A Minimax Theorem for Vector-Valued Functions

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Abstract. In this work, as usual in vector-valued optimization, we consider the partial ordering induced in a topological vector space by a closed and convex cone. In this way, we define maximal and minimal sets of a vector-valued function and consider minimax problems in this setting. Under suitable hypotheses (continuity, compactness, and special types of convexity), we prove that, for every

$$\alpha \in \operatorname{Max} \bigcup_{s \in X_0} \operatorname{Min}_w f(s, Y_0),$$

there exists

$$\beta \in \operatorname{Min} \bigcup_{t \in Y_0} \operatorname{Max} f(X_0, t),$$

such that $\beta \leq \alpha$ (the exact meanings of the symbols are given in Section 2).

Key Words. Minimax theorems, vector-valued optimization.

1. Introduction

Minimax theorems for real-valued (or extended real-valued) functions $f: X_0 \times Y_0 \to R$ [or $R \cup \{-\infty, +\infty\}$] state that, under suitable hypotheses of compactness, convexity, and continuity, the equality

$$\inf_{y \in Y_0} \sup_{x \in X_0} f(x, y) = \sup_{x \in X_0} \inf_{y \in Y_0} f(x, y)$$

holds. References 1–4 discuss on this subject. See also Ref. 5 for a survey and extensive bibliographical references. The numerous studies of vector-valued optimization in recent years (e.g., see Refs. 6–9) seem to lead, in a natural way, to the investigation of minimax problems in this more general

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setting. Nevertheless, as far as we know, until now only a few papers have dealt with this subject.

In Ref. 10, a minimax result is stated for vector-valued functions satisfying a particular convexity condition so that the involved maximal or minimal sets reduce to single points. The vectorial structure of the problem may thus be considered lost. In Ref. 11, some interesting questions are raised and clarifying examples are presented; however, no general result is given. In Ref. 12, saddle points of bilinear vector-valued functions are investigated through scalarization methods.

In this paper, we state in Section 2 some preliminary results, mostly about convexity properties of vector-valued functions. In Section 3, we state a minimax theorem that is proved using separation theorems and the classical minimax results, as well as properties of compact-valued multifunctions. Theorem 3.1 provides perhaps an initial answer to some of the problems raised in Ref. 11, even though questions remain open, as noted in Section 4. In this last section, we also give an example and clarify some hypotheses of the theorem.

2. Notation and Preliminary Results

Let V be a real vector space and let Q be a cone in V, i.e., a set such that $\alpha v \in Q$ whenever $v \in Q$ and $\alpha \geq 0$. Given $v_1, v_2 \in V$, we shall write $v_1 \leq v_2$ if $v_2 - v_1 \in Q$. It is easily seen that the relation $\leq$ is reflexive (since $0 \in Q$) and moreover that $v_1 \leq v_2$ implies $v_1 + v \leq v_2 + v$ for every $v \in V$ and $\alpha v_1 \leq \alpha v_2$ for every $\alpha \geq 0$. If the cone Q is convex, the relation $\leq$ is also transitive. Moreover, if Q is pointed [i.e., $Q \cap (-Q) = \{0\}$], $\leq$ is antisymmetric. It follows that a pointed convex cone introduces in V a (partial) ordering $\leq$.

Let $B \subset V$ and let Q be a pointed convex cone in V. We say that $v_0 \in B$ is a minimal point of B if $B \cap (v_0 - Q) = \{v_0\}$. Thus, for every $v \in B$, we have $v_0 - v \notin Q \backslash \{0\}$; that is, either $v_0 \leq v$ or v_0 and v are not comparable through $\leq$. Min B will be the set of all minimal points of B. In an analogous way, we define the maximal points of B.

In all that follows, V will be a locally convex Hausdorff topological vector space and $C \subset V$ a closed, pointed, convex cone such that int $C \neq \phi$. We shall use also the pointed convex cone $C^\circ = (\text{int } C) \cup \{0\}$ and write $\leq$ to denote the ordering induced by C and $\leq_w$ to denote the one induced by C°. Min and Min$_w$ will be similarly defined. It should be remarked that, for any set $B \subset V$, we have that Min $B \subset$ Min$_w$ B. We also write $v_1 <_w v_2$ to mean $v_1 \leq_w v_2$ and $v_1 \neq v_2$.

We have the following result with regard to these definitions.

