Math. Proc. Camb. Phil. Soc. (2008), **144**, 97 © 2008 *Cambridge Philosophical Society*
doi:10.1017/S0305004107000655 *Printed in the United Kingdom*
First published online 17 January 2008

Banach spaces whose algebras of operators have a large group of unitary elements

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(*Received* 12 *September* 2006; *revised* 15 *December* 2006)

Abstract

We prove that a complex Banach space X is a Hilbert space if (and only if) the Banach algebra $\mathcal{L}(X)$ (of all bounded linear operator on X) is unitary and there exists a conjugate-linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T^{\bullet} = T^{-1}$ for every surjective linear isometry T on X. Appropriate variants for real spaces of the result just quoted are also proven. Moreover, we show that a real Banach space X is a Hilbert space if and only if it is a real JB^*-triple and $\mathcal{L}(X)$ is w'_{op}-unitary, where w'_{op} stands for the dual weak-operator topology.

1. *Introduction*

Unitary elements of a norm-unital normed (associative) algebra A are defined as those invertible elements u of A satisfying $\|u\| = \|u^{-1}\| = 1$. By a unitary normed algebra we mean a norm-unital normed algebra A such that the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. In the sequel we will denote by U_A the set of unitary elements of A. Relevant examples of unitary Banach algebras are all unital (complex) C^*-algebras and the real or complex discrete group algebras $\ell_1(G)$ for every group G. The reader is referred to [**1**, **2**, **7**, **9**, **10**, **12**, **25**] for a full view of the theory of unitary normed algebras. We remark that unital C^*-algebras and discrete group algebras, as well as those unitary Banach algebras which are finite-dimensional or commutative and semisimple, satisfy Property $(\mathcal{S})$:

$(\mathcal{S})$ There exists an algebra involution on the algebra, which is linear in the real case and conjugate-linear in the complex one, and maps each unitary element to its inverse.

We could expect all semisimple unitary Banach algebras to satisfy Property ($\mathcal{S}$). Indeed, according to [1], such a conjecture is equivalent to the one that every group G is "good" (which means that all primitive ideals of the complex Banach $*$-algebra $\ell_1(G)$ are $*$-invariant).

If X is a complex Hilbert space, then the algebra $\mathcal{L}(X)$ (of all bounded linear operators on X) is a C^*-algebra, and hence it is unitary. It seems to be an open problem whether or not all complex Banach spaces X such that $\mathcal{L}(X)$ is unitary are in fact Hilbert spaces. Some partial affirmative answers to this problem have been given in [7]. The present paper provides the reader with some new partial affirmative answers to this problem, formulates the actual variant of the problem for real spaces, and gives partial affirmative answers to such a variant.

We prove that a complex Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is unitary and satisfies Property ($\mathcal{S}$) (Theorem 2·4). Therefore, according to previous comments, if every group is good, then all complex Banach spaces X such that $\mathcal{L}(X)$ is unitary are in fact Hilbert spaces. Given a real or complex Banach space X, and a vector space topology τ on $\mathcal{L}(X)$ stronger than the weak-operator topology (in short, w_{op}), let us say that $\mathcal{L}(X)$ is τ-unitary if the τ-closed convex hull of $U_{\mathcal{L}(X)}$ is equal to the closed unit ball of $\mathcal{L}(X)$. It seems to be an unsolved problem whether or not $\mathcal{L}(X)$ is unitary whenever X is an infinite-dimensional real Hilbert space. Anyway, we prove that the problem just raised has an affirmative answer if, in its formulation, unitarity is replaced with w_{op}-unitarity (Corollary 2·6). On the other hand, if X is a complex Banach space such that $\mathcal{L}(X)$ is w_{op}-unitary and satisfies Property ($\mathcal{S}$), then X is a Hilbert space (see again Theorem 2·4). It turns out a reasonable conjecture that a real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w_{op}-unitary. We prove that a real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w_{op}-unitary, fulfils Property ($\mathcal{S}$), and the involution (say $\bullet$) given by such a property satisfies $T^\bullet \circ T \neq 0$ for some one-dimensional operator $T \in \mathcal{L}(X)$ (Theorem 2·7). Moreover, a variant of this last result, along the line of [15], is proven (Theorem 2·9).

It is shown in [7] that a complex Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is unitary and, for Y equal to X, X^* or X^{**}, there exists a biholomorphic automorphism of the open unit ball of Y which cannot be extended to a surjective linear isometry on Y. We note that the existence of such a biholomorphic automorphism of the open unit ball of a complex Banach space Y is easily guaranteed in the case that Y is a (complex) JB^*-triple [18]. We also note that complex Hilbert spaces are JB^*-triples. Keeping in mind these ideas, we extend to the setting of real spaces the results of [7] quoted above. Indeed, we prove the following facts:

(i) a real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w'_{op}-unitary (where w'_{op} means the dual weak-operator topology [16]) and X or X^{**} is a real JB^*-triple in the sense of [13] (Theorem 3·3);

(ii) a real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w''_{op}-unitary (where w''_{op} means the second dual weak-operator topology) and X^* is a real JB^*-triple (Theorem 3·6).

Incidentally, some of the new techniques developed in this section allows us also to complement the results of [7] in their original complex setting (see Theorem 3·9).

