Integr. equ. oper. theory 58 (2007), 315–339

0378-620X/030315-25, *published online* April 14, 2007
DOI 10.1007/s00020-007-1489-x

Integral Equations and Operator Theory

Collinear Systems and Normal Contractive Projections on JBW*-Triples

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Abstract. Given a family $\{x_k\}_{k\in K}$ of elements x_k in the predual A_* of a JBW*-triple A, such that the support tripotents e_k of x_k form a collinear system in the sense of [31], necessary and sufficient criteria for the existence of a contractive projection from A_* onto the subspace $\overline{\text{lin}\{x_k : k \in K\}}^n$ are provided. Preparatory to these results, and interesting in itself, is a set of necessary and sufficient algebraic conditions upon a contractive projection P on A for its range PA to be a subtriple. The results also provide criteria for the range of a normal contractive projection on A to be a Hilbert space.

Mathematics Subject Classification (2000). Primary 17C65; Secondary 47D27.

Keywords. JBW*-triple, collinearity, contractive projection.

1. Introduction

In recent years, considerable effort has been devoted to the investigations of contractive projections on an operator algebra and its related Banach spaces. The question of whether or not there exists a contractive projection onto a given subspace of a Banach space is of fundamental significance. In the presence of algebraic structure on the Banach space in consideration, it is natural to tie this question to algebraic conditions upon the given subspace. Among the extensive literature on these topics we refer the reader to [4] [6] [7] [8] [22] [30] [33] [38].

The aim of this article is to investigate normal contractive projections in connection with certain algebraic conditions on generalized operator algebras. It has been observed earlier that the structures known as JB*-triples and their weak*-closed analogues, JBW*-triples, provide a natural setting for studying contractive projections or normal contractive projections. As shown by Kaup in [27] and independently by Stachò in [34], the range PA of a contractive projection P on a

Supported by the Irish Research Council for Science, Engineering and Technology, Grant No. R 9854.

JB*-triple A is itself a JB*-triple in a canonical way. The algebraic structure on A is given by a ternary product $\{\ldots\} : A \times A \times A \to A$, that on PA by $P\{\ldots\}$. This generalizes earlier results by Choi and Effros [5], Effros and Størmer [18], Friedman and Russo [21] and others. For further reading on the subject and its connections to holomorphy see for example [23] [26] [28] [31] [35] [36].

In joint work with Edwards and Rüttimann [11] [12] contractive projections have been studied by the author for the case in which the global vector space is the Banach predual A_* of a JBW*-triple A. By standard Banach space theory, the adjoints of the contractive projections on A_* are precisely the normal contractive projections on A. A main result in [12] states that, for any family $\{P_k\}_{k\in K}$ of contractive projections on A_*, such that, for $k \neq j$, all elements x in $P_j(A_*)$ and y in $P_k(A_*)$ are *L-orthogonal* in that $\|x \pm y\| = \|x\| + \|y\|$, there exists a contractive projection onto the norm closed subspace $\overline{\bigcup_{k\in K} P_k(A_*)}^n$ of A_*. In particular, there is a contractive projection from A_* onto the norm-closed subspace spanned by any family $\{x_k\}_{k\in K}$ of pairwise L-orthogonal elements in A_*. Moreover, such a projection is explicitly given in terms of the support tripotents e_k in A of the elements x_k. By [17] and [19], the L-orthogonality of the elements x_j and x_k in A_* is equivalent to the algebraic orthogonality of their respective support tripotents e_j and e_k in A, or in the terminology of [31], to $\{e_k\}_{k\in K}$ being an *orthogonal system*. For details on these and related results, see also [25]. In this article we investigate the similar situation in which the support tripotents e_k of the elements x_k form a *collinear system* in the sense of [31]. Under this global assumption, we obtain criteria upon $\{e_k\}_{k\in K}$ and $\{x_k\}_{k\in K}$ which are equivalent to the existence of a contractive projection on A_* with range $G = \overline{\mathrm{lin}\{x_k : k \in K\}}^n$. One set of these conditions involves only the triple product on A and its duality with A_*. Moreover, a necessary condition is that G is a Hilbert space with dual $H = \overline{\{e_k : k \in K\}}^{w^*}$ and H is a subtriple of A. Further conditions are formulated in terms of GL-projections on A_*, which were introduced in [11]. Neal and Russo classified the atomic contractively complemented subspaces of $\mathcal{B}(H)$ up to complete isometry using the theory of operator spaces [30]. The subtriple H given above is an example of an atomic JBW*-triple. Our considerations proceed within the theory of of JBW*-triples and their preduals, and hence provide detailed information related to the general triple structure of these spaces.

The article is organized as follows. In Section 2 some well known facts about JB*-triples and JBW*-triples are reviewed. Section 3 is devoted to the triple structure of a Hilbert space H. It is shown that a certain set of isometries, obtained from the triple product on H, acts transitively on the unit-sphere of H. In Theorem 3.3, pivotal in proving the main theorems, that observation is generalized to JB*-triples which are generated by a collinear system of tripotents. Of similar significance is Theorem 4.1 in Section 4. It provides algebraic conditions on a contractive projection P on a JB*-triple A which are necessary and sufficient for PA to be a subtriple. This result draws heavily on the conditional expectation formulas in [20], and [27]. In Section 5, the principal question as outlined above is answered in Theorem 5.3, Theorem 5.2 and Theorem 5.4. Further connections between contractive projections and Hilbert spaces are explored in Theorem 5.7 and Theorem 5.8. When dealing with collinearity, the technical difficulties are considerably greater than those arising from orthogonality. In particular, the existence of the sought-after projections is automatically ensured by orthogonality, but not by collinearity of $\{e_k\}_{k\in K}$. Therefore, our main theorems provide non-trivial information, even in the case of finite dimensional spaces, as can be seen from Example 5. The relation of collinearity was considered for the purpose of a classification of Jordan-triples by Dang and Friedman [9], Horn [24] McCrimmon and Meyberg [29], and by Neher [31]. Moreover, Edwards and Rüttimann [16] and Wright [37] used collinearity to represent the relation of quantum-decoherence of states, and JBW*-triples are used to describe non-classical statistical systems.

2. Preliminaries

A JB*-triple is a complex Banach space A endowed with a triple product $\{.\,,\,.\,,\,.\} : A \times A \times A \to A$, which has the following properties, the axioms of the theory:

(A1) The expression $\{a,\ b,\ c\}$ is symmetric and linear in the variables a and c and is conjugate linear in b.

(A2) For all elements a and b of A, the linear maping $c \mapsto \{a,\ b,\ c\}$ on A denoted by $D(a,b)$, satisfies the *Jordan triple identity*,

$$D(a,b)\{c,\ d,\ e\} = \{D(a,b)c,\ d,\ e\} + \{c,\ b, D(a,b)e\} - \{c,\ D(b,a)d,\ e\},$$

(A3) For all elements a of A, the linear operator $D(a,a)$ on A has nonnegative spectrum (cf. [2]), and norm equal to $\|a\|^2$.

(A4) For all real numbers t, the linear operator $\exp(itD(a,a))$ is an isometry of A.

By [26] Proposition 5.5 and [24] Proposition 2.4, two JB*-triples are isometrically isomorphic as Banach spaces if and only if they are triple isomorphic. Hence the group of all triple automorphisms, denoted $\mathrm{Aut}(A)$, coincides with the group of all bijective linear isometries of the JB*-triple A. If A is also the dual of a Banach space A_*, then A is said to be a JBW *-triple. To enhance the clarity of notation in later calculations, we will henceforth write $(x \cdot a)$ for the dual pairing of the elements x in A_* and a in A.

A subspace B of A is a *subtriple* if $\{B,\ B,\ B\}$ is a subset of B, which is the case if and only if for all $b \in B$ the element $\{b,\ b,\ b\}$ lies in B. For an element b of A, the conjugate linear operator $Q(b) : A \to A$ is defined by $Q(b)a = \{b,\ a,\ b\}$. The operators $D(a,b)$ and $Q(b)$ are norm-continuous on the JB*-triples and weak*-continuous on JBW*-triples. On the latter, the triple product is separately weak*-continuous [1] [28].