Lemma 2.1. Let B be a compact set in V. Then,

(a) Min $B \neq \phi$;
(b) $B \subset \text{Min } B + C$;
(c) $B \subset \text{Min}_w B + C^0$.

Proof. The proof of (a) may be found in Ref. 13, Theorem 1; (b) is stated in Ref. 14, Theorem 4.2, in the finite-dimensional case, but the argument is valid also in our more general siutation. To prove (c), let $v \in B$. If $v \in \text{Min}_w B$, we have that $v \in \text{Min}_w B + C^0$; otherwise, there exists $v_0 \in B$ such that $v_0 <_w v$ and also, by part (b), there exists $v_0' \in \text{Min } B \subset \text{Min}_w B$ such that $v_0' \leq v_0$. So,

$$v = v_0' + v - v_0 + v_0 - v_0' \in v_0' + \text{int } C + C \subseteq \text{Min}_w B + C^0. \qquad \square$$

Now let Y be a vector space, Y_0 a convex set in Y, and $\phi: Y_0 \to V$. We say that ϕ is C-convex if

$$\phi(ty_1 + (1-t)y_2) \leq t\phi(y_1) + (1-t)\phi(y_2), \quad \text{for every } y_1, y_2 \in Y_0 \text{ and } t \in [0, 1];$$

ϕ will be said to be properly quasi C-convex if, for every $y_1, y_2 \in Y_0$ and $t \in [0, 1]$, we have

$$\text{either } \phi(ty_1 + (1-t)y_2) \leq \phi(y_1)$$
$$\text{or } \quad \phi(ty_1 + (1-t)y_2) \leq \phi(y_2).$$

We remark that a function may be C-convex and not properly quasi C-convex, and conversely (see Ref. 10, Proposition 4.2). It is easily seen that, if ϕ is either C-convex or properly quasi C-convex, then for every $v \in V$, the set $\{y \in Y_0: \phi(y) \leq v\}$ is convex.

If Y is a Hausdorff topological vector space and $Y_0 \subset Y$, ϕ will be said lower C-semicontinuous if, for every $v \in V$, the set $\{y \in Y_0: \phi(y) \leq v\}$ is closed in Y_0. It is easily seen that, if ϕ is a continuous function, then it is also lower C-semicontinuous.

The following results will be useful.

Lemma 2.2. Let Y be a Hausdorff topological vector space and let Y_0 be a compact set in Y. Let $\phi: Y_0 \to V$ be properly quasi C-convex and lower C-semicontinuous. Then, there exists $y_0 \in Y_0$ such that $\{\phi(y_0)\} = \text{Min } \phi(Y_0)$.

Proof. Min $\phi(Y_0)$ is nonempty by essentially the arguments of Ref. 15. Now let $y_1, y_2 \in Y_0$ such that

$$\phi(y_1), \phi(y_2) \in \text{Min } \phi(Y_0),$$

and let

$$y(t)=ty_1+(1-t)y_2, \qquad t\in[0, 1].$$

By the assumptions about ϕ, we have

$$[0, 1]=\{t: \phi(y(t))\leq \phi(y_1)\}\cup\{t: \phi(y(t))\leq \phi(y_2)\},$$

where the two sets in the right-hand side are nonempty and closed. It follows that there exists $t_0\in[0, 1]$ such that

$$\phi(y(t_0))\leq \phi(y_1) \quad\text{and}\quad \phi(y(t_0))\leq \phi(y_2).$$

Finally, by the choice of y_1 and y_2, we have

$$\phi(y(t_0))=\phi(y_1)=\phi(y_2).$$ □

Lemma 2.3. Let X, Y be Hausdorff topological vector spaces and let X_0, Y_0 be compact convex sets in X, Y respectively. Let f: $X_0\times Y_0\to V$ be a continuous function such that $-f(\cdot, y)$ is properly quasi C-convex for every $y\in Y_0$. Then, the function ϕ: $Y_0\to V$, defined by $\{\phi(y)\}=\text{Max} f(X_0, y)$, is continuous. Moreover, ϕ is C-convex if $f(x, \cdot)$ is so for every $x\in X_0$.