Notation. Let X be a real or complex normed space. Then the symbol $\mathcal{L}(X)$ (respectively, $\mathcal{K}(X)$, or $\mathcal{F}(X)$) will stand for the normed algebra of all bounded (respectively, compact, or finite-rank) linear operators on X. We denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the (topological) dual, respectively, of X. The normed space X will be regarded without notice as a subspace of its second dual X^{**}. For a bounded linear mapping T from X to another normed space Y, we denote by $T^*: Y^* \to X^*$ the transpose of T.

2. *The case that the algebra of operators has an involution*

Let X be a Banach space, and let x and f be in X and X^*, respectively. We denote by $x \otimes f$ the bounded linear operator on X defined by

$$(x \otimes f)(y) := f(y)x$$

for every $y \in X$.

LEMMA 2·1. *Let X be a Banach space, and let α be in X^{**} such that $h \otimes \alpha = T^*$ for some $h \in X^* \setminus \{0\}$ and $T \in \mathcal{L}(X)$. Then α lies in X.*

Proof. Take $x \in X$ such that $h(x) = 1$. Then, for every $g \in X^*$ we have

$$g(T(x)) = T^*(g)(x) = [(h \otimes \alpha)(g)](x) = \alpha(g)h(x) = \alpha(g).$$

Therefore, $\alpha = T(x) \in X$.

We recall that an algebra A of linear operator on a vector space X is said to be strictly dense if for every $k \in \mathbb{N}$ and arbitrary vectors $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ where $x_1, \ldots, x_k$ are linearly independent, there exists $T \in A$ such that $T(x_i) = y_i$ for all $i = 1, \ldots, k$. The following lemma is proved in [**22**, theorem 2·5·19] for complex spaces and linear algebra isomorphisms, but it proof works without changes in the case of real spaces, as well as in that of complex spaces and conjugate-linear algebra isomorphisms. Indeed, [**22**, theorem 2·5·19] is nothing other than an analytic specialization of [**14**, IV·9 and IV·11].

LEMMA 2·2. *Let X and Y be real (respectively, complex) Banach spaces, let A and B strictly dense Banach algebras of bounded linear operators on X and Y, respectively, containing some nonzero finite-rank operator, and let ϕ be a linear (respectively, conjugate-linear) algebra isomorphism from A onto B. Then there exists a bicontinuous linear (respectively, conjugate-linear) bijection $\psi : X \to Y$ such that $\phi(T) = \psi \circ T \circ \psi^{-1}$ for every $T \in A$.*

Let X and Y be complex Banach spaces, and let $\psi : X \to Y$ be a continuous conjugate-linear mapping. The transpose ψ^* of ψ is defined as the continuous conjugate-linear mapping from Y^* to X^* defined by

$$\psi^*(g)(x) := \overline{g(\psi(x))}$$

for every $(g, x) \in Y^* \times X$. We note that, if ψ is bijective, then the equality

$$\psi \circ (x \otimes f) \circ \psi^{-1} = \psi(x) \otimes (\psi^{-1})^*(f) \tag{2·1}$$

holds for every $(x, f) \in X \times X^*$.

The next proposition has a forerunner in [**17**, lemma 3]. Indeed, it is proved there that, if X is a complex Banach space, and if there exists a linear anti-automorphism ϕ of $\mathcal{L}(X)$, then X is reflexive, and there is a bicontinuous linear bijection ψ: $X \to X^*$ such that $\phi(T) = \psi^{-1} \circ T^* \circ \psi$ for every $T \in \mathcal{L}(X)$.

PROPOSITION 2·3. *Let X be a real (respectively, complex) Banach space. Then the following conditions are equivalent*:

(i) *there exists a linear (respectively, conjugate-linear) algebra involution $\bullet$ on $\mathcal{L}(X)$*;

(ii) *X is reflexive, and there exists a bicontinuous linear (respectively, conjugate-linear) bijection ψ: $X \to X^*$ such that $\psi^* = \pm\psi$ (respectively, $\psi^* = \psi$).*

When the above conditions are fulfilled, then the mappings $\bullet$ and ψ above are related by means of the equality $T^\bullet = \psi^{-1} \circ T^* \circ \psi$ for every $T \in \mathcal{L}(X)$.

Proof. (i) $\Rightarrow$ (ii) Let $\bullet$ be the linear (respectively, conjugate-linear) algebra involution on $\mathcal{L}(X)$ whose existence is assumed. Consider the algebras A and B of bounded linear operators on X and X^*, respectively, given by $A := \mathcal{L}(X)$ and $B := \{T^* : T \in \mathcal{L}(X)\}$, both endowed with their natural operator norms, and the linear (respectively, conjugate-linear) algebra isomorphism ϕ from A onto B defined by $\phi(T) := (T^\bullet)^*$. By Lemma 2·2, there exists a bicontinuous linear (respectively, conjugate-linear) bijection $\psi : X \to X^*$ such that

$$\phi(T) = \psi \circ T \circ \psi^{-1} \tag{2·2}$$

for every $T \in A$. Let x and f be in X and X^*, respectively. By (2·1) and (2·2), we have

$$\phi(x \otimes f) = \psi(x) \otimes (\psi^{-1})^*(f). \tag{2·3}$$

Since $\phi(x \otimes f)$ belongs to B, it follows from Lemma 2·1 that $(\psi^{-1})^*(f)$ lies in X. Since f is arbitrary in X^*, and the range of $(\psi^{-1})^*$ is X^{**}, we realize that X is reflexive. Now, from (2·3) and the definition of ϕ we derive $(x \otimes f)^\bullet = (\psi^{-1})^*(f) \otimes \psi(x)$, and hence

