X^2AY^2 + \beta XAY + A = 0$ is used to determine the structure of a projection P such that for some complex unit $\lambda \neq 1$ the mapping $P + \lambda(I - P)$ is $A \mapsto UAV$, $A \mapsto UA^*V$, or $A \mapsto UA^{\mathrm{t}}V$ for some fixed operators U, V. Some applications of these results are also given.

AMS classification: Primary: 47A65; Secondary: 47A62

Keywords: Generalized bicircular projections; Isometries; Preserving mappings

1. Introduction and preliminaries

Let $\mathscr{X}$ be a complex Banach space and let $P : \mathscr{X} \to \mathscr{X}$ be a linear projection, that is a linear mapping with the property $P^2 = P$. The projection P is called bicircular if the mapping $P + \lambda(I - P)$ is an isometry for *every* complex unit λ. The projection P is called generalized bicircular if the mapping $P + \lambda(I - P)$ is an isometry for *some* complex unit $\lambda \neq 1$. Recently, the question of determining the structure of (generalized) bicircular projections on a given Banach space has drawn attention of a number of mathematicians from different points of view.

☆ The author was supported by the Ministry of Science, Education and Sports of the Republic of Croatia (Project No. 037-0372784-2757).

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2008.05.035

Structure theorems for bicircular projections acting on the spaces of the full operator algebra, symmetric operators and antisymmetric operators, relying on the algebraic structure of these spaces, were given by Stachó and Zalar [12]. Algebraic techniques were also used by M. Fošner and the author in describing bicircular projections on C^*-algebras [3]. Jamison [7] proved that the bicircular projections were precisely the Hermitian projections which enabled him to characterize bicircular projections in numerous Banach spaces without a nice algebraic structure.

Fošner et al. [4] extended the concept of bicircular projections to the so-called generalized bicircular projections and determined the structure of such projections on several finite-dimensional normed spaces. Botelho and Jamison characterized generalized bicircular projections on a minimal norm ideal of operators [1].

This paper has been motivated by the paper [1] based on the use of [2, Theorem 1] (referred as the Fong–Sourour's theorem). However, the Fong–Sourour's theorem seems to be much powerful tool in determining the structure of bicircular projections on some complex Banach spaces. Namely, our Lemma 1.1 is almost a direct consequence of that theorem.

Throughout the paper, $\mathscr{X}$ and $\mathscr{H}$ denote a complex Banach space and a complex Hilbert space, respectively, $B(\mathscr{X})$ and $B(\mathscr{H})$ denote the algebra of bounded linear operators on $\mathscr{X}$ and $\mathscr{H}$, respectively.

Lemma 1.1. *Let nonzero $\alpha, \beta \in \mathbb{C}$ and nonzero $X, Y \in B(\mathscr{X})$ be such that*

$$\alpha X^2AY^2 + \beta XAY + A = 0 \quad (A \in B(\mathscr{X})). \tag{1}$$

Then X is a scalar operator or Y is a scalar operator.

Proof. First assume that Y^2 and Y are linearly independent. Application of the Fong–Sourour's theorem for $m = 3$, $n = 2$, $A_1 = \alpha X^2$, $B_1 = Y^2$, $A_2 = \beta X$, $B_2 = Y$, $A_3 = B_3 = I$ yields that X is a scalar operator.

If there exists a nonzero $\gamma \in \mathbb{C}$ such that $Y^2 = \gamma Y$, then (1) becomes

$$\alpha\gamma X^2AY + \beta XAY + A = 0 \quad (A \in B(\mathscr{X})). \tag{2}$$

Multiplying (2) by Y from the right we get

$$\alpha\gamma^2X^2AY + \beta\gamma XAY + AY = 0 \quad (A \in B(\mathscr{X})). \tag{3}$$

Comparing (2) and (3) yields $\gamma A = AY$ for every $A \in B(\mathscr{X})$. Thus $Y = \gamma I$. $\square$

The above lemma is the key lemma in this paper. As we are going to see, it simplifies the proofs of the results of [1] and enables the extensions of these results as well.