Proof. Since

$$\text{Max} f(X_0, y)=-\text{Min}(-f(X_0, y)),$$

ϕ is well defined by Lemma 2.2. Now let $y_0\in Y_0$ and $\{y_i\}$ be a net in Y_0 such that $y_i\to y_0$. Let $\{\phi(z_j)\}$ be a subnet of $\{\phi(y_i)\}$. Since $f(X_0, Y_0)$ is compact, there exists a subnet $\{\phi(w_k)\}$ of $\{\phi(z_j)\}$ and $v_0\in V$ such that $\phi(w_k)\to v_0$. The continuity of ϕ will be proved by showing $\phi(y_0)=v_0$. Let $c_0\in\text{int } C$, U_0 a neighborhood of $0\in V$ such that $c_0+U_0\subset C$. Given $\varepsilon>0$, we have $v_0-\phi(w_k)\in\varepsilon U_0$ for k greater than a suitable k_ε and also

$$\varepsilon c_0+v_0-\phi(w_k)\in\varepsilon c_0+\varepsilon U_0\subset C, \quad \text{for } k>k_\varepsilon.$$

Thus,

$$\phi(w_k)\leq\varepsilon c_0+v_0, \quad \text{for } k>k_\varepsilon,$$

and also

$$\phi(y_0)\leq\varepsilon c_0+v_0.$$

In fact, since f is continuous, it is also lower C-semicontinuous and so is ϕ from the fact that

$$\{y: \phi(y)\leq v\}=\bigcap_{x\in X_0}\{y: f(x, y)\leq v\}.$$

Finally, for $\varepsilon\to 0$, we obtain $\phi(y_0)\leq v_0$. Now we must prove $v_0\leq\phi(y_0)$. For suitable $x_k\in X_0$, we have $\phi(w_k)=f(x_k, w_k)$. Let $\{x_{k_m}\}$ be a subnet of $\{x_k\}$ such that $x_{k_m}\to x_0\in X_0$. Hence, we have

$$\phi(w_{k_m})=f(x_{k_m}, w_{k_m})\to v_0=f(x_0, y_0)\leq\phi(y_0),$$

which yields the continuity of ϕ. The last part of the proposition is an easy consequence of the definitions. □

We conclude this section with some definitions and results about set-valued mappings. Let G_1, G_2 be Hausdorff topological spaces and let $\Gamma: G_1\to G_2$ be a set-valued mapping with nonempty values. Γ is said to be upper semicontinuous if, for every $x_0\in G_1$ and for every open set N containing $\Gamma(x_0)$, there exists a neighborhood M of x_0 such that $\Gamma(M)\subset N$. Moreover, it may be easily proved by direct arguments (see Refs. 11 and 16) that, if Γ is compact-valued, then Γ is upper semicontinuous if and only if, for every net $\{x_i\}\subset G_1$ such that $x_i\to x_0\in G_1$ and for every $z_i\in\Gamma(x_i)$, there exist $z_0\in\Gamma(x_0)$ and a subnet $\{z_{i_j}\}$ of $\{z_i\}$ such that $z_{i_j}\to z_0$.

3. Minimax Theorem

The main result of the paper is now stated. Hypothesis (H), which appears in the theorem, will be discussed in the next section.

Theorem 3.1. Let X, Y be Hausdorff topological vector spaces; let X_0, Y_0 be compact convex sets in X, Y, respectively. Let f: $X_0\times Y_0\to V$ be a continuous function such that $f(x, \cdot)$ is C-convex for every $x\in X_0$ and $-f(\cdot, y)$ is properly quasi C-convex for every $y\in Y_0$. Moreover, we suppose that

$$\text{(H)}\quad \text{Max}\bigcup_{s\in X_0}\text{Min}_w f(s, Y_0)\subset\text{Min}_w f(x, Y_0)+C, \quad \text{for every } x\in X_0.$$

Then, for every

$$\alpha\in\text{Max}\bigcup_{s\in X_0}\text{Min}_w f(s, Y_0),$$

which is nonempty, there exists

$$\beta\in\text{Min}\bigcup_{t\in Y_0}\text{Max} f(X_0, t),$$

such that $\beta\leq\alpha$; i.e.,

$$\text{Max}\bigcup_{s\in X_0}\text{Min}_w f(s, Y_0)\subset\text{Min}\bigcup_{t\in Y_0}\text{Max} f(X_0, t)+C.$$

Proof. We write

$$\Gamma(x)=\mathrm{Min}_w f(x, Y_0), \qquad x\in X_0.$$

Since $f(x, Y_0)$ is compact, we have $\Gamma(x)\neq\phi$ for every $x\in X_0$ [see Lemma 2.1(a)]. Moreover, $\Gamma(x)$ is compact since it is contained in the compact set $f(x, Y_0)$ and is closed. In fact, let $\{z_i\}$ be a net in $\Gamma(x)$ such that $z_i\to z_0$; we have $z_i-z\notin \mathrm{int}\, C$ for every $z\in f(x, Y_0)$, and so $z_0-z\notin\mathrm{int}\, C$ for every $z\in f(x, Y_0)$. Thus, $z_0\in\Gamma(x)$.

