ELSEVIER

Topology and its Applications 104 (2000) 119–139

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

On the existence of constant selections

C.D. Horvath [1]

Département de Mathématiques, Université de Perpignan, 66860 Perpignan cedex, France

Received 18 December 1997; received in revised form 2 December 1998

Abstract

Given a relation $\Omega : X \to Y$ between topological spaces, we inquire whether it has a constant selection. This problem has been investigated from different points of view, purely topological or convex. We present here a synthesis of some of the most interesting results, with some generalizations and new insights. © 2000 Published by Elsevier Science B.V. All rights reserved.

Keywords: Multifunction; Relation; Selection; Fixed point; Minimax

AMS classification: 54C10; 54C55; 54C65; 54H25

1. Introduction

This work is a synthesis, with some simplifications and generalizations, of various results dealing with the constant selection problem for a relation. After defining some terms, we shall state the problem and provide some motivation.

By a *relation from a set X to a set Y* we mean a map from X to the power set of Y. Relations are also called multifunctions or correspondences. We will use the functional notation $\Omega : X \to Y$ to denote a relation from X to Y (they are just morphisms in the appropriate category), the set Ωx is *the image of the point x*. Sets of the form $\{x \in X\colon\ y \in \Omega x\}$ are called *fibers* and are denoted by $\Omega^{-1}y$. The complement of a fiber $\Omega^{-1}y$ is called a *cofiber*, it is denoted by $\Omega^* y$. To a relation $\Omega : X \to Y$ are therefore associated two relations, $\Omega^{-1} : Y \to X$, *the inverse of Ω*, and $\Omega^* : Y \to X$, *the dual of Ω*. We will use the same notation for a relation $\Omega : X \to Y$ and for its graph, the subset $\{(x, y) \in X \times Y\colon\ y \in \Omega x\}$ of $X \times Y$. We will say that Ω *has the finite intersection property* if the family of its values has the finite intersection property.

Given a relation $\Omega : X \to Y$ between topological spaces there are three problems that one can consider:

[1] Email: horvath@univ-perp.fr.

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00022-X

(1) The continuous selection problem. Is there a continuous map $f : X \to Y$ such that $f(x) \in \Omega x$ for each $x \in X$?

(2) The fixed point problem. Assuming that $X \subseteq Y$, is there a point $\overline{x} \in X$ such that $\overline{x} \in \Omega \overline{x}$?

(3) The constant selection problem. Is there a point $\overline{y} \in Y$ such that $\overline{y} \in \Omega x$ for each $x \in X$?

We will be mainly concerned with problem (3) which obviously asks if the set $\bigcap_{x\in X} \Omega x$ is not empty. In many cases, as we will see, (3) cannot be dissociated from problems (1) and (2). We next give some motivation for considering the constant selection problem.

The first and main motivation comes from minimax theory, going back to Von Neumann and the fundamental theorem of zero sum games. Given a function $f : X \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ defined on the product of two topological spaces X and Y, we would like to know if

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y).$$

We only need to establish the inequality

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) \leqslant \sup_{x\in X} \inf_{y\in Y} f(x, y),$$

we can therefore assume that

$$\sup_{x\in X} \inf_{y\in Y} f(x, y) \neq \infty.$$

In this setting, to a real number λ one can associate a relation $\Omega_\lambda : X \to Y$ defined as follows:

$$\Omega_\lambda x = \bigl\{y \in Y:\ f(x, y) \leqslant \lambda\bigr\}.$$

Then, one can see that $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$ if and only if $\bigcap_{x\in X} \Omega_\lambda x \neq \emptyset$ for each $\lambda > \sup_X \inf_Y f(x, y)$. A most important result, which has achieved the status of a reference point in minimax theory, is due to Maurice Sion [41].

Theorem 1 (Sion). *Let X and Y be convex compact subsets of topological vector spaces and $f : X \times Y \to \overline{\mathbb{R}}$ a function such that*:

(i) *for any $x \in X$ the function $f(x, \cdot)$ is quasi-convex and lower semicontinuous on Y,*

(ii) *for any $y \in Y$ the function $f(\cdot, y)$ is quasi-concave and upper semicontinuous on X.*

Then

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y).$$

Recall that a function $g : X \to \overline{\mathbb{R}}$ is *quasi-convex* if for any real number $\lambda \in \mathbb{R}$ the set $\{x \in X:\ g(x) \leqslant \lambda\}$ is convex, it is *quasi-concave* if $-g$ is quasi-convex.

There is an extensive literature on minimax theorems, a broad but partial review is given in the paper of Simons [40] (citing over one hundred and thirty references), where topological results are almost absent. Sion's proof of his theorem involved the theorem of Knaster, Kuratowski and Mazurkiewicz [10] which can be seen as a geometric form of Brouwer's fixed point theorem. Other minimax equalities can be derived from Hahn–Banach's Theorem. Both methods require a convex setting. Much efforts have been spend on trying to understand the nature of minimax theorems and on generalizations of Sion's theorem. Generalizations are of two different kinds. One can relax convexity, replacing it by some algebraic conditions (we do not go into details since we are not concerned here with this side of the problem for which Simons [40] can be consulted). Or we can look for topological substitutes of convexity. The second possible generalization is in the direction of continuity conditions weaker than lower or upper semicontinuity. We will be exclusively concerned with the last two problems.

Additional motivation comes from mathematical economics where $\Omega : X \to X$ represents a *preference relation* on a consumption set X. There are two possible interpretations for Ω. As a large preference relation, $y \in \Omega x$ is then interpreted as *y is prefered or equivalent to x*, in which case it is natural to assume that $x \in \Omega x$, or as a strict preference relation $y \in \Omega x$ is then interpreted as *y is strictly prefered to x*, in which case it is natural to assume that $x \notin \Omega x$. In the first case, if $\overline{y} \in \bigcap_{x\in X} \Omega x$ then $\overline{y}$ is a largest element with respect to Ω. In the second case, if $\overline{x} \in \bigcap_{y\in Y} \Omega^* y$ then $\overline{x}$ is a maximal element with respect to Ω. The basic ingredient in this kind of results is again Brouwer's fixed point theorem. For the role of relations and convexity in mathematical economics see Border [6] or Klein and Thomson [26].

Any constant selection problem can be interpreted as the search for a winning strategy in a zero sum game between two players. Indeed, let the strategy sets X and Y be given as well as a relation, which we identify with its graph $\Omega \subseteq X \times Y$. Player I picks a point $x \in X$ and player II a point $y \in Y$. If $(x, y) \in \Omega$ then players II wins, otherwise player I wins. If $\bigcap_{x\in X} \Omega x \neq \emptyset$ then player II has a winning strategy, if $\bigcap_{y\in Y} \Omega^* y \neq \emptyset$ then player I has a winning strategy. It is not difficult to see that

$$\bigcap_{x\in X} \Omega x \neq \emptyset \quad \text{or} \quad \bigcap_{y\in Y} \Omega^* y \neq \emptyset$$

if and only if

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y),$$

where $f : X \times Y \to \mathbb{R}$ is the characteristic function of the complement of Ω in $X \times Y$.

Now, we move on with a description of the paper.

There are two sections. The first one deals with the topological intersection theorems. The first topological substitute for convexity that comes to mind is connectedness. One might not expect much from such a simple property, so it came a bit as a surprise when it was finally understood that many minimax theorems, including Sion's theorem, could be proved using exclusively arguments based on connectedness. In recent years there has been an intense research on intersection theorems based merely on connectedness. Contributions were made by Kindler [21,22], König [27,28], Ricceri [39] and the author [17,18], extending previous important studies by Telkersen [46], Tuy [47,48] and Wu Wen

Tsün [49]. Contributors are too numerous to be given their due share of recognition here, but one should mention at least Joo [20] and Stacho [43]. We will not state a single minimax theorem, apart from Sion's result. The interested reader can consult the papers of Kindler or König, which, according to König, settle completely the question of which minimax theorems can be obtained from connectedness.

However, there is one result, Fan's interection theorem [12] which is given below, that one cannot expect to derive from methods based on connectedness, (some asked if it could be done), and there are a few reasons for that. From Fan's theorem one can derive Schauder–Tychonoff's fixed point theorem and the latter does not rely only on the fact that compact convex sets are connected, furthermore, all the results based on connectedness are proved by induction. It appears doubtful that even Brouwer's fixed point theorem could be proved by induction. Before going any further let us state Fan's theorem, and let us say that it is one of the central results of nonlinear analysis, see, for example, the book by Aubin and Ekeland [1].

