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Behavior of Solutions of Impulsively Perturbed Non-Halflinear Oscillator Equations

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Intermittently and instantaneously perturbed oscillator equations play an important role in theory and application. In this paper, we investigate the asymptotic behavior of solutions of the impulsive system $(\phi_\beta(x'))' + f(x) = 0$ for $t \neq t_n$, $x'(t_n + 0) = b_n x'(t_n)$, where $n = 1, 2, \ldots$, and $\phi_\beta(u) = |u|^\beta \operatorname{sgn} u$ for $\beta > 0$. In the special case $f(u) = \phi_\beta(u)$, we obtain the so-called half-linear system, which exhibits similar behavior to the linear case. First, we prove attractivity results, and then apply our theorems to the nonautonomous equation $(\phi_\beta(x'))' + q(t)f(x) = 0$, where $q(t)$ is a step-function.

1. INTRODUCTION

Consider the impulsively damped nonlinear system

$$\begin{aligned} \left(\phi_\beta(x')\right)' + f(x) &= 0, \qquad t \neq t_n, \\ x(t_n + 0) &= x(t_n), \\ x'(t_n + 0) &= b_n x'(t_n), \end{aligned} \tag{1}$$

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where $t_n \to \infty$ as $n \to \infty$, $\phi_\beta(u) = |u|^\beta \operatorname{sgn} u$ for $\beta > 0$, f: $\mathbf{R} \to \mathbf{R}$ is continuous, and $uf(u) > 0$ $u \neq 0$. For the sake of simplicity, we assume that f is an odd function.

Equation (1) with $0 \leq b_n \leq 1$ is the impulsive analogue of the damped oscillator equation

$$\left(\phi_\beta(x')\right)' + a(t)h(x') + f(x) = 0, \tag{2}$$

where $a(t) \geq 0$ and $uh(u) > 0$ for $u \neq 0$. The analogy was investigated in [9, 10] for the case $\phi_\beta(u) = u$. The special case $h(u) = f(u) = \phi_\beta(u)$ is called the half-linear equation since, if $x(t)$ is a solution, then $cx(t)$ is also a solution. The behavior of the solutions is quite similar to that of a linear equation (see [8]). Note that a negative b_n in system (1) results in a beating effect (see the discussion for the case $\phi_\beta(u) = u$ in [10, 12]), which has no continuous analogue.

In this paper, we apply the method used in [9, 16] to obtain attractivity theorems for (1). We also consider the important special case $f(u) = \phi_\alpha(u)$. We will show that the asymptotic behavior of the solutions is completely different if $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$. As an application of our results, in Section 4 we investigate the attractivity properties of the equation

$$\left(\phi_\beta(x')\right)' + q(t)f(x) = 0, \tag{3}$$

where $q(t)$ is a step-function.

2. DEFINITIONS AND LEMMAS

We say that the zero solution of (1) is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0)| + |x'(0)| < \delta$ implies $|x(t)| + |x'(t)| < \varepsilon$ for $t \geq 0$. The zero solution is *asymptotically stable* (a.s.) if there exists $\delta > 0$ such that $|x(0)| + |x'(0)| < \delta$ implies $\lim_{t\to\infty}(x(t), x'(t)) = (0, 0)$. The asymptotic stability is *global* (g.a.s.) if $\delta = \infty$.

Throughout the paper, we assume that every solution can be continued to ∞. Obviously, the solutions are piecewise differentiable and $\phi_\beta(x'(t))$ is piecewise continuous and continuous from the left at every $t > 0$. We introduce the functions

$$\Phi(y) := y\phi_\beta(y) - \int_0^y \phi_\beta(s)\,ds = \frac{\beta}{\beta+1}|y|^{\beta+1} \quad \text{and}$$
$$F(x) := \int_0^x f(s)\,ds, \tag{4}$$

and investigate the solutions using the energy function

$$V(x, y) = \Phi(y) + F(x). \tag{5}$$

To simplify the formulation of our results, we assume that

$$\lim_{u \to \pm\infty} F(u) = \infty. \tag{6}$$

It is easy to verify that $V(x(t), x'(t))$ is constant along the solutions of the equation without impulses

$$\left(\phi_\beta(x')\right)' + f(x) = 0. \tag{7}$$

Next, we calculate the change in the energy along the solutions of (1). Using the notation $V(t) = V(x(t), x'(t))$, we obtain

$$\begin{aligned} V(t_{n+1}) - V(t_n) &= V(t_n + 0) - V(t_n) \\ &= \Phi\left(x'(t_n + 0)\right) + F\left(x(t_n + 0)\right) \\ &\quad - \Phi\left(x'(t_n)\right) - F\left(x(t_n)\right) \\ &= -\Phi\left(x'(t_n)\right)\left(1 - |b_n|^{\beta+1}\right) = -a_n \Phi\left(x'(t_n)\right), \end{aligned} \tag{8}$$

where $a_n := 1 - |b_n|^{\beta+1}$.

Note that $V(t)$ is nonincreasing if $|b_n| \le 1$, independently of the sign of b_n, and is constant between any t_n and t_{n+1}. The case $b_n = 0$ ($a_n = 1$) is critical since we lose the uniqueness of the solutions at t_n from the left. Moreover, there exist solutions that are identically zero for $t > t_n$. To guarantee uniqueness, we require the more restrictive condition $0 < b_n \le 1$, $n = 1, 2, \ldots$.

Using equality (8) repeatedly, we obtain

$$V(t) = V(0) - \sum_{t_n < t} a_n \Phi\left(x'(t_n)\right) \tag{9}$$

and

$$\begin{aligned} V(t) = V(t_n + 0) &= V(t_n)\left(1 - a_n \frac{\Phi\left(x'(t_n)\right)}{V(t_n)}\right) \\ &= V(0) \prod_{t_i < t} \left(1 - a_i \frac{\Phi\left(x'(t_i)\right)}{V(t_i)}\right) \end{aligned}$$

for $t_n \le t < t_{n+1}$ along the solutions of (1). A lower estimate is

$$V(t) \ge V(t_n)(1 - a_n) = V(0) \prod_{t_i < t} (1 - a_i) = V(0) \Big(\prod_{t_i < t} b_i \Big)^{\beta + 1} \tag{10}$$

since $b_n > 0$ here. Using the equality (2), it is easy to prove the following basic lemma.