The compact-valued mapping $\Gamma: X_0\to V$ is also upper semicontinuous. The following proof of this fact is similar to the finite-dimensional proof in Ref. 11. Let $x_0\in X_0$, $\{x_i\}$ be a net such that $x_i\to x_0$ and $z_i\in\Gamma(x_i)$. For suitable $y_i\in Y_0$, we have $z_i=f(x_i, y_i)\in f(X_0, Y_0)$; and, from compactness, there exist $y_0\in Y_0$, $z_0\in V$ and $\{y_{i_j}\}$, $\{z_{i_j}\}$, subnets of $\{y_i\}$, $\{z_i\}$, respectively, such that $y_{i_j}\to y_0$, $z_{i_j}\to z_0$. By contradiction, suppose that $z_0\notin\Gamma(X_0)$. We have

$$z_0=\lim_j f(x_{i_j}, y_{i_j})=f(x_0, y_0);$$

and, by Lemma 2.1(c), there exist $y'\in Y_0$ and $c\in\mathrm{int}\, C$ such that $z_0=f(x_0, y')+c$. Hence, we may write

$$z_{i_j}-f(x_{i_j}, y')=c-(z_0-z_{i_j}+f(x_{i_j}, y')-f(x_0, y')),$$

where the right-hand side lies in $\mathrm{int}\, C$ for j large enough; consequently, $z_{i_j}\notin\Gamma(x_{i_j})$ for such values of j, which is a contradiction. Thus, Γ is upper semicontinuous.

From Proposition 1.1.3 in Ref. 17, we now obtain that

$$\Gamma(X_0)=\bigcup_{s\in X_0}\mathrm{Min}_w f(s, Y_0)$$

is a compact set. This yields [see again Lemma 2.1(a)] that $\mathrm{Max}\,\Gamma(X_0)\neq\phi$. Next, let $\phi: Y_0\to V$ be defined as in Lemma 2.3; recall that ϕ is continuous and C-convex. Let $\alpha\in V$ and suppose that $\alpha\notin\phi(Y_0)+C$. $\phi(Y_0)$ is compact and so $\phi(Y_0)+C$ is closed and convex. Then, as a consequence of standard separation theorems in locally convex Hausdorff topological vector spaces, there exist $\sigma\in R$, $\epsilon>0$ and a linear and continuous mapping $\lambda_0: V\to R$ such that

$$\lambda_0(\alpha)\leq\sigma-\epsilon<\sigma\leq\lambda_0(\phi(y)+c),$$

$$\text{for every } y\in Y_0 \text{ and } c\in C.$$

So,

$$\lambda_0(c)\geq\lambda_0(\alpha-\phi(y)), \qquad \text{for every } y\in Y_0 \text{ and } c\in C,$$

which implies that, since C is a cone, $\lambda_0(c)\geq 0$ for every $c\in C$. We obtain also, for $c=0$,

$$\lambda_0(\alpha)\leq\sigma-\epsilon<\sigma\leq\lambda_0(\phi(y)), \qquad \text{for every } y\in Y_0.$$

Consider now the continuous function

$$g=\lambda_0\circ f: X_0\times Y_0\to R.$$

By the properties of λ_0 and f, it is easy to derive that the real-valued function g has the following properties: $g(x,\cdot)$ is convex in Y_0 for every $x\in X_0$ and $-g(\cdot, y)$ is quasi convex in X_0 for every $y\in Y_0$. By minimax results in the scalar case (see Ref. 4), it follows that

$$\min_{y\in Y_0}\max_{x\in X_0} g(x, y)=\max_{x\in X_0}\min_{y\in Y_0} g(x, y).$$

Moreover, by the definition of ϕ and the properties of λ_0, we have

$$g(x, y)=\lambda_0(f(x, y))\leq\lambda_0(\phi(y)), \qquad \text{for every } x\in X_0 \text{ and } y\in Y_0.$$