2. Main results

Theorem 2.1. *Let $P : B(\mathscr{X}) \to B(\mathscr{X})$ be a nontrivial linear projection, λ a complex unit not equal to 1, and $U, V \in B(\mathscr{X})$. Then $P(A) + \lambda(A - P(A)) = UAV$ for every $A \in B(\mathscr{X})$ if and only if one of the following holds*:

(i) *$\lambda \neq -1$ and there exists a projection $Q \in B(\mathscr{X})$ such that $P(A) = AQ$ for every $A \in B(\mathscr{X})$, or $P(A) = QA$ for every $A \in B(\mathscr{X})$,*
(ii) *$\lambda = -1$, $U^2 = \frac{1}{\mu}I$, $V^2 = \mu I$ for some nonzero $\mu \in \mathbb{C}$ and $P(A) = \frac{1}{2}(UAV + A)$ for every $A \in B(\mathscr{X})$.*

Proof. If $T = P + \lambda(I - P)$, then $T^2 - (\lambda + 1)T + \lambda I = 0$. Hence,

$$\bar{\lambda}U^2AV^2 - (1 + \bar{\lambda})UAV + A = 0 \quad (A \in B(\mathscr{X})).$$

Assume that $\lambda \neq -1$. According to Lemma 1.1, U is a scalar operator or V is a scalar operator. Hence there exists $R \in B(\mathscr{X})$ such that $T(A) = AR$ for every $A \in B(\mathscr{X})$, or $T(A) = RA$ for every $A \in B(\mathscr{X})$. Since $P = \frac{T-\lambda I}{1-\lambda}$, there exists $Q \in B(\mathscr{X})$ such that $P(A) = AQ$ for every $A \in B(\mathscr{X})$, or $P(A) = QA$ for every $A \in B(\mathscr{X})$. Since P is a projection, $Q^2 = Q$.

If $\lambda = -1$, then $U^2AV^2 - A = 0$ for every $A \in B(\mathscr{X})$, so the Fong–Sourour's theorem yields $V^2 = \mu I$ for some nonzero $\mu \in \mathbb{C}$. Then $U^2 = \frac{1}{\mu}I$. □

The following two results are put in the setting of operators on Hilbert spaces; A^* denotes the adjoint of A (Theorem 2.2) and A^{t} denotes the transpose of A relative to a fixed orthonormal basis of $\mathscr{H}$ (Theorem 2.3).

Theorem 2.2. *Let $P : B(\mathscr{H}) \to B(\mathscr{H})$ be a nontrivial linear projection, λ a complex unit not equal to 1, and $U, V : \mathscr{H} \to \mathscr{H}$ bounded conjugate-linear operators. Then $P(A) + \lambda(A - P(A)) = UA^*V$ for every $A \in B(\mathscr{H})$ if and only if $\lambda = -1$, $V = \mu(U^*)^{-1}$ for some complex unit μ, and $P(A) = \frac{1}{2}(UA^*V + A)$ for every $A \in B(\mathscr{H})$.*

Proof. Again, if $T = P + \lambda(I - P)$, then $T^2 - (\lambda + 1)T + \lambda I = 0$. Since P is nontrivial, T is not a scalar operator. We have

$$UV^*AU^*V - (\lambda + 1)UA^*V + \lambda A = 0 \quad (A \in B(\mathscr{H})). \tag{4}$$

If we assume that $\lambda \neq -1$, this implies

$$UA^*V = \frac{UV^*AU^*V + \lambda A}{\lambda + 1},$$

thus

$$V^*AU^* = \frac{\lambda V^*UA^*VU^* + A^*}{\lambda + 1}. \tag{5}$$

Inserting (5) in the first summand of (4) we arrive at

$$A = \frac{1}{\lambda(\lambda + 1)}((\lambda^2 + \lambda + 1)UA^*V - UV^*UA^*VU^*V),$$

so

$$A^* = \frac{1}{\lambda + 1}((\lambda^2 + \lambda + 1)V^*AU^* - \lambda^2V^*UV^*AU^*VU^*). \tag{6}$$

Now we insert (6) in (4) and finally get

$$\bar{\lambda}^2UV^*UV^*AU^*VU^*V - (\bar{\lambda}^2 + 1)UV^*AU^*V + A = 0 \quad (A \in B(\mathscr{H})). \tag{7}$$

If $\lambda \notin \{-i, i\}$, since $UV^*, U^*V \in B(\mathscr{H})$, we can apply Lemma 1.1 to conclude that UV^* is a scalar operator or U^*V is a scalar operator. If UV^* is a scalar operator, then VU^* is also a scalar operator; if U^*V is a scalar operator, then V^*U is also a scalar operator. Using this in (7), we conclude that both UV^* and U^*V are scalar operators. Then T^2 is a scalar operator. Since $T = \frac{T^2+\lambda I}{1+\lambda}$, then T is also a scalar operator; a contradiction.