Theorem 2 (Fan). *Let $\Omega : X \to X$ be a relation from a convex subset of a topological vector space to itself such that the following conditions hold*:

(A) *for each $x \in X$ the set Ωx is closed and $x \in \Omega x$,*

(B) *for each $y \in X$ the set $\Omega^* y$ is convex,*

(C) *there is at least one point $x_0 \in X$ for which Ωx_0 is compact.*

Then

$$\bigcap_{x \in X} \Omega x \neq \emptyset.$$

Section 1 presents a topological version of Fan's theorem due to the author. The formulation of that result involves topological conditions on arbitrary intersections of cofibers, and a simple example shows that the theorem is in some way optimal. If one restricts the domain X of the relation Ω to be finite-dimensional then one can do with topological conditions on the individual cofibers, or images, of the relation. This is the subject matter of the last part of the first section. We introduce there a new class of relations, which we call *Serre relations*, the name being justified by the fact that the graph of Ω with the projection on the domain X is a Serre fibration. The idea of looking at relations as fibrations goes back to Michael [36], the same point of view was exploited in a series of papers by McClendon [30–32]. We do not define Serre relations directly as Serre fibrations, our definition is much simpler, and so are the proofs. We dwell on Serre relations just enough to develop their fundamental properties, the basic results of McClendon are proved, the continuity properties of Serre relations are investigated. To prove that a relation has a constant selection one sometimes has to combine a fixed point theorem with a continuous selection theorem, therefore fixed point theorems and selection theorems for Serre relations are also proved. We believe that Serre relations form an interesting class and we hope that our presentation will contribute to carry further the initial ideas of Michael and McClendon.

The second section is entirely devoted to results of Greco and its associates [11,4,15].

With the results of Bassanezi and Greco or of Greco and Moschen we are back in the convex setting, but with continuity conditions weaker than lower or upper semicontinuity.

They obtained their results assuming that the domain is a finite-dimensional convex set and they ask if that condition could be removed. We give a partial answer to their question. Again, the analytic interpretation of the constant selection theorems are not presented here.

2. Topological intersection theorems

In this section X and Y are topological spaces and $\Omega : X \to Y$ is a relation. The results are of two kinds. Some give sufficient conditions for Ω to have the finite intersection property while others show directly that Ω has a constant selection. Recall that $\Omega : X \to Y$ is lower semicontinuous if for any open set $V \subseteq Y$ the set $\{x \in X\colon\ \Omega x \cap V \neq \emptyset\}$ is open in X. It is upper semicontinuous if for any open set $V \subseteq Y$ the set $\{x \in X\colon\ \Omega x \subseteq V\}$ is open in X.

Theorem 3 combines results of Kindler [21] and of the author [18], Corollary 1 is from [18].

Theorem 3. *Let X be connected topological space and assume that $\Omega : X \to Y$ has nonempty values and also the following properties*:

(A) *for any nonempty finite subset $B \subseteq X$ the set $\bigcap_{x\in B} \Omega x$ is connected,*

(B) *for any nonempty subset $A \subseteq Y$ the set $\bigcap_{y\in A} \Omega^* y$ is connected.*

Then, in any of the following three cases, Ω has the finite intersection property.

(1) *Ω is lower semicontinuous and the values are open.*

(2) *Ω is upper semicontinuous and the values are closed.*

(3) *The values are closed and the fibers are open.*

Proof. First, we show that $\Omega x_1 \cap \Omega x_2 \neq \emptyset$ for any $x_1, x_2 \in X$.

Before proceeding with a proof of the claim, notice that the set

$$[\![x_1, x_2]\!] = \bigcap \bigl\{\Omega^* y\colon\ \{x_1, x_2\} \subseteq \Omega^* y\bigr\}$$

is nonempty and connected, if $\{y\colon\ \{x_1, x_2\} \subseteq \Omega^* y\}$ is empty then $Y = \Omega x_1 \cup \Omega x_2$, in this case we let $[\![x_1, x_2]\!] = X$.

We establish the claim by contradiction. Assume that $\Omega x_1 \cap \Omega x_2 = \emptyset$.

Let $A_i = \{x \in [\![x_1, x_2]\!]\colon\ \Omega x \subseteq \Omega x_i\}$, $i = 1, 2$. Since all the Ωx are connected and nonempty, and also all open, or all closed, we have, from $\Omega([\![x_1, x_2]\!]) \subseteq \Omega x_1 \cup \Omega x_2$, $A_i = \{x \in [\![x_1, x_2]\!]\colon\ \Omega x \cap \Omega x_i \neq \emptyset\}$.

Notice that $A_i \neq \emptyset$, that $A_1 \cap A_2 = \emptyset$, and $[\![x_1, x_2]\!] = A_1 \cup A_2$. If we show that A_1 and A_2 are both open, or both closed, we have a contradiction.

If (1) or (3) is the case, then Ω is lower semicontinuous.

If (1) is the case, then Ωx_i is open and $A_i = \{x \in [\![x_1, x_2]\!]\colon\ \Omega x \cap \Omega x_i \neq \emptyset\}$, it is therefore open.

If (3) is the case, then Ωx_i is closed and $A_i = \{x \in [\![x_1, x_2]\!]\colon\ \Omega x \subseteq \Omega x_i\}$, it is therefore closed.

If (2) is the case, then Ω is upper semicontinuous, Ωx_i is closed and, by connectedness, $A_i = \{x \in [\![x_1, x_2]\!]\colon\ \Omega x \cap \Omega x_i \neq \emptyset\}$, it is therefore closed.

We have shown that $\Omega x_1 \cap \Omega x_2 \neq \emptyset$ for any $x_1, x_2 \in X$.

To complete the proof, we proceed by induction. Assume that $\bigcap_{x\in B} \Omega x \neq \emptyset$ if $B \subseteq X$ is a finite nonempty subset with at most n elements, where $2 \leqslant n$. Fix a finite subset $\{x_1, \ldots, x_{n+1}\} \subseteq X$ and let

$$\widetilde{\Omega} x = (\Omega x) \cap \left(\bigcap_{i=3}^{n+1} \Omega x_i \right).$$

We have to see that $\widetilde{\Omega} x_1 \cap \widetilde{\Omega} x_2 \neq \emptyset$.

It is obvious that (A) holds for $\widetilde{\Omega}$. As for (B) notice that $\widetilde{\Omega}^* y = \Omega^* y$ if $\{x_3, \ldots, x_{n+1}\} \cap \Omega^* y = \emptyset$, and $\widetilde{\Omega}^* y = X$ otherwise. Indeed, $\{x_3, \ldots, x_{n+1}\} \cap \Omega^* y = \emptyset$ is equivalent to $y \in \bigcap_{i=3}^{n+1} \Omega x_i$, and also to $\{x_3, \ldots, x_{n+1}\} \subseteq \Omega^{-1} y$. From this, we see that (B) holds for $\widetilde{\Omega}$.

If the values of Ω are open and if Ω is lower semicontinuous, then $\widetilde{\Omega}$ is lower semicontinuous with open values. Also, if the values of Ω are closed and if Ω is upper semicontinuous, then $\widetilde{\Omega}$ is upper semicontinuous with closed values. We have one last case to look at (3). From $\widetilde{\Omega}^{-1} y = \Omega^{-1} y$ if $\{x_3, \ldots, x_{n+1}\} \subseteq \Omega^{-1} y$, and $\widetilde{\Omega}^{-1} y = \emptyset$ otherwise, we conclude that $\widetilde{\Omega}^{-1} y$ is open if $\Omega^{-1} y$ is open.

From the first part of the proof we can infer that $\widetilde{\Omega} x_1 \cap \widetilde{\Omega} x_2 \neq \emptyset$. □

Corollary 1. *Let X be a connected topological space and $\Omega : X \to Y$ closed graph relation with nonempty compact values. If properties* (A) *and* (B) *below hold then* $\bigcap_{x\in X} \Omega x \neq \emptyset$.

(A) *for any nonempty finite subset $B \subseteq X$ the set $\bigcap_{x\in B} \Omega x$ is connected,*

(B) *for any nonempty subset $A \subseteq Y$ the set $\bigcap_{y\in A} \Omega^* y$ is connected.*

Proof. We keep the notation of Theorem 3. First we show that $\Omega x_1 \cap \Omega x_2 \neq \emptyset$ for any $x_1, x_2 \in X$.

