JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 113, No. 3, pp. 513–536, June 2002 (© 2002)

A Topological Minimax Theorem

G. H. GRECO [1] AND C. D. HORVATH [2]

Communicated by J. P. Crouzeix

Abstract. We present a topological minimax theorem (Theorem 2.2). The topological assumptions on the spaces involved are somewhat weaker than those usually found in the literature. Even when reinterpreted in the convex setting of topological vector spaces, our theorem yields nonnegligible improvements, for example, of the Passy–Prisman theorem and consequently of the Sion theorem, contrary to most results on topological minimax. This work is part of our ongoing effort to elaborate a coherent theory of minimax.

Key Words. Minimax theorems, connectedness, lower interconnected functions, lower connected functions, upper connected functions, PP functions, interconnected pairs of multifunctions, topologically concave pairs of multifunctions.

1. Introduction

In this paper, we continue our investigation of minimax theorems, more specifically topological minimax theorems. The beginning of our investigations was announced in Ref. 1 and was initiated in Ref. 2.

We will say that a function $f: X \times Y \to \bar{\mathbb{R}}$ defined on the product of two topological spaces and taking values in $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is a minimax function[3] if

$$\sup_X \inf_Y f = \inf_Y \sup_X f.$$

Our aim is simply stated: to find in a given context conditions (topological in the present work) on the spaces X and Y and on the lower and upper levels of a function $f: X \times Y \to \bar{\mathbb{R}}$ that will guarantee that it is a minimax

[1]Professor, Dipartimento di Matematica, Universitá di Trento, Povo, Italy.

[2]Professor, Département de Mathématiques, Université de Perpignan, Perpignan, France.

[3]The expression $\sup_X \inf_Y f$ stands for $\sup_{x\in X} \inf_{y\in Y} f(x,y)$, and similarly for $\inf_Y \sup_X f$. The maximization variable will be always in X, and the minimization variable will be always in Y.

0022-3239/02/0600-0513/0

function. Our approach is through the intersection theorems for multifunctions, like in most other works on this topic. But, whereas one usually deduces the minimax theorems from the fact that the intersection of the values of a multifunction are nonempty, we found it necessary to consider a given family $\mathbb{F}$ of multifunctions and to give criteria that imply that all of the multifunctions in $\mathbb{F}$ have the finite intersection property. These criteria do not bear on the individual members of $\mathbb{F}$; in that case, there would be no difference with what is done usually, but on how the members of the family $\mathbb{F}$ are interrelated.

The paper is organized as follows. In Section 2, the Passy–Prisman minimax theorem is recalled and our main result (Theorem 2.2), which extends the Passy–Prisman minimax theorem, is stated. We introduce the notion of interconnectedness, which evolved from our topological reading of the condition of P-convexity in the small considered by Passy and Prisman.

In Section 3, we present the basic concepts from which, through a sequence of simple lemmas and some remarks, we are led in three steps to an intersection theorem for a family of multifunctions. First, we establish a local pointed binary intersection property (Lemma 3.1). From the local pointed binary intersection property, we go to the binary intersection property (Lemma 3.2), and from there, with the unavoidable heredity conditions in such a general context, we move to the finite intersection property (Theorem 3.1).

In Section 4, we prove our main result, Theorem 2.2. We show in Section 5 that, in the context of Theorem 2.2, interconnectedness is not only sufficient but also necessary. Furthermore, we show also that none of the five conditions of Theorem 2.2 can be dropped. In Section 6, we will see that Theorem 2.2 extends the Sion minimax theorem in several ways. Moreover, we establish a link with our previous works (Refs. 1–2) and in the process we single out a very large class of functions (PP functions) to which Theorem 2.2 applies. In Section 7, we will improve Theorem 1.1 below, due to König; as a further illustration of our method, we will conclude with a result (Theorem 7.3), which encompasses the König general minimax theorem, which we will recall in due time.

Theorem 1.1. See König (Ref. 3). Let [X be connected[4] and] Y be compact. A function f, both lower semicontinuous on Y and upper semicontinuous on X, is a minimax function if, for every real number

[4]With respect to König's statement, connectedness of X has to be added to avoid counterexamples.

λ,[5] the two conditions below are satisfied:

$$\text{(i)} \quad \begin{cases} \text{either} & \text{(a)} \quad \bigcap_{x\in F} \{f<\lambda\}\, x \text{ is connected for any nonempty finite } F\subset X, \\ \text{or} & \text{(b)} \quad \bigcap_{x\in F} \{f\leq\lambda\}\, x \text{ is connected for any nonempty finite } F\subset X, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \text{either} & \text{(a}'\text{)} \quad \bigcap_{y\in A} \{f>\lambda\}\, y \text{ is connected for any nonempty } A\subset Y, \\ \text{or} & \text{(b}'\text{)} \quad \bigcap_{y\in A} \{f\geq\lambda\}\, y \text{ is connected for any nonempty } A\subset Y, \end{cases}$$

and in case of the combination (a) and (a′) the space Y is Hausdorff.

We close this section with some definitions and terminological conventions to which we will adhere throughout the paper. In the sequel, X and Y will always be nonempty topological spaces; f will always denote a function from $X\times Y$ to $\bar{\mathbb{R}}$; $\bar{\mathbb{R}}$ denotes the extended real line.

We will say that a multifunction $\Omega: X \rightarrow\!\!\!\!\rightarrow Y$ has the binary (or finite) intersection property, if the family of its values $\{\Omega x: x\in X\}$ has it. We do not impose a priori that a multifunction should have nonempty values. The symbol dom Ω denotes the set $\{x\in X: \Omega x\neq\varnothing\}$.

Given a subset $A\subset X$, we denote by Ω_A the multifunction defined by

$$\Omega_A x := \Omega x \cap \left(\bigcap_{z\in A} \Omega z\right).$$

We set

$$\bigcup_{x\in\varnothing} \Omega x := \varnothing \quad \text{and} \quad \bigcap_{x\in\varnothing} \Omega x := Y;$$

---

[5]More precisely, König takes

$$\lambda > \sup_X \inf_Y f =: a.$$

But this clearly less demanding condition makes really no difference. In fact, assume $a\neq+\infty$, define the function $f\vee a$ by

$$[f\vee a](x) := \max\{f(x), a\},$$

and consider a function $\psi: [a,+\infty]\rightarrow\bar{\mathbb{R}}$ which is continuous and strictly increasing. Let

$$f' := \psi \circ (f\vee a)$$

and observe that:

(a) if conditions (i) and (ii) hold for f, whenever $\lambda > a$, then they hold for f', whenever $\lambda\in\mathbb{R}$;

(b) if f' is a minimax function, then f is a minimax function.

hence,

$$\Omega_\emptyset = \Omega.$$

For a subset $A \subset X$, we define

$$\Omega(A) := \bigcup_{x \in A} \Omega x.$$

If Γ is another multifunction from X to Y, the statement $\Gamma \subset \Omega$ means that $\Gamma x \subset \Omega x$, for each $x \in X$.

Given a subset A of X, the function $f_A : X \times Y \to \bar{\mathbb{R}}$ is defined by

$$f_A(x, y) := \max\{f(x, y), \sup_{z \in A} f(z,y)\}.$$

As usual, the supremum [resp. infimum] of the empty set is assumed to be equal to $-\infty$ [resp. $+\infty$]; hence, $f_\emptyset = f$.

