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Minimax Type Theorems for n-Valued Functions (*) (**).

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Summary. – *We consider in* $\boldsymbol{R}^n$ *the ordering induced by a cone and give suitable definitions of* Sup, Inf, *convexity and lower semicontinuity relative to n-valued functions; in this framework we can prove minimax theorems for such a class of functions.*

1. – Introduction.

Minimax problems have been investigated by several authors in the last fifty years.

As it is well-known the problem is the following: find conditions about $f\colon X\times Y\to \boldsymbol{R}$ in such a way that

$$\sup_{x\in X}\inf_{y\in X} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y);$$

moreover the previous equality is closely related to the existence of saddle-points for f and plays a fundamental role in the theory of zero sum two-person games.

Very deep results in the subject may be found in [2], [6], [9], [11]. Extensive bybliographycal references and a survey of the most important results and technics are explained in [1].

In the present paper we approach the minimax problem for vector-valued functions.

As far as we know this seems to be a new object of investigation: only in [8] vector-valued functions are treated, but our approach is much different since among other things so are the meanings of Sup and Inf in our work and in [8].

In Section 2 we study the properties of orderings in $\boldsymbol{R}^n$; in Section 3 we define generalized saddle-points and saddle-values for n-valued functions and investigate their relations with minimax equalities or inequalities; in Section 4 we study problems of convexity and lower semicontinuity for n-valued functions. An alternative approach to this kind of problems may be found in [3] (see also [4], [5]).

Finally in Section 5 we state our minimax theorems: a deep result concerning properly quasi-convex functions (see Definition 4.1) and a weaker one concerning

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convex functions. The ideas of the proof of Theorem 5.1 are derived from the elementary approach to the minimax problem given in [11] while Theorem 5.2 is in some way related to the methods introduced in [7].

2. – Cones and orderings in R^n.

As usual we say that a set $C \subset R^n$ is a cone if $aC \subset C$ for every $a \geqslant 0$. It is well known that a cone C is convex if and only if $x + y \in C$ whenever $x, y \in C$.

In what follows we shall use a cone K which satisfies the following condition:

(H_1) K is a closed and convex cone such that

$$K \cap (-K) = \{0\}.$$

Further assumptions on K will be introduced in the sequel.

We define a partial ordering in R^n in the following way: if $x, y \in R^n$ we put $x \leqslant y$ if and only if $y - x \in K$; $x \geqslant y$ means $y \leqslant x$ while $x < y$ means $x \leqslant y$ and $x \neq y$, that is $y - x \in K - \{0\}$.

This ordering in R^n is said to be the ordering induced by K.

The same symbols ($\leqslant, <, \geqslant, >$) are used in this work for the usual ordering in R and for the ordering induced in R^n by K: nevertheless we think no confusion will derive.

Given $x \in R^n$ we may define

$$C(x) = \{y \in R^n : y \leqslant x \text{ or } x \leqslant y\} = \{y \in R^n : y - x \in K \cup (-K)\};$$

$C(x)$ is said to be the set of all elements which are comparable with x. We note that $C(x)$ is a closed set; indeed let $\{y_m\} \subset C(x)$ be a sequence such that $\lim_{m\to+\infty} y_m = y_0$; we have $y_m - x \in K \cup (-K)$ for every m and, since K is closed, we derive $y_0 - x \in K \cup (-K)$.

As usual we say that b is an upper bound of a given set $A \subset R^n$ if $b \geqslant a$ for every $a \in A$ (in this case we say that A is upper bounded); b is said to be a maximum of A if b is an upper bound of A and $b \in A$; finally b is said to be a supremum of A if b is an upper bound of A and $b \leqslant c$ whenever c is an upper bound of A.

In an analogous way we define lower bound, minimum and supremum of a set A.

Since K is closed it is easily seen that A and $\operatorname{cl} A$ have the same set of upper bounds. The condition $K \cap (-K) = \{0\}$ implies that if $y \leqslant x$ and $x \leqslant y$ then $y = x$; from this it is easily seen that there is at most one supremum for a set and at most one infimum. The maximum, if it exists, agrees with the supremum.