We have $\Omega([\![x_1, x_2]\!]) \subset \Omega x_1 \cup \Omega x_2$. Let $\Gamma : [\![x_1, x_2]\!] \to \Omega x_1 \cup \Omega x_2$ be the restriction of Ω to $[\![x_1, x_2]\!]$. It is a closed graph relation, and therefore upper semicontinuous because $\Omega x_1 \cup \Omega x_2$ is compact. If $y \in \Omega x_1 \cup \Omega x_2$ then $\Gamma^* y = [\![x_1, x_2]\!] \cap \Omega^* y$. Also, $\Gamma x = \Omega x$ if $x \in [\![x_1, x_2]\!]$. This shows that Γ verifies conditions (A) and (B-2) from Theorem 3. Therefore $\Omega x_1 \cap \Omega x_2 \neq \emptyset$.

Now, fix $x_1 \in X$ and consider the relation $\Omega_1 x = \Omega x_1 \cap \Omega x$ from X to Ωx_1. It is upper semicontinuous and it has nonempty compact values. If $y \in \Omega x_1$ then $\Omega_1^* y = \Omega^* y$. From Theorem 3 we conclude that $\Omega_1 : X \to \Omega x_1$ has the finite intersection property, and therefore Ω also. Since the values are compact, the proof is complete. □

Connectedness is a rather weak condition and it is surprising that so much of the theory of minimax can be based on it. Traditional proofs rely on the Hahn–Banach theorem or on some form of Brouwer's fixed point theorem. But we can not expect to recover everything using only connectedness.

To prove the results of this section we had to use two kinds of very strong hypotheses:

(i) the relation Ω is semicontinuous, the values and the fibers are closed or open,

(ii) intersections of the values, or of the cofibers, must be connected.

An intersection of connected sets is rarely connected, so it would be desirable to have that kind of condition on the images or the fibers only and not on their intersections. This will be subject matter of the second half of this section. For now, we turn our attention to conditions that are strong enough to yield Fan's like results, Theorems 4 and 5.

A topological property shared by all convex sets is contractibility. We recall that a topological space X is *contractible* if there exists a continuous map $H:[0,1]\times X\to X$ and a point $x_0\in X$, such that for any $x\in X$, $H(0,x)=x_0$ and $H(1,x)=x$. Any starshaped set is contractible. A subset X of a topological vector space is *starshaped* if there is a point $x_0\in X$ such that for any point $x\in X$ the interval $[x_0,x]$ is contained in X. There is a notion weaker than contractibility. A nonempty topological space X is *homotopically trivial* if for any natural number n and any continuous map $g:\partial\Delta_n\to X$, defined on the boundary of an n-dimensional euclidean simplex, there exists a continuous map $f:\Delta_n\to X$ whose restriction to $\partial\Delta_n$ is g. We will see that this is an adequate notion for our purpose.

The first theorem is a topological version of Fan's intersection theorem. Its proof relies on the following lemma which is a particular case of results from [19].

Lemma 1. *In a topological space X let $\{A_j:\ j\in J\}$ be a family of sets, all closed or all open. Let $\langle J\rangle$ denote the family of nonempty finite subsets of J. Then $\bigcap_{j\in J}A_j\neq\emptyset$ if and only if there exists a family $\{C_L:\ L\in\langle J\rangle\}$ of nonempty homotopically trivial subsets of X such that*:

(A) $C_{L_1}\subseteq C_{L_2}$ *if* $L_1\subseteq L_2$,
(B) $C_L\subseteq\bigcup_{j\in L}A_j$ *for any* $L\in\langle J\rangle$.

Theorem 4. *Let X be a homotopically trivial space and $\Omega:X\to X$ a relation with nonempty values such that*:

(A) *all the values are open, or all the values are closed,*
(B) *for all* $x\in X$, $x\in\Omega x$,
(C) *for any subset* $A\subseteq X$ *the set* $\bigcap_{y\in A}\Omega^*y$ *is homotopically trivial, or empty.*

Then Ω has the finite intersection property.

Proof. To each nonempty finite subset $B\subseteq X$ let us associate the following set:

$$\Delta_\Omega(B)=\bigcap\bigl\{\Omega^*y:\ B\subseteq\Omega^*y\bigr\},$$

if $\{y:\ B\subseteq\Omega^*y\}=\emptyset$ let $\Delta_\Omega(B)=X$. It is easily verified that:

(1) $\Delta_\Omega(B)$ is homotopically trivial,
(2) if $B\subseteq B'$ then $\Delta_\Omega(B)\subseteq\Delta_\Omega(B')$,
(3) $\Delta_\Omega(B)\subseteq\bigcup_{x\in B}\Omega x$.

A straightforward application of Lemma 1 shows that Ω has the finite intersection property. □

In that theorem we still impose a condition on arbitrary intersections of cofibers and not only on the cofibers themselves. But the following simple example shows that the assumption on intersections of cofibers might be impossible to remove if we want to stay at the level of generality of Theorem 4.

Let X be a Banach space which is infinite-dimensional and put $\Omega x = \{x\}$ for each $x \in X$. The values of Ω are convex and compact. If $A \subseteq X$ is a compact subset then $\bigcap_{y\in A} \Omega^* y = X \setminus A$ is homotopically trivial, since it is homeomorphic to the whole space [5]. But, obviously, Ω does not have the finite intersection property.

Evidently condition (B) in Theorem 4 provides a continuous selection for $\Omega : X \to X$. It is not too difficult to see from Theorem 4 that $\Omega : X \to Y$ has the finite intersection property if (A) and (C) hold, and if $\Omega : X \to Y$ has a continuous selection. This is the idea behind the next result.

Theorem 5. *Let $\Omega : X \to Y$ be a relation with nonempty values from a homotopically trivial and paracompact space X to a space Y which is homotopically trivial. Assume that*:

(A) *all the values are open, or all the values are closed,*

(B) *for any subset $B \subseteq X$, the set $\bigcap_{x\in B} \Omega x$ is homotopically trivial, or empty,*

(C) *for any subset $A \subseteq Y$ the set $\bigcap_{y\in A} \Omega^* y$ is homotopically trivial, or empty,*

(D) $X = \bigcup_{y\in Y} int\, \Omega^{-1} y$.

Then Ω has the finite intersection property.

Proof. If $A \subseteq Y$ is a nonempty finite subset, let $\boldsymbol{C}_\Omega(A) = \bigcap_{A\subseteq\Omega x} \Omega x$, $\boldsymbol{C}_\Omega(A) = Y$ if there is no $x \in X$ such that $A \subseteq \Omega x$. Then $\boldsymbol{C}_\Omega(A)$ is homotopically trivial, nonempty, and $\boldsymbol{C}_\Omega(A) \subseteq \boldsymbol{C}_\Omega(A')$ if $A \subseteq A'$. Furthermore, if $A \subseteq \Omega x$ then $\boldsymbol{C}_\Omega(A) \subseteq \Omega x$.

Now, $\{int\, \Omega^{-1} y\colon\ y \in Y\}$ is an open covering of the paracompact space X, take a locally finite and finer open covering $\mathcal{V}$. For each $V \in \mathcal{V}$ choose $y(V) \in Y$ such that $V \subseteq int\, \Omega^{-1} y(V)$. We claim that there exists a continuous map $f : X \to Y$ such that

$$f(x) \in \boldsymbol{C}_\Omega\big(\big\{y(V)\colon\ x \in V\big\}\big).$$

Such a map is a selection of Ω. Indeed, if $x \in V$ then $x \in int\ \Omega^{-1} y(V)$, therefore $\{y(V)\colon\ x \in V\} \subseteq \Omega x$ and finally, by definition of $\boldsymbol{C}_\Omega$,

$$\boldsymbol{C}_\Omega\big(\big\{y(V)\colon\ x \in V\big\}\big) \subseteq \Omega x.$$

Let $\widetilde{\Omega} x = f^{-1}(\Omega x)$. We have $x \in \widetilde{\Omega} x$ since f is a continuous selection of Ω, and from the continuity of f and hypothesis (A), the values of $\widetilde{\Omega}$ are either all closed or all open. We also have $\widetilde{\Omega}^* x = \Omega^* f(x)$, and consequently $\bigcap_{x\in B} \widetilde{\Omega}^* x$ is homotopically trivial, or empty, for any subset B of X.