For a real number $\lambda \in \mathbb{R}$, sets of the form

$$\{y \in Y : f(x, y) \leq \lambda\} \quad \text{and} \quad \{y \in Y : f(x, y) < \lambda\}$$

are referred to as large and strict lower levels of f at x. Sets of the form

$$\{x \in X : f(x, y) \geq \lambda\} \quad \text{and} \quad \{x \in X : f(x, y) > \lambda\}$$

are referred to as large and strict upper levels of f at y. Notice that lower level sets are taken with respect to the minimization variable, and that upper level sets are taken with respect to the maximization variable.

To a given function f and a real number $\lambda \in \mathbb{R}$, one can associate four multifunctions:

(a) $\{f \leq \lambda\}$ and $\{f < \lambda\}$ from X to Y, which associate to each $x \in X$ the corresponding lower level;
(b) $\{f \geq \lambda\}$ and $\{f > \lambda\}$ from Y to X, which associate to each $y \in Y$ the corresponding upper level.

Notice that

$$\{f_A < \lambda\} = \{f < \lambda\}_A, \qquad \text{if } A \text{ is finite},$$

and

$$\{f_A \leq \lambda\} = \{f \leq \lambda\}_A, \qquad \text{for any } A.$$

Let $\mathscr{P}$ be a property. We say that a function f is arbitrarily $\mathscr{P}$ [resp. finitely $\mathscr{P}$], if $\mathscr{P}$ is a property of f_A for every subset [resp. for every finite subset] A of X.

2. Minimax Theorem for Interconnected Functions

Our aim is not to present yet another generalization of a minimax theorem, but rather to present an approach to topological minimax theorems and their proofs which takes us further than what has been achieved by other means. Let us emphasize also that the Passy–Prisman original result (Theorem 2.1), is the core from which our investigations were born and around which our methods were developed. We see it as much more fundamental than the Sion theorem. We believe that the methods and results of this paper for topological minimax theorems and those of Ref. 4 for classical minimax theorems speak in favor of our point of view.

Now, we recall the minimax theorem of Passy–Prisman which will motivate the introduction of some new definitions.

Theorem 2.1. See Passy–Prisman (Ref. 5). Let X and Y be two finite-dimensional convex sets equipped with the Euclidean topology, and let Y be compact. A function f: $X\times Y\to\mathbb{R}$ is a minimax function, if the following properties hold:

(i) $\forall x\in X$, the function $y\mapsto f(x,y)$ is quasiconvex and lower semicontinuous,

(ii) $\forall y\in Y$, the function $x\mapsto f(x,y)$ is quasiconcave,

(iii) f is P-convex in the small in Y.

The definition of property (iii) as given by Passy and Prisman is the following.

Definition 2.1. A function f: $X\times Y\to\mathbb{R}$ is *P*-convex in the small in Y if, for any $\mu\in\mathbb{R}$ and any $(\bar{x},\bar{y})\in X\times Y$ such that $f(\bar{x},\bar{y})<\mu$, there exists $\epsilon>0$ such that, for any $(x,y)\in X\times Y$, if $|x-\bar{x}|<\epsilon$ and $f(x,y)<\mu$, then for each $\tilde{y}\in[\bar{y},y]$ there exists $\tilde{x}\in[\bar{x},x]$ such that $f(\tilde{x},\tilde{y})<\mu$.

Remark 2.1. Taking into account the quasiconcavity of f, Theorem 2.1(iii) was reformulated in Ref. 1 as follows:

(i) For any real number μ, any $\bar{x}\in X$, and any $\bar{y}\in Y$ with $\bar{y}\in\{f<\mu\}\bar{x}$, there exists a neighborhood V of $\bar{x}$ such that, for every $x\in V$,

$$[\bar{y},\{f<\mu\}x]\subset\{f<\mu\}\bar{x}\cap\{f<\mu\}x,$$

where $[\bar{y},\{f<\mu\}x]$ is the segment-join.[6]

[6]If A and y are a subset and a point of a convex set Y, the segment-join $[y,A]$ is defined by

$$[y,A]:=\bigcup_{y'\in A\cup\{y\}}[y,y'],$$

where $[y,y']$ is the segment

$$\{(1-t)y+ty': 0<t<1\}.$$

See Section 6 for the relationship between property (iv) and PP functions.

Remark 2.2. For a quasi-convex-concave function f, which is P-convex in the small, we have that:

(i) f is arbitrarily P-convex in the small;
(ii) for any pair of real numbers μ and λ with $\mu < \lambda$ and any $\bar{x} \in X$ with $\{f < \mu\}\bar{x} \neq \emptyset$, there exists a neighborhood V of $\bar{x}$ such that, for every $x \in V$, there exists a convex subset D of Y verifying

$$\{f < \mu\}x \subset D \subset \{f \leq \lambda\}\bar{x} \cup \{f \leq \lambda\}x \quad \text{and} \quad D \cap \{f \leq \lambda\}\bar{x} \neq \emptyset.$$

After an obvious topological transcription Remark 2.2(ii) will be retained as our interconnectedness condition.

Definition 2.2. A function f is said to be lower interconnected on Y if Remark 2.2(ii) holds, when "convex" is replaced by "connected".

For further uses and also to be able to compare different conditions where a mixture of strict and large levels occurs, let us introduce the following notions.

A function f is said to be lower connected on Y if, for every $x \in X$ and for every pair of real numbers μ, λ with $\mu < \lambda$, there is a connected subset D of Y such that

$$\{f < \mu\}x \subset D \subset \{f \leq \lambda\}x. \tag{1}$$

Clearly, a function f is arbitrarily lower connected [resp. finitely lower connected] on Y if and only if, for every nonempty subset [resp. every nonempty finite subset] A of X and for every pair of real numbers μ, λ with $\mu < \lambda$, there is a connected subset D of Y such that

$$\bigcap_{x \in \Lambda} \{f < \mu\}x \subset D \subset \bigcap_{x \in \Lambda} \{f \leq \lambda\}x. \tag{2}$$

Proposition 2.1. A function f is lower connected [resp. finitely or arbitrarily lower connected] on Y if and only if, for each $\mu \in \mathbb{R}$ and for each nonempty subset E of X which is a singleton [resp. finite or arbitrary], the subset $\bigcap_{x \in E}\{f < \mu\}x$ of Y is connected.[7]

Proof. The "if part" of this claim is obvious. To prove the "only if part", observe that, by definition of lower connectedness [resp. finitely,

[7]Let Y be the set of naturals numbers equipped with the cofinite topology. Consider the function f defined by $f(x, y) := y$, if $y \in \{0, 1\}$, and $f(x, y) = 1/y$, otherwise. Observe that f is arbitrarily lower connected on Y; but the large sublevel sets $\{f \leq 0\}x$, which are always equal to $\{0, 1\}$, are not connected.

arbitrarily lower connectedness], for every pair (μ, λ) with $\mu < \lambda$, there is a connected subset $D(\mu, \lambda)$ of Y such that

$$\bigcap_{x \in E} \{f < \mu\}x \subset D(\mu, \lambda) \subset \bigcap_{x \in E} \{f \leq \lambda\}x.$$

Therefore, we can find an increasing sequence $\{C(\mu_n, \mu_{n+1})\}_n$ of connected sets such that

$$\bigcup_n C(\mu_n, \mu_{n+1}) = \bigcap_{x \in E} \{f < \mu\}x.$$

Hence, the set $\bigcap_{x \in E} \{f < \mu\}x$ is connected. □

Dually, we give the notions of upper connected functions on X. A function f is said to be upper connected [resp. finitely and arbitrarily upper connected] on X if $-f$ is lower connected [resp. finitely and arbitrarily lower connected] on X.

