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ON SOME SPECIAL PSEUDOCONVEX SPACES

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Abstract. We present two ways of constructing pseudoconvex spaces. The first theorem states that an interval space is pseudoconvex if the intervals are range sets of continuous functions and a cone erected to a convex set is convex. Secondly, if H is pseudoconvex and in the Cartesian product $H \times \mathbf{R}$ the intervals are defined by going first along the H-segment, then along the $\mathbf{R}$-segment (the other coordinates fixed) then $H \times \mathbf{R}$ is pseudoconvex.

In the existence theorems of Nash equilibrium points of games the fixed point theorems of Browder [1] or Kakutani [6] are in general used (see e.g. in Horváth, Sövegjártó [3]). When game theory is extended using a more general convexity concept, we need generalized versions of these fixed point theorems. Here we mention first the work of Komiya [7] who introduced the following structures.

DEFINITION. Let X be an arbitrary set. A *convex hull operation on* X is a mapping $\langle\cdot\rangle : \wp(X) \to \wp(X)$ *satisfying the conditions*

$$\langle\emptyset\rangle = \emptyset, \quad \big\langle\{x\}\big\rangle = \{x\} \quad \text{for } x \in X; \tag{1}$$

$$\langle A\rangle = \cup\big\{\langle F\rangle : F \subset A \text{ is a finite subset}\big\}; \tag{2}$$

$$\big\langle\langle A\rangle\big\rangle = \langle A\rangle. \tag{3}$$

The set $A \subset X$ is called *convex* if $\langle A\rangle = A$.

DEFINITION. A *convex space* is a triplet $\big(X, \langle\cdot\rangle, \Phi\big)$ where X is a topological space, $\langle\cdot\rangle$ is a convex hull operation on it and $\Phi := \{\varphi_F : F \subset X \text{ is finite}\}$ is a set of mappings

$$\varphi_F : \langle F\rangle \to \mathbf{R}^n, \quad \operatorname{card}(F) = n$$

for which

$$\varphi_F \text{ is a homeomorphic imbedding of } \langle F\rangle, \tag{4}$$

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and

$$(5)\qquad \begin{cases} \varphi_F \text{ is convex hull-preserving in the following sense:} \\ A \subset \langle F\rangle \Rightarrow \varphi_F\big(\langle A\rangle\big) = \mathrm{co}\,\big(\varphi_F(A)\big) \end{cases}$$

where $\mathrm{co}\,(\cdot)$ denotes the usual convex hull in $\mathbf{R}^n$.

Komiya proved the Browder fixed point theorems in convex spaces. I. Joó [4] further generalized this notion.

DEFINITION [4]. A *pseudoconvex space* is a triplet $\big(X, \langle\cdot\rangle, \Phi\big)$ such that
a) X is a topological space and $\langle\cdot\rangle$ is a convex hull operation on it,
b) $\Phi := \{\varphi_F \,:\, F \subset X \text{ is finite}\}$, where

$$\varphi_F \,:\, \Delta^n \to \langle F\rangle, \quad \mathrm{card}\,(F) = n+1,$$

for which $\Delta^n := \mathrm{co}\,(e_0, \dots, e_n)$ is the standard simplex in $\mathbf{R}^n$, $e_0 := (0, \dots, 0)$, $e_1 := (1, 0, \dots, 0)$, $\dots$, $e_n := (0, \dots, 0, 1)$ and

$$(4')\qquad \varphi_F \quad \text{is continuous,}$$

$$(5')\qquad \varphi_F\big(\,\mathrm{co}\,(e_{i_0}, \dots, e_{i_k})\big) = \langle x_{i_0}, \dots, x_{i_k}\rangle$$

for all subsimplexes $\mathrm{co}\,(e_{i_0}, \dots, e_{i_k})$.