$$x \otimes f = (\psi^{-1})^*(\psi(x)) \otimes \psi((\psi^{-1})^*(f))$$

(because the mapping $\bullet$ is involutive). Since x and f are arbitrary in X and X^*, respectively, this implies that all elements in X are eigenvectors of $(\psi^{-1})^* \circ \psi$, and that all elements of X^* are eigenvectors of $\psi \circ (\psi^{-1})^*$, so that there exists in fact a nonzero real (respectively, complex) number λ satisfying $(\psi^{-1})^* \circ \psi = \lambda I_X$ and $\psi \circ (\psi^{-1})^* = \lambda^{-1} I_{X^*}$, where I_X and I_{X^*} stand for the identity mapping on X and X^*, respectively. Then, in the real case we have $\lambda^{-1}(\psi^{-1})^* = \lambda(\psi^{-1})^* = \psi^{-1}$, and hence $\psi^* = \pm\psi$. To conclude the proof of the present implication, let us consider the complex case. Then we have $\lambda^{-1}(\psi^{-1})^* = \overline{\lambda}(\psi^{-1})^* = \psi^{-1}$, and hence $|\lambda| = 1$ and $\psi^* = \overline{\lambda}\psi$. Taking $\mu \in \mathbb{C}$ with $\mu^2 = \overline{\lambda}$, we have $(\mu\psi)^* = \psi^*\mu = \overline{\lambda}\psi\mu = \overline{\lambda\mu}\psi = \mu\psi$. Since (2·2) determines ψ up to a nonzero complex multiple, the proof is concluded by replacing ψ with $\mu\psi$.

(ii) $\Rightarrow$ (i) Assume that Condition (ii) is fulfilled. Then we straightforwardly realize that the mapping $T \to T^\bullet := \psi^{-1} \circ T^* \circ \psi$ from $\mathcal{L}(X)$ to itself becomes a linear (respectively, conjugate-linear) algebra involution.

Let X be a Banach space. We put $\mathcal{G}_X := U_{\mathcal{L}(X)}$, and note that the elements of $\mathcal{G}_X$ are precisely the surjective linear isometries on X. We say that X is almost transitive if, for every $x \in S_X$, $\mathcal{G}_X(x)$ is dense in S_X. We say that X is convex-transitive if, for every $x \in S_X$, the convex hull of $\mathcal{G}_X(x)$ is dense in B_X. The weak-operator topology on $\mathcal{L}(X)$ (denoted by w_{op}) is defined as the initial topology on $\mathcal{L}(X)$ relative to the family of functionals

$$W := \{T \to f(T(x)) : (x, f) \in X \times X^*\}. \tag{2·4}$$

Now, let τ be a vector space topology on $\mathcal{L}(X)$ stronger than w_{op}. Then, since $B_{\mathcal{L}(X)}$ is w_{op}-closed, it is τ-closed, and hence contains the τ-closed convex hull of $\mathcal{G}_X$. We say that $\mathcal{L}(X)$ is τ-unitary if the containment just pointed out becomes an equality.

THEOREM 2·4. *Let X be a complex Banach space such that there exists a conjugate-linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$. Then the* *following conditions are equivalent*:

(i) $\mathcal{L}(X)$ *is unitary*;

(ii) $\mathcal{L}(X)$ *is* w_{op}*-unitary*;

(iii) X *is convex-transitive*;

(iv) X *is almost transitive*;

(v) X *is a Hilbert space.*

Proof. (i) $\Rightarrow$ (ii) Since the weak-operator topology is weaker than the norm topology.

(ii) $\Rightarrow$ (iii) By the right part of [**25**, theorem 5] (see Remark 2·10.(b) below).

(iii) $\Rightarrow$ (iv) Since X is reflexive (by Proposition 2·3), and reflexive Banach spaces are Asplund spaces, it follows from the assumption (iii) and [**5**, corollary 3·3] that X is almost transitive.

(iv) $\Rightarrow$ (v) By Proposition 2·3, X is reflexive, and there is a bicontinuous conjugate-linear bijection $\psi : X \to X^*$ satisfying $\psi^* = \psi$ and $\psi^{-1} \circ T^* \circ \psi = T^{-1}$ for every $T \in \mathcal{G}_X$. For $x, y \in X$, put $(x|y) := \psi(y)(x)$. It follows that $(\cdot|\cdot)$ is a continuous nondegenerate hermitian sesquilinear form on X satisfying

$$(T(x)|T(x)) = (x|x) \tag{2·5}$$

for every $x \in X$. By multiplying $(\cdot|\cdot)$ by a suitable real number if necessary, we may assume that the continuous nondegenerate hermitian sesquilinear form $(\cdot|\cdot)$ satisfies $(x_0|x_0) = 1$ for some $x_0 \in S_X$. Then, applying (2·5) and the assumption (iv), we derive $\|x\|^2 = (x|x)$ for every $x \in X$. Therefore X is a Hilbert space.

(v) $\Rightarrow$ (i) This is well known.

It is worth mentioning that Theorem 2·4 contains the known fact that complex Banach spaces X such that $\mathcal{L}(X)$ is a C^*-algebra (for some involution) are Hilbert spaces [**11**].