An element u of A is said to be a *tripotent* if $\{u,\ u,\ u\} = u$. The set of tripotents of A is denoted by $\mathcal{U}(A)$. Let j, k and l be equal to 0, 1 or 2. For each tripotent u of A, the norm- and weak*-continuous projections

$$\begin{aligned} P_2(u) &= Q(u)^2, \\ P_1(u) &= 2(D(u,u) - Q(u)^2), \\ P_0(u) &= \mathrm{id_A} - 2D(u,u) + Q(u)^2 \end{aligned}$$

are referred to as the *Peirce projections* corresponding to u. It can be seen that $P_0(u) + P_1(u) + P_2(u)$ equals the identity id_A on A and that if $j \neq k$, then $P_j(u)P_k(u)$ equals zero. The ranges, $A_k(u)$ of $P_k(u)$ are weak*-closed subtriples of A, referred to as the *Peirce spaces* of u. Moreover, for all elements a of A,

$$a \in A_k(u) \qquad \text{if and only if} \qquad D(u,u)a = \frac{k}{2}a. \tag{2.1}$$

Extensive use will be made of the *Peirce rules*,

$$\{A_j(u),\ A_k(u),\ A_l(u)\} \subseteq \begin{cases} A_{j-k+l}(u) & \text{if } j-k+l \in \{0,1,2\} \\ \{0\} & \text{else} \end{cases} \tag{2.2}$$

$$\{A_2(u),\ A_0(u),\ A\} = \{A_0(u),\ A_2(u),\ A\} = \{0\}. \tag{2.3}$$

A pair u, v of tripotents in A is said to be *orthogonal*, denoted $u \perp v$, if $u \in A_0(v)$ and $v \in A_0(u)$. The relation $\perp$ is symmetric, and $u \perp v$ is equivalent to $D(u,u)v = 0$, to $D(v,v)u = 0$ and to $D(u,v)$ being identically zero on A. Moreover v is said to be *less than or equal to* u, denoted $v \leq u$, if $(u - v) \in \mathcal{U}(A)$ and $(u - v) \perp v$. The relation $\leq$ is a partial order on the set $\mathcal{U}(A)$ [28]. If A is a JBW*-triple, then, for each element x in the predual A_* of A, there exists the *support tripotent* e_x of x which is the smallest of all tripotents u in A with the property that $(u \cdot x) = \|x\|$. Also, x lies in $A_2(e_x)_*$ [19].

For a non-empty subset G of A_*, the *support space* $s(G)$ of G is defined to be the weak*-closed subspace $\overline{\mathrm{lin}\{e_x \ : \ x \in G\}}^{w^*}$ of A [11].

A pair u, v of tripotents is said to be *collinear*, denoted $u \top v$, if $u \in A_1(v)$ and $v \in A_1(u)$. By (2.1) the relation $u \top v$ holds if and only if

$$D(u,u)v = \frac{1}{2}v \quad \text{and} \quad D(v,v)u = \frac{1}{2}u. \tag{2.4}$$

A family $\{u_k\}_{k \in K}$ of tripotents is said to be a *collinear system* if the relation $u_k \top u_l$ holds whenever $k \neq l$. Observe that a collinear system not equal to $\{0\}$ is linearly independent.

When G and H are Hilbert spaces, the set $\mathcal{B}(G,H)$ of bounded linear operators from G to H is a JBW*-triple with triple product, defined for elements a, b, c in $\mathcal{B}(G,H)$ by

$$\{a,\ b,\ c\} = \frac{1}{2}(ab^*c + cb^*a). \tag{2.5}$$

A JBW*-triple isomorphic to $\mathcal{B}(G,H)$ is said to be *rectangular*. When G and H have finite dimension, i.e. A is represented as a matrix algebra, examples of collinear tripotents are provided by distinct matrix units $u_{i,j}$ and $u_{k,l}$, i.e. matrices with entry 1 at position (i,j) and (k,l), respectively, and zero elsewhere. Then $u_{i,j} \top u_{k,l}$ if either $i = k$ or $j = l$. Further examples of JBW*-triples are provided by W*-algebras and JBW*-algebras, the *spin factors*, obtained from Hilbert spaces equipped with a conjugation, or by the *bi-Cayley numbers* $C^8_{1,2}$. The latter are the 1 by 2 matrices with entries in the Cayley numbers C^8 over the field of complex numbers. For more details on these structures, see for example [9] [24] [29] [31].

We conclude this section with a result which is easily obtained from the aforesaid.

Proposition 2.1. *Let u be a tripotent element in the JBW*-triple A. For $k = 0, 1, 2$, denote by P_k the pre-adjoint $P_k(u)_*$ of the Peirce projection $P_k(u)$. Then the range $P_k A_*$ of P_k is the eigenspace of the pre-adjoint $D(u,u)_*$ of $D(u,u)$, with corresponding eigenvalue $k/2$.*

Proof. Using the Peirce-rules it can be seen that $D(u,u)$ commutes with $P_k(u) = P_k^*$. Let x be an element of $P_k A_*$. By (2.1), for all a in A,

$$\begin{aligned}(D(u,u)_* x \cdot a) &= (P_{k*} x \cdot D(u,u)a) \;=\; (x \cdot P_k^* D(u,u)a) \;=\; (x \cdot D(u,u) P_k^* a) \\ &= \frac{k}{2}(x \cdot a).\end{aligned}$$

Therefore, x lies in the eigenspace of $D(u,u)_*$ with eigenvalue $k/2$.

Conversely, suppose that $D(u,u)_* x$ equals $(k/2)x$. Notice that $D(u,u)_* = P_2 + \frac{1}{2} P_1$ and that $P_2 + P_1 + P_0$ is the identity on A_*. Hence,

$$P_2 x + \frac{1}{2} P_1 x \;=\; \frac{k}{2}(P_2 x + P_1 x + P_0 x).$$

Since $P_i P_j = 0$, setting k equal to 0, 1 or 2 implies, in each case, that $x = P_k x$, as required. $\square$

3. Hilbert spaces

Since the triple structure of Hilbert spaces plays a significant part in subsequent considerations, it is necessary to establish some preliminary results on Hilbert spaces. Let H be a complex Hilbert space with scalar product $\langle . \, , \, . \rangle \,:\, H \times H \to H$ which is linear in the first and conjugate linear in the second variable. For elements a, b and c in H, let the triple product $\{a,\ b,\ c\}$ be defined by

$$\{a,\ b,\ c\} \;=\; \frac{1}{2}\left(\langle a,\ b\rangle c \;+\; \langle c,\ b\rangle a\right). \tag{3.1}$$

In this way, the Hilbert space H is a JBW*-triple. We remark that another triple product can be defined on H to obtain what is known as a spin triple. This, however, requires the presence of additional structure, such as a conjugation, as well as a new norm on H. We will only be working with the triple product described above. Notice that (3.1) concurs with (2.5) for the case in which G equals $\mathbb{C}$, i.e. H is identyfied with the rectangular triple $\mathcal{B}(\mathbb{C}, H)$.

Denote the unit sphere of H by $S_1(H)$. It is straightforward from Equation (3.1) that $S_1(H) \cup \{0\}$ coincides with the set $\mathcal{U}(H)$ of tripotents of H, and that the relation of collinearity on $\mathcal{U}(H)$ is the usual Hilbert-orthogonality restricted to $\mathcal{U}(A)$. For an non-empty set S, let $l^2(S)$ be the Hilbert space of all functions $f : S \to \mathbb{C}$, such that $\sum_{s\in S} |f(s)|^2 < \infty$, equipped with the inner product $\langle f, g\rangle = \sum_{s\in S} f(s)\overline{g}(s)$, for $f, g \in l^2(S)$.

The following lemma provides an algebraic characterization of Hilbert spaces among the JB*-triples and JBW*-triples. Similar results may be obtained from classification theory, as carried out e.g. in [31] and [9].

Lemma 3.1. *Let C be a collinear system in the JB*-triple A. Denote by H the closed subspace $\overline{\mathrm{lin} C}^n$ spanned by C. Then the following conditions are equivalent:*

(1.) *The subspace H is a subtriple of A.*

(2.) *Either $|C| \le 2$, or, for any three distinct elements u, v and w of C, $\{u,\ v,\ w\} = 0$.*

(3.) *The subspace H is a Hilbert space with orthonormal basis C, i.e. $H \cong l^2(C)$, and the restriction of the triple product of A to H coincides with the triple product given by* (3.1) *on H.*

If these hold, and if in addition A is a JBW-triple, then H is also weak*-closed, hence a JBW*-subtriple of A.*

Proof. (1.) $\Rightarrow$ (2.): Suppose that C has at least three elements, and choose u, v and w in C to be distinct. The assumption of collinearity of C implies that, $H \subseteq \mathbb{C}v \oplus A_1(v) \subseteq A_2(v) \oplus A_1(v)$. Since H is a subtriple, $\{u,\ v,\ w\} \in A_2(v) \oplus A_1(v)$. The Peirce rules (2.2), (2.3) imply that $\{u,\ v,\ w\} \in A_0(v)$. It follows that $\{u,\ v,\ w\} = \{0\}$.

(2.) $\Rightarrow$ (1.): It is easily seen that the arguments can be simplified when $|C| \le 2$. Therefore, assume again that $|C| \ge 3$. An element a in H can be written as the norm convergent sum $a = \sum_{k\in\mathbb{N}} \alpha_k u_k$, for $u_k \in C$. The assumptions and (2.2),(2.3) imply that, for distinct indices j, k and l in $\mathbb{N}$,

$$\{u_j,\ u_k,\ u_l\} \ = \ \{u_k,\ u_j,\ u_k\} \ = \ 0, \tag{3.2}$$
$$2\,\{u_j,\ u_l,\ u_l\} \ = \ 2\,\{u_l,\ u_l,\ u_j\} \ = \ \{u_j,\ u_j,\ u_j\} \ = \ u_j. \tag{3.3}$$

Since $\{a,\ a,\ a\}$ is a (possibly infinite) linear combination of these expressions, and is well defined in A, it follows that $\{a,\ a,\ a\}$ lies in H. Hence H is a subtriple of A, which proves (1.).