From Theorem 4 it follows that $\widetilde{\Omega}$ has the finite intersection property, and therefore also Ω.

Now, we have to prove our claim.

Denote by $\mathcal{N}(\mathcal{V})$ the nerve of the covering $\mathcal{V}$, by $|\mathcal{N}(\mathcal{V})|$ its geometric realization and by $|\mathcal{N}^k(\mathcal{V})|$ the kth skeleton. Denote by $p_V \in |\mathcal{N}^0(\mathcal{V})|$ the vertex associated to $V \in \mathcal{V}$. Starting from the map $\eta_0 : |\mathcal{N}^0(\mathcal{V})| \to Y$ which associates to a vertex $p_V \in |\mathcal{N}^0(\mathcal{V})|$ the point

$y(V) \in Y$ and using condition (B) a skeleton by skeleton construction yields a continuous map $\eta : |\mathcal{N}(\mathcal{V})| \to Y$ such that for any simplex $[p_{V_0}, \ldots, p_{V_n}]$ of $|\mathcal{N}(\mathcal{V})|$ one has

$$\eta\left(\bigcap_{i=0}^{i=n} St(p_{V_i})\right) \subseteq \boldsymbol{C}_{\Omega}\bigl(\bigl\{y(V_i)\colon\ i = 0, \ldots, n\bigr\}\bigr),$$

where $St(p_{V_i}) \subset |\mathcal{N}(\mathcal{V})|$ denotes the star of the vertex p_{V_i}. This is where condition (B) is used, details can be found in [16] Theorem 1.

From a partition of unity $\{\chi_V\colon\ V \in \mathcal{V}\}$ subortinated to V one has a continuous map $\chi : X \to |\mathcal{N}(\mathcal{V})|$ such that $\chi^{-1}(St(p_V)) \subseteq V$ for each $V \in \mathcal{V}$.

The map $\eta \circ \chi : X \to Y$ fulfils the claim since $\chi(x) \in \bigcap\{St(p_V)\colon\ \chi_V(x) > 0\}$ and therefore

$$\eta \circ \chi(x) \in \boldsymbol{C}_{\Omega}\bigl(\bigl\{y(V)\colon\ \chi_V(x) > 0\bigr\}\bigr) \subseteq \boldsymbol{C}_{\Omega}\bigl(\bigl\{y(V)\colon\ x \in V\bigr\}\bigr). \qquad \square$$

The example given after Theorem 4 shows that it might be hard to improve any of the previous two theorems without adding some strong conditions. The example uses in an essential way the fact that the space X is not finite-dimensional, and, as we will see, this is no coincidence. When X is finite-dimensional the assumptions on the intersections of images and cofibers can be replaced with assumptions on the images or the cofibers themselves. The results will be stated for a new class of relations.

A relation $\Omega : X \to Y$ between two topological spaces is a *Serre relation* if the following condition holds:

Let $h : \Delta_n \to X$ be a continuous map from an euclidean simplex Δ_n into X and $F_{n-1} \subseteq \Delta_n$ one of its $(n-1)$-dimensional faces. Then any continuous selection $g : F_{n-1} \to Y$ of the restriction of $\Omega \circ h$ to F_{n-1}, can be extended to a continuous selection of $\Omega \circ h : \Delta_n \to Y$.

Nothing is said here about the values of Ωx, but notice that if X is pathconnected and if a Serre relation Ω has at least one nonempty value, then all its values are nonempty. We will also see that being a Serre relation implies some kind of continuity. The denomination itself will soon be justified.

Lemma 2 (Homotopy extension). *$\Omega : X \to Y$ is a Serre relation if and only if for any homotopy $H : \Delta_n \times [0,1] \to X$ and for any continuous selection $g : \Delta_n \times \{0\} \to Y$ of $\Omega \circ H|_{\Delta_n \times \{0\}}$ there exists a continuous selection $f : \Delta_n \times [0,1] \to Y$ of $\Omega \circ H$ extending g.*

Proof. Let v_0 be one of the vertices of the $(n+1)$-dimensional simplex $\Delta_{n+1} \subseteq \mathbb{R}^{n+1}$. The face opposite v_0 is identified with $\Delta_n \times \{0\}$. Let $\theta_0 : \partial(\Delta_n \times [0,1]) \to \partial\Delta_{n+1}$ be any homeomorphism such that $\theta_0(\Delta_n \times \{0\}) = \Delta_n \times \{0\}$. A point p in the interior of $\Delta_n \times [0,1]$ is of the form $(1-t)\cdot(b_0, \frac{1}{2}) + t \cdot p'$, where $p' \in \partial(\Delta_n \times [0,1])$ with $0 \leqslant t < 1$ where b_0 is the barycenter of Δ_n. Call b_1 the barycenter of Δ_{n+1}, and extend θ_0 to a homeomorphism $\theta : \Delta_n \times [0,1] \to \Delta_{n+1}$ by taking $\theta(b_0, \frac{1}{2}) = b_1$ and $\theta(p) = (1-t)\cdot b_1 + t \cdot \theta_0(p')$.

The equivalence easily follows from the fact that θ is bijective and from $\theta(\Delta_n \times \{0\}) = \Delta_n \times \{0\}$. $\square$

This lemma is the justification for calling such relations Serre relations. Indeed, consider Ω as a subset of $X \times Y$, in other words identify the relation with its graph and let $p : \Omega \to X$ be the projection which sends $(x, y) \in \Omega$ to $x \in X$. Lemma 2 says that $\Omega : X \to Y$ is a Serre relation if for any homotopy $H : \Delta_n \times [0, 1] \to X$ and any continuous function $G_0 : \Delta_n \times \{0\} \to \Omega$ such that $p \circ G_0 = H(-, 0)$ there exists an homotopy $G : \Delta_n \times [0, 1] \to \Omega$ extending G_0 and such that $p \circ G = H$. In other words, the function $p : \Omega \to X$ has the homotopy lifting property with respect to simplices, it is a *weak fibration*, or a *Serre fibration*, see, for example, Switzer [44].

Lemma 3. *If $\Omega : X \to Y$ is a Serre relation with nonempty values, then for any continuous map $h : \Delta_n \to X$ from a simplex into X, the relation $\Omega \circ h : \Delta_n \to Y$ has a continuous selection.*

Proof. If $n = 0$ the conclusion is obviously true. Assume that the lemma holds if the dimension of the simplex is at most m. Given a continuous map $h : \Delta_{m+1} \to X$, the restriction of $\Omega \circ h$ to an m-dimensional face has a continuous selection. By definition of a Serre relation, that selection can be extended to Δ_{m+1}. $\square$

The following theorem should be compared with Corollary 7.

Theorem 6 (Existence of selection on AR). *Any Serre relation with nonempty values whose domain X is a compact finite-dimensional* AR *has a continuous selection.*

Proof. We can assume that X is contained in a simplex Δ_n. Since X is an AR there is a continuous retraction $r : \Delta_n \to X$. By Lemma 3, $\Omega \circ r : \Delta_n \to Y$ has a continuous selection. The restriction to X gives a continuous selection of Ω. $\square$

Corollary 2. *If X is a compact finite-dimensional* AR *then any Serre relation $\Omega : X \to X$ has a fixed point.*

Proof. By Theorem 6, Ω has a continuous selection, and a compact AR has the fixed point property. $\square$

We are now ready to prove a Fan like intersection theorem for Serre relations, Theorem 8. The nice thing is that the topological assumptions are on the individual images and cofibers, exactly as in Fan's theorem. On the other hand there is a strong continuity asumption on the relation, it is upper semicontinuous, the domain is also restricted, it has to be finite-dimensional. We will need a theorem of McClendon, (3.3) in [31], which we state now.

Theorem 7 (McClendon). *Let $\Omega : X \to Y$ be an open graph relation. If all the values are homotopically trivial then Ω is a Serre relation.*

We can now generalized Fan's intersection theorem, Theorem 2, to relations with homotopically trivial values. As a corollary we get Fan's theorem in $\mathbb{R}^n$ for relations with starshaped values.

Theorem 8 (Fan's Theorem on AR). *Let X be a compact finite-dimensional* AR *and $\Omega : X \to X$ a closed graph relation such that*:

(A) *$x \in \Omega x$ for each $x \in X$,*

(B) *the cofibers are homotopically trivial, or empty.*

Then $\bigcap_{x\in X} \Omega x \neq \emptyset$.

