Acta Math. Hungar., **118 (4)** (2008), 299–306.
DOI: 10.1007/s10474-007-6209-0
First published online September 26, 2007

SOME FUNCTIONAL EQUATIONS ON STANDARD OPERATOR ALGEBRAS

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(Received October 11, 2006; revised February 12, 2007; accepted February 20, 2007)

Abstract. The main purpose of this paper is to prove the following result. Let H be a complex Hilbert space, let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H, and let $\mathcal{A}(H) \subset \mathcal{B}(H)$ be a standard operator algebra which is closed under the adjoint operation. Suppose that $T : \mathcal{A}(H) \to \mathcal{B}(H)$ is a linear mapping satisfying $T(AA^*A) = T(A)A^*A - AT(A^*)A + AA^*T(A)$ for all $A \in \mathcal{A}(H)$. Then T is of the form $T(A) = AB + BA$ for all $A \in \mathcal{A}(H)$, where B is a fixed operator from $\mathcal{B}(H)$. A result concerning functional equations related to bicircular projections is proved.

Throughout, R will represent an associative ring. Given an integer $n \geqq 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. As usual we write $[x, y]$ for $xy - yx$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let $\mathcal{A}$ be an algebra over the real or complex field and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. A linear mapping $D : \mathcal{B} \to \mathcal{A}$ is called a linear derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in \mathcal{B}$. In case we have a ring R, an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs

Key words and phrases: ring, *-ring, prime ring, semiprime ring, Banach space, Hilbert space, standard operator algebra, derivation, Jordan derivation, bicircular projection.
2000 Mathematics Subject Classification: 13N15, 16E99, 39B05, 46K15.

$x, y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [x, a]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [8] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's theorem can be found in [4]. Cusack [5] generalized Herstein's theorem to 2-torsion free semiprime rings (see [2] for an alternative proof). Let X be a real or complex Banach space and let $\mathcal{B}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{B}(X)$, respectively. An algebra $\mathcal{A}(X) \subset \mathcal{B}(X)$ is said to be standard in case $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn–Banach theorem. Let X be a complex Banach space. A projection $P \in \mathcal{B}(X)$ is bicircular if all mappings of the form $e^{i\alpha}P + e^{i\beta}(I - P)$, where I denotes the identity operator, are isometric for all pairs of real numbers α and β. Stachó and Zalar [11] investigated bicircular projections on the C^*-algebra $\mathcal{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H (see also [12]).

Let us start with the following result proved by Brešar [3]: Let R be a 2-torsion free semiprime ring and let $D: R \to R$ be an additive mapping satisfying

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{0.1}$$

for all pairs $x, y \in R$. In this case D is a derivation.

One can easily prove that any Jordan derivation on an arbitrary 2-torsion free ring satisfies (0.1) (see [4] for the details), which means that the above result generalizes Cusack's generalization of Herstein's theorem we have mentioned above.

Recently, the second named author of the present paper, Kosi-Ulbl, and Eremita [15] have proved the following result: Let R be a 2-torsion free semiprime ring and let $D: R \to R$ be an additive mapping satisfying

$$T(xyx) = T(x)yx - xT(y)x + xyT(x) \tag{0.2}$$

for all pairs $x, y \in R$. In this case T is of the form $2T(x) = qx + xq$ for all $x \in R$, where q is some fixed element from the symmetric Martindale ring of quotients, which will be denoted by $Q_S(R)$.

In case of a *-ring we obtain, putting $y = x^*$ in (0.1) and (0.2), the relations

$$D(xx^*x) = D(x)x^*x + xD(x^*)x + xx^*D(x), \quad x \in R, \tag{0.3}$$

and

$$T(xx^*x) = T(x)x^*x - xT(x^*)x + xx^*T(x), \quad x \in R. \tag{0.4}$$

Recently, the first named author of the present paper [13] has proved the following result, which is related to (0.3): Let H be a complex Hilbert space and let $\mathcal{A}(H)$ be a standard operator algebra which is closed under the adjoint operation. Let $D : \mathcal{A}(H) \to \mathcal{B}(H)$ be a linear mapping satisfying

$$D(AA^*A) = D(A)A^*A + AD(A^*)A + AA^*D(A)$$

for all $A \in \mathcal{A}(H)$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(H)$ and some fixed operator $B \in \mathcal{B}(H)$, which means that D is a derivation.

The following result is related to (0.4).