PROPOSITION 2·5. *Let H be a real Hilbert space. Then $B_{\mathcal{K}(H)}$ is contained in the norm-closed convex hull of $\mathcal{G}_H$.*

Proof. It is enough to show that $B_{\mathcal{F}(H)}$ is contained in $\overline{co}(\mathcal{G}_H)$. Let $T = \sum_{i=1}^n x_i \otimes y_i$ be in $B_{\mathcal{F}(H)}$ (where, for $x, y \in H$, $x \otimes y$ denotes the operator $z \to (z|y)x$). Let H_1 stand for the linear hull of $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and let H_2 be the ortogonal of H_1 in H. Then T is diagonal relative to the decomposition $H = H_1 \oplus H_2$, and the restriction of T to H_2 is zero. Now, let A denote the set of those elements in $\mathcal{L}(H)$ which are diagonal relative to the decomposition $H = H_1 \oplus H_2$, and whose restrictions to H_2 are real multiples of the identity operator on H_2. Then A is a subalgebra of $\mathcal{L}(H)$ isometrically isomorphic to $\mathcal{L}(H_1) \oplus_\infty \mathbb{R}$. Since $\mathcal{L}(H_1)$ is unitary (by [**2**, remark 2·9]), it follows from [**2**, proposition 2·8] that A is unitary. Since T lies in B_A, we deduce that $T \in \overline{co}(U_A)$. Finally, note that, since the identity mapping on H belongs to A, we have $U_A \subseteq \mathcal{G}_H$.

Let X be a Banach space. The ultraweak-operator topology on $\mathcal{L}(X)$ (denoted by $\overline{w_{op}}$) is defined as the initial topology on $\mathcal{L}(X)$ relative to the family of all functionals in the norm-closed linear hull in $(\mathcal{L}(X))^*$ of the set W defined by (2·4). It is well known that, if X is reflexive, then the Banach space $\mathcal{L}(X)$ can be naturally identified with $(X\widehat{\otimes}_\pi X^*)^*$ (where $\widehat{\otimes}_\pi$ denotes the complete projective tensor product) in such a way that $\overline{w_{op}}$ becomes the natural weak* topology (i.e., the weak topology on $\mathcal{L}(X)$ relative to the duality with its predual $X\widehat{\otimes}_\pi X^*$) [**8**, proposition 42·13].

COROLLARY 2·6. *Let H be a real Hilbert space. Then $\mathcal{L}(H)$ is $\overline{w_{op}}$-unitary.*

Proof. Keeping in mind Proposition 2·5, and the fact that the ultraweak-operator topology on $\mathcal{L}(H)$ is weaker that the norm topology, we deduce that the $\overline{w_{op}}$-closed convex hull of $\mathcal{G}_H$ contains $B_{\mathcal{K}(H)}$. On the other hand, since $(\mathcal{K}(H))^* = H\widehat{\otimes}_\pi H$, and $(H\widehat{\otimes}_\pi H)^* = \mathcal{L}(H)$, and the weak* topology on $\mathcal{L}(H)$ coincides with $\overline{w_{op}}$, we have that $B_{\mathcal{K}(H)}$ is $\overline{w_{op}}$-dense in $B_{\mathcal{L}(H)}$ (by Goldstine's theorem).

THEOREM 2·7. *Let X be a real Banach space such that there exists a linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T_0^\bullet \circ T_0 \neq 0$ for some one-dimensional operator $T_0 = x_0 \otimes f_0 \in \mathcal{L}(X)$, and $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$. Then the following conditions are equivalent*:

(i) *$\mathcal{L}(X)$ is $\overline{w_{op}}$-unitary*;
(ii) *$\mathcal{L}(X)$ is w_{op}-unitary*;
(iii) *X is convex-transitive*;
(iv) *X is almost transitive*;
(v) *X is a Hilbert space.*

Proof. (i) $\Rightarrow$ (ii) Since the weak-operator topology is weaker than the ultraweak-operator topology.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) in the present theorem are the same as the corresponding ones in Theorem 2·4, and are proved in the same way.

(iv) $\Rightarrow$ (v) By Proposition 2·3, X is reflexive, and there is a bicontinuous linear bijection $\psi : X \to X^*$ satisfying $\psi^* = \pm\psi$ and $T^\bullet = \psi^{-1} \circ T^* \circ \psi$ for every $T \in \mathcal{L}(X)$. Assume that $\psi^* = -\psi$. Then, for every $x \in X$ we have $\psi(x)(x) = 0$, and hence

$$\begin{aligned} T_0^\bullet \circ T_0 &= \psi^{-1} \circ (x_0 \otimes f_0)^* \circ \psi \circ (x_0 \otimes f_0) = \psi^{-1} \circ (f_0 \otimes x_0) \circ \psi \circ (x_0 \otimes f_0) \\ &= (\psi^{-1}(f_0) \otimes \psi^*(x_0)) \circ (x_0 \otimes f_0) = \psi(x_0)(x_0)(\psi^{-1}(f_0) \otimes f_0) = 0, \end{aligned}$$

which is not possible. Therefore we have that $\psi^* = \psi$. For $x, y \in X$, put $(x|y) := \psi(y)(x)$. It follows that $(\cdot|\cdot)$ is a continuous nondegenerate symmetric bilinear form on X satisfying $(T(x)|T(x)) = (x|x)$ for all $x \in X$ and $\in \mathcal{G}_X$. Then, that X is a Hilbert space follows from the assumption (iv) as in the proof of the implication (iv) $\Rightarrow$ (v) in Theorem 2·4.

(v) $\Rightarrow$ (i) By Corollary 2·6.