Proof. If $\bigcap_{x\in X} \Omega x = \emptyset$ then $\Omega^* : X \to X$ has nonempty values, which are homotopically trivial by (B). Furthermore, the graph of Ω^* is open, by Theorem 7 Ω^* is a Serre relation. Finally, by Corollary 2 there is a point $\overline{x} \in X$ such that $\overline{x} \in \Omega^*\overline{x}$, but this means $\overline{x} \notin \Omega\overline{x}$, and it contradicts (A). □

As we have seen, this is false if the restriction on the dimension of X is dropped.

Corollary 3. *Let $X \subseteq \mathbb{R}^n$ a compact starshaped subset which is the union of a finite number of convex sets and $\Omega : X \to X$ a closed graph relation such that*:

(A) *$x \in \Omega x$ for each $x \in X$,*

(B) *the cofibers are starshaped, or empty.*

Then $\bigcap_{x\in X} \Omega x \neq \emptyset$.

One should notice that the spirit of the proof is here entirely different from what it was in all the preceding theorems. We did not show first that Ω has the finite intersection property, we showed that the intersection can not be empty. The selection given by the identity map played a crucial role, as did the fixed point theorem. The same technique, a selection theorem combined with a fixed point theorem, has been used by Greco and Moschen [15] as we will see in the next section. The proof of the next theorem is a good example of that method. In the previous results Ω was a relation from a set X to itself, now we consider relations between a priori different sets. Even in a convex framework one can not always reduce the second case in any obvious way to the first, and radically new methods might be required, as in [14], for example. Here again, topological assumptions are made on individual images and cofibers of the relation, not on intersections as in Theorem 3.

Theorem 9. *Let X be an arbitrary topological space, Y a compact finite-dimensional* AR *and $\Omega : X \to Y$ a closed graph relation with nonempty values such that*:

(A) *the images are homotopically trivial,*

(B) *the cofibers are homotopically trivial, or empty.*

Then $\bigcap_{x\in X} \Omega x \neq \emptyset$

Proof. Let us proceed by contradiction, as in the previous theorem. If $\bigcap_{x\in X} \Omega x = \emptyset$ then Ω^* is a Serre relation. By Theorem 6 there is a continuous map $f : Y \to X$ such that $f(y) \in \Omega^* y$, for each $y \in Y$.

Let us consider the composition $\Omega \circ f : Y \to Y$. It is a closed graph relation, with nonempty values, from a compact finite-dimensional AR to itself. The values $\Omega \circ f(x)$ are homotopically trivial, and therefore acyclic. By Eilenberg–Montgomery's fixed point theorem, there is a point $\overline{y} \in Y$ such that $\overline{y} \in \Omega \circ f(\overline{y})$. This is a contradiction, since $y \notin \Omega \circ f(y)$ for each $y \in Y$. □

Again, this is false if the restriction on the dimension of Y is dropped.

Corollary 4. *Let X be an arbitrary topological space, $Y \subseteq \mathbb{R}^n$ a compact starshaped set which is the union of a finite number of convex sets and $\Omega : X \to Y$ a closed graph relation with nonempty values such that*:

(A) *the images are starshaped,*

(B) *the cofibers are starshaped, or empty.*

Then $\bigcap_{x \in X} \Omega x \neq \emptyset$.

We have seen that a Serre relation, with nonempty values, whose domain is a compact finite-dimensional AR has a continuous selection, Theorem 6. We will now look at existence and extension of continuous selections on polyhedrons and finite-dimensional ANR.

By *polyhedron*, not necessarily finite, we mean the geometric realization of a simplicial scheme with the Whitehead topology.

Theorem 10 is a reformulation of a standard result on Serre fibrations (see [42, Theorem 6, p. 375]). It can also be proved from Lemma 2 with a skeleton by skeleton construction.

Theorem 10. *Let $\Omega : X \to Y$ be a Serre relation, P a polyhedron and $P_0 \subseteq P$ a subpolyhedron, which could be empty. Given a map $H : P \times [0,1] \to X$ and a continuous selection g of $\Omega \circ H$ restricted to $(P \times \{0\}) \cup (P_0 \times [0,1])$, there exists a continuous selection of $\Omega \circ H$ extending g.*

Corollary 5. *Let $\Omega : X \to Y$ be a Serre relation, P a polyhedron and $h_i : P \to X$, $i = 1, 2$ two continuous maps. If $\Omega \circ h_0 : P \to Y$ has a continuous selection, and if h_0 and h_1 are homotopic, then $\Omega \circ h_1 : P \to Y$ has a continuous selection.*

Corollary 6. *If $\Omega : X \to Y$ is a Serre relation with nonempty values then the following holds*:

(A) *if X is contractible then, for any polyhedron P and any continuous map $h : P \to X$ the relation $\Omega \circ h$ has a continuous selection,*

(B) *if P is a contractible polyhedron and $h : P \to X$ a continuous map, then the relation $\Omega \circ h$ has a continuous selection.*

Proof. To prove the first part, consider an homotopy $H : X \times [0,1]\ X$ such that $H(-,0)$ is a constant map, let us say x_0, and $H(-,1)$ is the identity map. Then the map $(p,t) \mapsto$

$H(h(p), t)$ is a homotopy between the constant map h_0 given by $p \mapsto x_0$ and h. Since $\Omega x_0 \neq \emptyset$, the relation $\Omega \circ h_0$ has a selection. Consequently, $\Omega \circ h$ has a selection.

To prove the second part, notice that $\Omega \circ h : P \to Y$ is a Serre relation. Now, from part (A) applied to the identity map of P we have the conclusion. $\square$

On a noncontractible polyhedron P, existence of a continuous selection will follow either from existence of a continuous selection on a subpolyhedron which is strong deformation retract of P, Theorem 11, or from the assumption that the relation has homotopically trivial values, Theorem 12.

Theorem 11. *Let $\Omega : X \to Y$ be a Serre relation with nonempty values, P a polyhedron and $P_0 \subseteq P$ a subpolyhedron which is a strong deformation retract of P. Given a continuous map $h : P \to X$ and a selection $g_0 : P_0 \to Y$ of $\Omega \circ h_{|P_0}$ there exists a selection $g : P \to Y$ of $\Omega \circ h$ extending g_0.*

Proof. Let $\rho : P \times [0,1]\ P$ be a strong deformation retraction of P onto P_0, ($\rho(p,1) = p$ for each $p \in P$, $\rho(p,t) = p$ for each $(p,t) \in P_0 \times [0,1]$, and $\rho(-,0)$ is a retraction of P onto P_0.

Consider the homotopy $H : P \times [0,1] \to X$ given by $(p,t) \mapsto h(\rho(p,t))$. Notice that $g_0(\rho(p,t)) \in \Omega \circ h(\rho(p,t))$ if $(p,t) \in P_0 \times [0,1]$ and $g_0(\rho(p,0)) \in \Omega \circ h(\rho(p,0))$ for any $p \in P$. In otherwords, we have a selection of $\Omega \circ H$ restricted to $(P \times \{0\}) \cup (P_0 \times [0,1])$. By Theorem 10 there exists a selection $G : P \times [0,1] \to Y$ of $\Omega \circ H$ such that $G(p,t) = g_0(\rho(p,t))$ if $(p,t) \in P_0 \times [0,1]$, and $G(p,0) = g_0(\rho(p,0))$ for any $p \in P$. The map $p \to G(p,1)$ is a selection of $\Omega \circ h$ extending g_0. $\square$

Theorem 12 and Corollary 7 were proved by McClendon for what he calls r-open relations (relations which are fibrewise retracts of an open set, [33, Definition 1.1]), [33, Theorems 2.1 and 2.2]. We follow his proof closely.