LEMMA 1. *If* $0 \le b_n \le 1$, $n = 1, 2, \dots$, *then* $V(t)$ *is nonincreasing for every solution. The zero solution of* (1) *is stable, every solution is bounded on* $[t_0, \infty)$, *and* $\liminf_{t \to \infty} \Phi(x'(t)) = 0$.

If $0 < b_n \le 1$, $n = 1, 2, \dots$, we obtain a Gronwall–Bellman type inequality for $V(t)$ from (2). Since $\ln(1 - y) \le -y$ for $y < 1$, we have

$$\begin{aligned} \ln V(t) &= \ln V(0) + \sum_{t_i < t} \ln \left(1 - a_i \frac{\Phi(x'(t_i))}{V(t_i)} \right) \\ &\le \ln V(0) - \sum_{t_i < t} a_i \frac{\Phi(x'(t_i))}{V(t_i)}. \end{aligned}$$

We have thus proved the following lemma.

LEMMA 2. *Suppose that* $0 < b_n \le 1$, $n = 1, 2, \dots$. *If* $x(t)$ *is a solution of* (1), *then*

$$V(t) \le V(0) \exp \left\{ - \sum_{t_n < t} a_n \frac{\Phi(x'(t_i))}{V(t_n)} \right\}. \tag{11}$$

Since $\ln(1 - a_n) \ge -((\beta + 1)|\ln b|)a_n/(1 - b^{\beta+1})$ provided $b_n \ge b > 0$, we can formulate the following necessary condition for the attractivity of the zero solution that is analogous to the case of distributed damping.

THEOREM 1. *Suppose that* $0 \le b_n \le 1$, $n = 1, 2, \dots$, *and let* $x(t)$ *be a solution of* (1). *If either*

$$\prod_{n=1}^{\infty} b_n > 0, \tag{12}$$

or

$$\liminf_{n \to \infty} b_n > 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n < \infty, \tag{13}$$

then $\lim_{t \to \infty} V(t) > 0$.

The theorem is not true if $b_n = 0$ for some n since in that case there are solutions that are identically zero for $t > t_n$.

The following lemma is fundamental in the estimation of the energy; it is a consequence of Corollary 2.4 in [4].

LEMMA 3. *Let $x(t)$ be a nonzero solution of* (1) *and $T \geq 0$. Then*

$$\phi_\beta(x'(t)) = \phi_\beta(x'(T)) \prod_{T \leq t_i < t} b_i^\beta - \int_T^t \Big(\prod_{s \leq t_i < t} b_i^\beta \Big) f(x(s))\, ds. \quad (14)$$

Remark 1. Clearly, the product in the integrand in (14) can be taken for $s < t_i < t$.

Remark 2. With the notation

$$\Psi(s,t) := \prod_{s \leq t_i < t} b_i^\beta,$$

(14) can then be written in the form

$$\phi_\beta(x'(t)) = \phi_\beta(x'(T))\Psi(T,t) - \int_T^t \Psi(s,t) f(x(s))\, ds. \quad (15)$$

The next lemma classifies the solutions of the impulsive system (1).

LEMMA 4. *Suppose $0 \leq b_n \leq 1$, $n = 1, 2, \ldots$, and let $x(t)$ be a solution of* (1) *that is not identically zero on any interval $[T, \infty)$, and let s_1 and s_2 be consecutive zeros of $x'(t)$. Then there exists $\tilde{t} \in (s_1, s_2)$ such that $x(\tilde{t}) = 0$. Hence, solutions of* (1) *are either oscillatory or monotonic on $[T, \infty)$.*

The proof is similar to the proof of Lemma 13 in [9] and so we omit the details here. As indicated above, if $b_n = 0$, then the solution can become zero from t_n forward. This lemma is not true if $b_n < 0$ for some n, since we can obtain nonoscillatory "saw-tooth" solutions.

To obtain asymptotic stability properties, we investigate the variation of V along solutions. It is easy to see from (8) that the critical places for impulses are the zero positions of the derivative of the solutions. Thus, we need to formulate conditions that assure that infinitely many t_n's avoid these places. Consequently, the key to our method is to estimate these zeros as well as possible. Hence, knowledge of the oscillatory behavior of the solutions of the equation (7) without impulses is essential to our analysis.

Every nonzero solution of (7) is periodic, and the trajectory is a level curve of V in the plane. Since f and ϕ_β are odd, the length of the time intervals on which $x(t)x'(t) \geq 0$ or $x(t)x'(t) \leq 0$ are equal. For a solution

$x(t)$ with $V(x(t), x'(t)) \equiv r$, denote the distance between two consecutive zeros of $x'(t)$, i.e., the halfperiod, by $\Delta(r)$. In the case $\phi_\beta(u) = u$, the behavior of the solutions was investigated, for example, in [19]. The quantity $\Delta(r)$ can be expressed in the form

$$\Delta(r) = 2\int_0^{F^{-1}(r)} \frac{dx}{\Phi^{-1}(r - F(x))} = 2r\int_0^1 \frac{dv}{f(F^{-1}(r(1-v))\Phi^{-1}(rv)}, \tag{16}$$

where $u = r - F(x)$, $v = u/r$, and F^{-1} and Φ^{-1} are the inverses of F and Φ on $[0, \infty)$, respectively. In the special case where f is a homogeneous function of order α, we obtain that

$$\Delta(r) = 2r^{1/(\alpha+1) - 1/(\beta+1)}\int_0^1 \frac{dv}{f(F^{-1}(1-v))\Phi^{-1}(v)}.$$

If $f(x) = \phi_\alpha(x)$, then

$$\begin{aligned}\Delta_{\alpha,\beta}(r) &= r^{1/(\alpha+1)-1/(\beta+1)} \\ &\quad\times \frac{2\beta^{1/(1+\beta)}\Gamma(1/(1+\alpha))\Gamma(\beta/(1+\beta))}{(1+\alpha)^{\alpha/(1+\alpha)}(1+\beta)^{1/(1+\beta)}\Gamma((1+2\beta+\alpha\beta)/(1+\alpha+\beta+\alpha\beta))} \\ &=: M_{\alpha,\beta} r^{1/(\alpha+1)-1/(\beta+1)}, \end{aligned} \tag{17}$$

and if, moreover, $\alpha = \beta$, then $\Delta_{\alpha,\beta}(r)$ is the constant

$$\begin{aligned}\Delta_\alpha &= \frac{2\alpha^{1/(1+\alpha)}\Gamma(1/(1+\alpha))\Gamma(\alpha/(1+\alpha))}{1+\alpha} \\ &= \frac{2\alpha^{1/(1+\alpha)}}{(1+\alpha)}\frac{\pi}{\sin(\pi/(1+\alpha))}.\end{aligned}$$

In particular, $\Delta_\alpha = \pi$ for $\alpha = 1$.