Suppose that the following additional assumptions fulfil:
i) X is an M_1-space, i.e. every point x has a countable neighbourhood basis $G(x,n)$, $n \in \mathbf{N}$;
ii) $y \in G(x,n) \Leftrightarrow x \in G(y,n)$;
iii) For any x, N and $x_n \to x$ there exists N_1 such that

$$n > N_1 \Rightarrow G(x_n, n) \subset G(x, N).$$

Finally we assume that the convex hull operation is continuous in the following sense. Denote for $A \subset X$

$$G(A,n) := \cup\big\{G(x,n) \,:\, x \in A\big\},$$

then

$$(\mathrm{C})\qquad \forall A\ \forall N\ \exists n \quad \text{such that} \quad B \subset G(A,n) \Rightarrow \langle B\rangle \subset G\big(\langle A\rangle, N\big).$$

The aim of this paper is to introduce two types of spaces which will be proved to be pseudoconvex. We first define a special class of interval spaces in the sense of Stachó [8].

DEFINITION [8]. An *interval space* is a topological space X and a mapping $[\cdot,\cdot] : X \times X \to \wp(X)$ with the properties:

a) the set $[x_1, x_2] \subset X$ is connected,
b) $x_1, x_2 \in [x_1, x_2]$,

for all $x_1, x_2 \in X$.

A set $H \subset X$ is called *convex* if

$$x_1, x_2 \in H \Rightarrow [x_1, x_2] \subset H.$$

A special class of interval spaces can be defined through a continuous function

$$g : X \times X \times [0,1] \to X, \quad g(x,y,0) = x, \quad g(x,y,1) = y$$

if we define the intervals by

$$[x_1, x_2] := \{ g(x_1, x_2, t) : t \in [0,1] \}.$$

Introduce the following axiom which tells us how to construct the convex hull:

$$\text{(A)} \qquad \begin{cases} \text{If } K \subset X \text{ is convex and } x_0 \notin K, \text{ then the set} \\ \cup \{ [x_0, x] : x \in K \} \text{ is convex.} \end{cases}$$

Now we can prove the following

THEOREM 1. *If the interval space generated by a continuous function g as above is a compact topological space and satisfies* (A), *then it is a pseudoconvex space.*

PROOF. Define inductively the continuous functions $g_k : X^{k+1} \times \Delta^k \to X$ by the following rules:

$$g_1(x,y,\lambda) := g(x,y,\lambda_1) \quad \big(\lambda = (\lambda_0, \lambda_1),\ \lambda_0 + \lambda_1 = 1,\ \lambda_0 \geqq 0,\ \lambda_1 \geqq 0\big);$$

$$g_{k+1}(x_0, \dots, x_{k+1}, \lambda) := g\left(g_k \left(x_0, \dots, x_k, \frac{(\lambda_0, \dots, \lambda_k)}{\lambda_0 + \dots + \lambda_k} \right), x_{k+1}, \lambda_{k+1} \right)$$
$$\text{if} \quad \lambda_{k+1} < 1;$$
$$:= x_{k+1} \quad \text{if} \quad \lambda_{k+1} = 1.$$

We know that g_1 is continuous. If g_k is continuous then in the points (x, λ), $x = (x_0, \dots, x_{k+1})$, $\lambda_{k+1} < 1$ the function g_{k+1} is defined by a continuous formula. To show the continuity in the points (x, λ), $\lambda_{k+1} = 1$ we need the following

LEMMA 1. *Assume the conditions of Theorem* 1. *Then for every* y_0 *and for every neighbourhood* $y_0 \in U$ *there exists a number* $\delta > 0$ *such that*

$$t > 1 - \delta \Rightarrow g(x, y_0, t) \in U \quad \forall x.$$

PROOF. Indirectly suppose that there exists a neighbourhood $y_0 \in U$ and $t_n > 1 - \frac{1}{n}$, $x_n \in X$ such that $g(x_n, y_0, t_n) \notin U \quad \forall n$. Since X is a compact space, we have a point $x^* \in X$ in every neighbourhood of which there are infinitely many x_n. Since $g(x^*, y_0, 1) = y_0$ and g is continuous, there exists a neighbourhood $x^* \in V$ and a number $\delta > 0$ such that