Theorem 12 (McClendon). *Let $\Omega : X \to Y$ be a Serre relation with nonempty homotopically trivial values. Then, for any polyhedron P and any continuous map $h : P \to X$ the relation $\Omega \circ h$ has a continuous selection.*

Proof. Identify Ω with its graph $\Omega \subseteq X \times Y$ and let $p : \Omega \to X$ be the projection. Fix points $x_0 \in Y$ and $y_0 \in \Omega x_0$. For each $n \geqslant 1$, the projection $p : \Omega \to X$ induces a group homomorphism of homotopy groups $p_* : \Pi_n(\Omega, (x_0, y_0)) \to \Pi_n(X, x_0)$. From Lemma 2, we know that $p : \Omega \to X$ is a weak fibration whose fibers $p^{-1}(x) = \{x\} \times \Omega x$ are homeomorphic to Ωx. By hypothesis, each of the groups $\Pi_n(\Omega x_0, y_0)$ is trivial, and therefore also each of the groups $\Pi_n(p^{-1}(x_0), (x_0, y_0))$. From the exact homotopy sequence of the weak fibration $p : \Omega \to X$ (Switzer [44, p. 56]), and the triviality of the groups $\Pi_n(\Omega x_0, y_0)$ we conclude that $p_* : \Pi_n(\Omega, (x_0, y_0)) \to \Pi_n(X, x_0)$ is a groups isomorphism for each $n \geqslant 1$.

Denoting by $[P, \Omega]$ and $[P, X]$ the sets of homotopy classes of maps and by $[f]$ the homotopy class of a map, we have an onto map $p_* : [P, \Omega] \to [P, X]$ induced by p

(Switzer [44, Theorem 6.31]). There is therefore a map $G : P \to \Omega$ such that $[p \circ G] = [h]$. Now, let $q : \Omega \to Y$ be the projection of the graph onto Y, and put $g = q \circ G$. Clearly $g : P \to Y$ is a selection of $\Omega \circ p \circ G$ and $p \circ G$ and h are homotopic. By Corollary 5, $\Omega \circ h$ has a continuous selection. □

Corollary 7 (Existence of selections on ANR). *A Serre relation $\Omega : X \to Y$ with homotopically trivial and nonempty values whose domain X is a compact finite-dimensional* ANR *has a continuous selection.*

Proof. We can assume that X is embedded in some euclidean space. Since X is compact any of its neighborhood has a subneighborhood which is a polyhedron. There is therefore a retraction of a polyhedron onto X. □

Next comes the question of regularity of Serre relations. We show that on finite-dimensional ANR they are lower semicontinuous. The next lemma could be obtained as a variant of Lemma 3, it is also a simple consequence of Theorem 11. By a *polytope* we mean the convex hull of a finite set of points.

Lemma 4. *If $\Omega : X \to Y$ be a Serre relation with nonempty values then for any polytope P and any continuous map $h : P \to X$, the relation $\Omega \circ h : P \to Y$ is lower semicontinuous.*

Proof. Take a point $p_0 \in P$ and a point $\Omega \circ h(p_0)$. The one point set $\{p_0\}$ is a subpolytope of P (considering any triangulation of P having p_0 as a vertex), and we have an obvious selection $g_0 : \{p_0\} \to Y$ of $\Omega \circ h$ restricted to $\{p_0\}$. There is therefore a continuous selection $g : P \to Y$ of $\Omega \circ h$ such that $g_0(p_0) = y_0$. This proves the lower semicontinuity of $\Omega \circ h$, by Proposition 2.2 of [34]. □

Theorem 13 (Lower semicontinuity of Serre relations). *If X is a compact finite-dimensional* ANR, *then any Serre relation $\Omega : X \to Y$ is lower semicontinuous.*

Proof. We can assume that X is embedded in $\mathbb{R}^n$. There is a finite polyhedron Q which is a neighborhood of X and there is a retraction $r : Q \to X$ of P onto X. Take a finite family of polytopes $\{P_i:\ i = 1, \ldots, m\}$ such that $Q = \bigcup_{i=1}^{i=m} P_i$. For each $i = 1, \ldots, m$ the relation $\Omega \circ r|_{P_i} : P_i \to Y$ is lower semicontinuous. Therefore $\Omega \circ r : Q \to Y$ is also lower semicontinuous. Finally, Ω which is the restriction of $\Omega \circ r$ to X is lower semicontinuous. □

At this point, one could ask for examples of Serre relations, or criteria for a given relation to be a Serre relation. We have already seen McClendon's Theorem 7 which gives one class of Serre relations. As a matter of fact the class of Serre relations is rather large. We start with a few obvious examples and then we proceed with nontrivial classes one of which is derived from a selection theorem of Michael.

Any continuous map $f : X \to Y$ is a Serre relation, the constant relation $\Omega : X \to Y$ whose graph is the cartesian product $X \to Y$ is a Serre relation.

Let us assume that $\Omega : X \to Y$ is a Serre relation, that $\Delta \subseteq \Omega$ and that there exists a *vertical retraction* (a continuous retraction $r : \Omega \to \Delta$ of the form $(x, y) \mapsto (x, \widetilde{r}(x, y)))$, then Δ is a Serre relation. This is straightforward from the definition. This, in connection with McClendon's theorem raises an interesting question:

Let X and Y be topological spaces, even ANR or AR, which subsets $\Omega \subseteq X \times Y$ of the cartesian product are vertical retracts of an open neighborhood $U \supseteq \Omega$ with Ux homotopically trivial for each $x \in X$?

One thing is clear, such an Ω must share all the properties that a neighborhood retract of $X \times Y$ would have.

Let us say that $\Omega : X \to Y$ is a *locally trivial relation* if there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X, a topological space Z and a family of continuous maps $\phi_\lambda : U_\lambda \times Z \to Y$ such that:

(a) $\phi_\lambda(x, z) \in \Omega x$ for each $(x, z) \in U_\lambda \times Z$,

(b) for each $\lambda \in \Lambda$ the map $(x, z) \mapsto (x, \phi_\lambda(x, z))$ is a homeomorphism from $U_\lambda \times Z$ onto $(U_\lambda \times Y) \cap \Omega$.

We have seen that a trivial relation $\Omega : X \to Y$ is a Serre relation. Less obvious, is the fact that a locally trivial relation $\Omega : X \to Y$ is a Serre relation. This is a restatement of a standard result on fibre bundles, a proof of which can be found in Switzer [44, Proposition 4.10]. It is also a consequence of the fact that a local Serre fibration is a Serre fibration.

For our last result we need to recall a definition.

For subsets $A \subseteq B \subseteq Y$ of a topological space Y, the notation $A \sqsubseteq_n B$ means that for each natural number $m \leqslant n$ and each continuous map $g : \partial \Delta_{m+1} \to A$ there exists a continuous map $f : \Delta_{m+1} \to B$ whose restriction to $\partial \Delta_{m+1}$ is g.

A family $\mathcal{F}$ of nonempty subsets of Y is *equi*-LC^n if for any $y \in \bigcup \mathcal{F}$ and any neighborhood U of y there exists a neighborhood V of y such that $F \cap V \sqsubseteq_n F \cap U$ for any $F \in \mathcal{F}$. The family is equi-LC$^\infty$ if it is equi-LC^n for each n. A family of convex sets in a locally convex topological vector space is equi-LC$^\infty$. Notice that a subfamily of an equi-LC^n is also equi-LC^n.

Theorem 14. *Let Y be a complete metric space and $\Omega : X \to Y$ a lower semicontinuous relation with nonempty, closed and homotopically trivial values. If the set of values $\{\Omega x\colon x \in X\}$ is equi-*LC$^\infty$ *then Ω is a Serre relation.*

Proof. Consider a continuous map $h : \Delta_n \to X$ and a continuous selection $g : F_{n-1} \to Y$ on an $(n-1)$-dimensional face of Δ_n of $\Omega \circ h|_{F_{n-1}}$. From Theorem 1.2 in [35], g can be extended to a continuous selection of $\Omega \circ h$. $\square$

In [36] Michael gives a theorem on Serre fibrations similar to Theorem 14. Our proof is somewhat more direct. Theorem 14 and Corollary 6 yield right away a selection theorem.