If $\alpha > \beta$, then $\lim_{r\to 0}\Delta_{\alpha,\beta}(r) = \infty$, and if $\alpha < \beta$, then $\lim_{r\to 0}\Delta_{\alpha,\beta}(r) = 0$. The case $\alpha = \beta$ is called halflinear [8], so we may refer to the other cases as super- and sub-half-linear, respectively. In general, we do not make a homogeneity assumption on f. Hence, we introduce the notation

$$\Delta^0 = \liminf_{r\to 0} \Delta(r) \quad \text{and} \quad \Delta^\infty = \liminf_{r\to\infty}\Delta(r). \tag{18}$$

It is easy to see that the oscillatory properties of the small solutions essentially depend on the nonlinearity of f and ϕ_β. The following lemma is fundamental in our paper since it shows that the above properties of Eq. (7) are inherited by system (1).

LEMMA 5. *Let $x(t)$ be a solution of* (1) *such that* $\lim_{t\to\infty} V(t) = r > 0$. *Then for any $\varepsilon > 0$ and $\delta > 0$ with $\delta < r$ there exists $T > 0$ such that if $T < \tau_1 < \tau_2$, $F(x(\tau_1)) = F(x(\tau_2)) = r - \delta$, and $F(x(t)) < r - \delta$ for $t \in (\tau_1, \tau_2)$, then*

$$\tau_2 - \tau_1 \geq \int_{-F^{-1}(r-\delta)}^{F^{-1}(r-\delta)} \frac{dx}{\Phi^{-1}((1+\varepsilon)r - F(x))}.$$

In particular, if $x(t)$ is oscillatory and $\{s_n\}_{n=1}^{\infty}$ is a sequence of zeros of $\dot{x}(t)$, then

$$\liminf_{n\to\infty} (s_{n+1} - s_n) \geq \Delta(r).$$

Proof. Let $x(t)$ be a solution of (1) for which $\lim_{t\to\infty} V(t) = r > 0$. Let $\varepsilon > 0$ be given and choose T such that

$$r \leq \Phi(x'(t)) + F(x(t)) < r(1+\varepsilon)$$

for $t > T$. Then

$$\Phi^{-1}(r - F(x(t))) < |x'(t)| < \Phi^{-1}(r(1+\varepsilon) - F(x(t))),$$

so

$$\frac{|x'(t)|}{\Phi^{-1}(r(1+\varepsilon) - F(x(t)))} < 1 < \frac{|x'(t)|}{\Phi^{-1}(r - F(x(t)))}.$$

Let $0 < \delta < r$ and let $T < \tau_1 < \tau_2$ with $F(x(\tau_1)) = F(x(\tau_2)) = r - \delta$ and $F(x(t)) < r - \delta$ for $t \in (\tau_1, \tau_2)$. It follows from Lemma 4 that $x'(t)$ does not change sign on (τ_1, τ_2). Hence, integrating the above inequality, we obtain

$$\tau_2 - \tau_1 \geq \int_{-F^{-1}(r-\delta)}^{F^{-1}(r-\delta)} \frac{dx}{\Phi^{-1}(r(1+\varepsilon) - F(x))}.$$

If s_1 and s_2 are zeros of $x'(t)$ such that $s_1 < \tau_1 < \tau_2 < s_2$, then

$$s_2 - s_1 \geq \int_{-F^{-1}(r)}^{F^{-1}(r)} \frac{dx}{\Phi^{-1}(r(1+\varepsilon) - F(x))},$$

since s_1, s_2 are independent of δ.

Now, let $x(t)$ be oscillatory and let $\{s_n\}$ be a sequence of the zeros of $x'(t)$. Then letting $n \to \infty$, we have

$$\liminf_{n\to\infty} (s_{n+1} - s_n) \geq \int_{-F^{-1}(r)}^{F^{-1}(r)} \frac{dx}{\Phi^{-1}(r(1+\varepsilon) - F(x))}.$$

Since ε is arbitrary, we have

$$\liminf_{n\to\infty} (s_{n+1} - s_n) \geq \Delta(r).$$

In addition, we can obtain the upper estimate

$$\tau_2 - \tau_1 < \int_{-F^{-1}(r-\delta)}^{F^{-1}(r-\delta)} \frac{dx}{\Phi^{-1}(r - F(x))} < \Delta(r).$$

∎

Note that the property $\liminf_{n\to\infty}(s_{n+1} - s_n) \geq \Delta(r)$ in Lemma 5 is called $\Delta(r)$-discreteness (see [14]).

3. ASYMPTOTIC STABILITY

Before stating our main theorem, we formulate a simple result that shows we cannot expect to prove results on the asymptotic behavior of solutions without taking into account the properties of f and ϕ_β. Similarly, although $\lim_{u\to\pm\infty}\Phi(u) = \lim_{u\to\pm\infty} F(u) = \infty$, the globality of the asymptotic stability cannot be expected either.

THEOREM 2. *Let $t_n = np$, $0 \leq b_n \leq 1$, and let $D_0 = \{r\colon\ \Delta(r) = p/k,$ $k = 1, 2 \dots\}$. The solutions of* (1) *with the initial conditions $F(x(t_1)) = r \in D_0$, $x'(t_1) = 0$ are periodic.*

The proof of Theorem 2 is obvious. The following corollary is a simple, but important, consequence of Theorem 2.

COROLLARY 3. *Let $0 \leq b_n \leq 1$ and $t_n = np$ with $p > 0$.*

(a) *If $\lim_{r\to 0}\Delta(r) = 0$, then the zero solution cannot be asymptotically stable.*

(b) *If $\lim_{r\to\infty}\Delta(r) = 0$, then the asymptotic stability cannot be global.*

The following property was first formulated for the special case of system (1) with $\phi_\beta(u) = u$. It is useful in determining the value of $\lim_{t\to\infty} V(t)$ as Theorem 4 below shows.