In addition, for every $y_0\in Y_0$, there exists $x_0\in X_0$ such that

$$f(x_0, y_0)=\phi(y_0),$$

which implies that

$$g(x_0, y_0)=\lambda_0(\phi(y_0))$$

and

$$\max_{x\in X_0} g(x, y_0)\geq\lambda_0(\phi(y_0)).$$

Thus, we have proved that

$$\max_{x\in X_0} g(x, y)=\lambda_0(\phi(y))\geq\sigma>\sigma-\epsilon\geq\lambda_0(\alpha), \qquad \text{for every } y\in Y_0,$$

from which

$$\min_{y\in Y_0}\max_{x\in X_0} g(x, y)>\lambda_0(\alpha).$$

By the minimax property of g, we obtain that

$$\max_{x\in X_0}\min_{y\in Y_0} g(x, y)>\lambda_0(\alpha).$$

Hence, there exists $x'\in X_0$ such that

$$\min_{y\in Y_0} g(x', y)>\lambda_0(\alpha);$$

that is,

$$\lambda_0(f(x', y)-\alpha)>0, \qquad \text{for every } y\in Y_0.$$

This means that $f(x', y) - \alpha \not\in -C$, for every $y \in Y_0$; i.e., $\alpha \not\in f(x', Y_0) + C$, in contradiction to hypothesis (H), if $\alpha \in \operatorname{Max} \Gamma(X_0)$. For every $\alpha \in \operatorname{Max} \Gamma(X_0)$, it follows that $\alpha \in \phi(Y_0) + C$ to conclude the proof, since

$$\phi(Y_0) + C = \operatorname{Min} \phi(Y_0) + C,$$

by Lemma 2.1(b). □

4. Condition (H) and Remarks

Assumption (H) in Theorem 3.1 may be regarded as a condition that controls the movement of the sets $\operatorname{Min} f(x, Y_0)$ when x varies. (H) always holds if f is real-valued, i.e., $V = R$. The following simple example shows that (H) is not implied by the continuity and convexity hypotheses on f required in Theorem 3.1 and also that the result is not true without this condition.

Example 4.1. Let

$$X = Y = R, \qquad V = R^2,$$

$$C = \{(v_1, v_2): v_2 \geq |v_1|\},$$

$$X_0 = Y_0 = [0, 1],$$

$$f(x, y) = (y, 0), \qquad \text{if } y \leq x,$$

$$f(x, y) = (y, 2(y - x)), \qquad \text{if } y \geq x.$$

Then, f is continuous in $[0, 1] \times [0, 1]$, $f(x, \cdot)$ is C-convex for every $x \in [0, 1]$, $-f(\cdot, y)$ is properly quasi C-convex for every $y \in [0, 1]$, (H) does not hold,

$$\operatorname{Max} \bigcup_{s \in X_0} \operatorname{Min}_w f(s, Y_0) \not\subset \operatorname{Min} \bigcup_{t \in Y_0} \operatorname{Max} f(X_0, t) + C$$

$$= \operatorname{Min}_w \bigcup_{t \in Y_0} \operatorname{Max} f(X_0, t) + C.$$

Moreover, consider the family of functions defined by

$$f_\theta(x, y) = (y, \theta(1 - x)), \qquad \text{if } y \leq x,$$

$$f_\theta(x, y) = (y, \theta(1 - x) + 2(y - x)), \qquad \text{if } y \geq x.$$

It can be easily seen that all of the hypotheses of Theorem 3.1 are satisfied by f_θ when $\theta \geq 1$, while neither (H) nor the conclusion of Theorem 3.1 are satisfied when $0 \leq \theta < 1$.

As to the hypotheses of Theorem 3.1, it should be noted that the minimax problem has already been investigated in Ref. 10 in the case that both $f(x, \cdot)$ and $-f(\cdot, y)$ are properly quasi C-convex. In this situation, however, the involved minimal and maximal sets are single points, as noted in Section 1. On the contrary, it would be interesting to investigate the problem in the case that both $f(x, \cdot)$ and $-f(\cdot, y)$ are C-convex. Example 4.1 gives no answer to this problem since $-f(\cdot, y)$ is not C-convex there. Other open questions include the following: whether there are reasonable conditions that assure the conclusions of Theorem 3.1 and also the validity of

$$\operatorname{Min} \bigcup_{t \in Y_0} \operatorname{Max} f(X_0, t) \subset \operatorname{Max} \bigcup_{s \in X_0} \operatorname{Min}_w f(s, Y_0) - C.$$

We conclude this section with some simple conditions sufficient for (H) to hold.