3. Convex intersection theorems

This section offers three intersection theorems. They all deal with concave-convex relations. A relation $\Omega : X \to Y$ between two convex sets is *convex* if the values are convex, it is *concave* if the cofibers are convex, this is equivalent to $\Omega([x_1, x_2]) \subseteq \Omega x_1 \cup \Omega x_2$ for any $x_1, x_2 \in X$. It is *concave-convex* if it is both concave and convex. The name is due to Greco [13]. The first theorem below is a slight generalization of a result of Greco and Moschen [15], it gives a partial answer to a question raised in their paper. The proof, which relies on a selection theorem of Michael, follows the original idea of Greco and Moschen, to combine a selection theorem with a fixed point theorem. Their method has already been used in the previous section. Their theorem is not of the finite-intersection type. The second theorem is due to Bassanezi and Greco [4]. They ask if the finite-dimensionality of the space is unavoidable. Unfortunately, we are not able to answer, but we can show that a similar result holds in infinite dimension. The great interest of the theorems of Bassanezi–Greco and Greco–Moschen is that the values of Ω are not closed, this has been a sine qua non condition with everyone else. In functional terms, it means that semicontinuity (upper in one variable, lower in the other), of the function $f : X \times Y \to \overline{\mathbb{R}}$ can be replaced with marginal semicontinuity. Since the seminal paper of Sion [41], it has been more or less taken for granted by everyone working on minimax that one can not do without semicontinuity. The third theorem, the proof of which is too involved to be given here, is due to Flåm and Greco [11]. It gives a nontrivial necessary and sufficient condition for a concave-convex multifunction with compact values to have a constant selection. It is without a doubt one of the most original results of its kind. It has up till now resisted all efforts to give a simple proof, or to show that it can be derived from, or linked to other classical results, like the theorem of Knaster, Kuratowski and Mazurkiewicz, or some selection theorem. Also, the nature of the Simplex Condition is not so clear.

Let us now recall a theorem of Michael in [37] which will be the main ingredient of the proofs.

A *face of a closed convex set* $C \subset E$ contained in a Banach space E is a closed convex subset $F \subset C$ such that any segment in C, which has an interior point in F, must be contained in F. In other words, if $[x_0, x_1] \subseteq C$ and $]x_0, x_1[\cap F \neq \emptyset$ then $[x_0, x_1] \subseteq F$. Let $\mathcal{I}(C)$ be the complement in C of the union of the faces of C. If C is separable then $\mathcal{I}(C) \neq \emptyset$, Lemma 5.1 in [34].

Theorem 15 (Michael). *Let X be a metric space, E a Banach space and $\Omega : X \to E$ a lower semicontinuous relation with nonempty closed convex values. If, for each $x \in X$, Ωx is separable then the relation $x \mapsto \mathcal{I}(\Omega x)$ has a continuous selection.*

From the Hahn–Banach theorem we have:

Lemma 5. *Let $C \subseteq E$ be a convex subset of a Banach space E. If C is finite-dimensional, or if $\operatorname{int} C \neq \emptyset$ then $\mathcal{I}(\overline{C}) \subseteq C$.*

The following result was obtained by Greco and Moschen [15] under the assumtion that X and Y are finite-dimensional convex spaces. They asked if that assumption could be removed.

Theorem 16. *Let X and Y be convex subsets of Banach spaces E_1 and E_2, with either X or Y compact, and $\Omega : X \to Y$ a relation with convex cofibers. For Ω to have a constant selection it is sufficient that the following three conditions hold*:

(A) *$\Omega^* y$ is separable and $\mathcal{I}(\overline{\Omega^* y}) \subseteq \Omega^* y$ for any $y \in Y$,*

(B) *$\bigcap_{x \in U} \Omega x$ is closed in Y for any open subset $U \subseteq X$,*

(C) *there exists a lower semicontinuous relation with nonempty separable convex values $\Delta : X \to Y$ such that, for each $x \in X$, $\mathcal{I}(\overline{\Delta x}) \subseteq \Delta x \subseteq \Omega x$.*

Proof. We proceed by contradiction. If $\bigcap_{x \in X} \Omega x = \emptyset$ then $\Omega^* y \neq \emptyset$ for each $y \in Y$. Condition (B) says that $y \mapsto \Omega^* y$ is a lower semicontinuous relation from X to Y. The relation $y \mapsto \overline{\Omega^* y}$ from X to E_2 is lower semicontinuous with convex separable values. From Michael's theorem, $y \mapsto \mathcal{I}(\overline{\Omega^* y})$ has a continuous selection, from (A) it is also a continuous selection of Ω^*.

In conclusion, we have a continuous map $\gamma : Y \to X$ such that $y \notin \Omega \gamma(y)$ for each $y \in Y$.

Now, consider $\Delta : X \to Y$. Michael's theorem implies that $x \mapsto \mathcal{I}(\overline{\Delta x})$ has a continuous selection, from (C) it is also a continuous selection of Ω. We have a continuous map $\delta : X \to Y$ such that $\delta(x) \in \Omega x$ for each $x \in X$.

By hypothesis, either X or Y is compact. If Y is compact, consider the continuous map $\delta \circ \gamma : Y \to Y$. By Schauder's fixed point theorem there exists $\overline{y} \in Y$ such that $\delta \circ \gamma(\overline{y}) = \overline{y}$.

We have $\delta \circ \gamma(\overline{y}) \in \Omega \gamma(\overline{y})$, and therefore $\overline{y} \in \Omega \gamma(\overline{y})$. But $y \notin \Omega \gamma(y)$ for each $y \in Y$, we have reached a contradiction.

If X is compact, one proceeds similarly with the map $\gamma \circ \delta : X \to X$ to obtain a point $\overline{x} \in X$ such that $\delta(\overline{x}) \notin \Omega \overline{x}$. $\square$

Obviously, some of the separability assumptions can be removed, depending on which space is assumed to be compact.

Let us see in which ways we have generalized the theorem of Greco and Moschen. If X and Y are finite-dimensional convex sets, then the sets $\Omega^* y$ and Δx are automatically separable and from Lemma 5 we see that condition (A) is verified, as well as the left hand side of the inclusion in (C). If we do not assume that X and Y are finite-dimensional, Lemma 5 shows that it is enough to assume that the values Δx and the cofibers $\Omega^* y$ are finite-dimensional, or that the values Δx are finite-dimensional and the fibers $\Omega^{-1} y$ are closed.

The following fixed point theorem, which is an obvious generalization of Schauder's fixed point theorem, is worth mentioning. The proof is clear from Michael's selection theorem and Schauder's fixed point theorem. The finite-dimensional version is explicitely stated in Greco and Moschen.

Theorem 17. *Let $X \subseteq E$ be a compact convex set in a Banach space E, and $R : X \to X$ a lower semicontinuous relation with nonempty convex values. If the values are finite-dimensional, or have nonempty interior, or more generally if $\mathcal{I}(\overline{Rx}) \subseteq Rx$ for each $x \in X$, then there exists $\overline{x} \in X$ such that $\overline{x} \in R(\overline{x})$.*

Is that result true in locally convex topological vector spaces?

Bassanezi and Greco gave in [4] the following intersection theorem for relations with finite-dimensional codomain.

Theorem 18 (Bassanezi and Greco). *Let X be a convex set in a locally convex topological vector space, Y a compact convex finite-dimensional set. A relation $\Omega : X \to Y$ has a constant selection if the following poperties hold*:

(A) *for any open set $U \subseteq X$ the set $\bigcap_{x \in U} \Omega x$ is closed in Y*,

(B) *there is a lower semicontinuous concave-convex relation with nonempty values $\Delta : X \to Y$ such that $\Delta x \subseteq \Omega x$ for each $x \in X$.*

It is still unknown wether that result holds without the finite-dimensional assumption on Y. At first one could be tempted to prove the theorem of Bassanezi and Greco as we proved the theorem of Greco and Moschen, by contradiction. We would have a lower semicontinuous relation $\Omega^* : Y \to X$ with nonempty values. From $\Omega^* y \subseteq \Delta^* y$ we can take the convex hull of $\Omega^* y$ to get a lower semicontinuous relation $\Gamma y \subseteq \Delta^* y$. Now Δ has a continuous selection, since the values are finite-dimensional. But the difficulty is that we do not know that $\Gamma : Y \to X$ has a continuous selection, it is a lower semicontinuous relation with nonempty convex values. To use Michael's theorem we would have to know that $\mathcal{I}(\overline{\Gamma y}) \subseteq \Gamma y$.

We conclude with the theorem of Flåm and Greco [11] whose topological nature is still, as we said, somewhat mysterious.

First, let us introduce what they called *The Simplex Property*:

A relation $\Omega : X \to Y$ between two convex sets has the simplex property if the following condition holds: for any simplex $S \subseteq X$ of dimension at least one and for any vertex $v \in S$,

$$\text{if} \bigcap_{x \in S \setminus \{v\}} \Omega x \neq \emptyset \text{ then } \bigcap_{x \in S} \Omega x \neq \emptyset.$$

Theorem 19 (Flåm and Greco). *Let $\Omega : X \to Y$ be a concave-convex multifunction with nonempty compact values. Then*

$$\bigcap_{x \in X} \Omega x \neq \emptyset$$

if and only if the simplex property holds.