(2.) $\Rightarrow$ (3.): The equalities (3.2) and (3.3) determine the triple product on H completely. In particular, for any finite subset $F \subseteq C$, $\mathrm{lin}F$ is a JB*-triple, and as such is isomorphic to $l^2(F)$. Isomorphic JB*-triples are also isometrically isomorphic as Banach spaces. Hence, for any finite subset F of C, lin$F$ is isometric to the Hilbert space $l^2(F)$. This entails the norm of an arbitrary $a \in H$ to be

$$\|a\| = \|\sum_{k\in\mathbb{N}} \alpha_k u_k\| \ = \ (\sum_{k\in\mathbb{N}} |\alpha_k|^2)^{\frac{1}{2}}.$$

It follows that $H \cong l^2(C)$, proving (3.).

(3.) $\Rightarrow$ (2.): This is obvious from (3.1).

Suppose that A is a JBW*-triple and that (1.), (2.) and (3.) hold. Then, for all elements a and b in H,

$$Q(a)(b) \ = \ \{a,\ b,\ a\} \ = \ \langle a, b\rangle a \ \in \ \mathbb{C}a. \tag{3.4}$$

By separate weak*-continuity of the triple product, (3.4) is preserved when a and b are chosen in the weak*-closure $\overline{H}^{w*}$ of H. When a and b are linearly independent, (3.1) and (3.4) also imply that $\mathbb{C}a \oplus \mathbb{C}b$ is a subtriple of $\overline{H}^{w*}$. Up to triple-isomorphism, there are only two different JBW*-triples of dimension 2, namely $\mathbb{C}^2$ with the componentwise triple product, and $\mathbb{C}^2$ as a Hilbert space. The former contains elements for which (3.4) does not hold. Therefore $\mathbb{C}a \oplus \mathbb{C}b$ is (isometrically) equal to the Hilbert space $\mathbb{C}^2$. It follows that every two dimensional subspace of $\overline{H}^{w*}$ is a Hilbert space. Clearly $\overline{H}^{w*}$ is complete, hence is itself a Hilbert space. To see that $H = \overline{H}^{w*}$, extend C to an orthonormal basis $C \cup C'$ of $\overline{H}^{w*}$. For every $u \in C \cup C'$, the linear functional x_u, defined for $a \in \overline{H}^{w*}$ by $(x_u \cdot a) = \langle a, u\rangle$ is weak*-continuous on $\overline{H}^{w*}$. Therefore, x_u has a weak*-continuous extension $\tilde{x}_u$ on A. We remark that by [3], $\tilde{x}_u$ can even be chosen to have the same norm as x_u. However, if u is chosen in C', then $\tilde{x}_u$ annihilates H, hence also $\overline{H}^{w*}$. This contradicts the obvious equality $(\tilde{x}_u \cdot u) = 1$. Consequently, C' is empty and H coincides with $\overline{H}^{w*}$. □

The *inner derivations* of a JB*-triple A, that is the mappings of the form $D(a, b) - D(b, a)$ and their exponentials, the *inner automorphisms*, have been investigated in numerous works. See e.g. [28], [35]. In the following lemma, a symmetry property of the unit sphere $S_1(H)$ of the Hilbert space H with respect to inner automorphisms of the form $\exp itD(a, a)$ is established. The result may be deduced from the general theory of inner automorphisms. Since we need this result only for the special case of Hilbert spaces, we state it separately and give an independent, more elementary proof.

Lemma 3.2. *Let H be a Hilbert space, equipped with the triple product, given by (3.1). Let $S_1(H)$ and $S_1(\mathbb{C})$ be the unit spheres of H and of the complex plane $\mathbb{C}$ respectively. Then, the set*

$$\mathcal{E} \ = \ \{\sigma \exp itD(a, a) \ : \ \sigma \in S_1(\mathbb{C}),\ a \in H,\ t \in \mathbb{R}\}$$

consists of linear isometries of H and acts transitively on $S_1(H)$.

Proof. Suppose first that H is two-dimensional. Let $\mathcal{B}_s = \{b_1, b_2\}$ be the standard basis of H, coordinatized in the usual way. For any complex number λ of modulus one, let $a = a(\lambda)$ be the element of H, defined by

$$a \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

Denote the operator $D(a, a)$ by D_λ, to indicate its dependence of λ. An elementary caclulation shows that, with respect to $\mathcal{B}_s$, the operator $\exp itD_\lambda$, is given by

$$\exp itD_\lambda \quad = \quad \frac{1}{2} \begin{pmatrix} e^{it} + e^{it/2} & \bar{\lambda}(e^{it} - e^{it/2}) \\ \lambda(e^{it} - e^{it/2}) & e^{it} + e^{it/2} \end{pmatrix}.$$

In particular, the unit vector b_1 is mapped to

$$(\exp itD_\lambda)(b_1) \;=\; (\exp itD_\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; \frac{1}{2} \begin{pmatrix} e^{it} + e^{it/2} \\ \lambda(e^{it} - e^{it/2}) \end{pmatrix}. \tag{3.5}$$

This vector is of norm one, for all reals t. The modulus of its first component is $|e^{it} + e^{\frac{i}{2}t}|/2$ and attains all values between 0 and 1 when t runs through $\mathbb{R}$. If c is any vector with components γ_1 and γ_2, and if c has norm one in H, then $|\gamma_2|$ equals $\sqrt{1 - |\gamma_1|^2}$ and t can be chosen such that

$$\frac{1}{2}|e^{it} + e^{it/2}| = |\gamma_1|, \qquad \text{and} \qquad \frac{1}{2}|e^{it} - e^{it/2}| = |\gamma_2|. \tag{3.6}$$

If $\gamma_1 = 0$, then $\sigma(\exp\ 2\pi i D_\lambda)b_1$ equals c, for some $\sigma \in S_1(\mathbb{C})$. Otherwise set $\lambda = \gamma_2(e^{it} + e^{it/2})/\gamma_1(e^{it} - e^{it/2})$. The equations (3.6) and (3.5) imply that

$$|\lambda| \;=\; 1 \;=\; \left| \frac{2\gamma_1}{e^{it} + e^{it/2}} \right|, \quad \text{and} \quad \frac{2\gamma_1}{e^{it} + e^{it/2}} \,(\exp itD_\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; c.$$

This shows that, $\exp itD_\lambda$ acts transitively on $S_1(H)$, up to multiplication by a complex number of modulus one.

The operators $D(a, a)$ and $\exp itD(a, a)$ are defined on any Hilbert space, in fact, on any JB*-triple containing the element a. In the case when H is of arbitrary dimension, and b and c are any two elements of H such that $\|b\| = \|c\|$, the above arguments can applied to the subspace $\mathrm{lin}\{b, c\}$ which is also a subtriple of H. Hence, there exist elements t of $\mathbb{R}$, σ of $S_1(\mathbb{C})$, and a of $\mathrm{lin}\{b, c\}$, such that $\sigma(\exp\ itD(a, a))b = c$, and $\sigma\exp\ itD(a, a)$ is an isometry of the whole space H. □

The two foregoing lemmas can be combined as follows.

Theorem 3.3. *Let C be a collinear system in a JB*-triple A, such that the subspace $H = \overline{\mathrm{lin} C}^n$ is a subtriple of A. Then,*

(1.) *The subtriple H is a Hilbert space with orthonormal basis C, and the triple product on H given by* (3.1) *conicides with the restriction of the triple product of A to H.*

(2.) *The set* $\mathcal{E} = \{\sigma \exp itD(a,a) : \sigma \in S_1(\mathbb{C}), a \in H\}$ *consists of linear isometric triple isomorphisms on the whole space* A *and acts transitively on the unit sphere* $S_1(H)$ *of* H.

(3.) *Either* $|C| \le 2$, *or, for any three distinct elements* u, v, w *of* C, *the product* $\{u, v, w\}$ *vanishes.*

If A *is a JBW***-triple, then* H *is also weak***-closed in* A*, and therefore, is a JBW***-subtriple of* A.

Proof. For any element a of H, the linear mapping $\exp(itD(a,a))$ is defined on the whole space A, and, by (A4), is an isometry on A. The remaining assertions follow from Lemma 3.1 and Lemma 3.2. □

4. Contractive projections

By a *projection* we always mean a linear mapping P on a vector space E which is such that $P^2 = P$. When E is endowed with a norm $\|.\|$, and if P is such that, for all elements x of E, $\|Px\| \le \|x\|$, then P is said to be *contractive.* In this case, the range P^*E^* of the adjoint projection P^* is isometrically isomorphic to the dual $(PE)^*$ of the range PE of P. Observe also that, if P and S are continuous projections on E, such that $PE = SE$ and $P^*E^* = S^*E^*$, then P and S coincide. The set of all bounded linear mappings on E is denoted by $\mathcal{B}(E)$. For an arbitrary set K, let K^f denote the set of all finite subsets of K, partially ordered by set-inclusion. Given a family $\{P_k\}_{k\in K}$ of projections, or of general elements in $\mathcal{B}(E)$, the formal sum $\sum_{k\in K} P_k$ is said to be convergent in the *strong operator topology* or *SOT-convergent* in $\mathcal{B}(E)$ if there exists an element P of $\mathcal{B}(E)$ such that, for each x of E, the net $\{\sum_{k\in F} P_k x : F \in K^f\}$ converges to Px in norm.