A set $B \subset R^n$ is said to be a chain if it is totally ordered, that is if for every $x, y \in B$ we have either $x \leqslant y$ or $y \leqslant x$.

THEOREM 2.1. – Let $A \subset R^n$; if A is upper bounded then:

(i) for every $a_0 \in R^n$ the set $\{a \in A : a \geqslant a_0\}$ is bounded in norm;

(ii) if $B \subset A$ is a chain then there exists $\sup B \in \operatorname{cl} A$;

(iii) if we put $B = \{b_\alpha\}_{\alpha \in \mathcal{E}}$ where $\mathcal{E}$ is a totally ordered set such that $b_{\alpha_1} \leqslant b_{\alpha_2}$ whenever $\alpha_1 \leqslant \alpha_2$, there exists $\lim_\alpha b_\alpha = \sup B$.

PROOF. – Let us choose $a_0 \in R^n$ and let $A_0 = \{a \in A : a \geqslant a_0\}$. If A_0 is non-empty we fix an upper bound c of A and put $X_0 = \{x = c - a : a \in A_0\}$; A_0 is bounded in norm if and only if X_0 is so.

Assume that X_0 is not bounded in norm and let $\{x_n\} \subset X_0$ be a sequence such that $\lim_{h\to+\infty} \|x_h\| = +\infty$ and $\|x_h\| \geqslant 1$ for every h.

Since $X_0 \subset K$ and $K \cap (-K) = \{0\}$ we have $x_h \notin -K$: now let $r_h \in R$, $r_h > 0$, be such that $\|r_h x_h\| = 1$; we have $r_h x_h \notin -K$.

The cone $-K$ is closed and convex and so there exists the projection z_h of $r_h x_h$ on $-K$. We have

$$0 < \|r_h x_h - z_h\| \leqslant \|r_h x_h - z\|, \quad \text{for every } z \in -K,$$

and so

$$\|x_h - r_h^{-1} z_n\| \leqslant \|x_h - r_h^{-1} z\|, \quad \text{for every } z \in -K.$$

Since $-K$ is a cone it follows

$$\|x_h - r_h^{-1} z_h\| \leqslant \|x_h - z\|, \quad \text{for every } z \in -K.$$

From this we derive

$$(*) \quad \|x_h - r_h^{-1} z_h\| = d(x_h, -K) = r_h^{-1} d(r_h x_h, -K) \geqslant$$
$$\geqslant r_h^{-1} \min\{d(x, -K) : x \in K \cap S(0;1)\} = r_h^{-1} \varrho,$$

where $S(0;1)$ is the sphere with center at the origin and radius 1 and ϱ is a suitable positive number.

By the definition of r_h and the properties of $\{x_h\}$ we have $\lim_{h\to+\infty} r_h = 0$ and so $(*)$ yields $\lim_{h\to+\infty} d(x_h, -K) = +\infty$. On the other hand we have $x_h - x_0 \in -K$, where $x_0 = c - a_0$, and so

$$d(x_h, -K) \leqslant \|x_h - (x_h - x_0)\| = \|x_0\|,$$

which is a contradiction.

(ii) Let $b_0 \in B$; by (i) $\{b \in B: b \geqslant b_0\}$ is bounded in norm and so there exists a subsequence $\{b_\beta\}_{\beta\in\mathcal{K}}$ of $\{b_\alpha\}_{\alpha\in\mathcal{E}}$ which is convergent (here we use the notations introduced in (iii)). Let b' be its limit; if c is an upper bound of B we have $c - b_\beta \in K$ for every β and so $c - b' \in K$, that is $b' \leqslant c$ for every upper bound c of B. Afterwards if $b_\alpha \in B$ there exists $\beta \in \mathcal{K}$ such that $b_\beta \geqslant b_\alpha$ and so $b' \geqslant b_\alpha$; this means that b' is an upper bound of B and so $b' = \sup B$.

(iii) With a standard argument, by the uniqueness of $\sup B$ and by (ii) we obtain the result. □

From now on we need a further condition about K. So throughout this paper we shall assume that the fixed cone K satisfies the following condition:

(H_2) K is a closed and convex cone such that $K \cap (-K) = \{0\}$ and $\operatorname{int} K \neq \emptyset$.