LEMMA 1. *Let H be a complex Hilbert space and let $\mathcal{A}(H) \subset \mathcal{B}(H)$ be a standard operator algebra which is closed under the adjoint operation. Let $T : \mathcal{A}(H) \to \mathcal{B}(H)$ be a linear mapping satisfying*

$$T(AA^*A) = T(A)A^*A - AT(A^*)A + AA^*T(A) \tag{0.5}$$

for all $A \in \mathcal{A}(H)$. In this case T is of the form $T(A) = AB + BA$ for all $A \in \mathcal{A}(H)$ and some fixed operator $B \in \mathcal{B}(H)$.

PROOF. Let us define

$$L(X, Y, Z) = T(XY^*Z) - \big(T(X)Y^*Z - XT(Y^*)Z + XY^*T(Z)\big)$$

for all $X, Y, Z \in \mathcal{A}(H)$. Note that the above form is 3-sesquilinear. Assumption (0.5) implies

$$L(X, X, X) = 0 \tag{0.6}$$

for all operators $X \in \mathcal{A}(H)$. Putting $X + Y$ for X in (0.6) we get

$$L(X, X, Y) + L(X, Y, X) + L(X, Y, Y)$$
$$+ L(Y, X, X) + L(Y, X, Y) + L(Y, Y, X) = 0.$$

Putting $-X$ for X and comparing with the above relation we get

$$L(X, X, Y) + L(X, Y, X) + L(Y, X, X) = 0. \tag{0.7}$$

Further, if we put iY for Y in (0.7) and compare with the relation (0.7) we get $L(X, Y, X) = 0$ for all $X, Y \in \mathcal{A}(H)$. This yields

$$L(X, Y^*, X) = T(XYX) - \big(T(X)YX - XT(Y)X + XYT(X)\big) = 0$$

for all $X, Y \in \mathcal{A}(H)$. Applying [15, Theorem 2.1] it follows that T is of the form $T(A) = AB + BA$ for all $A \in \mathcal{A}(H)$ and some fixed operator $B \in Q_S\big(\mathcal{A}(H)\big)$. Since $Q_S\big(\mathcal{A}(H)\big) = \mathcal{B}(H)$ (this is a direct consequence of [1, Theorem 4.3.8] and [9, p. 78, Example 5]) it follows that $B \in \mathcal{B}(H)$. $\square$

The next corollary includes Lemma 1 as well as Theorem 1 in [13].

COROLLARY 2. *Let H be a complex Hilbert space and let $\mathcal{A}(H)$ be a standard operator algebra which is closed under the adjoint operation. Let $D, G : \mathcal{A}(H) \to \mathcal{B}(H)$ be linear mappings satisfying*

$$D(AA^*A) = D(A)A^*A + AG(A^*)A + AA^*D(A) \tag{0.8}$$

and

$$G(AA^*A) = G(A)A^*A + AD(A^*)A + AA^*G(A) \tag{0.9}$$

for all $A \in \mathcal{A}(H)$. In this case D and G are of the form

$$D(A) = A(B + C) + (C - B)A, \quad G(A) = A(B - C) - (B + C)A$$

for all $A \in \mathcal{A}(H)$ and some fixed operators $B, C \in \mathcal{B}(H)$.

PROOF. Combining (0.8) with (0.9) we obtain

$$F(AA^*A) = F(A)A^*A + AF(A^*)A + AA^*F(A), \quad A \in \mathcal{A}(H), \tag{0.10}$$

where $F(A)$ stands for $D(A) + G(A)$. On the other hand subtracting (0.9) from (0.8) we arrive at

$$H(AA^*A) = H(A)A^*A - AH(A^*)A + AA^*H(A), \quad A \in \mathcal{A}(H), \tag{0.11}$$

where $H(A)$ denotes $D(A) - G(A)$. According to (0.10) all the requirements of Theorem 1 in [13] are fulfilled, which means that $D(A) + G(A) = [A, B]$ holds for all $A \in \mathcal{A}(H)$, where $B \in \mathcal{B}(H)$ is some fixed operator. On the other hand it follows from (0.11) and Lemma 1 that

$$D(A) - G(A) = AC + CA$$

holds for all $A \in \mathcal{A}(H)$ and some fixed operator $C \in \mathcal{B}(H)$. The last two relations yield that $2D(A) = A(B + C) + (C - B)A$ and $2G(A) = A(B - C) - (B + C)A$ holds for all $A \in \mathcal{A}(H)$, where $B, C \in \mathcal{B}(H)$ are some fixed operators. □

According to Proposition 3.4 in [11] every bicircular projection $P : \mathcal{B}(H) \to \mathcal{B}(H)$, where H is a complex Hilbert space, satisfies the functional equation

$$P(ABA) = P(A)BA - AP(B^*)^*A + ABP(A) \tag{0.12}$$

for all pairs $A, B \in \mathcal{B}(H)$. Fošner and Iliševič [6] investigated the above functional equation in 2-torsion free semiprime *-ring. They expressed the solution of the equation (0.12) in terms of derivations and so-called double centralizers (see also [14]).