A pair of elements x, y in E is said to be *L-orthogonal*, denoted $x \diamond y$, if $\|x \pm y\|$ equals $\|x\| + \|y\|$. The *L-complement* $F^\diamond$ of a non-empty subset F of E is defined to be the set

$$F^\diamond = \{y \in E : x \diamond y \ \forall x \in F\}.$$

A contractive projection P on E is said to be a *GL-projection* if the L-complement $(PE)^\diamond$ of its range PE is a subset of its kernel $\ker P$. The set of all GL-projections on E is denoted by $\mathcal{GL}(E)$.

The results of [27] and [34] show that the range PA of a contractive projection P on a JB*-triple A is itself a JB*-triple when equipped with the *restricted triple product* $\{\ldots\}_{PA}$, defined for elements a, b and c in PA by

$$\{a,\ b,\ c\}_{PA} = P\{a,\ b,\ c\}. \tag{4.1}$$

When A is a JBW*-triple with predual A_*, and if P is the adjoint $P = R^*$ of a contractive projection R on A_*, then PA is a JBW*-triple with product given by

(4.1). Further algebraic relations, referred to as *conditional expectation formulas* hold.

$$P\{Pa,\ Pb,\ Pc\} = P\{Pa,\ b,\ Pc\}, \tag{4.2}$$
$$P\{Pa,\ Pb,\ Pc\} = P\{a,\ Pb,\ Pc\}. \tag{4.3}$$

The equality (4.2) was proved in [27], and (4.3) was proved in [20]. As shown next, the conditional expectation formulas provide some interesting connections between contractive projections and norm-closed subtriples of JB*-triples.

Theorem 4.1. *Let A be a JB*-triple, and let P be a contractive projection on A. Then, the following conditions are equivalent.*

(1.) *The range PA of P is a subtriple of A.*

(2.) *For each element b in PA, the operator $Q(b)$ on A commutes with P.*

(3.) *For each element b in PA, the operator $D(b,b)$ on A commutes with P.*

(4.) *For elements a and b in PA, the operator $D(a,b)$ on A commutes with P.*

Proof. Suppose that PA is a subtriple of A. Consider elements b of PA and a of A. Then, by (4.2),

$$\begin{aligned} Q(b)Pa &= \{b,\ Pa,\ b\} = \{Pb,\ Pa,\ Pb\} \\ &= P\{Pb,\ Pa,\ Pb\} = P\{Pb,\ a,\ Pb\} \\ &= P\{b,\ a,\ b\} \\ &= PQ(b)a, \end{aligned}$$

which proves (2.). Conversely, if b lies in PA and P commutes with $Q(b)$ then

$$P\{b,\ b,\ b\} = PQ(b)b = Q(b)Pb = Q(b)b = \{b,\ b,\ b\}.$$

It follows that, for all elements b of PA, the product $\{b,\ b,\ b\}$ lies in PA which is therefore a subtriple of A.

The equivalence of (1.), (3.) and (4.) can be derived in a similar way using (4.3). □

The following results relate collinear systems in a JB*-triple A with contractive projections on A.

Lemma 4.2. *Let $\{u_1, ..., u_n\}$ be a collinear system in a JB*-triple A, For $k = 1, ..., n$, let P_k be a contractive projection onto the one-dimensional subspace $\mathbb{C}u_k$ of A, and define the linear map P by $P = \sum_{k=1}^n P_k$. Then, for $j \neq k$, the product P_jP_k vanishes, and P is a projection.*

Proof. Clearly, $\sum_{k=1}^n P_k$ is a continuous linear mapping on A. Consider two distinct indices k and j in $\{1, ..., n\}$. By Lemma 3.1, $H = \mathbb{C}u_k \oplus \mathbb{C}u_j$ is a Hilbert space. The restrictions $P_k|_H$ and $P_j|_H$ of P_k and P_j to H are orthogonal orthoprojections on H. Therefore, the products P_kP_j and P_jP_k vanish on A. This implies that $\sum_{k=1}^n P_k$ is a projection. □

Theorem 3.3 and Theorem 4.1 can be combined to obtain an algebraic criterion for contractivity of the continuous projection described in Lemma 4.2. The corollary below serves as a key ingredient in the proof of the main theorems in the next section.

Corollary 4.3. *Under the conditions of Lemma* 4.2*, suppose that* $\mathrm{lin}\{u_1, ..., u_n\}$ *is a subtriple of* A*. Then the projection* P *is contractive if and only if it commutes with* $D(u_j, u_k)$*, for all* $j, k = 1, ..., n$*.*

Proof. If $\mathrm{lin}\{u_1, ..., u_n\}$ is a subtriple and $P = \sum_{k=1}^{n} P_k$ is contractive, then by Theorem 4.1 P commutes with $D(u_j, u_k)$.

Conversely, suppose that P and $D(u_j, u_k)$ commute. By linearity and continuity, P commutes also with $D(b, b)$ and with $\exp itD(b, b)$, for all elements b of H. From Theorem 3.3 it follows that, for all $a \in A$ and a fixed $j \in \{1, ..., n\}$, there exist elements b of H and t of $\mathbb{R}$ such that the isometry $\phi = \exp itD(b, b)$ of A satisfies,

$$\phi P(a) = P\phi(a) \in P_j A = \mathbb{C}u_j.$$

Therefore, $P_j \phi P(a) = \phi P(a)$. Moreover, from Lemma 4.2 it can be seen that $P_j P = P_j$. It follows that for all $a \in A$,

$$\begin{aligned} \|P(a)\| &= \|\phi P(a)\| = \|P_j \phi P(a)\| = \|P_j P \phi(a)\| = \|P_j \phi(a)\| \\ &\leq \|\phi(a)\| = \|a\|. \end{aligned}$$

Hence, P is contractive. □

5. Normal contractive projections

Attention is now turned to the case when A is a JBW*-triple. The set $\mathcal{GL}(A_*)$ of GL-projections on the predual A_* of A will be a valuable tool. Several characterizations of $\mathcal{GL}(A_*)$ can be given in terms of the support space $s(PA_*)$ of the range PA_* of P. Theorem 4.6 in [11] states that, for a contractive projection P on A_*, the following conditions are equivalent

$$P \in \mathcal{GL}(A_*), \tag{5.1}$$
$$s(PA_*) = P^* A, \tag{5.2}$$
$$s(PA_*) \supseteq P^* A, \tag{5.3}$$
$$s(PA_*) \subseteq P^* A. \tag{5.4}$$

In fact, the condition $s(PA_*) \subseteq P^* A$ forces P to be in $\mathcal{GL}(A_*)$, even when P is any projection defined on A_* [12]. Moreover, for any contractive projection Q on A_*, the subspace $s(QA_*)$ is a subtriple of A, and there exists a unique element P in $\mathcal{GL}(A_*)$ such that $PA_* = QA_*$ [11]. The Hahn-Banach theorem implies the existence of a contractive projection onto any one-dimensional subspace of A_*. The corresponding unique GL-projection can be explicitly described as follows [12] [25]. Given an element $x \neq 0$ of A_*, define the mapping $P_x : A_* \to A_*$ by

$$z \mapsto P_x(z) \;=\; \frac{(e_x \cdot z)}{\|x\|} x. \tag{5.5}$$

Then, P_x is an element of $\mathcal{GL}(A_*)$, and it is easy to see that the adjoint P_x^* of P_x on A is given by $P_x^* a = (a \cdot x) e_x$. For $x = 0$, the mapping P_x is defined to be the zero-projection on A_*.

Corollary 5.1. *Let $(x_k)_{k \in K}$ a family of elements of the predual A_* of a JBW*-triple A, such that the corresponding support tripotents e_k form a collinear system. Then, for distinct indices j and k in K,*

$$(e_j \cdot x_k) \;=\; \delta_{j,k} \|x_k\|.$$

Proof. Let P_k be the unique element of $\mathcal{GL}(A_*)$ with range $\mathbb{C}x_k$. Observe that $(e_j \cdot x_j) = \|x_j\|$, by definition of e_j. For $j \neq k$ the desired equality follows either by combining Lemma 4.2 with Equation (5.5), or from the Peirce rules which imply that $(e_j \cdot x_j) = (P_1(e_k) e_j \cdot P_2(e_k)_* x_k) = 0$. □

It is now possible to address the main problem of this article. First, in Theorem 5.2, we focus on the properties and the explicit description of the various projections involved. The properties (2.) and (4.) given therein, are obtained from the connection between GL-projections, support spaces and support tripotents.