$$x \in V, t > 1 - \delta \Rightarrow g(x, y_0, t) \in U.$$

In particular

$$x_n \in V, n \geqq n_0 \Rightarrow g(x_n, y_0, t_n) \in U.$$

Since there exists such an index n, we get a contradiction. □

Returning to the proof of Theorem 1 we see from Lemma 1 that g_{k+1} is continuous also if $\lambda_{k+1} = 1$. Since $g(x, y, \lambda)$ runs over the segment $[x, y]$, we see by induction on k that

$$g_k(x_0, \ldots, x_k, \Delta^k) = \langle x_0, \ldots, x_k \rangle.$$

Even more is true. Namely let $\Delta^* = \mathrm{co}\,(e_{i_0}, \ldots, e_{i_k}) \subset \Delta^n$ be a subsimplex of Δ^n, then

$$g_n(x_0, \ldots, x_n, \Delta^*) = \langle x_{i_0}, \ldots, x_{i_k} \rangle.$$

Indeed, use induction on k. If $k = -1$ then the construction shows that

$$g_n(x, \lambda) = g_n\big(x, (0, \ldots, 0, \underset{i_0}{1}, 0, \ldots, 0)\big) = g_{i_0}\big(x_0, \ldots, x_{i_0}, (0, \ldots, 0, 1)\big) = x_{i_0}.$$

If the case $k - 1$ is proved, then (in case $\lambda_{i_k} < 1$)

$$g_n(x, \lambda) = g_{i_k}\big(x_0, \ldots, x_{i_k}, (\lambda_0, \ldots, \lambda_{i_k})\big)$$
$$= g\left(g_{i_k - 1}\left(x_0, \ldots, x_{i_k - 1}, \frac{(\lambda_0, \ldots, \lambda_{i_k - 1})}{\lambda_0 + \ldots + \lambda_{i_k - 1}}\right), x_{i_k}, \lambda_{i_k}\right)$$
$$= g\left(g_{i_{k-1}}\left(x_0, \ldots, x_{i_{k-1}}, \frac{(\lambda_0, \ldots, \lambda_{i_{k-1}})}{\lambda_0 + \ldots + \lambda_{i_{k-1}}}\right), x_{i_k}, \lambda_{i_k}\right).$$

By induction hypothesis

$$y = g_{i_{k-1}}\left(x_0, \ldots, x_{i_{k-1}}, \frac{(\lambda_0, \ldots, \lambda_{i_{k-1}})}{\lambda_0 + \ldots + \lambda_{i_{k-1}}}\right)$$

runs over $\langle x_{i_0}, \dots, x_{i_{k-1}}\rangle$. Then the values $g(y, x_{i_k}, \lambda_{i_k})$, $0 \leqq \lambda_{i_k} < 1$, run over the segments $[y, x_{i_k}]$ with eventual exception of the points x_{i_k}. However,

$$x_{i_k} = g_n\big(x, (0, \dots, 0, \overset{i_n}{\smile}{1}, 0, \dots, 0)\big),$$

hence the axiom (A) implies that $g_n(x, \Delta^*) = \langle x_{i_0}, \dots, x_{i_k}\rangle$ as asserted. Theorem 1 is proved. □

REMARK 1. In the above proof the compactness of the space is used only locally in the sense that every point x is an inner point of a compact set. So if compactness is changed to local compactness, Theorem 1 remains true.

REMARK 2. The question arises whether in the pseudoconvexity of Theorem 1 the continuity axiom (C) holds or not. If we drop (A) and suppose only that g is continuous, then (C) may not be true. Here we provide an example where the convex hull of a ball is the whole space. Namely let $X := \mathbf{R}^2$ and for $x, y \in \mathbf{R}^2$ let the "segment" $[x, y]$ be the union of ordinary segments $[x, 2x]$, $[2x, 2y]$, $[2y, y]$. We define $g(x, y, t)$ such that it goes linearly from x to y through the above polygon when t varies from 0 to 1. It is clear from the construction that g is continuous and that the convex hull of any disk with center at the origin is the whole plane $\mathbf{R}^2$. So (C) fails to hold; but the segments are not convex and neither is (A) true. We formulate

PROBLEM 1. Is there a continuous function g such that the related convexity satisfies (A) but not (C)?