COROLLARY 2·8. *Let X be a real Banach space such that there exists a linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T_0^\bullet \circ T_0 \neq 0$ for some one-dimensional operator $T_0 \in \mathcal{L}(X)$, and $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$. If $\mathcal{L}(X)$ is unitary, then X is a Hilbert space.*

Let A be a norm-unital normed algebra. We say that A is maximal if, whenever $|||\cdot|||$ is an equivalent norm on A converting A into a norm-unital normed algebra and satisfying $U_A \subseteq U_{(A,|||\cdot|||)}$, we have that $U_A = U_{(A,|||\cdot|||)}$. In general, maximality and unitarity of A are independent conditions. However, it is well known that, in the case that $A = \mathcal{L}(X)$ for some normed space X, unitarity is strictly stronger than maximality (see for example [**1**, remark 2·6·(d)]). An involution $*$ on A is said to be proper if $x^*x \neq 0$ for every $x \in A \setminus \{0\}$. A joint variant of Theorems 2·4 and 2·7 is the following result in the spirit of [**15**].

THEOREM 2·9. *Let X be a real (respectively, complex) Banach space. Then the following assertions are equivalent*:

(i) *$\mathcal{L}(X)$ is maximal, and there exists a proper linear (respectively, conjugate-linear) algebra involution $\bullet$ on $\mathcal{L}(X)$ such that $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$*;
(ii) *X is a Hilbert space.*

Proof. (i) $\Rightarrow$ (ii) Let ψ be the linear (respectively, conjugate-linear) bijection from X to X^* given by Proposition 2·3 because of the existence of the involution $\bullet$ on $\mathcal{L}(X)$, and for $x, y \in X$ put $(x|y) := \psi(y)(x)$. We know that $(\cdot|\cdot)$ is a symmetric or antisymmetric bilinear form (respectively, a hermitian sesquilinear form) on X satisfying

$$(T(x)|y) = (x|T^\bullet(y)) \tag{2·6}$$

for every $T \in \mathcal{L}(X)$ and all $x, y \in X$, and that for $(x, f) \in X \times X^*$ we have

$$(x \otimes f)^\bullet \circ (x \otimes f) = (x|x)(\psi^{-1}(f) \otimes f). \tag{2·7}$$

Now, the assumption that the involution $\bullet$ is proper, together with (2·7), gives $(x|x) \neq 0$ for every $x \in X \setminus \{0\}$ (which implies in the real case that $(\cdot|\cdot)$ cannot be antisymmetric). It follows from the connectedness of $X \setminus \{0\}$ (the case $X = \mathbb{R}$ is trivial) and the continuity of the mapping $x \to (x|x)$ from X to $\mathbb{R}$ that, by multiplying $(\cdot|\cdot)$ by a suitable real number if necessary, there is no loss of generality in assuming that $(\cdot|\cdot)$ is an inner product on X satisfying $(x_0|x_0) = 1$ for some prefixed $x_0 \in S_X$. Let $|\cdot|$ denote the pre-Hilbertian norm associated to $(\cdot|\cdot)$. We claim that $|\cdot|$ and $\|\cdot\|$ are equivalent norms on X. Indeed, for every $x \in X$ we have

$$|x|^2 = \psi(x)(x) \leqslant \|\psi(x)\|\|x\| \leqslant \|\psi\|\|x\|^2,$$

and hence $|\cdot| \leqslant \sqrt{\|\psi\|}\|\cdot\|$ on X. Moreover, for $x \in X$ we can find $f \in S_{X^*}$ with $f(x) = \|x\|$, so that

$$\begin{aligned}\|x\| = f(x) = (x|\psi^{-1}(f)) \leqslant |x||\psi^{-1}(f)| \leqslant |x|\sqrt{\|\psi\|}\|\psi^{-1}(f)\| \\ \leqslant |x|\sqrt{\|\psi\|}\|\psi^{-1}\|\|f\| = |x|\sqrt{\|\psi\|}\|\psi^{-1}\|,\end{aligned}$$

and therefore $\|\cdot\| \leqslant \sqrt{\|\psi\|}\|\psi^{-1}\||\cdot|$ on X. Now that the claim has been proved, we invoke the assumption that $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$, together with (2·6), to realize that $\mathcal{G}_X \subseteq \mathcal{G}_{(X,|\cdot|)}$. In this way, denoting by $|||\cdot|||$ the operator norm on $\mathcal{L}(X)$ corresponding to the norm $|\cdot|$ on X, it turn out that $|||\cdot|||$ is an equivalent algebra norm on $\mathcal{L}(X)$ converting $\mathcal{L}(X)$ into a norm-unital normed algebra and satisfying $U_{\mathcal{L}(X)} \subseteq U_{(\mathcal{L}(X),|||\cdot|||)}$. It follows from the assumption that $\mathcal{L}(X)$ is maximal that $U_{\mathcal{L}(X)} = U_{(\mathcal{L}(X),|||\cdot|||)}$, or equivalently $\mathcal{G}_X = \mathcal{G}_{(X,|\cdot|)}$. Since $(X, |\cdot|)$ is almost transitive, and x_0 belongs to $S_X \bigcap S_{(X,|\cdot|)}$, it follows that $|\cdot| = \|\cdot\|$ on X.

(ii) $\Rightarrow$ (i) This is well known. Indeed, the maximality of $\mathcal{L}(X)$, for a Hilbert space X, follows from the almost transitivity of X, together with [**24**, theorem 9·6·3] and [**25**, lemma 1 and theorem 1].