Theorem 5.2. *Let A be a JBW*-triple with predual A_*, and let $\{x_k\}_{k \in K}$ be a family of non-zero elements of A_*, such that the corresponding support tripotents form a collinear system $\{e_k\}_{k \in K}$. Let the subspaces H and G of A and A_* be defined by $H = \overline{\mathrm{lin}\{e_k \;:\; k \in K\}}^{w*}$ and $G = \overline{\mathrm{lin}\{x_k \;:\; k \in K\}}^{n}$ respectively. For k in K, let P_k be the GL-projection onto $\mathbb{C}x_k$, described by (5.5). Then, the following conditions are equivalent.*

(1.) *There exists a contractive projection P on A_* with range G.*

(2.) *For any finite subset F of K, the projection $P_F = \sum_{k \in F} P_k$ on A_* is contractive. In this case, P_F is also the unique element of $\mathcal{GL}(A_*)$ with range* $\mathrm{lin}\{x_k \;:\; k \in F\}$.

(3.) *The formal sum $\sum_{k \in K} P_k$ is SOT-convergent in $\mathcal{B}(A_*)$ and is a contractive projection on A_*.*

(4.) *The formal sum $\sum_{k \in K} P_k$ is SOT-convergent in $\mathcal{B}(A_*)$ and is the unique GL-projection on A_* with range G.*

(5.) *When x is a nonzero element of G such that $x = \sum_{k \in K} \alpha_k x_k$, and $\sum_{k \in K} |\alpha_k|^2 = ||x||^2$, then the support tripotent e_x of x is*

$$e_x \;=\; \frac{\sum_{k \in K} \overline{\alpha_k} e_k}{(\sum_{k=1}^{n} |\alpha_k|^2)^{\frac{1}{2}}}.$$

All the sums converge in norm in the corresponding spaces.

(6.) *For all finite subsets F of K, and any element $x \in \operatorname{lin}\{x_j : j \in F\}$ the support tripotent e_x of x lies in $\operatorname{lin}\{e_j : j \in F\}$.*

If these conditions hold, then H is a subtriple of A, isometrically isomorphic to a Hilbert space, and is equal to the dual G^* and to the support space $s(G)$ of G. Moreover, H and G have orthonormal basis $\{e_k\}_{k\in K}$ and $\{x_k/\|x_k\| \ : \ k \in K\}$ respectively.

Proof. (1.) $\Rightarrow$ (2.): Let P be a contractive projection with range $PA_* = G$. By [12] Theorem 3.1, it can be assumed that P lies in $\mathcal{GL}(A_*)$. Then, by (5.1) the subspaces $s(PA_*)$ and P^*A of A coincide. Since P is contractive, P^*A and PA_* form a dual pair. Denote by $H_\circ$ and $(H_\circ)^\circ$ the annihilator and the bi-annihilator of H in PA_* and P^*A respectively. Corollary 5.1 shows that if $j \neq k$ then $x_j \in \{x_k\}_\circ$. Hence,

$$H_\circ \ \subseteq \ (\{x_k\}_{k\in K})_\circ \ = \ \bigcap_{k\in K} \{x_k\}_\circ \ = \ \{0\}.$$

Since H is a weak*-closed subspace of $s(PA_*) = P^*A$, it follows that

$$H \ = \ (H_\circ)^\circ \ = \ \{0\}^\circ \ = \ P^*A. \tag{5.6}$$

By [14] Lemma 5.1, H is a subtriple of A, and by Theorem 3.3, H is a Hilbert space. Therefore, also G is a Hilbert space of the same dimension as H.

Let K^f denote the of all finite subsets of K. For every set $F \in K^f$, let the projection P_F on A_*, be defined by

$$P_F \ = \ \sum_{k\in F} P_k.$$

Its range $P_F A_*$ is a subspace of the Hilbert space $G = PA_*$, with finite dimension $|F|$. Hence, $P_F A_*$ is the range of a contractive projection on A_*. As argued before, there is a unique element Q_F of $\mathcal{GL}(A_*)$ such that

$$Q_F A_* \ = \ P_F A_*, \qquad \text{and} \qquad Q_F^* A \ = \ s(Q_F A_*) \ = \ \operatorname{lin}\{e_k : k \in F\} \ = \ P_F^* A.$$

It follows that P_F and Q_F coincide.

(2.) $\Rightarrow$ (3.): Repeating the above argument with K replaced by F shows that if P_F is contractive, then $P_F A_*$ and $P_F^* A$ are Hilbert spaces, both having dimension $|F| < \infty$. Moreover, $P_F \in \mathcal{GL}(A_*)$, which implies that $s(P_F A_*) = P_F^* A$. The restriction $P_k|_{P_F A_*}$ of P_k to $P_F A_*$ is an orthoprojecion on $P_F A_*$. By Lemma 4.2, P_k annihilates x_j, if $j \neq k$. It follows that the elements x_j and x_k are Hilbert-orthogonal in $P_F A_*$ and that, for all elements x of A_*,

$$\|\sum_{i\in F} P_i x\|^2 \ = \ \sum_{i\in F} \|P_i x\|^2 \ = \ \|P_F x\|^2 \ \leq \ \|x\|^2. \tag{5.7}$$

Therefore, $\{P_F x\}_{F\in K^f}$ is a Cauchy net with respect to the norm topology on A_*, its norm limit exists for every x in A_*, and $\sum_{k\in K} P_k$ converges in the strong operator topology to a linear projection P which, by (5.7), is also contractive. This also shows that $\{x_k/\|x_k\|\}_{k\in K}$ is an orthonormal basis of G.

(3.) $\Rightarrow$ (4.): Set $P = \sum_{k\in K} P_k$. It is clear that P has range G. That P is a GL-projection is verified as follows. For each k in K, the L-complement $(P_k A_*)^\diamond$ of $P_k A_*$ is a subset of $\ker P_k$. It follows that

$$\begin{aligned}(PA_*)^\diamond \;&=\; (\overline{\mathrm{lin}\bigcup_{k\in K} P_k A_*}^{n})^\diamond \;\subseteq\; (\bigcup_{k\in K} P_k A_*)^\diamond \;=\; \bigcap_{k\in K}(P_k A_*)^\diamond \\ &\subseteq\; \bigcap_{k\in K}\ker P_k \;=\; \{x\in A_* \;:\; P_k(x) \;=\; 0,\ \forall k\in K\} \\ &\subseteq\; \ker P.\end{aligned}$$

Since P is also assumed to be contractive, it lies in $\mathcal{GL}(A_*)$. The uniqueness of P among the GL-projections with given range is obtained from Corollary 4.7 in [11]. Hence, (3.) implies (4.).

(4.) $\Rightarrow$ (1.): Since GL-projections are contractive by definition, this implication is obvious.

(4.) $\Rightarrow$ (5.): If (4.) holds then, as it has already been shown, $\{x_k/\|x_k\|\}_{k\in K}$ is an orthonormal basis of G. For any $x\in G$ there are coefficients α_k in $\mathbb{C}$, such that

$$x \;=\; \sum_{k\in K}\alpha_k\frac{x_k}{\|x_k\|}, \qquad\text{and}\qquad \|x\|^2 \;=\; \sum_{k\in K}|\alpha_k|^2,$$

Since by (5.1) the support tripotent e_x of x lies in the subtriple $H = s(PA_*)$, and since $\{e_k\}_{k\in K}$ is an orthonormal basis of H, it follows that there are coefficients β_k, $(k\in K)$ with

$$e_x \;=\; \sum_{k\in K}\beta_k e_k. \qquad\text{and}\qquad \sum_{k\in K}|\beta_k|^2 \;=\; \|e_x\|^2 \;=\; 1.$$

Here it is assumed that x is not zero, and hence that e_x is not zero. We identify H and G with $l^2(K)^*$ and $l^2(K)$ respectively, with the conventional inner product $\langle(\alpha_k)_{k\in K},(\beta_k)_{k\in K}\rangle = \sum_{k\in K} a_k\overline{\beta_k}$. Using Corollary 5.1 and the Cauchy-Bunyakowsky-Schwarz (CBS) inequality,

$$\begin{aligned}\|x\|^2 \;&=\; (e_x\cdot x)^2 \;=\; (\sum_{k\in K} a_k\beta_k)^2 \;=\; \langle(\alpha_k)_{k\in K}\;,\;(\overline{\beta_k})_{k\in K}\rangle^2 \\ &\le\; (\sum_{k\in K}|\alpha_k|^2)(\sum_{k\in K}|\beta_k|^2) \;\le\; \|x\|^2.\end{aligned}$$

This shows that equality holds in (CBS), and it follows that there exists a complex number z such that $\alpha_k = z\overline{\beta_k}$, for all $k\in K$. Hence,

$$\|x\|^2 \;=\; (\sum_{k\in K}\alpha_k\beta_k)^2 \;=\; (\sum_{k\in K} z\overline{\beta_k}\beta_k)^2 \;=\; |z|^2,$$

and

$$\beta_k \; = \; \frac{\overline{\alpha_k}}{|z|^2} \; = \; \frac{\overline{\alpha_k}}{\|x\|^2}.$$

This proves (5.).

(5.) $\Rightarrow$ (6.) This is trivial.