Putting A^* for B in (0.12) one obtains

$$P(AA^*A) = P(A)A^*A - AP(A)^*A + AA^*P(A), \quad A \in \mathcal{B}(H). \tag{0.13}$$

This together with Theorem 1 in [14] was the inspiration for our last result.

THEOREM 3. *Under the conditions of Corollary* 2, D *and* G *are of the form*

$$D(A) = [A, B + C] + (P + Q)A + A(P + Q), \tag{0.14}$$

$$G(A) = [A, B - C] + (P - Q)A + A(P - Q) \tag{0.15}$$

for all $A \in \mathcal{A}(H)$ *and some fixed operators* $B, C, P, Q \in \mathcal{B}(H)$. *Besides,* $B^* + B = \lambda I$, $C^* - C = \mu I$, $P^* = -P$, $Q^* = Q$, *where* λ *and* μ *are some fixed complex numbers.*

PROOF. The proof goes through in several steps.

First step. Let us first assume that $D = G$. In this case

$$F(AA^*A) = F(A)A^*A + AF(A)^*A + AA^*F(A)$$

for all $A \in \mathcal{A}(H)$. It is our aim to prove that F is of the form

$$F(A) = [A, B] + PA + AP$$

for all $A \in \mathcal{A}(H)$, where B and P are some fixed operators from $\mathcal{B}(H)$. Besides, $B^* + B = \lambda I$ for some fixed complex number λ and $P^* = -P$. Let us introduce the mappings $d : \mathcal{A}(H) \to \mathcal{B}(H)$ and $f : \mathcal{A}(H) \to \mathcal{B}(H)$ by

$$d(A) = F(A) + F(A^*)^* \quad \text{and} \quad f(A) = F(A) - F(A^*)^*,$$

and prove that

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A) \tag{0.16}$$

and

$$d(A^*)^* = d(A) \tag{0.17}$$

hold for all $A \in \mathcal{A}(H)$. We have

$$\begin{aligned} d(AA^*A) &= F(AA^*A) + F(A^*AA^*)^* = F(A)A^*A + AF(A)^*A + AA^*F(A) \\ &+ \big(F(A^*)AA^* + A^*F(A^*)^*A^* + A^*AF(A^*)\big)^* = F(A)A^*A + AF(A)^*A \\ &+ AA^*F(A) + AA^*F(A^*)^* + AF(A^*)A + F(A^*)^*A^*A \\ &= \big(F(A) + F(A^*)^*\big)A^*A + A\big(F(A^*) + F(A)^*\big)A + AA^*\big(F(A) + F(A^*)^*\big) \\ &= d(A)A^*A + Ad(A^*)A + AA^*d(A), \end{aligned}$$

which proves (0.16). We also have

$$d(A^*)^* = \big(F(A^*) + F(A)^*\big)^* = F(A) + F(A^*)^* = d(A),$$

which proves (0.17). Similarly one proves that

$$f(AA^*A) = f(A)A^*A - Af(A^*)A + AA^*f(A) \tag{0.18}$$

and

$$f(A^*)^* = -f(A) \tag{0.19}$$

hold for all $A \in \mathcal{A}(H)$. (0.16) tells us that all the assumptions of Theorem 1 in [13] are fulfilled, which means that d is of the form $d(A) = [A, B]$ for all $A \in \mathcal{A}(H)$, where B is some fixed operator from $\mathcal{B}(H)$. Further, (0.17) tells us that $[A, B] = [A^*, B]^*$ holds for all $A \in \mathcal{A}(H)$, which means that we have $[A, B + B^*] = 0$ for all $A \in \mathcal{A}(H)$. Since $B + B^*$ commutes with all operators from $\mathcal{F}(H)$ one can conclude that $B + B^* = \lambda I$, where λ is some fixed complex number. From (0.18) it follows that all the assumptions of Lemma 1 are fulfilled, which means that f is of the form $f(A) = AP + PA$ for all $A \in \mathcal{A}(H)$, where P is some fixed operator from $\mathcal{B}(H)$. According to (0.19) we have $(A^*P + PA^*)^* = -(AP + PA)$, which gives