Condition Attr(L). Suppose that there exists a number $L \geq 0$ and a sequence of intervals $\{[s_n, s_n + i_n]\}$ such that $s_n \to \infty$ as $n \to \infty$, $i_n > 0$, $s_{n+1} \geq s_n + i_n$, and

$$\limsup_{n \to \infty} i_n \leq L. \tag{19}$$

Let the impulses satisfy the following properties:

$$\liminf_{n \to \infty} \prod_{s_n < t_j < s_n + i_n} b_j > 0; \tag{20}$$

For every sequence $\{u_n\}$ with $s_n \leq u_n \leq s_n + i_n$

$$\sum_{n=1}^{\infty} \left(\sum_{s_n \leq t_k \leq s_n + i_n} a_k \, \mu(t_k, u_n) \right) = \infty, \tag{21}$$

where $\mu(t, u) = \min\{1, |t - u|^{1 + (1/\beta)}\}$.

The difference in the inequalities in the range of the product in (20) and the sum in (21) is essential as we will see later in formulating some special cases. The condition Attr(L) can be formulated without (20), but in that case, the definition of μ_n is somewhat more complicated. Our main result is the following.

THEOREM 4. *Suppose that* $0 \leq b_n \leq 1$, $n = 1, 2, \ldots,$ *and* Attr(L) *holds for some* $L \geq 0$. *Let* $x(t)$ *be a solution of Eq.* (1) *for which* $\lim_{t \to \infty} V(t) = r$. *Then either* $r = 0$ *or* $\Delta(r) \leq L$.

Now we consider some special cases and applications of the above results. We will prove Theorem 4 at the end of this section. Success in applying Theorem 4 depends on how the sequences $\{s_n\}$ and $\{i_n\}$ are chosen. Then, if condition Attr(L) is satisfied, the attractivity of the zero solution depends on the local and global properties of $\Delta(r)$. Here, we formulate corollaries for some important cases. The following conditions imply Attr(L). Let $s_n = t_{k_n}$ and $s_n + i_n = t_{k_n + l_n}$.

*Condition Attr*1(L). Let a subsequence $\{t_{k_n}\}$ of $\{t_n\}$ and the sequence of positive integers $\{l_n\}$ satisfy the following properties:

$$\limsup_{n \to \infty} \left(t_{k_n + l_n} - t_{k_n} \right) \leq L, \qquad 0 \leq L < \infty; \tag{22}$$

$$\liminf_{n \to \infty} \prod_{t_{k_n} < t_j < t_{k_n + l_n}} b_j > 0; \tag{23}$$

$$\sum_{n=1}^{\infty} \left(t_{k_n + l_n} - t_{k_n} \right)^{1 + 1/\beta} \min\left\{ a_{k_n}, a_{k_n + l_n}) \right\} = \infty. \tag{24}$$

Note that condition (24) in Attr1(L) means that the impulses at t_{k_n} and $t_{k_n+l_n}$ are significant, and the impulses between them can be neglected in estimating V. In the special case $l_n \equiv 1$, we have the following condition.

*Condition Attr*2(*L*). Let a subsequence $\{t_{k_n}\}$ of $\{t_n\}$ satisfy the conditions

$$\limsup_{n\to\infty}\left(t_{k_n+1}-t_{k_n}\right)\le L, \qquad 0\le L<\infty, \tag{25}$$

$$\sum_{n=1}^{\infty}\left(t_{k_n+1}-t_{k_n}\right)^{1+1/\beta}\min\{a_{k_n},a_{k_n+1}\}=\infty. \tag{26}$$

This last condition formulates a very useful strategy that can be used in applications, that is, the strategy of "twin-effects." The effect working at a time τ has to be repeated at $\tau+\delta$. This strategy guarantees that we avoid problems occurring from a wrong choice of τ.

To prove that Attr1(L) implies Attr(L), we have only to show that (21) is satisfied. For any $u_n\in[t_{k_n},t_{k_n+l_n}]$, we obtain

$$\begin{aligned}
&\min\left\{\left(t_{k_n}-u_n\right)^{1+1/\beta},1\right\}a_{k_n}+\min\left\{\left(t_{k_n+l_n}-u_n\right)^{1+1/\beta},1\right\}a_{k_n+l_n}\\
&\quad\ge\left(\min\left\{\left(t_{k_n}-u_n\right)^{1+1/\beta},1\right\}+\min\left\{\left(t_{k_n+l_n}-u_n\right)^{1+1/\beta},1\right\}\right)\\
&\qquad\times\min\{a_{k_n},a_{k_n+l_n}\}\\
&\quad\ge\frac{1}{2^{1+1/\beta}}\left(t_{k_n+l_n}-t_{k_n}\right)^{1+1/\beta}\min\{a_{k_n},a_{k_n+l_n}\}
\end{aligned}$$

since one of the distances $(t_{k_n+l_n}-u_n)$ or $(u_n-t_{k_n})$ is greater than or equal to $(t_{k_n+l_n}-t_{k_n})/2$ and the sequence $\{t_{k_n+l_n}-t_{k_n}\}$ is bounded.

To obtain attractivity criteria, we need to recall that the energy is decreasing along the solutions, so if $V(0)=r_0$, then $\lim_{t\to\infty}V(t)<r_0$. Hence, r_0 can be compared to a number L for which one of the conditions Attr(L), Attr1(L), or Attr2(L) holds. In this way, we can estimate the attractivity region. Now, the following theorem easily follows from Theorem 4.

THEOREM 5. *Assume that* Attr(L) *holds for Eq.* (1) *with* $L\ge 0$. *If* $0\le L\le\Delta^0$, *then the zero solution is asymptotically stable. If, in addition,* $0\le L\le\Delta^\infty$, *then the asymptotic stability is global.*

Note that if $\Delta^0=0$ and $L>0$, then the above theorem cannot guarantee asymptotic stability. If $\Delta^\infty=0$ and $L>0$, then the globality cannot be assured. Choosing $f(u)=\phi_\alpha(u)$ provides us the appropriate counterexamples. Combining Theorem 5 with condition Attr2(L) and Theorem 2, we obtain the following corollary.