(6.) $\Rightarrow$ (2.) When F is a finite subset of K, then by Lemma 4.2, $\sum_{j\in F} P_j$ is a projection on A_* with adjoint $\sum_{j\in F} P_j^*$ on A. The assumption implies that $s(\sum_{j\in F} P_j A_*)$ is a subset of $\sum_{j\in F} P_j^* A$. Theorem 3.18 in [12] shows that $\sum_{j\in F} P_j$ lies in $\mathcal{GL}(A_*)$, and is therefore contractive. This completes the proof. □

Having explicitly described the GL-projections involved, it is now possible to turn the attention to algebraic conditions. The case in which the index set K consists of two elements differes in some details from all other cases and is therefore treated separately.

Theorem 5.3. *Let A be a JBW**-triple with predual A_*, and let x and y be elements of A_* the support tripotents e_x and e_y of which are collinear. Let P_x and P_y the GL-projections onto $\mathbb{C}x$ and $\mathbb{C}y$ defined by* (5.5)*. Then the following conditions are equivalent.*

(1.) *There exists a contractive projection P from A_* onto $\mathbb{C}x \oplus \mathbb{C}y$.*

(2.) *The projection $P_x^* + P_y^*$ commutes with $D(e_x, e_y)$, $D(e_y, e_x)$, $D(e_x, e_x)$ and with $D(e_y, e_y)$.*

(3.) *The elements x and y satisfy*

$$\begin{aligned} 2D(e_x, e_y)_* x \;&=\; 2D(e_x, e_x)_* y \;=\; y, \\ 2D(e_y, e_x)_* y \;&=\; 2D(e_y, e_y)_* x \;=\; x. \end{aligned}$$

In particular, x lies in $A_1(e_y)_*$ and y lies in $A_1(e_x)_*$.

Proof. (1.) $\Leftrightarrow$ (2.): By Lemma 4.2, $P_x + P_y$ and $P_x^* + P_y^*$ are projections. The range $\mathbb{C}e_x \oplus \mathbb{C}e_y$ of $P_x^* + P_y^*$ is a subtriple of A. Combining Theorem 5.2 with Corollary 4.3 gives the equivalence of (1.) and (2.), as required.

(2.) $\Leftrightarrow$ (3.): To obtain this equivalence, the commutators in (2.) are to be calculated explicitely. For all elements a of A,

$$\begin{aligned} (P_x^* + P_y^*) D(e_x, e_x) a \;&=\; (x \cdot \{e_x\, e_x\, a\}) e_x + (y \cdot \{e_x\, e_x\, a\}) e_y \\ &=\; (D(e_x, e_x)_* x \cdot a) e_x + (D(e_x, e_x)_* y \cdot a) e_y \\ &=\; (x \cdot a) e_x + (D(e_x, e_x)_* y \cdot a) e_y \end{aligned}$$

and

$$\begin{aligned} D(e_x, e_x)(P_x^* + P_y^*) a \;&=\; D(e_x, e_x)((x \cdot a) e_x + (y \cdot a) e_y) \\ &=\; (x \cdot a) e_x + (y \cdot a)\{e_x\, e_x\, e_y\} \\ &=\; (x \cdot a) e_x + \frac{1}{2}(y \cdot a) e_y. \end{aligned}$$

Hence $P_x^* + P_y^*$ commutes with $D(e_x, e_x)$, if and only if

$$D(e_x, e_x)_* y = \frac{1}{2} y. \tag{5.8}$$

By Proposition 2.1, the equality (5.8) is equivalent to y lying in the predual $A_1(e_x)_*$ of the Peirce space $A_1(e_x)$ of e_x. Exchanging x and y in these calculations shows that $(P_x + P_y)^*$ commutes with $D(e_y, e_y)$ if and only if x lies in $A_1(e_y)_*$, or, equivalently,

$$D(e_y, e_y)_* x = \frac{1}{2} x. \tag{5.9}$$

To obtain the required expressions for $D(e_x, e_y)_* x$ and $D(e_y, e_x)_* y$, observe that, by the relation $e_x \top e_y$ and the Peirce rules,

$$D(e_y, e_x)_* x = D(e_x, e_y)_* y = 0, \tag{5.10}$$

$$\{e_x,\ e_y,\ e_x\} = \{e_y,\ e_x,\ e_y\} = 0. \tag{5.11}$$

From Equation (5.10) it follows that, for any element a in A,

$$\begin{aligned}(P_x + P_y)^* D(e_x, e_y)(a) &= (P_x + P_y)^* \{e_x, e_y, a\} \\ &= (x \cdot \{e_x, e_y, a\}) e_x + (y \cdot \{e_x, e_y, a\}) e_y \\ &= (x \cdot \{e_x, e_y, a\}) e_x \\ &= (D(e_x, e_y)_* x \cdot a) e_x,\end{aligned}$$

and from (2.1) and (5.11) that

$$\begin{aligned}D(e_x, e_y)(P_x + P_y)^*(a) &= \{e_x, e_y, (P_x + P_y)^* a\} \\ &= (x \cdot a)\{e_x, e_y, e_x\} + (y \cdot a)\{e_x, e_y, e_y\} \\ &= \frac{1}{2}(y \cdot a) e_x.\end{aligned}$$

This shows that $P_x^* + P_y^*$ and $D(e_x, e_y)$ commute if and only if

$$D(e_x, e_y)_* x = \frac{1}{2} y. \tag{5.12}$$

Similarly, $D(e_y, e_x)$ and $(P_x + P_y)^*$ commute if and only if

$$D(e_y, e_x)_* y = \frac{1}{2} x. \tag{5.13}$$

The relations (5.8), (5.9), (5.12) and (5.13) provide the equivalence of (2.) and (3.). This completes the proof. □

In order to pass to arbitrary families of elements in A_*, a careful re-examination of the above proof is required. In particular, further conditions need to be added to those in (3.) of Theorem 5.3, and it must be proved that the stated algebraic criterias imply the convergence of the formal sums as described in Theorem 5.2.

Theorem 5.4. *Let A be a JBW*-triple with predual A_*, and let $\{x_k\}_{k\in K}$ with $|K| \geq 3$ be a family of elements of A_*, such that the corresponding support tripotents form a collinear system $\{e_k\}_{k\in K}$. Let H, G and P_k be defined as in Theorem 5.2. Then, the following conditions are equivalent.*

(1.) *There exists a contractive projection on A_* with range G.*

(2.) *The formal sum $\sum_{k\in K} P_k^*$ is SOT-convergent in $\mathcal{B}(A)$ and commutes with $D(e_j, e_k)$, for all elements j, k in K.*

(3.) *For distinct indices j, k, l in K, the following relations hold;*

$$(i) \quad 2D(e_j, e_j)_* x_k \ = \ x_k, \qquad (ii) \quad 2D(e_k, e_j)_* x_k \ = \ x_k,$$

$$(iii) \quad D(e_j, e_k)_* x_l \ = \ 0, \qquad (iv) \quad \{e_j,\ e_k,\ e_l\} \ = \ 0.$$

In particular, x_k lies in $A_1(e_j)_*$ and H is a subtriple of A that is isometrically isomorphic to a Hilbert space with orthonormal basis $\{e_k\}_{k\in K}$.

(4.) *For any three distinct indices j, k and l in K, there exists a contractive projection on A_* with range* $\mathrm{lin}\{x_j,\ x_k,\ x_l\}$.

Proof. (1.) $\Rightarrow$ (2.): Theorem 5.2 (4.) shows that $P = \sum_{k\in K} P_k$ lies in $\mathcal{GL}(A_*)$. By (5.1) $s(PA_*)$ and P^*A coincide, and hence P^*A is a weak*-closed subtriple of A. Theorem 4.1 shows that, for a, b in H, P^* commutes with $D(a, b)$.

(2.) $\Rightarrow$ (3.): Following the strategy used in the proof of Theorem 5.3 we calculate the commutators explicitly. Recall that x_j lies in $A_2(e_j)_*$ and that $D(e_j, e_j)_* x_j$ equals x_j. Hence, for any element a in A,

$$\begin{aligned}
(\sum_{k\in K} P_k^*) D(e_j, e_j) a &= (\sum_{k\in K} P_k^*)\{e_j,\ e_j,\ a\} = \sum_{k\in K} (x_k \cdot \{e_j,\ e_j,\ a\}) e_k \\
&= (x_j \cdot \{e_j,\ e_j,\ a\}) e_j + \sum_{k\in K\setminus\{j\}} (x_k \cdot \{e_j,\ e_j,\ a\}) e_k \\
&= (x_j \cdot a) e_j + \sum_{k\in K\setminus\{j\}} (x_k \cdot \{e_j,\ e_j,\ a\}) e_k.
\end{aligned}$$

The relation $e_j \top e_k$ implies that

$$\begin{aligned}
D(e_j, e_j)(\sum_{k\in K} P_k^*) a &= D(e_j, e_j) \sum_{k\in K} (x_k \cdot a) e_k = \sum_{k\in K} (x_k \cdot a)\{e_j,\ e_j,\ e_k\} \\
&= (x_j \cdot a)\{e_j,\ e_j,\ e_j\} + \sum_{k\in K\setminus\{j\}} (x_k \cdot a)\{e_j,\ e_j,\ e_k\}) \\
&= (x_j \cdot a) e_j + \sum_{k\in K\setminus\{j\}} \frac{1}{2} (x_k \cdot a) e_k.
\end{aligned}$$

Therefore, the operators $D(e_j, e_j)$ and $\sum_{k\in K} P_k^*$ commute if and only if, for all $a \in A$,

$$\sum_{k\in K\setminus\{j\}} \frac{1}{2}(x_k \cdot a)e_k \;=\; \sum_{k\in K\setminus\{j\}} (x_k \cdot \{e_j,\ e_j,\ a\})e_k. \tag{5.14}$$

From the linear independence of the set $\{e_k\}_{k\in K}$ and Equation (5.14) we conclude that $(x_k \cdot \{e_j,\ e_j,\ a\})$ equals $(x_k \cdot a)/2$, i.e.

$$D(e_j, e_j)_* x_k \;=\; \frac{1}{2} x_k. \tag{5.15}$$

Therefore, (3.)(i) holds.