The new condition we introduced about K may be expressed in an alternative way, as it is shown in the following proposition:

PROPOSITION 2.2. – Let K be a closed and convex cone such that $K \cap (-K) = \{0\}$. Then we have $\operatorname{int} K \neq \emptyset$ if and only if $K - K = \boldsymbol{R}^n$.

PROOF. – Let $\operatorname{int} K \neq \emptyset$; then there exist $k_0 \in K$ and $\delta > 0$ such that $k_0 + y \in K$ for every $y \in B(0; \delta)$, where $B(0; \delta)$ is the ball with center at the origin and radius δ. So we have $y = k_0 + y - k_0 \in K - K$ for every $y \in B(0; \delta)$; hence $K - K$ contains a neighborhood of the origin and since $K - K$ is a cone we have $K - K = \boldsymbol{R}^n$.

Now we prove the «if part» of the proposition.

If $K - K = \boldsymbol{R}^n$ for every $a_1, a_2 \in \boldsymbol{R}^n$ there exists a_0 such that $a_0 \geqslant a_1$ and $a_0 \geqslant a_2$: indeed from $a_1 - a_2 \in \boldsymbol{R}^n = K - K$ we derive $(a_1 + K) \cap (a_2 + K) \neq \emptyset$.

By induction it is easily proved that every finite set has an upper bound. Now let $A = \{0, e_1, e_2, \dots, e_n\}$, where $\{e_i: i = 1, \dots, n\}$ is a basis for $\boldsymbol{R}^n$. There exists $b_0 \in \boldsymbol{R}^n$ such that $b_0 - 0 \in K$ and $b_0 - e_i \in K$, $i = 1, \dots, n$.

By the convexity of K we obtain $b_0 - a \in K$ for every $a \in \operatorname{co} A$, that is $a \in b_0 - K$ for every $a \in \operatorname{co} A$. The proof is complete since $\operatorname{int}(\operatorname{co} A) \neq \emptyset$. □

THEOREM 2.3. – Let $A \subset \boldsymbol{R}^n$ be a norm bounded set. Then A has upper and lower bounds.

PROOF. – By the hypotheses there exists $\delta_0 > 0$ such that $A \subset B(0; \delta_0)$; moreover there exist $k_0 \in K$ and $\delta > 0$ such that $B(k_0; \delta) \subset K$. So we have

$$A \subset B(0; \delta_0) = \delta_0 \delta^{-1} B(0; \delta) \subset \delta_0 \delta^{-1}(-k_0 + K) = -\delta_0 \delta^{-1} k_0 + K .$$

We obtain $a \geqslant -\delta_0 \delta^{-1} k_0$ for every $a \in A$ and so A has a lower bound. In an analogous way it may be proved that A has an upper bound. □

Now we give some definitions.

DEFINITION 2.4. – Let $A \subset \boldsymbol{R}^n$; we say that $a_0 \in \boldsymbol{R}^n$ is a maximal element of A if $a_0 \in A$ and $a_0 \geqslant a$ for every $a \in A \cap C(a_0)$. In an analogous way we define the minimal elements of A. □

DEFINITION 2.5. – Let $A \subset \boldsymbol{R}^n$; we say that $a_0 \in \boldsymbol{R}^n$ is a supremal element of A if a_0 is a maximal element of $\operatorname{cl} A$; we say that $+\infty$ is a supremal element of A if there exists a chain in $\operatorname{cl} A$ which has no upper bound.

We write $\operatorname{Sup} A$ for the set of all supremal elements of A. In an analogous way we define the infimal elements of A and $\operatorname{Inf} A$.

Moreover we put $\operatorname{Max} A = A \cap \operatorname{Sup} A$ and $\operatorname{Min} A = A \cap \operatorname{Inf} A$. □

It will be useful the following

LEMMA 2.6. – Let $A \subset \boldsymbol{R}^n$ and $a_0 \in A$.

Then either there exists $a' \in \boldsymbol{R}^n \cap \operatorname{Sup} A$ such that $a_0 \leqslant a'$ or a_0 is an element of a chain in $\operatorname{cl} A$ which has no upper bound.