COROLLARY 6. *Let* $f(u) = \phi_\alpha(u)$, *and* $t_{n+1} - t_n = L > 0$, *and assume that* (26) *holds with* $k_n = n$.

Case (*a*). $\alpha = \beta$. If $L < \Delta$, then the zero solution is globally asymptotically stable. If $L = \Delta$, then the zero solution is not asymptotically stable.

Case (*b*). $\alpha > \beta$. The zero solution is asymptotically, but not globally, stable. A region of attractivity is $\{(x_0, x_0'): F(x_0) + \Phi(x_0') < \Delta^{-1}(L)\}$.

Case (*c*). $0 < \alpha < \beta$. The zero solution is not asymptotically stable. Every solution satisfies the property $\lim_{t\to\infty} V(t) < \Delta^{-1}(L)$.

Remark 3. In the half-linear case ($\alpha = \beta$), if $t_{n+1} - t_n = L > \Delta_\alpha$, we can remove any interval of length Δ_α from the intervals (t_n, t_{n+1}). Consequently,

$$(t_{n+1} - t_n) \bmod \Delta_\alpha$$

can be used in conditions Attr1(L) and Attr2(L) instead of $(t_{n+1} - t_n)$.

Finally, let us prove Theorem 4. In order to make use of (9), an estimate of the derivative of a solution is needed. The following lemma is useful in this regard.

LEMMA 6. *Let* $x(t)$ *be a solution of* (1) *for which* $\lim_{t\to\infty} V(t) = r$, *and let* $0 < \delta < r$ *and* $0 < s_1 < s_2$ *be given. The following statements hold.*

(a) *If* $F(x(t)) \le r - \delta$ *for* $t \in [s_1, s_2]$, *then* $|x'(t)| \ge \Phi^{-1}(\delta)$.

(b) *If* $F(x(t)) > r - \delta$ *and* $x(t)x'(t) \ge 0$ *for* $t \in [s_1, s_2]$, *then* $|\phi_\beta(x'(t))| \ge \delta_1 |t - s_2|$.

(c) *If* $F(x(t)) > r - \delta$ *and* $x(t)x'(t) \le 0$ *for* $t \in [s_1, s_2]$, *then*

$$\left|\phi_\beta(x'(t))\right| \ge \delta_1 \int_{s_1}^t \Psi(s,t)\, ds = \delta_1 \int_{s_1}^t \left(\prod_{s<t_j<t} b_j^\beta \right) ds,$$

where $\delta_1 = \inf\{|f(u)|: F(u) > r - \delta, |u| < \sup_{t\ge 0}|x(t)|\} > 0$.

Remark 4. Observe that if $\Psi(s,t) \ge Le^{-K(t-s)}$ for $t > s$ and $t, s \in (s_1, s_2)$, then $|\phi_\beta(x'(t))| \ge \varepsilon\delta_1 \min\{1, t - s_1\}$ for some $0 < \varepsilon < 1$. The constants L, K, ε, and δ_1 in the above estimates depend only on δ and r.

Proof. Let $x(t)$ be a solution of (1) for which $\lim_{t\to\infty} V(t) = r > 0$, and let $r > \delta > 0$ and the interval $[s_1, s_2]$ be given.

(a) If $F(x(t)) \le r - \delta$ for $t \in [s_1, s_2]$, we see that $|x'(t)| \ge \Phi^{-1}(\delta)$ since $F(x(t)) + \Phi(x'(t)) > r$.

(b) Let $F(x(t)) > r - \delta$ and $x'(t)x(t) \geq 0$ for $t \in [s_1, s_2]$. Assume that $x'(t) \geq 0$. By Lemma 3, we have

$$0 \leq \phi_\beta(x'(s_2)) = \phi_\beta(x'(t))\Psi(t, s_2) - \int_t^{s_2} \Psi(s, s_2) f(x(s))\, ds$$

and so

$$\phi_\beta(x'(t)) \geq \int_t^{s_2} \frac{\Psi(s, s_2)}{\Psi(t, s_2)} f(x(s))\, ds \geq \delta_1(s_2 - t),$$

where $\delta_1 = \inf\{|f(u)|: F(u) > r - \delta,\ u < \sup_{t \geq 0}|x(t)|\} > 0$.

(c) Let $F(x(t)) > r - \delta$ and $x(t)x'(t) \leq 0$ for $t \in [s_1, s_2]$. We can assume that $x'(t) \geq 0$ and $x(t) < 0$. Again by Lemma 3, we have

$$\begin{aligned}\phi_\beta(x'(t)) &= \phi_\beta(x'(s_1))\Psi(s_1, t) - \int_{s_1}^{t} \Psi(s, t) f(x(s))\, ds \\ &\geq \delta_1 \int_{s_1}^{t} \Psi(s, t)\, ds = \delta_1 \int_{s_1}^{t} \left(\prod_{s < t_j < t} b_j^\beta \right) ds.\end{aligned}$$

As observed earlier, replacing $\prod_{s \leq t_j < t}$ by $\prod_{s < t_j < t}$ does not modify the integral. ∎

Proof of Theorem 4. Suppose that Attr(L) is satisfied with $L \geq 0$. Let $x(t)$ be a solution of (1) such that $\lim_{t \to \infty} V(x(t), x'(t)) = r > 0$. Without loss of generality, we can assume that for any $n > 0$ the intervals $[s_n, s_n + i_n]$ and $[s_{n+1}, s_{n+1} + i_{n+1}]$ are disjoint. If this is not the case, but there exists k_0 such that for every n the intervals $[s_n, s_n + i_n]$ and $[s_{n+k_0}, s_{n+k_0} + i_{n+k_0}]$ are disjoint, then (9) can be written as

$$V(t) = V(0) - \sum_{t_n < t} a_n \Phi(x'(t_n)) = \sum_{t_n < t} \sum_{j=1}^{k_0} \frac{a_n \Phi(x'(t_n))}{k_0},$$

and we can estimate the expressions $a_n \Phi(x'(t_n))/k_0$ on any $[s_n, s_n + i_n]$ independently of the other intervals.