If $\lambda \in \{-i, i\}$, then $UV^*UV^*AU^*VU^*V = A$ for every $A \in B(\mathscr{H})$. The Fong–Sourour's theorem yields that U^*VU^*V is a scalar operator. Inserting UV^*A instead of A in (4), we get

$$UV^*UV^*AU^*V - (\lambda + 1)UA^*VU^*V + \lambda UV^*A = 0. \tag{8}$$

Multiplying (4) by UV^* from the left, we get

$$UV^*UV^*AU^*V - (\lambda + 1)UV^*UA^*V + \lambda UV^*A = 0. \tag{9}$$

Comparing (8) and (9), we obtain $UA^*VU^*V = UV^*UA^*V$, that is $V^*UV^*AU^* = V^*AU^*VU^*$. Multiplying this by U from the left and VU^*V from the right, we conclude that T^2 is a scalar operator. Then T is a scalar operator as well, contradicting the assumption that P is nontrivial.

If $\lambda = -1$, then $UV^*AU^*V - A = 0$ for every $A \in B(\mathscr{H})$. From the Fong–Sourour's theorem it follows that $U^*V = \mu I$ for some nonzero $\mu \in \mathbb{C}$. Since $UV^*AU^*V = A$, we have $\mu UV^* = I$, so $\bar{\mu}VU^* = I$. Hence U^* is invertible and $V = \mu(U^*)^{-1}$ with $|\mu| = 1$. □

Theorem 2.3. *Let $P : B(\mathscr{H}) \to B(\mathscr{H})$ be a nontrivial linear projection, λ a complex unit not equal to 1, and $U, V \in B(\mathscr{H})$. Then $P(A) + \lambda(A - P(A)) = UA^{\mathrm{t}}V$ for every $A \in B(\mathscr{H})$ if and only if $\lambda = -1$, $V = \pm(U^{\mathrm{t}})^{-1}$, and $P(A) = \frac{1}{2}(UA^{\mathrm{t}}V + A)$ for every $A \in B(\mathscr{H})$.*

Proof. Let $T = P + \lambda(I - P)$. We have

$$UV^{\mathrm{t}}AU^{\mathrm{t}}V - (\lambda + 1)UA^{\mathrm{t}}V + \lambda A = 0 \quad (A \in B(\mathscr{H})). \tag{10}$$

We follow the proof of Theorem 2.2 and for $\lambda \neq -1$ we arrive at

$$\bar{\lambda}^2 UV^{\mathrm{t}}UV^{\mathrm{t}}AU^{\mathrm{t}}VU^{\mathrm{t}}V - (\bar{\lambda}^2 + 1)UV^{\mathrm{t}}AU^{\mathrm{t}}V + A = 0 \quad (A \in B(\mathscr{H})).$$

If $\lambda \notin \{-i, i\}$, then Lemma 1.1 implies UV^{t} is a scalar operator or $U^{\mathrm{t}}V$ is a scalar operator. Hence both UV^{t} and $U^{\mathrm{t}}V$ are scalar operators. Then T^2 is a scalar operator, so T is also a scalar operator; a contradiction.

If $\lambda \in \{-i, i\}$, then $UV^{\mathrm{t}}UV^{\mathrm{t}}AU^{\mathrm{t}}VU^{\mathrm{t}}V = A$ for every $A \in B(\mathscr{H})$, so the Fong–Sourour's theorem implies that $U^{\mathrm{t}}VU^{\mathrm{t}}V$ is a scalar operator. As in the proof of Theorem 2.2, comparing the identity obtained from (10) inserting $UV^{\mathrm{t}}A$ instead of A, and the identity obtained multiplying (10) by UV^{t} from the left, we first conclude that T^2 is a scalar operator and then that T is a scalar operator; a contradiction.