PROOF. – If there is no element $a' \in \operatorname{Sup} A$ such that $a_0 \leqslant a'$ then there exists $a_1 \in \operatorname{cl} A$ such that $a_0 < a_1$ (otherwise a_0 would be an element of $\operatorname{Sup} A$); moreover we may choose a_1 such that $\|a_1\| > 1$. In an analogous way we may choose $a_2 \in \operatorname{cl} A$ such that $a_1 < a_2$ and $\|a_2\| > 2$. In this way we obtain a chain $\{a_m\} \subset \operatorname{cl} A$ such that $a_0 \in \{a_m\}$ and $\|a_m\| \to +\infty$; by Theorem 2.1 we derive that $\{a_m\}$ has no upper bound. □

DEFINITION 2.7. – Let $A \subset \boldsymbol{R}^n$; we say that $a_0 \in \boldsymbol{R}^n$ is a proper supremal element of A if $a_0 \in \operatorname{Sup} A$ and there exists $a \in A$ such that $a \leqslant a_0$; we say that $+\infty$ is a proper supremal element of A if there exists an element in A belonging to a chain in $\operatorname{cl} A$ which has no upper bound.

We write $\operatorname{Sup}_0 A$ for the set of all proper supremal elements of A. In an analogous way we define the proper infimal elements of A and $\operatorname{Inf}_0 A$. □

THEOREM 2.8. – Let $A \subset \boldsymbol{R}^n$; then we have

$$\operatorname{Sup} A \supset \operatorname{Sup}_0 A \neq \emptyset$$

and

$$\operatorname{Max} A = A \cap \operatorname{Sup} A = A \cap \operatorname{Sup}_0 A .$$

PROOF. – If $+\infty \notin \operatorname{Sup}_0 A$, given $a_0 \in A$ by Lemma 2.6 there exists $a' \in \boldsymbol{R}^n \cap \operatorname{Sup} A$ such that $a_0 \leqslant a'$ and so $a' \in \operatorname{Sup}_0 A$; this means $\operatorname{Sup}_0 A \neq \emptyset$. The other parts of the theorem are trivial. □

The following proposition may be easily proved.

PROPOSITION 2.9. – Let $A \subset \mathbf{R}^n$; we have:

(i) if $a_0 = \max A$ then $\{a_0\} = \operatorname{Max} A = \operatorname{Sup}_0 A = \operatorname{Sup} A$;

(ii) if $a_0 = \max \operatorname{cl} A$ then $\{a_0\} = \operatorname{Sup}_0 A = \operatorname{Sup} A$;

(iii) if $\{a_0\} = \operatorname{Sup} A$ and $a_0 \neq +\infty$ then $a_0 = \max \operatorname{cl} A$; if in addition $a_0 \in A$ then $a_0 = \max A$. □

Now we introduce some notations which we shall use in the next sections. Given two sets X, Y and a function $f: X \times Y \to \mathbf{R}^n$ we shall write $\operatorname{Sup}_{x\in X} f(x, y)$ instead of $\operatorname{Sup}\{f(x, y): x \in X\}$; in an analogous way we shall use the symbols Sup_0, Max, Inf, Inf_0, Min. Moreover we shall write $\operatorname{Inf}_{y\in Y} \operatorname{Sup}_{x\in X} f(x, y)$ instead of $\operatorname{Inf} \bigcup_{y\in Y} \operatorname{Sup}\{f(x, y): x \in X\}$ and in a similar way we shall use the other related symbols. Also we use the following convention: if $+\infty \in \operatorname{Sup}_{x\in X} f(x, y)$ for some y we put

$$\operatorname{Inf}_{y\in Y} \operatorname{Sup}_{x\in X} f(x, y) = \{+\infty\}, \quad \text{if } \{+\infty\} = \bigcup_{y\in Y} \operatorname{Sup}_{x\in X} f(x, y),$$

and otherwise

$$\operatorname{Inf}_{y\in Y} \operatorname{Sup}_{x\in X} f(x, y) = \operatorname{Inf}_{y\in Y} \Big(\bigcup_{y\in Y} \operatorname{Sup}\{f(x, y): x \in X\} \setminus \{+\infty\}\Big),$$

an analogous convention we adopt for $\{-\infty\}$.