We will now show that $x(t)$ is oscillatory on $[0, \infty)$. Suppose $x(t)$ is nonoscillatory. Since $x(t)$ and $x'(t)$ are bounded, $\lim_{t \to \infty} x'(t) = 0$. We can apply Lemma 6 on each interval $[s_n, s_n + i_n]$ to estimate $x'(t)$ for sufficiently large $n > N$. It follows from the boundedness of $x'(t)$ and part (b) of Lemma 6 that $x(t)x'(t) \leq 0$ for t large enough, so we can assume that N is chosen large enough for this to be the case. Now, since $\{i_n\}$ is bounded and condition (20) holds, it follows from Remark 4 that $|\phi_\beta(x'(t))| \geq \varepsilon_1 \min\{1, t - s_n\}$, for some $\varepsilon_1 > 0$. Then, from (9), we obtain that

$$V(t) \le V(s_n) - \sum_{\substack{n>N,\\ s_n+i_n\le t}} \left[\sum_{t_k\in[s_n,s_n+i_n]} a_k \varepsilon_2 \min\left\{1, \Phi\left(\phi_\beta^{-1}(t_k - s_n)\right)\right\} \right]$$

for some positive constant ε_2. By the definitions of ϕ_β and Φ, (21) implies that the right hand side tends to $-\infty$ as $t \to \infty$, and so we have a contradiction.

Now, let $x(t)$ be oscillatory on some half-ray $[t_0, \infty)$. Assume that $0 \le L < \Delta(r)$. Let

$$\Delta'(r, \delta, \varepsilon) := \int_{-F^{-1}(r-\delta)}^{F^{-1}(r-\delta)} \frac{dx}{\Phi^{-1}((1+\varepsilon)r - F(x))}.$$

It follows from the Lebesque dominated convergence theorem that

$$\lim_{\varepsilon\to 0} \left(\lim_{\delta\to 0} \Delta'(r, \delta, \varepsilon) \right) = \Delta(r).$$

Hence, there exist δ, ε, and N such that $i_n < \Delta'(r, \delta, \varepsilon)$ if $n > N$, and $V(t) < r(1+\varepsilon)$ if $t > s_N$. We can apply Lemma 6 with this $\delta > 0$. From Lemma 5, we know that the lengths of the intervals where $F(x(t)) \le r - \delta$ are not smaller than $\Delta'(r, \delta, \varepsilon)$. Hence, there can be at most one zero of $x'(t)$ in $[s_n, s_n + i_n]$, and there are three possible cases to consider.

Case 1. The interval $(s_n, s_n + i_n)$ does not contain a zero of $x'(t)$ and $x'(t)x(t) \ge 0$ for $t \in (s_n, s_n + i_n)$. Then, parts (a) and (b) of Lemma 6 imply that

$$\phi_\beta(x'(t)) \ge \min\left\{\phi_\beta(\Phi^{-1}(\delta)), \delta_1|t - s_n - i_n|\right\} > \varepsilon_3 \min\{1, |t - s_n - i_n|\}$$

for some $\varepsilon_3 > 0$.

Case 2. The interval $(s_n, s_n + i_n)$ does not contain a zero of $x'(t)$ and $x(t)x'(t) \le 0$ for $t \in (s_n, s_n + i_n)$. Now, parts (a) and (c) of Lemma 6 and (20) imply that

$$\left|\phi_\beta(x'(t))\right| \ge \min\left\{\phi_\beta(\Phi^{-1}(\delta)), \varepsilon_4(t - s_n)\right\} > \varepsilon_5 \min\{1, t - s_n\}$$

for some ε_4, ε_5, which are independent of n.

Case 3. There is a zero u_n of $x'(t)$ in $(s_n, s_n + i_n)$. Then the intervals (s_n, u_n) and $(u_n, s_n + i_n)$ belong to Cases 1 and 2, respectively.

We note that since $x'(t)$ is continuous from the left, $x'(t) = 0$ at an extremum of $x(t)$, while $x'(t+0) = 0$ can occur without there being an extremum since $b_n = 0$ is not ruled out for any n.

Summarizing the above cases, we see that there is a sequence $\{u_n\}$ with $s_n \le u_n \le s_n + i_n$, such that one of the properties $u_n = s_n$, $u_n = s_n + i_n$, or $x'(u_n) = 0$ holds. For $x'(t)$, we obtain

$$\left|\phi_\beta(x'(t))\right| \ge \varepsilon_6 \min\{1, |t - u_n|\}$$

with $\varepsilon_6 > 0$ independent of n. We then have the estimate

$$V(t) \le V(s_N) - \varepsilon_7 \sum_{\substack{n>N, \\ s_n+i_n<t}} \left[\sum_{t_k \in [s_n, s_n+i_n]} a_k \min\left\{1, \Phi\left(\phi_\beta^{-1}(|t_k - u_n|)\right)\right\} \right],$$

where $\varepsilon_7 > 0$ is a constant. Then, by the definition of ϕ_β and Φ, (21) implies that the right hand side tends to $-\infty$ as $t \to \infty$. ∎

4. ATTRACTIVITY OF THE EQUATION $(\phi_\beta(x'))' + q(t)f(x) = 0$

In this section, we apply the results of the previous section to the equation

$$\left(\phi_\beta(x')\right)' + q(t)f(x) = 0. \tag{27}$$

The function f satisfies the same general conditions as before, $q(t)$ is a nondecreasing step-function such that $q(t) = q_n$ if $t \in [t_n, t_{n+1})$, where $0 < q_n \le q_{n+1}$ and $0 \le t_n < t_{n+1}$ for every $n = 1, 2, \ldots$, and $t_n \to \infty$ as $n \to \infty$. Note that this equation includes the equation

$$\left(p(t)\phi_\beta(x')\right)' + q(t)f(x) = 0$$

if $p(t)$ is also a step-function. Here, we give results which guarantee the property

$$\lim_{t\to\infty} x(t) = 0 \tag{28}$$

for the solutions of Eq. (27).

Since the publication of the famous Armellini–Sansone–Tonelli theorem [7, 20], there have been a large number of papers devoted to this problem in the case $\phi_\beta(u) = u$ (see [1, 2, 5, 13, 14, 18] and the references contained therein). There have also been some extensions to other more general systems such as the half-linear case of Eq. (27) (see, for example [6]).