It follows from the above proof that a real (respectively, complex) Banach space is isomorphic to a Hilbert space if (and only if) there exists a proper linear (respectively, conjugate-linear) algebra involution $\bullet$ on $\mathcal{L}(X)$. The real case of this fact is one of the main results in [**15**].

Remark 2·10. (*a*) For a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, consider the following conditions:

(i) X is a Hilbert space;

(ii) $\mathcal{L}(X)$ is unitary.

We already know that, if $\mathbb{K} = \mathbb{C}$ or X is finite-dimensional, then (i) implies (ii). It is also known that, if X is finite-dimensional, then (ii) implies (i) (see Part (*b*) of the present remark), so that (i) is actually equivalent to (ii) in the finite-dimensional setting. However, the following problems seem to remain still open:

(P1) Does (i) imply (ii) when $\mathbb{K} = \mathbb{R}$ and X is infinite-dimensional?

(P2) Does (ii) imply (i) when X is infinite-dimensional?

Partial affirmative answers to (P2) are those given by Corollaries 2·8 and 3·7 (for $\mathbb{K} = \mathbb{R}$) and Theorems 2·4 and 3·9 (for $\mathbb{K} = \mathbb{C}$). Nevertheless, if the answer to (P1) were completely negative, then Corollaries 2·8 and 3·7 would become only characterizations of finite-dimensional real Hilbert spaces, and the following problem would merit a special consideration:

(P3) Is there an infinite-dimensional real Banach space X such that $\mathcal{L}(X)$ is unitary?

(b) It is well known that

(♮) Convex-transitive finite-dimensional real or complex Banach spaces are Hilbert spaces

(see [**24**, theorem 9·7·1 and proposition 9·6·1]). It follows from (♮) and Lemma 3·1 below that, *if X is a finite-dimensional real or complex Banach space such that $\mathcal{L}(X)$ is unitary, then X is a Hilbert space*. The result just formulated seems to have been stated first in [**20**]. Our favorite proof consists of putting together (♮) and the general fact that, *if X is a real or complex Banach space such that $\mathcal{L}(X)$ is unitary (or merely w_{op}-unitary), then X is convex transitive* [**9**, theorem 6·4] (see also [**25**, theorem 5]). By the way, in both [**9**] and [**25**] it is claimed that, conversely, if X is a convex-transitive Banach space, then $\mathcal{L}(X)$ is w_{op}-unitary. However, the proof of such a claim contains a gap which seems to us difficult to overcome. Indeed, w_{op}-continuous linear functionals on $\mathcal{L}(X)$ need not be of the form $T \to f(T(x))$ for some $(x, f) \in X \times X^*$. Since Hilbert spaces are convex-transitive, our criticism above gives special interest to Corollary 2·6.

3. *The case of real JB^*-triples*

Let X be a Banach space. Following [**16**], we define the dual weak-operator topology on $\mathcal{L}(X)$ as the initial topology on $\mathcal{L}(X)$ relative to the family of functionals

$$W' := \{T \to \alpha(T^*(f)) : (f, \alpha) \in X^* \times X^{**}\},$$

and we denote it by w'_{op}. We also consider the topology $\overline{w'_{op}}$ on $\mathcal{L}(X)$, defined as the initial topology on $\mathcal{L}(X)$ relative to the family of all functionals in the norm-closed linear hull of W' in $(\mathcal{L}(X))^*$. Since $W \subseteq W'$, where W is defined by (2·4), we have $w_{op} \leqslant w'_{op}$ and $\overline{w_{op}} \leqslant \overline{w'_{op}}$. Moreover, the two inequalities above become equalities whenever X is reflexive.

LEMMA 3·1. *Let X be a Banach space such that $\mathcal{L}(X)$ is w'_{op}-unitary. Then, for every f in S_{X^*}, we have*

$$\overline{co}\{T^*(f) : T \in \mathcal{G}_X\} = B_{X^*}.$$

Proof. Let f be in S_{X^*}, let g be in B_{X^*}, and let $-1 < \delta < 1$. Choose $x \in B_X$ with $f(x) = \delta$, and denote by F the operator on X defined by $F(y) := g(y)x$. Then there exists a net $\{F_\lambda\}$ in the convex hull of $U_{\mathcal{L}(X)}$ converging to F in the dual weak-operator topology. Therefore, $\{\alpha(F^*_\lambda(f))\}$ converges to $\alpha(F^*(f)) = \alpha(\delta g)$ for every $\alpha \in X^{**}$. In other words, $\{F^*_\lambda(f)\}$ converges to δg in the weak topology of X^*, and hence δg belongs to the weak-closed convex hull of $\{T^*(f) : T \in \mathcal{G}_X\}$. Letting $\delta \to 1$, and keeping in mind that weakly closed convex subsets of X^* are norm-closed, the arbitrariness of g in B_{X^*} yields that

$$B_{X^*} \subseteq \overline{co}\{T^*(f) : T \in \mathcal{G}_X\}.$$

We recall that a complex JB^*-triple is a complex Banach space X with a continuous triple product $\{\cdots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(i) for all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum;

(ii) the main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X;

(iii) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning Condition (i) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be hermitian if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Following [**13**], we define real JB^*-triples as norm-closed real subtriples of complex JB^*-triples. Here, by a subtriple we mean a subspace which is closed under triple products of its elements. An element e of a real JB^*-triple is said to be a tripotent if $\{eee\} = e$. Real JBW^*-triples where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately weak*-continuous (see [**13**, definition 4·1 and theorem 4·4]). Later, it has been shown in [**19**] that the requirement of separate w^*-continuity of the triple product is superabundant.