Now assume j and l to be distinct indices in K. Equation (5.10) implies that

$$\begin{aligned}
(\sum_{k\in K} P_k^*)D(e_j, e_l)a \;&=\; (\sum_{k\in K} P_k^*)\{e_j,\ e_l,\ a\} \;=\; \sum_{k\in K}(x_k \cdot \{e_j,\ e_l,\ a\})e_k \\
&= (x_j \cdot \{e_j,\ e_l,\ a\})e_j + (x_l \cdot \{e_j,\ e_l,\ a\})e_l + \sum_{k\in K\setminus\{j,l\}} (x_k \cdot \{e_j,\ e_l,\ a\})e_k \\
&= (x_j \cdot \{e_j,\ e_l,\ a\})e_j + \sum_{k\in K\setminus\{j,l\}} (x_k \cdot \{e_j,\ e_l,\ a\})e_k,
\end{aligned}$$

and Equation (5.11) implies that

$$\begin{aligned}
D(e_j, e_l)(\sum_{k\in K} P_k^*)a \;&=\; D(e_j, e_l)\sum_{k\in K}(x_k \cdot a)e_k \\
&= (x_j \cdot a)\{e_j,\ e_l,\ e_j\} + (x_l \cdot a)\{e_j,\ e_l,\ e_l\} + \sum_{k\in K\setminus\{j,l\}} (x_k \cdot a)\{e_j,\ e_l,\ e_k\} \\
&= \frac{1}{2}(x_l \cdot a)e_j + \sum_{k\in K\setminus\{j,l\}} (x_k \cdot a)\{e_j,\ e_l,\ e_k\}.
\end{aligned}$$

Hence, $D(e_j, e_l)$ and $\sum_{k\in K} P_k^*$ commute if and only if, for all $a \in A$,

$$(x_j \cdot \{e_j,\ e_l,\ a\})e_j + \sum_{k\in K\setminus\{j,l\}} (x_k \cdot \{e_j,\ e_l,\ a\})e_k \tag{5.16}$$

$$=\; \frac{1}{2}(x_l \cdot a)e_j + \sum_{k\in K\setminus\{j,l\}} (x_k \cdot a)\{e_j,\ e_l,\ e_k\}. \tag{5.17}$$

Choose an index $m \in K$, distinct from j and l, and set $a = e_m$. Using again Corollary 5.1, the right hand side (5.17) of the above equation becomes $\{e_j,\ e_l,\ e_m\}$. The Peirce rules show that,

$$\{e_j,\ e_l,\ e_m\} \;\in\; A_2(e_j) \cap A_0(e_l) \cap A_2(e_m). \tag{5.18}$$

By Equation (5.15), x_k lies in $A_1(e_j)_* \cap A_1(e_m)_*$ when $k \neq j, m$, and x_j lies in $A_1(e_m)_*$. Therefore, the left hand side (5.16) vanishes, and we obtain,

$$\{e_j,\ e_l,\ e_m\} \;=\; 0. \tag{5.19}$$

This proves (3.)(iv). By inserting (5.19) back into (5.17), the equality (5.17) = (5.16) simplifies as

$$(x_j \cdot \{e_j,\ e_l,\ a\})e_j + \sum_{k \in K \setminus \{j,l\}} (x_k \cdot \{e_j,\ e_l,\ a\})e_k \ = \ \frac{1}{2}(x_l \cdot a)e_j.$$

The linear independence of $\{e_k\}_{k \in K}$ shows that $(x_j \cdot \{e_j,\ e_l,\ a\}) = (x_l \cdot a)/2$, and $(x_k \cdot \{e_j,\ e_l,\ a\}) = 0$, which implies that

$$D(e_j, e_l)_* x_j \ = \ \frac{1}{2} x_l, \qquad \text{and} \qquad D(e_j, e_l)_* x_k \ = \ 0.$$

This proves (3.)(ii) and (3.)(iii). The condition (3.)(iv) implies also that H is a Hilbert space and a subtriple of A, and that $\{e_k\}_{k \in K}$ is and orthonormal basis of H.

(3.) $\Rightarrow$ (1.): It is enough to prove that the condition (2.) of Theorem 5.2 holds. For any finite subset F of K, set $H_F = \mathrm{lin}\{e_k : k \in F\}$. By Lemma 4.2, $P_F = \sum_{k \in F} P_k$ is a projection on A_*. Theorem 3.3 and (3.iv) show H_F to be a subtriple of A isomorphic to the Hilbert space with triple product (2.5). Reinspecting the relations (5.14), (5.16) and (5.17) shows that, for j, l in F, the projection $P_F^* = \sum_{k \in F} P_k^*$ commutes with $D(e_j, e_l)$ and with $D(e_j, e_j)$ on A. Hence, by Corollary 4.3, P_F is contractive, as required.

(3.) $\Leftrightarrow$ (4.): This follows from the equivalence of (1.) and (3.) and the fact that (3.) is formulated in terms of all possible triples $(j, k, l) \in K \times K \times K$. The proof is now complete. □

Let $\{x_k\}_{k \in K}$ a family of elements in A_* such that the corresponding support tripotents $\{e_k\}_{k \in K}$ form a collinear system. Then $\{x_k\}_{k \in K}$ is said to be *projectively collinear* if there exists a contractive projection onto $\overline{\mathrm{lin}\{x_k : k \in K\}}^n$.

Corollary 5.5. *Let $\{x_k\}_{k \in K}$ be an arbitrary family of elements of A_*. Then, $\{x_k\}_{k \in K}$ is projectively collinear if and only if the conditions* (3.i)–(3.iv) *in Theorem* 5.4 *hold for $\{x_k\}_{k \in K}$ and $\{e_k\}_{k \in K}$. In particular, these conditions are sufficient for $\{e_k\}_{k \in K}$ to be a collinear system.*

Proof. The condition (3.i) and Proposition 2.1 imply that $x_k \in A_1(e_l)_*$ $(k \neq l)$. By Proposition 3 in [19], e_k lies in $A_1(e_l)$ and e_l lies in $A_1(e_k)$, i.e. $\{e_k\}_{k \in K}$ is a collinear system. Theorem 5.4 completes the proof. □

At this stage, two questions have to be clarified. First, is there a non-trivial example of a projectively collinear system, and, secondly, is there an example of a family of elements in A_* with pairwise collinear support tripotent which is not projectively collinear. A negative answer to either of the questions would render our main results redundant. The following example answers both questions affirmatively.

Example. Let A be the W*-algebra $\mathcal{B}(\mathbb{C}^4)$, represented by 4×4-matrices. Let τ be the normalized trace on A, and let m_x, m_y and m_z be the matrices

$$m_x = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; m_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \; m_z = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Define the linear functionals x, y and z, for any element a of A with entries a_{ij} $(i, j = 1, 2, 3, 4)$ by

$$\begin{aligned} (x \cdot a) &= \tau(m_x a) = \frac{1}{2}(a_{11} + a_{22}), \\ (y \cdot a) &= \tau(m_y a) = \frac{1}{2}(a_{13} + a_{24}), \\ (z \cdot a) &= \tau(m_z a) = \frac{1}{4}(2a_{11} + a_{21} + a_{12} + 2a_{22}). \end{aligned}$$

Then the elements u and v of A, given by

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

form a collinear pair of tripotents of A and are such that $u = e_x = e_z$ and $v = e_y$. The mapping P defined for a linear functional s on A, by $s \mapsto (u \cdot s)x + (v \cdot s)y$ is a contractive projection and is also the unique GL-projection from A_* onto $\mathbb{C}x \oplus \mathbb{C}y$, whereas there is no contractive projection from A_* onto $\mathbb{C}z \oplus \mathbb{C}y$.