It is known that the condition $\lim_{t\to\infty} q(t) = \infty$ is necessary but not sufficient for the zero solution to be attractive [17]. The Armellini–Sansone–Tonelli theorem says that if ln $q(t)$ grows "regularly" to infinity, then the zero solution of the linear equation ($f(u) = \phi_\beta(u) = u$) is globally attractive. In order to describe the concept of "regular growth" we first define the *density* of the interval system (a_i, b_i), $i = 1, 2, \ldots$ to be

$$\varepsilon = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n}(b_i - a_i)}{b_i}.$$

Then we say that $q(t) \to \infty$ *irregularly* as $t \to \infty$, if for every $\varepsilon > 0$ there exists a system of intervals $\{(a_i, b_i)\}$ of density ε, such that the increase in $q(t)$ on $(0, \infty) \setminus \bigcup_i (a_i, b_i)$ is finite; we say that $q(t) \to \infty$ *regularly* as $t \to \infty$, if it does not grow irregularly.

The notion of regular growth roughly means that $q(t)$ cannot tend to infinity on a set of very small measure. Although this condition is not easy to verify, and is not very sharp, it has remained a "milestone," a starting point, for subsequent results. Several conditions require higher differentiability of $q(t)$ to avoid the difficulties due to the concept of the irregular growth (for details and references see [13]). A more natural approach is to follow and improve the original irregular growth concept. Such results are called "sequence-of-intervals criteria," and give finer restrictions on the measure and the distribution of the irregularly growing parts of $q(t)$. Since the oscillatory behavior of the solutions is often the main concern, and this behavior can be investigated easily in the case of linear equations, most such results are for this case [1, 14, 16, 18, 20]. Equation (27) with $\phi_\beta(u) = u$ and with a very irregularly growing step-function $q(t)$ was treated in [11]. Hatvani [15] investigated the step-function case of linear equations from a statistical point of view.

Unlike most earlier results, our method is essentially based on, and hence the attractivity properties mainly depend on, the nonlinear structure of the functions ϕ_β and f. Here, we generalize the results proved in [11] to the equation (27). Using the generalized Liouville transformation, we will transform Eq. (27) to an impulsive system (1).

Let us apply the transformation

$$\tau = \int_0^t q^{1/(\beta+1)}(u)\, du, \tag{29}$$

to Eq. (27), and use the notation "$\cdot$" $:= d/d\tau$. Let $T > 0$ be given, and apply (29) to the equivalent integral equation

$$\phi_\beta(x'(t)) = \phi_\beta(x'(T)) - \int_T^t q(s) f(x(s))\, ds.$$

Since $x' = \dot{x}q^{1/(\beta+1)}(t)$, we obtain the equation

$$q^{\beta/(\beta+1)}(t)\,\phi_\beta(\dot{x}(\tau(t))) = q^{\beta/(\beta+1)}(t)\,\phi_\beta(\dot{x}(\tau(T))) - \int_{\tau(T)}^{\tau} q^{1/(\beta+1)}(s) f(x(s(u)))\,du.$$

Let t_n be a jump point of $q(t)$, $\delta > 0$, $t = t_n + \delta$, and $T = t_n - \delta$. Since the integral on the right hand side is continuous, if $\delta \to 0$, we obtain the equality

$$q^{\beta/(\beta+1)}(t_n+0)\,\phi_\beta(\dot{x}(\tau_n+0)) = q^{\beta/(\beta+1)}(t_n-0)\,\phi_\beta(\dot{x}(\tau_n-0)),$$

where $\tau_n = \tau(t_n)$. Thus, we obtain the jump condition for $\dot{x}(\tau)$ at τ_n, namely,

$$\phi_\beta(\dot{x}(\tau_n+0)) = \left(\frac{q_{n-1}}{q_n}\right)^{\beta/(\beta+1)} \phi_\beta(\dot{x}(\tau_n-0)) = \phi_\beta\left(\left(\frac{q_{n-1}}{q_n}\right)^{1/(\beta+1)} \dot{x}(\tau_n-0)\right)$$

due to the form of ϕ_β. On any interval (t_{n-1}, t_n), the transformation (29) yields the equation (7) without impulses.

Consequently, (29) transforms Eq. (27) into the system with impulse damping

$$\begin{gathered}(\phi_\beta(\dot{x}))^{\cdot} + f(x) = 0, \qquad \tau \neq \tau_n,\\ x(\tau_n+0) = x(\tau_n-0),\\ \dot{x}(\tau_n+0) = \left(\frac{q_{n-1}}{q_n}\right)^{1/(\beta+1)} \dot{x}(\tau_n-0), \qquad n = 1,2,\ldots,\end{gathered} \tag{30}$$

where

$$\tau_n = \int_0^{t_n} q^{1/(\beta+1)}(u)\,du \quad \text{and} \quad \tau_{n+1} - \tau_n = q_n^{1/(\beta+1)}(t_{n+1} - t_n).$$

Using the notation from the previous section, we have

$$b_n = \left(\frac{q_{n-1}}{q_n}\right)^{1/(\beta+1)} \quad \text{and} \quad a_n = 1 - \frac{q_{n-1}}{q_n},$$

and it is clear that $0 \le b_n \le 1$. The energy V defined in (5) for the impulsive system (1) has the form

$$V(\tau(t)) = \Phi(\dot{x}(\tau)) + F(x(\tau)) = \frac{\Phi(x'(t))}{q(t)} + F(x(t))$$

for the system (27).

Now, we are ready to apply our results in the previous sections. It follows from Theorem 1 that no solution of (27) can tend to zero if

$$\lim_{N\to\infty} \prod_{n=1}^{N} \frac{q_{n-1}}{q_n} > 0, \qquad \text{i.e.,} \lim_{N\to\infty} q_n < \infty. \tag{31}$$

If $f(u) = \phi_\beta(u)$ and

$$\prod_{n=1}^{\infty} \frac{q_{n-1}}{q_n} = 0, \tag{32}$$

then Atkinson and Elbert [3] show that there exists a solution tending to zero.

Using condition Attr1(L), we can formulate an analog of Theorem 4.