Proposition 4.1. Let f be as in Lemma 2.3. Then, (H) holds if f satisfies any of the following conditions:

(i) for every $x', x'' \in X_0$ and $y', y'' \in Y_0$, if $f(x', y') \leq_w f(x', y'')$, then $f(x'', y') \leq_w f(x'', y'')$;

(ii) for every $x', x'' \in X_0$ and $y', y'' \in Y_0$, if $(x', y') <_w f(x', y'')$, then there exists $y_0 \in Y_0$ such that $f(x'', y_0) \leq_w f(x'', y'')$;

(iii) for every $x', x'' \in X_0$ and $\gamma(x') \in \Gamma(x')$, there exist $\gamma(x'') \in \Gamma(x'')$, $t_0 \in [0, 1]$, $\gamma(x(t_0)) \in \Gamma(x(t_0))$ such that $\gamma(x') \leq \gamma(x(t_0))$ and $\gamma(x'') \leq \gamma(x(t_0))$, where $x(t) = tx' + (1 - t)x''$;

(iv) for every $x', x'' \in X_0$ and $y', y'' \in Y_0$, if $f(x', y') \leq_w f(x', y'')$, then either $f(x'', y') \leq_w f(x'', y'')$ or there exists $y_0 \in Y_0$ such that $f(x'', y_0) \leq_w f(x', y')$ and $f(x'', y_0) \leq_w f(x'', y'')$.

Moreover, we have (i)⇒(ii)⇒(iii) and (i)⇒(iv).

Proof. (i)⇒(ii). Given $x', x'' \in X_0$ and $y', y'' \in Y_0$, suppose that

$$f(x', y') <_w f(x', y'').$$

Then, by (i),

$$f(x'', y') \leq_w f(x'', y'').$$

If

$$f(x'', y') = f(x'', y''),$$

again by (i) we obtain

$$f(x', y'') \leq_w f(x', y'),$$

which is a contradiction. Thus,

$$f(x'', y') <_w f(x'', y''),$$

and (ii) holds.

(ii)⇒(iii). Let

$$f(x', y') = \gamma(x') \in \Gamma(x').$$

If there exists $y'' \in Y_0$ such that

$$f(x'', y'') <_w f(x'', y'),$$

then there exists $y_0 \in Y_0$ such that

$$f(x', y_0) <_w f(x', y'),$$

which is a contradiction. So $f(x'', y') \in \Gamma(x'')$, and more generally this means also that the set $\{y: f(x, y) \in \Gamma(x)\}$ is independent of x. Now, since $-f(\cdot, y')$ is properly quasi C-convex, we have

$$[0, 1] = \{t: f(x', y') \leq f(x(t), y')\} \cup \{t: f(x'', y') \leq f(x(t), y')\},$$

where the sets in the right-hand side are both nonempty and closed. This implies the existence of $t_0 \in [0, 1]$ such that

$$f(x', y') \leq f(x(t_0), y') \quad \text{and} \quad f(x'', y') \leq f(x(t_0), y').$$

The proof of this part is complete, since

$$f(x(t_0), y') \in \Gamma(x(t_0)).$$

(iii)⇒(H). Let

$$x', x'' \in X_0 \quad \text{and} \quad \gamma(x') \in \Gamma(x') \cap \operatorname{Max} \Gamma(X_0).$$

Then, there exist $\gamma(x'') \in \Gamma(x'')$, $t_0 \in [0, 1]$, $\gamma(x(t_0)) \in \Gamma(x(t_0))$ such that

$$\gamma(x') \leq \gamma(x(t_0)) \quad \text{and} \quad \gamma(x'') \leq \gamma(x(t_0)).$$

It follows that

$$\gamma(x'') \leq \gamma(x(t_0)) = \gamma(x').$$

(i)⇒(iv). The proof of this statement is trivial.