If $\lambda = -1$, then $UV^{\mathrm{t}}AU^{\mathrm{t}}V - A = 0$ for every $A \in B(\mathscr{H})$. According to the Fong–Sourour's theorem, $U^{\mathrm{t}}V = \mu I$ for some nonzero $\mu \in \mathbb{C}$. Since $UV^{\mathrm{t}}AU^{\mathrm{t}}V = A$, it follows that $\mu UV^{\mathrm{t}} = I$, thus $\mu VU^{\mathrm{t}} = I$. Hence U^{t} is invertible and $V = \mu(U^{\mathrm{t}})^{-1}$ with $\mu^2 = 1$. □

3. Applications

As stated in Section 1, a generalized bicircular projection is a linear projection P with the property $P + \lambda(I - P)$ is an isometry for some complex unit $\lambda \neq 1$. However, the mappings $A \mapsto UAV$, $A \mapsto UA^*V$, $A \mapsto UA^{\mathrm{t}}V$ appear not only in the setting of the isometries, but also in the setting of some other preserving mappings. Since there are no particular assumptions on the mappings U, V in Theorems 2.1, 2.2 and 2.3, these results can be also applied to a projection P such that the mapping $P + \lambda(I - P)$ for some complex unit $\lambda \neq 1$ has some other preserving property instead of preserving the norm. Thus we can extend the concept of a generalized bicircular projection by considering not only isometries but some other suitable preserving properties as well. In the sequel we give some of many examples.

Let $P : \mathscr{A} \to \mathscr{A}$ be a nontrivial linear projection, λ a complex unit not equal to 1 and $T = P + \lambda(I - P)$ (note that T is a bijection). Theorems 2.1, 2.2 and 2.3 enable us to determine the structure of P in e.g. the following cases:

- $\mathscr{A} = B(\mathscr{H})$, T is an isometry [8];
- $\mathscr{A}$ is a minimal norm ideal in $B(\mathscr{H})$ different from the Hilbert–Schmidt class, T is an isometry [11, Theorem 2];
- $\mathscr{A} = B(l^p)$ with $p \in (1, \infty)$, $p \neq 2$, T is an isometry [5, Theorem 2];
- $\mathscr{A} = B(\mathscr{H})$, k is a positive integer, T is a rank-k nonincreasing map which is weakly continuous on norm bounded sets [6, Theorem 1];
- $\mathscr{A} = B(\mathscr{H})$, k is a positive integer, T is a rank-k preserving map which is weakly continuous on norm bounded sets [6, Theorem 2];
- $\mathscr{A} = B(\mathscr{H})$, $\mathscr{H}$ is infinite-dimensional, k is a positive integer, T is weakly continuous on norm bounded sets and preserving corank-k operators in both directions [6, Theorem 3];
- $\mathscr{A}$ is an ideal of $B(\mathscr{H})$, $\dim \mathscr{H} > 1$, T preserves orthogonality in both directions [6, Theorem 4];
- $\mathscr{A} = B(\mathscr{H})$, $\mathscr{H}$ is infinite-dimensional, T preserves the extreme points of the unit ball of $B(\mathscr{H})$ [9, Corollary 1];
- $\mathscr{A} = M_n(\mathbb{C})$ (the algebra of all complex $n \times n$ matrices), T preserves the extreme points of the unit ball of $M_n(\mathbb{C})$ [9, Corollary 3];
- $\mathscr{A} \subseteq B(\mathscr{H})$ is a standard operator algebra containing an ideal of $B(\mathscr{H})$ different from $F(\mathscr{H})$ and $\mathscr{A}$ is linearly generated by the set of all of its positive elements, T is an order isomorphism (that is, T preserves positive operators in both directions) [10, Theorem 1];
- $\mathscr{A} \subseteq B(\mathscr{H})$ is a standard operator algebra, T is a triple isomorphism [10, Theorem 3]; etc.

Acknowledgement

The author wishes to express her thanks to the referee for pointing out an oversight in Theorem 2.2.

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