THEOREM 7. *Suppose that there exist a subsequence* $\{t_{k_n}\}$ *of* $\{t_n\}$ *and a sequence* $\{l_n\}$ *of positive integers such that* $t_{k_{n+1}} \ge t_{k_n+l_n}$. *Assume that the following conditions hold*:

$$\limsup_{n\to\infty} \sum_{i=k_n}^{k_n+l_n-1} q_i^{1/(\beta+1)}(t_{i+1} - t_i) = L < \infty; \tag{33}$$

$$\liminf_{l\to\infty} \frac{q_{k_n}}{q_{k_n+l_n-1}} > 0; \tag{34}$$

$$\sum_{n=1}^{\infty} \left(t_{k_n+l_n} - t_{k_n}\right)^{1+1/\beta} q_{k_n}^{1/\beta} \min\left\{1 - \frac{q_{k_n-1}}{q_{k_n}}, 1 - \frac{q_{k_n+l_n-1}}{q_{k_n+l_n}}\right\} = \infty. \tag{35}$$

Then for every solution $x(t)$ *of* (27) *either* $\lim_{t\to\infty} V(t) = r = 0$ *or* $\Delta(r) \le L$.

The sharpness of our results can be shown by choosing $p = q_n^{1/(\beta+1)}(t_{n+1} - t_n) = \text{const.}$ in Theorem 2. Moreover, from Theorem 5 we have that if the conditions of Theorem 7 hold and $0 \le L \le \Delta_0$, then the zero solution is asymptotically stable. If, in addition, $0 \le L \le \Delta_\infty$, then the asymptotic stability is global.

Because of the specific importance of the case where $f(x) = \phi_\alpha(x)$ for $\alpha > 0$, we formulate a consequence of Corollary 6.

COROLLARY 8. *Let* $f(u) = \phi_\alpha(u)$, *and* $q_n^{1/(\beta+1)}(t_{n+1} - t_n) = L > 0$. *Assume that*

$$\sum_{n=1}^{\infty} \min\left\{1 - \frac{q_{n-1}}{q_n}, 1 - \frac{q_n}{q_{n+1}}\right\} = \infty. \tag{36}$$

Then the conclusions of Corollary 6 *hold.*

Corollary 8 generalizes and improves a theorem by F. V. Atkinson [1, Theorem 3] for Eq. (27). Condition (36) can be written in the equivalent form

$$\sum_{n=1}^{\infty} \min\left\{\ln\frac{q_{n+1}}{q_n}, \ln\frac{q_n}{q_{n-1}}\right\} = \infty$$

that is used in Atkinson's theorem. This form expresses the importance of the growth of $\ln q(t)$.

5. GENERALIZATIONS

Our results here can also be derived under more general conditions. Instead of $\phi_\beta(u)$, we can consider functions $\phi(u)$ that are positive homogeneous. For such functions, Φ is defined as in (4), and the key estimate (8) for $V(t_n + 0)$ and Lemma 3 for ϕ can be easily proved. But now ϕ is not necessarily odd, and consequently, $f(u)$ is not assumed to be odd, either. The function Δ has to be defined for both positive and negative halfcycles, $\Delta_+(r)$ and $\Delta_-(r)$, respectively. Then, in this more general setting, $\Delta(r)$ can be defined by $\Delta(r) = \min\{\Delta_+(r), \Delta_-(r)\}$. The proofs of our results can be carried through under the same conditions, where the previous functions are replaced by these new ones.

Finally, note that since $\Delta(r)$ can only be calculated in special cases (for example, for power functions), computer methods and comparison results are often useful in estimating the value of $\Delta(r)$. The following lemma generalizes Theorem 3.1.3 in [19].

LEMMA 7. *Let* $r_0 > 0$ *be given. If* $|\hat{f}(u)| \geq |f(u)|$ *and* $|\underline{\phi}(u)| \leq |\phi(u)|$ *for* $u \in \{u\colon F(u) \leq r_0, \Phi(u) \leq r_0\}$, *then*

$$\Delta_{\hat{f}, \underline{\phi}}(\hat{F}(x)) = \int_0^x \frac{ds}{\underline{\Phi}^{-1}(\hat{F}(x) - \hat{F}(s))} \leq \int_0^x \frac{ds}{\Phi^{-1}(F(x) - F(s))}$$
$$= \Delta_{f,\phi}(F(x)) \tag{37}$$

for $x \in \{u\colon F(u) \leq r_0, \Phi(u) \leq r_0\}$.

If $\hat{f}(u) = \phi_\alpha(u)$ and $\underline{\phi}(u) = \phi_\beta(u)$, then from (37) we obtain that

$$\Delta_{f,\phi}\Big(F\big((\alpha+1)r\big)^{1/(\alpha+1)}\Big) \geq M_{\alpha,\beta} r^{(\beta-\alpha)/(\alpha+1)(\beta+1)},$$

where $M_{\alpha,\beta}$ is the constant in (17).

As an example of our remarks in this section, we formulate a corollary of Theorem 5 in this more general setting.

THEOREM 9. *Assume that* Attr(L) *holds for Eq.* (1) *with* $L \geq 0$, *and there exist* α, β, *and* r_0 *such that* $|\phi_\alpha(u)| \geq |f(u)|$ *and* $|\phi_\beta(u)| \leq |\phi(u)|$ *for* $u \in \{u\colon F(u) \leq r_0,\ \Phi(u) \leq r_0\}$. *The zero solution of system* (1) *is asymptotically stable if any one of the following holds*:

(a) $\alpha = \beta$ *and* $L < \Delta_\alpha$;

(b) $\alpha > \beta$ *and* L *is bounded*;

(c) $\alpha < \beta$ *and* $L = 0$.

A region of attractivity is $F(x(0)) + \Phi(x'(0)) \leq R$, *where* $R = r_0$ *for cases* (a) *and* (c), *and* $R = \min(r_0, F((\alpha+1)^{1/(\alpha+1)}(\frac{L}{M_{\alpha,\beta}})^{(\beta+1)/(\beta-\alpha)})$ *for case* (b).

Assume, in addition, that there exist $\alpha' \leq \beta'$ *such that* $|\phi_{\alpha'}(u)| \leq |f(u)|$ *and* $|\phi_{\beta'}(u)| \geq |\phi(u)|$ *for* $u \in \{u\colon F(u) \geq r_0,\ \Phi(u) \geq r_0\}$. *Then the asymptotic stability is global if either* $\alpha' < \beta'$ *and* L *is bounded, or* $\alpha' = \beta'$ *and* $L < \Delta_\alpha$.

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