The following lemma becomes a generalization of [**6**, corollary 2·6] to the real setting.

LEMMA 3·2. *Let X be an almost transitive real JBW^*-triple. Then X is a Hilbert space.*

Proof. Keeping in mind that extreme points of the closed unit ball of a real JB^*-triple are tripotents [**13**, lemma 3·3], the Krein–Milman theorem and the almost transitivity of X give us that the set of all nonzero tripotents of X is dense in S_X. Since the set of tripotents of X is closed, we derive that $\{xxx\} = \|x\|^2 x$ for every $x \in X$. Finally, arguing as in the proof of [**23**, lemma 1], we realize that X is a Hilbert space.

THEOREM 3·3. *Let X be a real Banach space. Then the following assertions are equivalent*:

(i) *X is a real JB^*-triple, and $\mathcal{L}(X)$ is $\overline{w'_{op}}$-unitary;*

(ii) *X is a real JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary;*

(iii) *X^{**} is a real JB^*-triple, and $\mathcal{L}(X)$ is $\overline{w'_{op}}$-unitary;*

(iv) *X^{**} is a real JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary;*

(v) *X is a Hilbert space.*

Proof. The implications (i) ⇒ (ii) and (iii) ⇒ (iv) hold because $w'_{op} \leqslant \overline{w'_{op}}$, whereas the ones (i) ⇒ (iii) and (ii) ⇒ (iv) follow from the fact that the bidual of every real JB^*-triple is a real JB^*-triple [**13**, lemma 4·2].

(iv) ⇒ (v) Since X^{**} is a real JBW^*-triple (by assumption), and B_{X^*} has extreme points (by the Krein–Milman theorem), it follows from [**21**, corollary 2·1] that X^{**} has a "minimal tripotent" (see [**21**] for a definition), which is a point of Fréchet-differentiability of the norm [**4**, lemma 3·1]. This implies that the norm of X^{**} is "non rough" (see [**5**] for a definition). On the other hand, since $\mathcal{L}(X)$ is w'_{op}-unitary (by assumption), Lemma 3·1 applies, giving that X^* is convex-transitive. It follows from the implication (4) ⇒ (1) in [**5**, theorem 3·2, remark 4·6] that X is reflexive and almost transitive. By Lemma 3·2, X is a Hilbert space.

(v) $\Rightarrow$ (i) Keeping in mind that Hilbert spaces are reflexive, it follows from the assumption (v) and Corollary 2·6 that $\mathcal{L}(X)$ is $\overline{w'_{op}}$-unitary. On the other hand, the fact that real Hilbert spaces are real JB^*-triples is well-known. Indeed, a possible choice of the triple product $\{\cdots\}$ is the one given by $\{xyz\} := ((x|y)z + (z|y)x)/2$.

Let X be a Banach space. We denote by w''_{op} the initial topology on $\mathcal{L}(X)$ relative to the family of functionals

$$W'' := \{T \to \Lambda(T^{**}(\alpha)) : (\alpha, \Lambda) \in X^{**} \times X^{***}\},$$

and by $\overline{w''_{op}}$ the initial topology on $\mathcal{L}(X)$ relative to the family of all functionals in the norm-closed linear hull of W'' in $(\mathcal{L}(X))^*$. We have $w_{op} \leqslant w'_{op} \leqslant w''_{op}$ and $\overline{w_{op}} \leqslant \overline{w'_{op}} \leqslant \overline{w''_{op}}$, with equalities instead of inequalities if X is reflexive.

LEMMA 3·4. *Let X be a Banach space such that $\mathcal{L}(X)$ is w''_{op}-unitary. Then, for every α in $S_{X^{**}}$, we have*

$$\overline{co}\{T^{**}(\alpha) : T \in \mathcal{G}_X\} \supseteq B_X.$$

Proof. Let α be in $S_{X^{**}}$, let x be in B_X, and let $-1 < \delta < 1$. Choose $f \in B_{X^*}$ with $\alpha(f) = \delta$, and denote by F the operator on X defined by $F(y) := f(y)x$. Then there exists a net $\{F_\lambda\}$ in the convex hull of $U_{\mathcal{L}(X)}$ converging to F in the topology w''_{op}, and hence $\{\Lambda(F_\lambda^{**}(\alpha))\}$ converges to $\Lambda(F^{**}(\alpha)) = \Lambda(\delta x)$ for every $\Lambda \in X^{***}$. Therefore, $\{F_\lambda^{**}(\alpha)\}$ converges to δx in the weak topology of X^{**}, and hence δx belongs to the weak-closed convex hull of $\{T^{**}(\alpha) : T \in \mathcal{G}_X\}$. Letting $\delta \to 1$, and keeping in mind the arbitrariness of x in B_X, we obtain that

$$B_X \subseteq \overline{co}\{T^{**}(\alpha) : T \in \mathcal{G}_X\}.$$

Let X be a Banach space. We say that X is L-embedded if there exists a linear projection p from X^{**} onto X satisfying

$$\|\alpha\| = \|p(\alpha)\| + \|\alpha - p(\alpha)\|$$

for every $\alpha \in X^{**}$. We note that, in such a case, $1 - 2p$ is an isometry on X^{**}. It is known that, if X satisfies the conclusion in Lemma 3·4, and if there exists a linear projection p from X^{**} onto X such that $1 - 2p$ is an isometry, then both X and X^* are superreflexive and almost transitive (see the proof of [**7**, proposition 2·3], and [**7**, remark 2·6]). Therefore we have the following.