$$(P^* + P)A + A(P^* + P) = 0 \tag{0.20}$$

for all $A \in \mathcal{A}(H)$. Putting in the above relation first CA for A, then multiplying (0.20) from the left side by C and subtracting one from another we obtain $[P^* + P, C]A = 0$ for all $A, C \in \mathcal{A}(H)$, whence it follows that $[P^* + P, C] = 0$. Thus we can replace in (0.20) $A(P^* + P)$ by $(P^* + P)A$, which gives $(P^* + P)A = 0$ for all $A \in \mathcal{A}(H)$ and finally $P^* + P = 0$. Combining $F(A) + F(A^*)^* = [A, B]$ with $F(A) - F(A^*)^* = AP + PA$ we obtain $2F(A) = [A, B] + AP + PA$, which completes the proof of the first step.

Second step. Let us assume that $D = -G$. Now we have

$$H(AA^*A) = H(A)A^*A - AH(A)^*A + AA^*H(A)$$

for all $A \in \mathcal{A}(H)$. In this case H is of the form

$$H(A) = [A, C] + QA + AQ$$

for all $A \in \mathcal{A}(H)$, where C and Q are some fixed operators from $\mathcal{B}(H)$. Besides $C^* - C = \mu I$, where μ is a fixed complex number and $Q^* = Q$. The proof of the second step will be omitted since it goes through using the same arguments as in the proof of the first step.

Third step. Now we are ready for the proof of the general case. Combining (0.8) with (0.9) we obtain

$$F(AA^*A) = F(A)A^*A + AF(A)^*A + AA^*F(A), \quad A \in \mathcal{A}(H),$$

where $F(A)$ stands for $D(A) + G(A)$. On the other hand subtracting (0.9) from (0.8) we arrive at

$$H(AA^*A) = H(A)A^*A - AH(A)^*A + AA^*H(A), \quad A \in \mathcal{A}(H),$$

where $H(A)$ denotes $D(A) - G(A)$. Now, according to the first and the second step we have

$$D(A) + G(A) = [A, B] + PA + AP$$

and

$$D(A) - G(A) = [A, C] + QA + AQ$$

for all $A \in \mathcal{A}(H)$. From the above relations one obtains

$$2D(A) = [A, B + C] + (P + Q)A + A(P + Q)$$

and

$$2G(A) = [A, B - C] + (P - Q)A + A(P - Q)$$

for all $A \in \mathcal{A}(H)$. $\square$

References

[1] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, *Rings with Generalized Identities,* Marcel Dekker, Inc. (New York, 1996).
[2] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.,* **104** (1988), 1003–1006.
[3] M. Brešar, Jordan mappings of semiprime rings, *J. Algebra,* **127** (1989), 218–228.
[4] M. Brešar and J. Vukman, Jordan derivations on prime rings, *Bull. Austral. Math. Soc.,* **37** (1988), 321–322.
[5] J. Cusack, Jordan derivations on rings, *Proc. Amer. Math. Soc.,* **53** (1975), 321–324.
[6] M. Fošner and D. Ilišević, On a class of projections on *-rings, *Comm. Algebra,* **33** (2005), 3293–3310.
[7] M. Fošner and J. Vukman, On some equations in prime rings, preprint.
[8] I. N. Herstein, Jordan derivations of prime rings, *Proc. Amer. Math. Soc.,* **8** (1957), 1104–1119.

[9] N. Jacobson, *Structure of Rings,* American Mathematical Society Colloquium Publications, Vol. 37. Revised edition American Mathematical Society (Providence, 1964).

[10] L. Molnár, On centralizers of an H^*-algebra, *Publ. Math. Debrecen,* **46** (1995), 89–95.

[11] L. L. Stachó and B. Zalar, Bicircular projections on some matrix and operator spaces, *Linear Algebra Appl.,* **384** (2004), 9–20.

[12] L. L. Stachó and B. Zalar, Bicircular projections and characterization of Hilbert spaces, *Proc. Amer. Math. Soc.,* **3** (2004), 3019–3025.

[13] J. Vukman, On derivations of algebras with involution, *Acta Math. Hungar.,* **112** (2006), 187–192.

[14] J. Vukman, On functional equations related to bicircular projections, *Glasnik Mat.,* **41** (2006), 51–56.

[15] J. Vukman, I. Kosi-Ulbl and D. Eremita, On certain equations in rings, *Bull. Austral. Math. Soc.,* **71** (2005), 53–60.