In the next part of the paper we investigate a generalization of the construction of the Joó–Stachó-type convexities [5]. First recall the ideas from Bezdek, Joó [2]. Suppose that we are given a set H, a set $\mathfrak{H}$ of subsets of H such that

a) $H_j \in \mathfrak{H} \Rightarrow \bigcap_{j \in J} H_j \in \mathfrak{H}$ for every index set J (finite or infinite);

b) A set $M \subset H$ belongs to $\mathfrak{H}$ if and only if $P, Q \in M \Rightarrow [P, Q] \subset M$; where

$$[P, Q] := \cap\{H_j \in \mathfrak{H} \ : \ P, Q \in H_j\}.$$

Now the sets of $\mathfrak{H}$ are called *convex subsets* of H. Remark that this convexity does not necessarily fulfil the axioms (1)–(3); for example, in case $\mathfrak{H} = \{H\}$. Consider the set $H^* := H \times \mathbf{R}$ and denote $P_t := (P, t) \in H^*$. Define the set

$$[P_{t_1}, Q_{t_2}]^* := [Q_{t_2}, P_{t_1}]^*$$
$$:= \{R_{t_1} \ : \ R \in [P, Q]\} \cup \{Q_t \ : \ t \in [t_1, t_2]\} \quad \text{if} \quad t_1 \leqq t_2.$$

Hence, we obtain a "product convexity" where along the "segment" $[P_{t_1}, Q_{t_2}]$ we first descend to the lower height, then inside the lower level we connect the points according to the convexity of H. This is precisely the convexity of [5] (in a much more general setting). Now we suppose that in H we are given a pseudoconvexity and investigate whether H^* inherits this property. We have

THEOREM 2. *Suppose that H is a pseudoconvex space, let*

$$[P,Q] := \cap\{K : K \subset H \text{ is convex, } P,Q \in K\},$$

and define a convexity in $H^* = H \times \mathbf{R}$ as described above. Suppose that H^* is endowed with the product topology and suppose that the following weakened injectivity property of the mappings φ_F is true on H:

$$\text{(I)}\quad \begin{cases} \text{if } F \subset H \text{ is finite, } F = A \cup B,\ A \cap B = \emptyset,\ x \in \langle A\rangle,\ y \in \langle B\rangle \\ \text{and for some } \lambda_1, \lambda_2, \mu \in \Delta^n,\ n = \operatorname{card}(F) - 1 \text{ we have} \\ \varphi_F(\lambda_1) = \varphi_F(\lambda_2) = x,\ \varphi_F(\mu) = y, \\ \text{then } \varphi_F\big(t\lambda_1 + (1-t)\mu\big) = \varphi_F\big(t\lambda_2 + (1-t)\mu\big) \text{ for } t \in [0,1]. \end{cases}$$

Under these assumptions H^* is also a pseudoconvex space.

PROOF. Consider the set $F^* := \{(P_0,t_0),\dots,(P_n,t_n)\}$. We have to construct a mapping $\varphi_{F^*} : \Delta^n \to \langle F^*\rangle$ with the desired properties of pseudoconvex spaces. Use induction on the number of different heights t_i. If there is only one height t then let $F := \{P_0,\dots,P_n\}$. The operator $A_t : P \mapsto (P,t)$ is an isomorphic imbedding of H into H^*, and it is convex hull preserving. Thus $\varphi_{F^*} := A_t \circ \varphi_F$ is a proper choice. In the general case denote t^* the lowest height in F^* and let $F^* = F_1^* \cup F_2^*$, $\operatorname{card}(F_i^*) = n_i + 1$, be the partition of F^* where F_2^* is the subset of points of height t^* and $F_1^* = F^* \setminus F_2^*$. By the induction hypothesis there exists a mapping $\varphi_{F_1^*} : \Delta^{n_1} \to \langle F_1^*\rangle$. The points $\lambda \in \Delta^n$, $\operatorname{card}(F^*) = n+1$ can be uniquely decomposed into the form

$$\lambda = t(\lambda_1,0) + (1-t)(0,\lambda_2), \quad \lambda_i \in \Delta^{n_i}, \quad t \in [0,1].$$