3. – Generalized saddle-points and saddle-values.

Let X, Y be two sets and

$$f: X \times Y \to \mathbf{R}^n.$$

It is well known that in the case $n = 1$ we have $\sup_{x\in X} \inf_{y\in Y} f(x, y) \leq \inf_{y\in Y} \sup_{x\in X} f(x, y)$. The same is not true in the case $n > 1$ if we deal with Sup, Inf instead of sup, inf. The following examples clarify this simple fact:

EXAMPLES. – Let $X = Y = [0, 1]$, $n = 2$, $K = \{(x, y): x \geq 0, y \geq 0\}$.

(i) Let $f(x, y) = (x, 1)$ if $0 < y \leq 1$, $f(x, y) = (\frac{1}{2}, \frac{1}{2})$ if $y = 0$.

We have $\operatorname{Min}_{y\in Y} \operatorname{Max}_{x\in X} f(x, y) = \{(\frac{1}{2}, \frac{1}{2})\}$ and

$$\operatorname{Sup}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y) = \operatorname{Sup}_0{}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y) = \{(\tfrac{1}{2}, 1)\}.$$

(ii) Let $f(x, y) = (x, 1)$ if $0 \leq y < 1$, $f(x, y) = (1, 0)$ if $y = 1$. We have

$$\operatorname{Min}_{y\in Y} \operatorname{Max}_{x\in X} f(x, y) = \{(1, 0)\} \quad \text{and} \quad \operatorname{Sup}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y) = \operatorname{Sup}_0{}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y) = \{(1, 1)\}.$$

(iii) Let $f(x, y) = (1, 0)$ if $0 \leq x < 1$ and $0 \leq y < 1$, $f(x, y) = (0, \frac{1}{2})$ if $x = 1$ and $0 \leq y < 1$, $f(x, y) = (0, 1)$ if $0 \leq x \leq 1$ and $y = 1$. We have

$$\operatorname{Max}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y) = \{(1, 0), (0, 1)\} \quad \text{and} \quad \operatorname{Min}_{y\in Y} \operatorname{Max}_{x\in X} f(x, y) = \{(1, 0), (0, \tfrac{1}{2})\}. \quad \square$$

In the following we shall say that $(x_0, y_0) \in X \times Y$ is a generalized saddle-point for f if $\gamma \leq f(x_0, y_0) \leq \delta$ for every $\gamma \in C(f(x_0, y_0)) \cap \operatorname{cl}\{f(x, y_0): x \in X\}$ and $\delta \in C(f(x_0, y_0)) \cap \operatorname{cl}\{f(x_0, y): y \in Y\}$.

By the definitions we have:

PROPOSITION 3.1. – f has a generalized saddle-point if and only if there exists $(x_0, y_0) \in X \times Y$ such that $f(x_0, y_0) \in \operatorname{Min}_{y\in Y} f(x_0, y) \cap \operatorname{Max}_{x\in X} f(x, y_0)$; in this case there exist $\beta \in \operatorname{Inf}_0{}_{y\in Y} \operatorname{Max}_{x\in X} f(x, y)$ and $\alpha \in \operatorname{Sup}_0{}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y)$ such that $\beta \leq f(x_0, y_0) \leq \alpha$ (it is allowed $\beta = -\infty$ or $\alpha = +\infty$). □

In analogy with Proposition 3.1 we shall say that $\eta_0 \in \mathbf{R}^n$ is a generalized saddle-value for f if there exists $(x_0, y_0) \in X \times Y$ such that $\eta_0 \in \operatorname{Inf}_{y\in Y} f(x_0, y) \cap \operatorname{Sup}_{x\in X} f(x, y_0)$; in this case we have $\gamma \leq \eta_0 \leq \delta$ for every $\gamma \in C(\eta_0) \cap \operatorname{cl}\{f(x, y_0): x \in X\}$ and $\delta \in C(\eta_0) \cap \operatorname{cl}\{f(x_0, y): y \in Y\}$; the converse is not true in general. An other necessary but not sufficient condition for the existence of a generalized saddle-value is that there exist $\beta \in \operatorname{Inf}_{y\in Y} \operatorname{Sup}_{x\in X} f(x, y)$ and $\alpha \in \operatorname{Sup}_{x\in X} \operatorname{Inf}_{y\in Y} f(x, y)$ such that $\beta \leq \alpha$.