Finally let us state our main result which extends the Passy–Prisman Theorem 2.1, and consequently the Sion theorem.

Theorem 2.2. Main Theorem. Let either X or Y be connected, and let Y be compact. A function f is a minimax function, if the following conditions are satisfied:

(i) f is lower semicontinuous on Y,
(ii) f is arbitrarily upper connected on X,
(iii) f is finitely lower interconnected on Y.

To appreciate the scope of this minimax theorem for (finitely lower) interconnected functions in comparison to the Passy–Prisman Theorem 2.1 and the Sion theorem, one can consider, in the standard topological convex setting, an arbitrary multifunction $\Omega: X \twoheadrightarrow Y$ which is concave–convex[8] with closed values, such that $\bigcap_{x \in X} \Omega x \neq \varnothing$. Clearly, its indicator function verifies properties (i)–(iii); on the other hand, it is a quasi-concave–convex lower semicontinuous function, but generally neither P-convex in the small nor upper semicontinuous.

3. Tools and Concepts

We introduce in this section the tools and concepts that will be used to establish the finite intersection property for a large class of multifunctions. Recall that we have assumed that X and Y are nonempty topological spaces.

[8]Following Ref. 6, Ω is said to be concave–convex, if X, Y and Ωx are convex for every $x \in X$ and if $\Omega(x) \subset \Omega(x_1) \cup \Omega(x_2)$, whenever $x \in [x_1, x_2] \subset X$. The indicator function of Ω is the function f defined by $f(x, y) := -\infty$, if $y \in \Omega x$, and by $f(x, y) := +\infty$, if $y \notin \Omega x$.

Definition 3.1. A pair (Γ, Ω) of multifunctions from X to Y is an interconnected pair if, for every $\bar{x} \in \operatorname{dom} \Gamma$, there exists a neighborhood V of $\bar{x}$ such that

(i) $\forall x \in V, \exists D \subset Y$ connected such that $\Gamma x \subset D \subset \Omega \bar{x} \cup \Omega x$ and $D \cap \Omega \bar{x} \neq \emptyset$.

Remark 3.1. Sandwich Property. Notice that, if (Γ, Ω) is an interconnected pair, there exists a multifunction $\Xi: X \rightrightarrows Y$ with connected values such that $\Gamma \subset \Xi \subset \Omega$.

Definition 3.2. A pair of multifunctions (Ω, Φ) from X to Y is topologically concave between two points x_1 and x_2 of X if there is a connected subset C of X such that

$$\{x_1, x_2\} \subset C \quad \text{and} \quad \Omega(C) \subset \Phi x_1 \cup \Phi x_2.$$

Moreover, (Ω, Φ) is said to topologically concave if it is topologically concave between any two points of X.

Notice that a pair (Ω, Φ) is topologically concave if and only if, for any $B \subset Y$, there is a connected subset C of X such that

$$\{x \in X: \Phi x \subset B\} \subset C \subset \{x \in X: \Omega x \subset B\}.$$

Observe that, for any topologically concave pair (Ω, Φ), we have $\Omega \subset \Phi$ (take $B := \Phi x$). It is clear that the existence of a topologically concave pair implies that X is connected (take $B := Y$).

Remark 3.2. Heredity of Topologically Concave Pairs. One readily sees from the definition that, if a pair (Ω, Φ) is topologically concave between two points x_1 and x_2 and if $\Omega \subset \Phi$, then so is the pair (Ω_A, Φ_A) for any subset $A \subset X$.

Remark 3.3. Flexibility for Interconnected and Topologically Concave Pairs. Let $\Psi, \Psi', \Sigma, \Sigma'$ be multifunctions from X to Y such that $\Psi' \subset \Psi$ and $\Sigma \subset \Sigma'$. Then, (Ψ', Σ') is interconnected [resp. topologically concave between two points x_1 and x_2] if (Ψ, Σ) is interconnected [resp. topologically concave between two points x_1 and x_2].

Lemma 3.1. Local Pointed Binary Intersection Property. Let (Γ, Ω) be an inter-connected pair of multifunctions from X to Y. If the values of Γ are not empty, then:

(i) $\forall \bar{x} \in X, \exists$ a neighborhood V of $\bar{x}$ such that $\overline{\Omega \bar{x}} \cap \overline{\Omega x} \neq \emptyset$ for any $x \in V$.

Proof. Since $\varnothing \neq \Gamma x$ and $\Gamma x \subset \Omega x$ for each $x \in X$, property (i) is an immediate consequence of Definition 3.1. □

Lemma 3.2. Binary Intersection Property. Let (Ω, Φ) be a pair of multifunctions from X to Y which is topologically concave between two points x_1 and x_2 of X and $\Omega \subset \Phi$. If the values of Ω are connected and if Lemma 3.1(i) is satisfied, then:

$$\overline{\Phi x_1} \cap \overline{\Phi x_2} \neq \varnothing.$$

Proof. For contradiction, assume that

$$\overline{\Phi x_1} \cap \overline{\Phi x_2} = \varnothing.$$

Let C be as in Definition 3.2. Consider the following sets:

$$A_i = \{x \in C \colon \Omega x \subset \overline{\Phi x_i}\},$$

where $i = 1, 2$. Since $\Omega \subset \Phi$, each set A_i is nonempty, because it contains x_i. On the other hand, if $x \in C$, Definition 3.2(i) entails that $\Omega x \subset \overline{\Phi x_1} \cup \overline{\Phi x_2}$; but Ωx is connected; therefore,

$$\Omega x \subset \overline{\Phi x_1} \quad \text{or} \quad \Omega x \subset \overline{\Phi x_2}.$$

This shows that

$$C = A_1 \cup A_2.$$

The set C is connected; therefore, $C \cap \overline{A_1} \cap \overline{A_2}$ is nonempty. Pick a point $\bar{x}$ in that set; we can assume that $\bar{x} \in A_1$. Therefore,

$$\bar{x} \in A_1 \cap \overline{A_2}.$$

From $\bar{x} \in A_1$, we get

$$\Omega \bar{x} \subset \overline{\Phi x_1}. \tag{3}$$

Now, let V be the neighborhood given by Lemma 3.1(i). Since $\bar{x} \in \overline{A_2}$, there is a point $\hat{x}$ in $V \cap A_2$. But from the definition of A_2 and Lemma 3.1(i), it follows respectively that

$$\Omega \hat{x} \subset \overline{\Phi x_2} \quad \text{and} \quad \overline{\Omega \bar{x}} \cap \overline{\Omega \hat{x}} \neq \varnothing. \tag{4}$$

Hence, (3) and (4) imply

$$\overline{\Phi x_1} \cap \overline{\Phi x_2} \neq \varnothing;$$

thus, we have reached a contradiction. □

Proposition 3.1. Binary Intersection Property for a Family of Multifunctions. Let $\mathbb{F}$ be a family of multifunctions from X to Y with closed nonempty values. Then, every multifunction in $\mathbb{F}$ has the binary intersection property if the following two conditions hold:

(i) for any $\Delta \in \mathbb{F}$ and any pair $x_1, x_2 \in X$, there exists $\Delta' \in \mathbb{F}$ with $\Delta' \subset \Delta$ such that the pair (Δ', Δ) is topologically concave between x_1 and x_2;
(ii) for any $\Delta \in \mathbb{F}$, there exists $\Delta' \in \mathbb{F}$ such that the pair (Δ', Δ) is interconnected.