In the paper of Flåm and Greco it is shown that one cannot replace the compactness assumption on all the values by the hypothesis that the values are closed and one is compact. So, this is not a finite intersection type theorem.

4. Conclusion

The conclusion will take us back to the beginning, to what has been the main motivation for most of these intersection results, the theory of minimax.

Given a function $f: X \times Y \to \overline{\mathbb{R}}$ we ask whether

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

To a real number λ one can associate a relation $\Omega_\lambda x = \{y \in Y\colon\ f(x, y) \leqslant \lambda\}$ for which we would like to show that $\bigcap_{x \in X} \Omega_\lambda x \neq \emptyset$. The translation of the conditions on Ω_λ in terms of analytical conditions on the function $f: X \times Y \to \overline{\mathbb{R}}$ are well understood in the classical theorems, Sion or Fan, or in the theorems of Bassanezi and Greco and Greco and Moschen. For the theorem of Flåm and Greco one knows natural analytical conditions on the function which imply that Ω_λ has the simplex property, but there is still no convincing analytic translation of the simplex property itself (by convincing we mean a condition that is not a simple tautology), but useful conditions that imply the simplex property are known.

The topological intersection theorems raise the same kind of questions. What analytical conditions on a function $f: X \times Y \to \overline{\mathbb{R}}$ defined, say on subsets of $\mathbb{R}^n$, will guarantee that the sets $\{y \in Y\colon\ f(x, y) \leqslant \lambda\}$ are homotopically trivial, or form an equi-LC^n family, or that the relation Ω_λ is locally trivial?

References

[1] J.P. Aubin, I. Ekeland, Applied Nonlinear Analysis, Wiley, New York, 1984.

[2] C. Bardaro, R. Ceppitelli, Some further generalizations of Knaster–Kuratowski–Mazurkiewicz theorem and minimax inequalities, J. Math. Anal. Appl. 132 (1988) 484–490.

[3] C. Bardaro, R. Ceppitelli, Applications of the generalized Knaster–Kuratowski–Mazurkiewicz theorem to variational inequalities, J. Math. Anal. Appl. 132 (1988) 484–490.

[4] R.S. Bassanezi, G.H. Greco, A miniamax theorem, Topol. Methods Nonlinear Anal. 5 (1995) 249–253.

[5] C. Bessaga, A. Pelczynski, Selected Topics in Infinite-Dimensional Topology, PWN, Warsaw, 1975.

[6] K.C. Border, Fixed Point Theorems With Applications to Economics and Game Theory, Cambridge University Press, Cambridge, 1985.

[7] S. Chang, Y. Ma, Generalized KKM theorem on H-spaces with applications, J. Math. Anal. Appl. 163 (1992) 406–421.

[8] S.S. Chang, Y.J. Cho, X. Wu, Y. Zhang, The topological versions of KKM theorems and Fan's matching theorem with applications, Topol. Methods Nonlinear Anal. 1 (1993) 231–245.

[9] A. Dold, Partitions of unity in the theory of fibrations, Ann. Math. 78 (1963) 223–255.

[10] J. Dugundji, A. Granas, KKM-maps and variational inequalities, Ann. Scuola Norm. Sup. Pisa (4) 5 (1978) 679–682.

[11] S.D. Flåm, G.H. Greco, Minimax and intersection theorems, in: M.A. Théra, J.B. Baillon (Eds.), Fixed Point Theory and Applications, Longman Scientific and Technical, 1990, pp. 123–140.
[12] K. Fan, A generalization of Tychonoff's fixed point theorem, Math. Ann. 142 (1961) 305–310.
[13] G.H. Greco, Minimax theorems and saddling transformations, J. Math. Anal. Appl. 147 (1990) 180–197.
[14] G.H. Greco, C.D. Horvath, A reconstruction of polytopes by convex pastings, Mathematika, to appear.
[15] G.H. Greco, M.P. Moschen, A minimax inequality for marginaly semicontinuous functions, in: B. Ricceri, S. Simons (Eds.), Minimax Theory and Applications, Kluwer Academic Publishers, Dordrecht, 1998, pp. 41–50.
[16] C.D. Horvath, Extensions and selection theorems in topological spaces with a generalized convexity structure, Annales Fac. Sci. Toulouse 2 (1993) 253–269.
[17] C.D. Horvath, Quelques théorèmes en théorie des mini-max, C. R. Acad. Sci. Paris 310 (1990) 269–272.
[18] C.D. Horvath, A connectivity approach to minimax inequalities, Unpublished.
[19] C.D. Horvath, M. Lassonde, Intersection of sets with n-connected unions, Proc. Amer. Math. Soc. 125 (1997) 1200–1214.
[20] I. Joó, A general minimax theorem, Publ. Math. Debrecen 41 (1992) 1–4.
[21] J. Kindler, Topological intersection theorems, Proc. Amer. Math. Soc. 117 (1993) 1003–1011.
[22] J. Kindler, Itersection theorems and minimax theorems based on connectedness, J. Math. Anal. Appl. 178 (1993) 529–546.
[23] J. Kindler, Intersecting sets in midset spaces I, Arch. Math. 62 (1994) 49–57.
[24] J. Kindler, Intersecting sets in midset spaces II, Arch. Math. 62 (1994) 168–176.
[25] J. Kindler, R. Trost, Minimax theorems for interval spaces, Acta Math. Hungar. 54 (1989) 39–49.
[26] E. Klein, A.C. Thomson, Theory of Correspondences, Wiley, New York, 1984.
[27] H. König, A general minimax theorem based on connectedness, Arch. Math. 59 (1992) 55–64.
[28] H. König, Addendum to "A general minimax theorem based on connectedness", Arch. Math. 64 (1995) 139–143.
[29] H. Kneser, Sur un théorème fondamental de la théorie des jeux, C. R. Acad. Sci. Paris (1952) 2418–2420.
[30] J.F. McClendon, Fixed point of subopen multifunctions, Proc. Amer. Math. Soc. 84 (1982) 425–428.
[31] J.F. McClendon, Open subsets of fibrations, Comp. Math. 49 (1983) 109–119.
[32] J.F. McClendon, Minimax theorems for ANR's, Proc. Amer. Math. Soc. 90 (1984) 149–154.
[33] J.F. McClendon, On non-contractible valued multifunctions, Pacific J. Math. 115 (1984) 155–163.
[34] E. Michael, Continuous selections I, Ann. of Math. 63 (1956) 361–382.
[35] E. Michael, Continuous selections II, Ann. of Math. 64 (1956) 562–580.
[36] E. Michael, Continuous selections III, Ann. of Math. 65 (1957) 575–590.
[37] E. Michael, Dense families of continuous selections, Fund. Math. 47 (1959) 173–178.
[38] E. Michael, A generalization of a theorem on continuous selections, Proc. Amer. Math. Soc. 105 (1989) 236–243.
[39] B. Ricceri, Some topological minimax theorems via an alternative principle for multifunctions, Arch. Math. 60 (1993) 367–377.
[40] S. Simons, Minimax theorems and their proofs, in: D.Z. Du, P.M. Pardalos (Eds.), Minimax and Applications, Kluwer, Dordrecht, 1995, pp. 1–23.
[41] M. Sion, On general minimax theorems, Pacific J. Math. 8 (1958) 171–176.
[42] E.H. Spanier, Algebraic Topology, Springer, Berlin, 1966.
[43] L.L. Stacho, Minimax theorems beyond topological vector spaces, Acta. Sci. Math. 42 (1980) 157–164.

[44] R.W. Switzer, Algebraic Topology—Homotopy and Homology, Springer, Berlin, 1975.
[45] E. Tarafdar, A fixed point theorem in H-spaces and related results, Bull. Austral. Math. Soc. 42 (1990) 133–140.
[46] F. Telkersen, Some minimax theorems, Math. Scand. 31 (1972) 405–413.
[47] H. Tuy, On a general minimax theorem, Soviet. Math. Dokl. 15 (1974) 1689–1693.
[48] H. Tuy, On the general minimax theorem, Colloq. Math. 33 (1975) 145–158.
[49] Wu Wen-Tsün, A remark on the fundamental theorem in the theory of games, Sci. Rec. New. Ser. 3 (1959) 229–233.