We have also:

PROPOSITION 3.2. – The following statements hold:

(i) If there exists $\eta_0 \in \operatorname{Min}_{y\in Y} \operatorname{Sup}_{x\in X} f(x, y) \cap \operatorname{Max}_{x\in X} \operatorname{Inf}_{y\in Y} f(x, y)$, then η_0 is a generalized saddle-value for f.

(ii) If there exists $\eta_0 \in \operatorname{Min}_{y\in Y} \operatorname{Max}_{x\in X} f(x, y) \cap \operatorname{Max}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y)$ then η_0 is a generalized saddle-value for f and also there exist (x', y'), $(x'', y'') \in X \times Y$ such that

$$\gamma \leq f(x', y') = \eta_0 = f(x'', y'') \leq \delta$$

for every $\gamma \in C(\eta_0) \cap \operatorname{cl}\{f(x, y'): x \in X\}$ and $\delta \in C(\eta_0) \cap \operatorname{cl}\{f(x'', y): y \in Y\}$.

(iii) In the hypotheses of (i), if in addition $\{\eta_0\} = \operatorname{Inf}_0\limits_{y\in Y} \operatorname{Sup}\limits_{x\in X} f(x, y)$ and f is upper bounded then f has a generalized saddle-point.

(iv) In the hypotheses of (ii) if in addition $\{\eta_0\} = \operatorname{Inf}_0\limits_{y\in Y} \operatorname{Sup}\limits_{x\in X} f(x, y)$ and f is upper bounded then f has a generalized saddle-point.

PROOF. – The proof of (i) is trivial.

(ii) By the definitions there exist $y' \in Y$ and $x'' \in X$ such that

$$\eta_0 \in \operatorname{Max}_{x\in X} f(x, y') \cap \operatorname{Min}_{y\in Y} f(x'', y)$$

and so there exist $x' \in X$ and $y'' \in Y$ such that

$$\eta_0 = f(x', y') = f(x'', y'') .$$

(iii) By the definitions and the hypotheses we have $\eta_0 \in \boldsymbol{R}^n$ and for suitable $x_0 \in X$ and $y_0 \in Y$,

$$\eta_0 \in \operatorname{Sup}_{x\in X} f(x, y_0) \cap \operatorname{Inf}_{y\in Y} f(x_0, y) .$$

If $\eta \in \operatorname{Sup}\limits_{x\in X} f(x, y_0) \setminus \{+\infty\}$ we have $\eta_0 \leqslant \eta$ and so $\eta_0 = \eta$; thus we have obtained $\{\eta_0\} = \operatorname{Sup}\limits_{x\in X} f(x, y_0) \setminus \{+\infty\}$ and also $\{\eta_0\} = \operatorname{Sup}\limits_{x\in X} f(x, y_0)$ since we suppose that f is upper bounded. So $\eta_0 \geqslant f(x_0, y_0)$ and from $\eta_0 \in \operatorname{Inf}\limits_{y\in Y} f(x_0, y)$ we derive $\eta_0 = f(x_0, y_0)$.

(iv) We argue as in the proof of (iii). □

The same arguments used in the proof of Proposition 3.2 (ii) yield:

PROPOSITION 3.3. – The following statements hold:

(i) If there exist $\beta \in \operatorname{Min}\limits_{y\in Y} \operatorname{Sup}\limits_{x\in X} f(x, y)$ and $\alpha \in \operatorname{Max}\limits_{x\in X} \operatorname{Inf}\limits_{y\in Y} f(x, y)$ such that $\beta \leqslant \alpha$ then there exist $x'' \in X$ and $y' \in Y$ such that

$$\gamma \leqslant \beta \leqslant \alpha \leqslant \delta$$

for every $\gamma \in C(\beta) \cap \operatorname{cl}\{f(x, y') \colon x \in X\}$ and $\delta \in C(\beta) \cap \operatorname{cl}\{f(x'', y) \colon y \in Y\}$.