Proof. Fix an element Φ of $\mathbb{F}$ and two points x_1 and x_2 of X. Taking into account Remark 3.3 and the sandwich property (Remark 3.1), it follows that there are $\Gamma \in \mathbb{F}$ and a multifunction Ω from X to Y such that:

(a) $\Gamma \subset \Omega \subset \Phi$;
(b) (Γ, Ω) is an interconnected pair, Ω has connected values, and (Ω, Φ) is a topologically concave pair between x_1 and x_2.

Indeed, pick $\Delta' \in \mathbb{F}$ with $\Delta' \subset \Phi$ such that (Δ', Φ) is topologically concave between x_1 and x_2. There is $\Delta'' \in \mathbb{F}$ such that (Δ'', Δ') is an interconnected pair. By the sandwich property (Remark 3.1), there is a multifunction Ξ with connected values such that $\Delta'' \subset \Xi \subset \Delta'$. Finally, pick $\Delta''' \in \mathbb{F}$ such that (Δ''', Δ'') is an interconnected pair. Now, let $\Gamma := \Delta'''$, $\Omega := \Xi$. That the choice is appropriate follows from Remark 3.3 on flexibility of interconnected and topologically concave pairs. Now, let us return to the main stream of the proof.

The values of Γ are nonempty, because $\Gamma \in \mathbb{F}$. The pair (Γ, Ω) is interconnected; therefore, from Lemma 3.1, we see that Ω fulfills Lemma 3.1(i). Since Ω has connected values and (Ω, Φ) is a topologically concave pair between x_1 and x_2, Lemma 3.2 and the fact that the values of Φ are closed yield that

$$\Phi x_1 \cap \Phi x_2 \neq \varnothing.$$

The points x_1 and x_2 of X and the multifunction Φ in $\mathbb{F}$ being arbitrary, we have that the family $\{\Phi x: x \in X\}$ has the binary intersection property, whenever $\Phi \in \mathbb{F}$. □

Theorem 3.1. Finite Intersection Property. Let $\mathbb{F}$ be a family of multifunctions from X to Y with closed nonempty values. Then, every multifunction in $\mathbb{F}$ has the finite intersection property if Proposition 3.1(i) and the following property hold:

(i) for any finite subset F of X and any $\Delta \in \mathbb{F}$, there exists $\Delta' \in \mathbb{F}$ such that the pair (Δ'_F, Δ_F) is interconnected.

Proof. We proceed by induction. Fix a natural number $n \geq 1$. Assume that $\bigcap_{x\in A} \Delta x \neq \emptyset$, for any $\Delta \in \mathbb{F}$ and any nonempty subset A of X having at most n points. Take an arbitrary subset $F \subset X$ of cardinality $n-1$. From Remark 3.2 and the induction hypothesis, we see that the family

$$\mathbb{F}_F := \{\Delta_F : \Delta \in \mathbb{F}\}$$

verifies all the conditions required by Proposition 3.1. Hence,

$$\Delta_F x_1 \cap \Delta_F x_2 \neq \emptyset, \qquad \text{for any } \Delta \in \mathbb{F} \text{ and any } x_1, x_2 \in X.$$

Therefore, $\bigcap_{x\in A} \Delta x \neq \emptyset$ for any $\Delta \in \mathbb{F}$ and any nonempty subset A of X having at most $n+1$ points. □

4. Proof of the Main Theorem 2.2

Topologically concave pairs of multifunctions are related to arbitrarily upper connected functions. In fact, we have the following lemma.

Lemma 4.1. Let X be connected. A function f is arbitrarily upper connected on X if and only if

(i) for every $\mu, \lambda \in \mathbb{R}$ with $\mu < \lambda$, the pair $(\{f < \mu\}, \{f \leq \lambda\})$ is topologically concave.

Proof. Let μ, λ be real numbers; let $B \subset Y$ and $C \subset X$. The following two conditions are equivalent:

(a) $\{x \in X: \{f \leq \lambda\}x \subset B\} \subset C \subset \{x \in X: \{f < \mu\}x \subset B\}$;
(b) $\bigcap_{y\in Y\setminus B}\{f > \lambda\}y \subset C \subset \bigcap_{y \in Y\setminus B}\{f \geq \mu\}y$.

Using the connectedness of X if $Y = B$, and taking into account the definitions of arbitrarily upper connected function and of topologically concave pair, the required equivalence follows immediately. □

Lemma 4.2. From Y Connectedness to X Connectedness. Let Y be connected and let f be a function which is both lower semicontinuous on Y and arbitrarily upper connected on X. Then, there is a nonempty connected subset X' of X such that the restriction of f on $X' \times Y$, call it f', verifies:

(i) $\inf_Y \sup_X f = \inf_Y \sup_{X'} f'$,
(ii) f' is arbitrarily upper connected on X'.

Moreover, if f is finitely lower interconnected on Y, then so is f'.

Proof.

Case 1. $\exists y \in Y$ such that, for every $x \in X, f(x, y) = -\infty$. Pick any point $\bar{x}$ in X and define $X' = \{\bar{x}\}$. Obviously, (ii) holds; moreover, (i) also holds, because

$$\inf_Y \sup_X f = \inf_Y \sup_{X'} f = -\infty.$$

Case 2. $\forall y \in Y$, there is $x \in X$ such that $f(x, y) > -\infty$. Now, define a multifunction Ω: $Y \twoheadrightarrow X$ by

$$\Omega y := \{x \in X: f(x, y) > -\infty\};$$

observe that Ω has nonempty values; moreover, the values of Ω are connected, because f is upper connected. Consider the family $\mathscr{H}$ of open subsets of Y defined by

$$\mathscr{H} := \{A \subset Y: A \text{ open and } \Omega(A) \text{ connected}\}.$$

Then:

(a) for every $y \in Y$, there is an open set $U \in \mathscr{N}(y)$ such that $U \in \mathscr{H}$.

In fact, fix $y \in Y$; f being lower semicontinuous on Y, there is an open set $U \in \mathscr{N}(y)$ such that $\bigcap_{z \in U} \Omega z \neq \varnothing$; hence, the connectedness of the values of Ω entails that U belongs to $\mathscr{H}$. Moreover:

(b) the union of any chain of open sets belonging to $\mathscr{H}$, belongs to $\mathscr{H}$;

(c) $A \cup B \in \mathscr{H}$, if $A, B \in \mathscr{H}$ and $A \cap B \neq \varnothing$.