Proof. It is obvious that $(x \cdot u) = (z \cdot u) = (y \cdot v) = 1$, and the norms of m_x, m_y and m_z are, respectively, $\|m_x\| = 2$, $\|m_y\| = 2$, and $\|m_z\| = 3$. The following trace inequality, obtained in [32], is useful in subsequent calculations. For all elements a, b in a C*-algebra A and a continuous linear functional τ on A, the condition

$$|\tau(ab)| \leq \|b\| \, \tau(|a|) \tag{5.20}$$

holds if and only if τ is a positive normalized trace on A. From (5.20) it follows that, for all b in A, and m equal to m_x, m_y, or to m_z,

$$|\tau(mb)| \leq \|b\| \, \tau(|m|) = \|b\|,$$

hence, that x, y and z are of norm one in A_*. Since x attains its norm at u, it follows that $e_x \leq u$. In particular, e_x lies in $A_2(u)$. Clearly, e_x is not zero. Consider any tripotent w of A with $w \leq u$. Observe that u is a partial isometry of rank two. If w is not equal to zero or to u, then w is a partial isometry of rank one, that is $\mathrm{trace}(|w|) = 1$. Then, by (5.20),

$$|(x \cdot w)| = |\tau(m_x w)| \leq \|m_x\| \, \tau(|w|) = \frac{1}{2}.$$

It follows that e_x has rank two. This and the condition that $e_x \leq u$, implies that $e_x = u$. Similarly, $(z \cdot u) = 1 = \|z\|$ implies that $e_z \leq u$, and if w is as above,

$$|(z \cdot w)| \;=\; |\tau(m_z w)| \;\leq\; \|m_z\|\; \tau(|w|) \;=\; \frac{3}{4}.$$

Hence $e_z = e_x = u$. Also v has rank two. The same arguments as those used above show that $e_y \leq y$. And, for any tripotent w with $w \leq v$ and $\mathrm{trace}(|w|) = 1$,

$$|(y \cdot w)| \;=\; |\tau(m_y w)| \;\leq\; \|m_y\|\; \tau(|w|) \;=\; \frac{1}{2}.$$

Hence $e_y = v$. We have shown that the subsets $\{x,\ y\}$ and $\{z,\ y\}$ satisify the assumptions of Theorem 5.3.

It can be seen from elementary calculations that for all $a \in A$,

$$\begin{aligned}
(D(u,u)_* y \cdot a) &= \frac{1}{4}(a_{13} + a_{24}) = \frac{1}{2}(y \cdot a),\\
(D(u,v)_* x \cdot a) &= \frac{1}{4}(a_{13} + a_{24}) = \frac{1}{2}(y \cdot a),\\
(D(v,u)_* y \cdot a) &= \frac{1}{4}(a_{11} + a_{22}) = \frac{1}{2}(x \cdot a),\\
(D(v,v)_* x \cdot a) &= \frac{1}{4}(a_{11} + a_{22}) = \frac{1}{2}(x \cdot a).
\end{aligned}$$

By Theorem 5.3, the mapping P defined for a linear functional s on A, by $s \mapsto (u \cdot s)x + (v \cdot s)y$ is a contractive projection and is also the unique GL-projection from A_* onto $\mathbb{C}x \oplus \mathbb{C}y$. On the other hand,

$$(D(u,v)_* z \cdot a) = \frac{1}{8}(2a_{13} + a_{23} + a_{14} + 2a_{24}), \tag{5.21}$$

$$(D(v,v)_* z \cdot a) = \frac{1}{8}(2a_{11} + a_{12} + a_{21} + 2a_{22}) = \frac{1}{2}(z \cdot a). \tag{5.22}$$

Equation (5.22) shows that z lies in $A_1(v)$, but (5.21) condradicts (3.) in Theorem 5.3. Hence, there is no contractive projection from A_* onto $\mathbb{C}z \oplus \mathbb{C}y$. $\square$

The pair $\{y,\ z\}$ in Example 5 shows that in general the condition that $\{x_k\}_{k\in K}$ be projectively collinear is not necessary for the existence of a normal contractive projection onto $H = \overline{\mathrm{lin}\{e_k : k \in K\}}^{w*}$. Given a normal contractive projecton R from A onto H, the range R_*A_* of its pre-adjoint R_* may be different from $G = \overline{\mathrm{lin}\{x_k : k \in K\}}^{n}$. In the remainder of this section, we assume the existence of a normal contractive projection R on A with range RA equal to the weak*-closed span of an arbitrary collinear system. We also investigate the properties of R_*.

A tripotent u in a JBW*-triple is said to be *σ-finite* if the following implication holds. Whenever B is an orthogonal system in $\mathcal{U}(A)$ with the property that

$B \le u$ then B is countable. The set of all σ-finite tripotents is denoted by $\mathcal{U}_\sigma(A)$. The results of [15] and [19], show that

$$\mathcal{U}_\sigma(A) \;=\; \{e_x : x \in A_*\}. \tag{5.23}$$

This characterizes the support tripotents in terms of algebraic criteria, and the next lemma can established from earlier results. By [11] Corollary 4.8 (iii), a contractive projection P on A_* lies in $\mathcal{GL}(A_*)$ if and only if

$$\mathcal{U}_\sigma(P^*A) \;=\; \mathcal{U}_\sigma(A) \cap s(PA_*). \tag{5.24}$$

Lemma 5.6. *Let A be a JBW*-triple, and let H be a subtriple of A isometrically isomorphic as subtriple to a Hilbert space. Denote by $S_1(H)$ the unit sphere of H, a subset of $\mathcal{U}(A)$. If $S_1(H) \cap \mathcal{U}_\sigma(A)$ is not empty, then $S_1(H) \subseteq \mathcal{U}_\sigma(A)$.*

Proof. Since σ-finiteness is an algebraic property, it is preserved under triple automorphisms of A. By Theorem 3.3, the set of triple automorphisms of A acts transitively on $S_1(H)$. This completes the proof. □

When R is any normal contractive projection on A, denote by R_* its pre-adjoint, a contractive projection on A_*. Recall that the range RA of R is a JBW*-triple when equipped with the restricted triple product $\{\ldots\}_{RA}$ defined in (4.1).

Theorem 5.7. *Let C be a collinear system in the JBW*-triple A. Denote by H_C the subspace $\overline{\mathrm{lin} C}^{w*}$ of A. Let R be a normal contractive projection on A with range H_C and with pre-adjoint R_*. Let P be the unique GL-projection on A_* such that $PA_* = R_*A_*$. Then, the following results hold,*

(1.) *There exists a projectively collinear system $\{x_k\}_{k\in K}$ in A_* such that the range R_*A_* of R_* is the norm closed span of $\{x_k\}_{k\in K}$.*

(2.) *The JBW*-triple H_C with the restricted triple product $\{\ldots\}_{RA}$ is isomorphic to the subtriple $H = \overline{\mathrm{lin}\{e_k : k \in K\}}^{w*}$ spanned by the support tripotents e_k of x_k. In particular, H_C is a Hilbert space, isomorphic to H.*

(3.) *The restrictions $R|_H$ and $P^*|_{H_C}$ of R and P^* to H and H_C are triple isomorphisms and inverse of each other.*

(4.) *When u, v w are distinct elements of C (provided that $|C| \ge 3$), then $\{u,\ v,\ w\}$ lies in* $\ker R$.

Proof. By the conditional expectation formulas (4.2) and (4.3), it can be seen that, for two different elements u and v of C.

$$\{u,\ u,\ v\}_{RA} \;=\; R\{u,\ u,\ v\} \;=\; \frac{1}{2}Rv \;=\; \frac{1}{2}v.$$

Therefore, the relation of collinearity holds also with respect to the restricted triple product. By [14] Lemma 5.1 and [19] Proposition 2.2, it can be seen that the triples $(RA, \{\ldots\}_{RA})$ and $(s(R_*A_*), \{\ldots\})$ are isomorphic. When P is the unique element of $\mathcal{GL}(A_*)$ with $PA_* = R_*A_*$, then by [11] Lemma 3.2 (iv), $P|_{RA}$ is a triple isomorphism and inverse to $R|_{R^*A}$. It follows that $P^*C = \{P^*u : u \in C\}$ is a collinear system in the Hilbert space H. Consequently, by Lemma 5.6, P^*C is a subset of $\mathcal{U}_\sigma(A)$. From (5.23) and (5.24), it can be seen that, for each u in C the element P^*u is the support tripotent of some x_u in R_*A_*. This proves (1.), (2.) and (3.).

For distinct elements u, v and w in C, the above arguments and Theorem 5.4 (3.iv) show that

$$\{P^*u,\ P^*v,\ P^*w\} \ = \ 0.$$

Since $R|_{P^*A}$ is a triple isomorphism with inverse of $P|_{RA}$, it follows that $R\{u,\ v,\ w\}$ is zero, proving (4.). □

The above Theorem can be generalized as follows.

Theorem 5.8. *Let Q be a normal contractive projection on a JBW*-triple A, with preadjoint Q_* on A_*. Then QA is isometrically a Hilbert space if and only if Q_*A_* is the norm-closed span of a projectively collinear system.*

Proof. Let P be the uniqe element of $\mathcal{GL}(A_*)$ with $PA_* = Q_*A_*$. Then, as argued before, $P^*|_{QA}$ and $Q|_{P^*A}$ are isometries and inverse of each other, and P^*A is a subtriple of A. Applying Theorem 5.7 gives the proof. □

The results presented in this article apply to the more restricted relation of rigid collinearity [13]. In that case, the *exchange automorphisms* introduced in [29] can be used to answer our main question, whether or not a contractive projection with the prescribed range exists. Details and proofs are part of current investigations, and will be presented in forthcoming publications.

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Submitted: March 27, 2006
Revised: February 27, 2007