The following operators are continuous:

a) $\lambda \mapsto t$, $\lambda \mapsto \lambda_1$, $\lambda \mapsto \lambda_2$, $\lambda_1 \mapsto a := \varphi_{F_1^*}(\lambda_1)$,

b) $a \mapsto \hat{a}$, $a \mapsto \tau$ if $a = (\hat{a},\tau) \in H \times \mathbf{R}$.

Now take any point $\hat{\lambda}_1 \in \Delta^{n_1}$ with $\hat{a} = \varphi_F(\hat{\lambda}_1, 0)$ and define $\varphi_{F^*}(\lambda)$ in the following way:

i) if $t \in \left[0, \frac{\tau - t^*}{1+\tau-t^*}\right]$ then $\varphi_{F^*}(\lambda)$ decreases linearly the height of the point a from τ to t^* (here λ_1 and λ_2 are kept fixed),

ii) if $t \in \left[\frac{\tau - t^*}{1+\tau-t^*}, 1\right]$ then $\varphi_{F^*}(\lambda) = (A_{t^*} \circ \varphi_F)\big(c(t)\big)$,

where $c(t)$ varies linearly from $(\hat{\lambda}_1, 0)$ to $(0, \lambda_2)$. Now the weak injectivity property (I) shows that φ_{F^*} is continuous. Clearly the points $\varphi_{F^*}(\lambda)$ in i) run over the whole part of $\langle F^*\rangle$ over the lowest height. On the other hand, since the values $\varphi_F(\hat\lambda_1, 0)$ cover $\langle F_1\rangle$, we see from (I) that the values $\varphi_{F^*}(\lambda)$ from ii) cover the lowest level of $\langle F^*\rangle$. Hence, $\langle F^*\rangle = \varphi_{F^*}(\Delta^n)$. Now consider a subset $\tilde F^* \subset F^*$, with card $(\tilde F^*) = \tilde n + 1$. The above method shows that $\langle \tilde F^*\rangle = \varphi_{F^*}(\Delta^{\tilde n})$. The proof is complete. □

REMARK 3. If there are only two levels in the set F^* then the mapping $\lambda \mapsto \hat\lambda_1$ can be made continuous so in this case the additional assumption (I) can be eliminated. For general sets F^* this question remains open.

REMARK 4. If in the pseudoconvex space H the convex hull operation is continuous in the sense of (C) then the same is true in H^*. This follows from the facts that

a) product topology is given in $H^* = H \times \mathbf{R}$;

b) if $A^* \subset H^*$ then the convex hull $\langle A^*\rangle$ can be obtained in the following way. Let t be any height for which A^* has points at heights $\leqq t$ and $\geqq t$. Then the set

$$\langle A^*\rangle_t := \{\, a \in \langle A^*\rangle \, : \, a \text{ has height } t \}$$

can be generated as follows: project all points of A^* with height $\geqq t$ to the "hyperplane" of height t and take the convex hull of this set corresponding to the H-convexity. Since the H-convex hull is continuous, it follows from a) and b) that this is true also for the H^*-convexity. In connection with this we formulate the following

PROBLEM 2. Prove or disprove that all pseudoconvex spaces have property (C).

REMARK 5. There are pseudoconvex spaces where the convex sets are not homeomorphic to "ordinary" convex sets in Euclidean spaces. For example, in the pseudoconvexity of [5] (see our Theorem 2) a pseudoconvex set is a rectangle and a segment over it, which is not homeomorphic to an ordinary convex set since inner points are invariant in homeomorphic transformations.

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