Now, using the Zorn lemma, from the connectedness of Y and from properties (a), (b), (c) we have that $Y \in \mathscr{H}$. Therefore, by definition of $\mathscr{H}$, the nonempty set $X' := \Omega(Y)$ is connected.

Condition (ii) holds because

$$\{f > \lambda\}y \subset X', \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } y \in Y.$$

Finally, for every $y \in Y$ and $x \in X \backslash X'$, we have $f(x, y) = -\infty$; hence,

$$\sup_{x \in X} f(x, y) = \sup_{x \in X'} f(x, y).$$

Therefore, (i) holds. □

Proof of the Main Theorem 2.2.

Case 1. Assume that X is connected. Consider the family

$$\mathbb{F} := \left\{ \{ f \leq \lambda \} : \lambda > \sup_X \inf_Y f \right\}.$$

Clearly, the multifunctions in $\mathbb{F}$ have nonempty values; moreover, by the lower semicontinuity of f, they have also closed values. The connectedness of X and the arbitrarily upper connectedness of f in conjunction with Lemma 4.1 and Remark 3.3 on flexibility of topologically concave pairs entails that:

(a) $\forall \lambda, \mu \in \mathbb{R}$ with $\mu < \lambda$, the pair $(\{ f \leq \mu \}, \{ f \leq \lambda \})$ is topologically concave.

On the other hand, f is finitely interconnected on Y, that is:

(b) $\forall \lambda, \mu \in \mathbb{R}$ with $\mu < \lambda$ and $\forall$ finite sets $F \subset X$, the pair $(\{ f \leq \mu \}_F, \{ f \leq \lambda \}_F)$ is interconnected.

Properties (a) and (b) tell us that we are in the situation described in Theorem 3.1. Now, apply that theorem to the family $\mathbb{F}$ and take into account the compactness of Y to conclude that

$$\bigcap_{x \in X} \{ f \leq \lambda \} x \neq \emptyset, \qquad \text{for any } \lambda > \sup_X \inf_Y f.$$

Case 2: Assume that Y is connected. Use Lemma 4.2 to reduce this case to Case 1. □

5. Comments on Properties Occurring in the Main Theorem 2.2

Necessity of the Interconnectedness Condition. We will see in Proposition 6.1 that lower interconnected functions occur naturally in the context of topological minimax theorems. One could ask to what extent such a condition is unavoidable. First, consider the rather standard topological setting described by the following two conditions.

(i) X is a connected topological space and Y is a compact space;
(ii) $f: X \times Y \to \bar{\mathbb{R}}$ is a function verifying the following properties:

  (a) f is lower semicontinuous on Y,
  (b) f is finitely lower connected on Y,
  (c) f is arbitrarily upper connected on X.

Proposition 5.1. Minimax Criterion. Let

$$a := \sup_X \inf_Y f.$$

Then, under assumptions (i) and (ii), f is a minimax function if and only if $f \vee a$ is finitely lower interconnected on Y.

Proof. If f verifies (i) and (ii), then so does $f \vee a$. Therefore, if $f \vee a$ is a finitely lower interconnected function on Y, from Theorem 2.2, it is a minimax function. In other words, f is a minimax function.

The other implication is a consequence of the following simple remarks. Assume that f is a minimax function. First, observe that

$$\bigcap_{x \in X} \{f < \lambda\} x \neq \varnothing, \qquad \text{for all } \lambda > a.$$

Therefore, by (ii)(b), we have that, $\forall \lambda > a, \forall$ finite subsets F of X, the multifunction $\{f < \lambda\}_F$ has connected values and

$$\bigcap_{x \in X} \{f < \lambda\}_F x \neq \varnothing.$$

Hence, $\forall \mu, \lambda$ with $a < \mu < \lambda$ and $\forall$ finite subsets F of X, the pair $(\{f < \mu\}_F, \{f < \lambda\}_F)$ is interconnected; therefore, $f \vee a$ is finitely lower interconnected on Y. □

Independence of the Properties Occurring in the Main Theorem 2.2. The examples below serve to show that none of the following five properties which occur in Theorem 2.2 can be dropped:

(P1) either X or Y is connected;
(P2) Y is compact;
(P3) f is lower semicontinuous on Y;
(P4) f is arbitrarily upper connected on X;
(P5) f is finitely lower interconnected on Y.

In each of the examples below, a multifunction $\Omega: X \twoheadrightarrow Y$ is given; its indicator function verifies all except one of the previous five assumptions of Theorem 2.2 and is not a minimax function, because the values of Ω are nonempty and $\bigcap_{x \in X} \Omega x = \varnothing$.

Example 5.1. $X := Y := \{0, 1\}$ are equipped with the discrete topology and Ω is defined by

$$\Omega x := \begin{cases} \{0\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1. \end{cases}$$

Clearly, neither X nor Y is connected; hence, (P1) does not hold.

Example 5.2. $X := Y := (0, 1]$ are equipped with the usual topology and Ω is defined by

$$\Omega x := (0, 1/(n+1)],$$

where $n \geq 1$ is the natural number such that

$$1/(n+1) < x \leq 1/n.$$

Clearly, Y is not compact; hence, (P2) does not hold.

Example 5.3. $X := Y := [0, 1]$ are equipped with the usual topology and Ω is defined by

$$\Omega x := \begin{cases} [0, 1/2), & \text{if } x = 0, \\ [0, 1/2], & \text{if } x \in (0, 1/2), \\ \{1/2\}, & \text{if } x = 1/2, \\ [1/2, 1], & \text{if } x \in (1/2, 1]. \end{cases}$$

Clearly, (P3) does not hold, because Ω has values which are not closed.

Example 5.4. $X := Y := [0, 1]$ are equipped with the usual topology and Ω is defined by

$$\Omega x := \begin{cases} \{0\}, & \text{if } x = 1/2, \\ \{1\}, & \text{if } x \in [0, 1/4] \cup [3/4, 1], \\ [0, 1], & \text{if } x \in (1/4, 1/2) \cup (1/2, 3/4). \end{cases}$$

Clearly, (P4) does not hold, because Ω has cofibers[9] which are not connected.

Example 5.5. $X := Y := [0, 1]$ are equipped with the usual topology and Ω is defined by

$$\Omega x := \begin{cases} \{0\}, & \text{if } x = 0, \\ \{1\}, & \text{if } x = 1, \\ \{0, 1\}, & \text{if } x \in (0, 1). \end{cases}$$

Clearly, (P5) does not hold, because Ω has values which are not connected.

---

[9]The cofibers of an arbitrary multifunction $\Omega: X \twoheadrightarrow Y$ are the sets $\{x \in X : y \notin \Omega x\}$, where $y \in Y$.

6. Extensions of the Sion Minimax Theorem: Cones and PP Functions

Proposition 6.1. Lower Interconnected Functions. The following properties holds:

(i) if f is lower interconnected on Y, then it is lower connected on Y;
(ii) if f is lower connected on Y and upper semicontinuous on X, then it is lower interconnected on Y.

Proof. Interconnectedness of f on Y amounts to:

(a) $\forall \lambda, \mu \in \mathbb{R}$ with $\mu < \lambda$, the pair $(\{f<\mu\}, \{f\leq\lambda\})$ is interconnected.

Hence, (i) follows immediately from (a) and the sandwich property (Remark 3.1) for interconnected pairs.