(iv)⇒(H). Let $x' \in X_0$ and $y' \in Y_0$ be such that

$$f(x', y') \in \Gamma(x') \cap \operatorname{Max} \Gamma(X_0),$$

and let $x \in X_0$. We take $y'' \in Y_0$ such that

$$f(x', y') \leq_w f(x', y'').$$

If there exists $y_0 \in Y_0$ satisfying

$$f(x, y_0) \leq_w f(x', y'),$$

we obtain by Lemma 2.1(c) that

$$f(x', y') \in \Gamma(x) + C^0 \subset \Gamma(x) + C.$$

Otherwise, for every $y'' \in Y_0$ such that

$$f(x', y') \leq_w f(x', y''),$$

we have that

$$f(x, y') \leq_w f(x, y'').$$

Let $\bar{y} \in Y_0$ such that

$$f(x, \bar{y}) \leq_w f(x, y') \quad \text{and} \quad f(x, \bar{y}) \in \Gamma(x)$$

[$\bar{y}$ exists by Lemma 2.1(c)]. If there exists $z \in Y_0$ such that

$$f(x', z) \leq_w f(x, \bar{y}) \quad \text{and} \quad f(x', z) \leq_w f(x', y'),$$

then

$$f(x', z) = f(x', y') \leq_w f(x, \bar{y}),$$

and so

$$f(x', y') \leq f(x, \bar{y}).$$

This implies that

$$f(x', y') = f(x, \bar{y}),$$

since

$$f(x', y') \in \operatorname{Max} \Gamma(X_0)$$

and by the choice of $\bar{y}$. This equality is a contradiction, so we have

$$f(x', \bar{y}) \leq_w f(x', y'),$$

and consequently

$$f(x', \bar{y}) = f(x', y').$$

Thus,

$$f(x, y') \leq_w f(x, \bar{y}).$$

At this point, it is useful to summarize what has been established thus far: given $x, x' \in X_0$ and $y' \in Y_0$ such that $f(x', y') \in \Gamma(x') \cap \operatorname{Max} \Gamma(X_0)$, either (a) there exists $y_0 \in Y_0$ such that $f(x, y_0) \leq_w f(x', y')$ [and so $f(x', y') \in \Gamma(x) + C^0$ and (H) holds] or (b) $f(x, y') \in \Gamma(x)$.

Consider the situation (b). Let $t_0 \in [0, 1]$ be such that

$$f(x', y') \leq f(x(t_0), y') \quad \text{and} \quad f(x, y') \leq f(x(t_0), y'),$$

where

$$x(t_0) = t_0 x' + (1 - t_0) x.$$

If

$$f(x(t_0), y') \in \Gamma(x(t_0)),$$

we obtain

$$f(x, y') \leq f(x', y') = f(x(t_0), y'),$$

and (H) holds. Otherwise, we have $t_0 \neq 1$ and, by the previous remark applied to $x(t_0)$, there exists $\tilde{y} \in Y_0$ such that

$$f(x(t_0), \tilde{y}) \leq_w f(x', y') \quad \text{and} \quad f(x(t_0), \tilde{y}) \in \Gamma(x(t_0)).$$

We have also that

$$\text{either } f(x, \tilde{y}) \leq f(x(t_0), \tilde{y}) \quad \text{or } f(x', \tilde{y}) \leq f(x(t_0), \tilde{y}).$$

In the former case, it follows that (H) holds; in the latter case, by the inequalities

$$f(x', \tilde{y}) \leq f(x(t_0), \tilde{y}) \leq_w f(x', y'),$$

we obtain

$$f(x(t_0), \tilde{y}) = f(x', y') \in \text{Max}\, \Gamma(X_0).$$

Consider next the set

$$A = \{t \in [0, 1]: f(x', y') = f(x(t), y_t) \in \Gamma(x(t)) \cap \text{Max}\, \Gamma(X_0), \text{ for some } y_t \in Y_0\},$$

which has the following properties: $A \neq \phi$, $0 \leq \inf A = t' < 1$, and $t' \in A$ by the upper semicontinuity of Γ. If $t' = 0$, we have

$$f(x', y') = f(x, y),$$

for a suitable $y \in Y_0$. If $t' > 0$, we may repeat the previous arguments and find that either

$$f(x', y') = f(x(t'), y_{t'}) \in \Gamma(x) + C$$

or the existence of $x_0 \in [x, x(t'))$ and $y_0 \in Y_0$ such that

$$f(x', y') = f(x(t'), y_{t'}) = f(x_0, y_0) \in \Gamma(x_0) \cap \text{Max}\, \Gamma(X_0).$$

Hence, t' would not be the infimum of A. The proof is now complete. □

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