COROLLARY 3·5. *Let X be an L-embedded Banach space over $\mathbb{K}$ such that $\mathcal{L}(X)$ is w''_{op}-unitary. Then both X and X^* are superreflexive and almost transitive.*

THEOREM 3·6. *Let X be a real Banach space. Then the following assertions are equivalent*:

(i) *X is the predual of a real JBW^*-triple, and $\mathcal{L}(X)$ is $\overline{w''_{op}}$-unitary*;
(ii) *X is the predual of a real JBW^*-triple, and $\mathcal{L}(X)$ is w''_{op}-unitary*;
(iii) *X is a Hilbert space.*

Proof. The implication (i) $\Rightarrow$ (ii) is clear, whereas the one (iii) $\Rightarrow$ (i) follows from Corollary 2·6 and the already commented fact that real Hilbert spaces are real JB^*-triples.

(ii) $\Rightarrow$ (iii) Since preduals of real JBW^*-triples are L-embedded [**3**, proposition 2·2], the assumption (ii), together with Corollary 3·5, yields that X^* is almost transitive. Then, since

X^* is a JBW^*-triple (by assumption), Lemma 3·2 applies, so that X^* (and hence X) is a Hilbert space.

The following corollary follows straightforwardly from Theorems 3·3 and 3·6.

COROLLARY 3·7. *Let X be a real Banach space such that $\mathcal{L}(X)$ is unitary. If X, X^* or X^{**} is a real JB^*-triple, then X is a Hilbert space.*

Remark 3·8. Looking at the proof of Lemma 3·1 (respectively, Lemma 3·4), we realize that its conclusion remains true if the assumption that $\mathcal{L}(X)$ is w'_{op}-unitary (respectively w''_{op}-unitary) is relaxed to the one that $B_{\mathcal{K}(X)}$ is contained in the w'_{op}- (respectively w''_{op}-) closed convex hull of $\mathcal{G}_X$. Then, keeping in mind Proposition 2·5, and arguing as in the proof of Theorem 3·3 (respectively, Theorem 3·6), we realize that, *for a real Banach space X, the following assertions are equivalent*:

(i) X is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the norm-closed convex hull of $\mathcal{G}_X$;
(ii) X is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the w'_{op}-closed convex hull of $\mathcal{G}_X$;
(iii) X^* a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the norm-closed convex hull of $\mathcal{G}_X$;
(iv) X^* is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the w''_{op}-closed convex hull of $\mathcal{G}_X$;
(v) X^{**} is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the norm-closed convex hull of $\mathcal{G}_X$;
(vi) X^{**} is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the w'_{op}-closed convex hull of $\mathcal{G}_X$;
(vii) X is a Hilbert space.

Some of the new techniques introduced in the present section allow us to complement the main results of [**7**]. Indeed, when the arguments of [**7**] involve the assumption on a Banach space X that $\mathcal{L}(X)$ is unitary, in fact they only use that such an assumption implies the conclusion in Lemma 3·4, that such a conclusion implies that of Lemma 3·1, and that the conclusion of Lemma 3·1 implies that X is convex-transitive [**7**, lemma 2.1]. Therefore, keeping in mind Lemmas 3·1 and 3·4, and looking carefully at the arguments in [**7**], we obtain Theorem 3·9 immediately below. For a Banach space X, we denote by Δ_X the open unit ball of X.

THEOREM 3·9. *Let X be a complex Banach space X. Then the following assertions are equivalent*:

(i) *X is a JB^*-triple, and $\mathcal{L}(X)$ is unitary*;
(ii) *X is a JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary*;
(iii) *there exists a nonlinear biholomorphic automorphism of Δ_X and $\mathcal{L}(X)$ is unitary*;
(iv) *there exists a nonlinear biholomorphic automorphism of Δ_X and $\mathcal{L}(X)$ is w'_{op}-unitary*;
(v) *X^* is a JB^*-triple and $\mathcal{L}(X)$ is unitary*;
(vi) *X^* is a JB^*-triple and $\mathcal{L}(X)$ is w''_{op}-unitary*;
(vii) *there exists a nonlinear biholomorphic automorphism of Δ_{X^*} and $\mathcal{L}(X)$ is unitary*;
(viii) *there exists a nonlinear biholomorphic automorphism of Δ_{X^*} and $\mathcal{L}(X)$ is w''_{op}-unitary*;
(ix) *X^{**} is a JB^*-triple and $\mathcal{L}(X)$ is unitary*;
(x) *X^{**} is a JB^*-triple and $\mathcal{L}(X)$ is w'_{op}-unitary*;
(xi) *there exists a nonlinear biholomorphic automorphism of $\Delta_{X^{**}}$ and $\mathcal{L}(X)$ is unitary*;
(xii) *there exists a nonlinear biholomorphic automorphism of $\Delta_{X^{**}}$ and $\mathcal{L}(X)$ is w''_{op}-unitary*;
(xiii) *X is a Hilbert space.*

Acknowledgments. The authors are grateful to G. Dales, M. Neumann, and A. M. Peralta for their interesting remarks concerning the matter of the paper. The present work is partially supported by Junta de Andalucía grants FQM 0199 and FQM 1215, and Project I+D MCYT MTM-2004-03882 and MTM-2006-15546-C02-02.

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