Now, we will show (ii). Let μ, λ be real numbers with $\mu<\lambda$. If $\{f<\mu\}\bar{x}\neq\emptyset$, by upper semicontinuity choose $\bar{y}\in\{f<\mu\}\bar{x}$ and a neighborhood V of $\bar{x}$ such that

$$\bar{y}\in\{f<\mu\}x, \qquad \text{for any } x\in V.$$

Since f is lower connected on Y, for each $x\in V$ there is a connected set D_x of Y such that

$$\{f<\mu\}x\subset D_x\subset\{f\leq\lambda\}x.$$

Hence, for $x\in V$, we have

$$\{f<\mu\}x\subset D_x\subset\{f\leq\lambda\}\bar{x}\cup\{f\leq\lambda\}x \quad \text{and} \quad \bar{y}\in D_x\cap\{f\leq\lambda\}\bar{x}.$$

This proves that the pair of multifunctions $(\{f<\mu\}, \{f\leq\lambda\})$ is interconnected. □

From the definition of arbitrarily lower (inter)connectedness, Proposition 6.1 entails the following corollary.

Corollary 6.1. Let f be upper semicontinuous on X. Then, the following properties are equivalent:

(i) f is arbitrarily lower connected on Y,
(ii) f is arbitrarily lower interconnected on Y.

From Corollary 6.1 and Theorem 2.2, the next theorem à la Sion follows in a straightforward manner.

Theorem 6.1. Let either X or Y be connected, and let either X or Y be compact. A function f is a minimax function if the two conditions below are satisfied:

(i) f is both lower semicontinuous and arbitrarily lower connected on Y;
(ii) f is both upper semicontinuous and arbitrarily upper connected on X.

That Theorem 6.1 is a topological version of the Sion theorem is clear if we recall the inclusions (2) [resp. the dual inclusions] to deduce from the quasiconvexity [resp. quasiconcavity] of a function its arbitrarily lower connectedness [resp. arbitrarily upper connectedness].

How to construct arbitrarily lower interconnected functions, without assuming upper semicontinuity as in Corollary 6.1 above?

Let us say that a pair (Γ, Ω) of multifunctions from X to Y is an arbitrarily [resp. finitely] interconnected pair if, for every set [resp. finite set] $A \subset X$, the pair (Γ_A, Ω_A) is interconnected.

Recognizing that a given pair is interconnected might be a difficult task, seeing that it is finitely [resp. arbitrarily] interconnected might be even more difficult. But, when the space Y has some geometric structure, we might be able to show in one step that a pair of multifunctions is arbitrarily interconnected. Convexity is one such structure; also, an interval structure[10] would do. But we prefer to express ourselves in another language.

The concept of a cone on a set, which we introduce now, is an instrumental and convenient tool for proving topological minimax theorems and for defining large classes of minimax functions.

Definition 6.1. See Greco–Horvath (Ref. 2, Definition 2). A function cn: $Y \times \mathscr{P}(Y) \to \mathscr{P}(Y)$ is said to be a cone on Y if, for every $y \in Y$, A and $B \in \mathscr{P}(Y)$, the following two conditions are satisfied:

(i) $\{y\} \cup A \subset \mathrm{cn}(y, A)$,
(ii) if $A \subset B$, then $\mathrm{cn}(y, A) \subset \mathrm{cn}(y, B)$.

The prototype of the cones is the segment-join

$$[y, A] := \bigcup_{y' \in A \cup \{y\}} [y, y'],$$

where $[y, y']$ is the segment

$$\{(1-t)y + ty' : 0 \le t \le 1\}.$$

[10]An interval structure on Y (see Stachó, Ref. 7) is a function $[\![\cdot, \cdot]\!]$ which maps each pair $(y, y') \in Y \times Y$ to a connected subset $[\![y, y']\!]$ of Y including the points y and y'.

Among the convex cones (i.e., cones with convex values), the standard convex cone is the smallest.

A cone on Y with connected values is said to be a connected cone. Clearly, if a convex set Y is equipped with a topology which turns segments into connected sets, then every convex cone is connected. To every interval structure $[\![\cdot, \cdot]\!]$ on Y, we can associate the connected cone, defined by

$$[\![y, A]\!] := \bigcup_{y' \in A \cup \{y\}} [\![y, y']\!].$$

Conversely, to any connected cone cn on Y, there corresponds the interval structure defined by

$$[\![y, y']\!] := \mathrm{cn}(y, \{y'\}).$$

The latter cone includes the former, i.e.,

$$[\![y, A]\!] \subset \mathrm{cn}(y, A), \qquad \text{for } y \in Y \text{ and } A \subset Y.$$

Lemma 6.1. See Greco–Horvath (Ref. 2, Definition 3). For a pair (Γ, Ω) of multifunctions from X to Y to be interconnected, it is sufficient that there exists a connected cone cn on Y such that:

(i) for any $\bar{x} \in X$ and $\bar{y} \in \Gamma \bar{x}$, there exists $V \in \mathcal{N}(\bar{x})$ such that, for any $x \in V$,

$$\mathrm{cn}(\bar{y}, \Gamma x) \subset \Omega \bar{x} \cup \Omega x.$$

The obvious proof is left to the reader. If conditions (i) is satisfied for some connected cone, we say that (Γ, Ω) is a PP pair.

Lemma 6.2. Heredity. Let (Γ, Ω) be a PP pair of multifunctions from X to Y. Then, for any subset $A \subset X$, the pair (Γ_A, Ω_A) is PP.

Proof. Let cn be a connected cone on Y with respect to which the pair (Γ, Ω) is PP. Fix a subset $A \subset X$. Now, let $\bar{x} \in X$ and $\bar{y} \in \Gamma_A \bar{x}$. Since $\bar{y} \in \Gamma \bar{x}$, from the definition of PP pair it follows that there is a neighborhood V of $\bar{x}$ such that

$$\mathrm{cn}(\bar{y}, \Gamma x) \subset \Omega \bar{x} \cup \Omega x, \qquad \text{for any } x \in V. \tag{5}$$

Notice that

$$\mathrm{cn}(y, \Gamma x) \subset \Omega x, \qquad \text{if } y \in \Gamma x.$$

Combined with the monotonicity of the cone, this yields

$$\mathrm{cn}(\bar{y}, \Gamma_A x) \subset \mathrm{cn}(\bar{y}, \Gamma x) \cap \left(\bigcap_{x' \in A} \Omega x' \right).$$

Finally, taking (5) into account we get that, for any $x \in V$,

$$\mathrm{cn}(\bar{y}, \Gamma_A x) \subset \Omega_A \bar{x} \cup \Omega_A x. \qquad \square$$

Let us say that a function f is a PP function if, for every $\lambda, \mu \in \mathbb{R}$ with $\mu < \lambda$, the pair $(\{f < \mu\}, \{f \leq \lambda\})$ is PP. It is immediate to check that the minimax functions of the Passy–Prisman Theorem 2.1 are PP functions with respect to the segment-join; recall Remark 2.1. Lemmas 6.1 and 6.2 lead to the following theorem.

Theorem 6.2. Every PP function is arbitrarily lower interconnected on Y. Therefore, Theorem 2.2 holds if (iii) is replaced by "f is a PP function".

For example, let X and Y be topological spaces with an interval structure $[\![\cdot, \cdot]\!]$ on Y. If f: $X \times Y \to \bar{\mathbb{R}}$ is a function such that, for every $\lambda, \mu \in \mathbb{R}$ with $\mu < \lambda$, at least one of the following seven pairs is PP with respect to the cone associated to the interval structure, then f is a PP function:

$$\begin{array}{lll} \text{(L)} & (\{f<\mu\}, \{f<\lambda\}), & (\{f<\mu\}, \{f\leq\lambda\}), \\ & (\{f\leq\mu\}, \{f<\lambda\}), & (\{f\leq\mu\}, \{f\leq\lambda\}), \\ & (\{f<\mu\}, \{f<\mu\}), & (\{f<\mu\}, \{f\leq\mu\}), \\ & (\{f\leq\mu\}, \{f\leq\mu\}). & \end{array}$$

As we have seen in Remark 2.1, the hypotheses of the Passy–Prisman Theorem 2.1 imply that, for each $\mu \in \mathbb{R}$, the pairs of multifunctions $(\{f < \mu\}, \{f < \mu\})$ are PP. The list (L) shows that one can mix strict and large inequalities as one likes. Concerning the version with strict inequalities, $(\{f < \mu\}, \{f < \mu\})$, and the version with large inequalities, $(\{f \leq \mu\}, \{f \leq \mu\})$, we showed in Ref. 1, Examples 2 and 3, that they are independent; that is, one cannot be derived from the other. One can also find in Ref. 2 explicit formulations in terms of interval spaces, for example in topological semilattices or hyperconvex metric spaces.

In the setting of upper semicontinuous functions, we have the following proposition, which offers a new insight on Theorem 6.1 (à la Sion) above.

Proposition 6.2. Let Y be connected, and let f be upper semicontinuous on X. The following properties are equivalent:

(i) f is arbitrarily lower connected,
(ii) f is arbitrarily lower interconnected,
(iii) f is a PP function.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was stated in Corollary 6.1. The implication (ii) $\Leftrightarrow$ (iii) is stated in Theorem 6.2. Now, assume (i). To show (iii), it is enough to prove that $(\{f<\mu\}, \{f<\mu\})$ is a PP pair for every real number μ. Let us fix μ.

For every $y, y' \in Y$, let

$$[\![y, y']\!] := \begin{cases} Y, & \text{if there is no } x \in X \text{ such that } \{y, y'\} \subset \{f<\mu\}x, \\ \cap \{\{f<\mu\}x: x \in X \text{ and } \{y, y'\} \subset \{f<\mu\}x\}, & \text{otherwise,} \end{cases}$$

and define a cone cn on Y by

$$\operatorname{cn}(y, A) := \bigcup_{y' \in A \cup \{y\}} [\![y, y']\!].$$

The sets $[\![y, y']\!]$ are connected, since f is arbitrarily lower connected on Y; hence, the cone cn is connected.

Observe that

$$\operatorname{cn}(y, \{f<\mu\}x) = \{f<\mu\}x, \qquad \text{if } y \in \{f<\mu\}x. \tag{6}$$

Now, to verify Lemma 6.1(i), let $\bar{x} \in X$ and $\bar{y} \in \{f<\mu\}\bar{x}$. By the upper semicontinuity of f, there is $V \in \mathcal{N}(\bar{x})$ such that, for every $x \in V$,

$$\bar{y} \in \{f<\mu\}x;$$

hence, by (6), we have

$$\operatorname{cn}(\bar{y}, \{f<\mu\}x) \subset \{f<\mu\}x \cup \{f<\mu\}\bar{x}, \qquad \text{for every } x \in V;$$

therefore, f is a PP function. □

In the usual context of interval spaces, interval structures are fixed, independently of any function (see the Stachó theorem below); in contrast, for each given PP function, there is a family of intervals [see (6) in the proof above], each one adapted to the family of strict sublevel sets $\{\{f<\mu\}x: x \in X\}$.

The first minimax theorem in the context of interval spaces (i.e., a space with a fixed interval structure) was given by Stachó[11] in Ref. 7. Given two interval spaces $(X, [\![\cdot,\cdot]\!])$ and $(Y, [\![\cdot,\cdot]\!])$, let us say that $f: X \times Y \to \mathbb{R}$ is quasiconcave in the sense of Stachó on the interval space $(X, [\![\cdot,\cdot]\!])$ if, for all

$x_1, x_2 \in X$ and for all $x \in [\![x_1, x_2]\!]$, one has $\min\{f(x_1, f(x_2)\} \le f(x)$. Quasiconvexity on the interval space $(Y, [\![\cdot, \cdot]\!])$ is defined similarly.

Theorem 6.3. See Stachó (Ref. 7). Let $(X, [\![\cdot, \cdot]\!])$ and $(Y, [\![\cdot, \cdot]\!])$ be two compact interval spaces. A continuous function f: $X \times Y \to \mathbb{R}$ is a minimax function if it is both quasiconcave on X and quasiconvex on Y in the sense of Stachó.

7. Improvements of the König Minimax Theorem

From Proposition 6.1 and the definition of finitely lower (inter)connectedness, we have the following corollary.

Corollary 7.1. Let f be upper semicontinuous on X. Then, the following properties are equivalent:

(i) f is finitely lower connected on Y,
(ii) f is finitely lower interconnected on Y.

The next theorem à la Sion follows in a straightforward way from this corollary and our main result, Theorem 2.2.

Theorem 7.1. Let either X or Y be connected, and let Y be compact. A function f is a minimax function if the two conditions below are satisfied:

(i) f is both lower semicontinuous and finitely lower connected on Y,
(ii) f is both upper semicontinuous and arbitrarily upper connected on X.

That Theorem 7.1 improves the König Theorem 1.1 is clear; there are no separation properties on Y, and the connectedness of Y can replace that of X. There were attempts by König to remove the Hausdorff condition on Y; he showed that it could be replaced by the weaker and unusual weak Hausdorff property; see Ref. 3 and references therein.

König gave two versions of his theorem, one in Ref. 3 cited here as Theorem 1.1 and another in Ref. 8, which we will recall as Theorem 7.2 below.

Let $a \in \bar{\mathbb{R}}$; a subset $B \subset \mathbb{R}$ is said to be a border set at a if a is the infimum of B and each $b \in B$ is strictly greater than a. Obviously, a convex border set is an interval.

Theorem 7.2. See König (Ref. 8). Let X be connected, Y compact, and f a function both lower semicontinuous on Y and upper semicontinuous on X. Let B and I be respectively a border [resp. border convex] set at $a := \sup_X \inf_Y f$. Then, f is a minimax function under each of the four assumptions below:

$$\text{(i)} \quad \begin{cases} \text{(a)} \quad \bigcap_{x\in F} \{f \leq \lambda\}x \text{ is connected}, \forall \lambda \in B, \forall \text{ nonempty finite } F \subset X, \\ \text{(b)} \quad \bigcap_{y\in H} \{f > \lambda\}y \text{ is connected}, \forall \lambda \in B, \forall \text{ nonempty subset } H \subset Y; \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \text{(a)} \quad \bigcap_{x\in F} \{f < \lambda\}x \text{ is connected}, \forall \lambda \in I, \forall \text{ nonempty finite } F \subset X, \\ \text{(b)} \quad \bigcap_{y\in H} \{f > \lambda\}y \text{ is connected}, \forall \lambda \in B, \forall \text{ nonempty subset } H \subset Y, \\ \text{(c)} \quad Y \text{ is Hausdorff}; \end{cases}$$

$$\text{(iii)} \quad \begin{cases} \text{(a)} \quad \bigcap_{x\in F} \{f < \lambda\}x \text{ is connected}, \forall \lambda \in B, \forall \text{ nonempty finite } F \subset X, \\ \text{(b)} \quad \bigcap_{y\in H} \{f \geq \lambda\}y \text{ is connected}, \forall \lambda \in B, \forall \text{ nonempty subset } H \subset Y; \end{cases}$$

$$\text{(iv)} \quad \begin{cases} \text{(a)} \quad \bigcap_{x\in F} \{f \leq \lambda\}x \text{ is connected}, \forall \lambda \in I, \forall \text{ nonempty finite } F \subset X, \\ \text{(b)} \quad \bigcap_{y\in H} \{f \geq \lambda\}y \text{ is connected}, \forall \lambda \in B, \forall \text{ nonempty subset } H \subset Y; \end{cases}$$

The König proof of Theorem 7.2 was not simple. The statement of the theorem itself is quite intricate. Now, we give an improvement of the König Theorem 7.2 by allowing a free mixture of large/strict lower/upper level sets.

Theorem 7.3. Let X be connected, and let Y be compact. Assume that f is a function which is both lower semicontinuous on Y and upper semicontinuous on X. Let $a := \sup_X \inf_Y f$. Then, f is a minimax function if the two conditions below are satisfied:

(i) For any $\lambda > a$ and for any finite subset $F \subset X$, there exists μ with $a < \mu < \lambda$ such that, for every $x \in X$,

$$\exists\, D \subset Y \text{ connected such that } \{f < \mu\}_F x \subset D \subset \{f \leq \lambda\}_F x.$$

(ii) For any $\lambda > a$ and for any nonempty subset $H \subset X$, there exists μ with $a < \mu < \lambda$ and

$$\exists C \subset X \text{ connected such that } \bigcap_{y\in H} \{f > \lambda\}y \subset C \subset \bigcap_{y\in H} \{f \geq \mu\}y.$$

Proof. We have to see that the family of multifunctions $\mathbb{F}=\{\{f\leq\lambda\}: \lambda>a\}$ has the finite intersection property. For that purpose, we will use Theorem 3.1. Hence, we have to show that Proposition 3.1(i) and Theorem 3.1(i) hold.

Claim 1. Proposition 3.1(i) holds. Fix $\lambda>a$ and x_1, x_2 in X. By the flexibility of topologically concave pairs (see Remark 3.3), it is enough to show that: for every $\lambda>a$, there exists μ with $a<\mu<\lambda$ such that the pair $(\{f<\mu\}, \{f\leq\lambda\})$ is topologically concave between x_1, x_2. Define

$$H:=\{y\in Y: \{x_1, x_2\}\subset\{f>\lambda\}y\}. \tag{7}$$

If $H=\emptyset$, choose a real number μ such that $a<\mu<\lambda$. Then, one has

$$Y=\{f\leq\lambda\}x_1\cup\{f\leq\lambda\}x_2;$$

in other words,

$$\{f<\mu\}x\subset\{f\leq\lambda\}x_1\cup\{f\leq\lambda\}x_2, \quad \text{for every } x\in X.$$

Hence, X being connected, from Definition 3.2 it follows that the pair $(\{f<\mu\}, \{f\leq\lambda\})$ is topologically concave between x_1, x_2.

Otherwise, in the case where $H\neq\emptyset$, take μ and C as in (ii). By (7), we have that

$$\{x_1, x_2\}\subset\bigcap_{y\in H}\{f>\lambda\}y;$$

hence, from (ii), it follows that $\{x_1, x_2\}\subset C$. On the other hand, from the inclusion

$$C\subset\bigcap_{y\in H}\{f\geq\mu\}y$$

in (ii) and from the definition (7) of H, it follows easily that

$$\{f<\mu\}x\subset\{f\leq\lambda\}x_1\cup\{f\leq\lambda\}x_2, \quad \text{for every } x\in C.$$

Hence, by Definition 3.2, we have that the pair $(\{f<\mu\}, \{f\leq\lambda\})$ is topologically concave between x_1, x_2.

Claim 2. Theorem 3.1(i) holds. Fix $\lambda>a$ and $F\subset X$ finite subset. By flexibility of the interconnected pairs (see Remark 3.3), it is enough to show that, for every $\lambda>a$, there exists μ with $a<\mu<\lambda$ such that the pair $(\{f<\mu\}_F, \{f\leq\lambda\}_F)$ is interconnected.

Hence, choose μ as in (i) and fix $\bar{x}\in X$ such that $\{f<\mu\}_F\bar{x}\neq\emptyset$. By the upper semicontinuity of f_F, choose $\bar{y}\in Y$ and a neighborhood V of $\bar{x}$ such that

$$\bar{y}\in\{f<\mu\}_F x, \qquad \text{for any } x\in V.$$

Property (i) entails that, for each $x \in V$, there exists a connected set D of Y such that

$$\{f < \mu\}_F x \subset D \subset \{f \leq \lambda\}_F x.$$

Hence, for $x \in V$, we have

$$\{f < \mu\}_F x \subset D \subset \{f \leq \lambda\}_F \bar{x} \cup \{f \leq \lambda\}_F x \quad \text{and} \quad \bar{y} \in D \cap \{f \leq \lambda\}_F \bar{x}.$$

Thus, the pair $(\{f < \mu\}_F, \{f \leq \lambda\}_F)$ is interconnected. □

References

1. Greco, G. H., and Horvath, C. D., *Passy–Prisman's Minimax Theorem*, Annales des Sciences Mathématiques du Quebec, Vol. 22, pp. 181–191, 1998.
2. Greco, G. H., and Horvath, C. D., *Topological Versions of Passy–Prisman's Minimax Theorem*, Optimization, Vol. 47, pp. 155–166, 2000.
3. König, H., *The Topological Minimax Theorem*, Séminaire Initiation à l'Analyse (G. Choquet, G. Godefroy, M. Rogalski, J. Saint Raymond), 1994/95.
4. Greco, G. H., and Horvath, C. D., *Toward a Geometric Theory of Minimax Equalities*, Optimization, Vol. 47, pp. 166–188, 2000.
5. Passy, U., and Prisman, E. Z., *A Duality Approach to Minimax Results for Quasisaddle Functions in Finite Dimensions*, Mathematical Programming, Vol. 55, pp. 81–98, 1992.
6. Greco, G. H., *Minimax Theorems and Saddling Transformations*, Journal of Mathematical Analysis and Applications, Vol. 147, pp. 180–197, 1990.
7. Stachó, L. L., *Minimax Theorems beyond Topological Vector Spaces*, Acta Scientiarum Mathematicarum, Vol. 42, pp. 157–164, 1980.
8. König, H., *A General Minimax Theorem Based on Connectedness*, Archiv der Mathematik, Vol. 59, pp. 55–64, 1992.