(ii) If there exist $\beta \in \operatorname{Min}\limits_{y\in Y} \operatorname{Max}\limits_{x\in X} f(x, y)$ and $\alpha \in \operatorname{Max}\limits_{x\in X} \operatorname{Min}\limits_{y\in Y} f(x, y)$ such that $\beta \leqslant \alpha$ then there exist $(x', y'), (x'', y'') \in X \times Y$ such that

$$\gamma \leqslant f(x', y') = \beta \leqslant \alpha = f(x'', y'') \leqslant \delta$$

for every $\gamma \in C(\beta) \cap \operatorname{cl}\{f(x, y') \colon x \in X\}$ and

$$\delta \in C(\alpha) \cap \operatorname{cl}\{f(x'', y) \colon y \in Y\} . \quad \square$$

4. – Convexity and lower semicontinuity of n-valued functions.

At first we introduce some concepts about convexity and semicontinuity for n-valued functions.

DEFINITION 4.1. – Let Y be a convex set in a vector space and $\varphi \colon Y \to \boldsymbol{R}^n$.

(i) We say that φ is convex in Y if for every $y_1, y_2 \in Y$ and $t \in [0, 1]$ we have

$$\varphi(ty_1 + (1-t)y_2) \leqslant t\varphi(y_1) + (1-t)\varphi(y_2) .$$

(ii) We say that φ is properly quasi-convex in Y if for every $y_1, y_2 \in Y$ and $t \in [0, 1]$ we have

$$\varphi(ty_1 + (1-t)y_2) \leqslant \varphi(y_1) \qquad \text{or} \qquad \varphi(ty_1 + (1-t)y_2) \leqslant \varphi(y_2) .$$

(iii) We say that φ is quasi-convex in Y if for every $\alpha \in \boldsymbol{R}^n$ the set $\{y \in Y \colon \varphi(y) \leqslant \alpha\}$ is convex or empty. □

In the case $n = 1$ it is well known that φ is properly quasi-convex if and only if φ is quasi-convex and that if φ is convex then it is also quasi-convex.
In the general case $n > 1$ the situation is rather different. In fact we have:

PROPOSITION 4.2. – The following statements hold:

(i) If φ is either convex or properly quasi-convex then φ is quasi-convex.

(ii) The conditions expressed in Definitions 4.1 (i) and 4.1 (ii) are mutually independent.

PROOF. – The proof of (i) is a standard one. As to (ii) we observe that the function $\varphi(y) = (y^3, y)$, $y \in \boldsymbol{R}$, is properly quasi-convex but not convex, while the function $\varphi(y) = (y, 1-y)$, $y \in \boldsymbol{R}$, is convex but not properly quasi-convex (it is understood $K = \{(x, y) \colon x \geqslant 0,\ y \geqslant 0\}$). □

Now we consider semicontinuity questions:

DEFINITION 4.3. – Let Y be a Hausdorff topological space and $\varphi \colon Y \to \boldsymbol{R}^n$; we say that φ is lower semicontinuous if for every $\alpha \in \boldsymbol{R}^n$ the set $\{y \in Y \colon \varphi(y) \leqslant \alpha\}$ is closed. □

It will be useful the following

PROPOSITION 4.4. – Let φ be lower semicontinuous on a Hausdorff space. If $\{y_i\}$ is a generalized sequence such that $\lim_i y_i = y_0$ and $\lim_i \varphi(y_i) = \lambda \in \boldsymbol{R}^n$, then we have $\varphi(y_0) \leqslant \lambda$.

PROOF. – Given $\delta > 0$ there exists i_0 such that $\lambda - \varphi(y_i) \in B(0; \delta)$ for $i \geqslant i_0$; moreover there exists $k_\delta \in K$ such that $B(k_\delta; \delta) \subset K$. Then

$$k_\delta + \lambda - \varphi(y_i) \in B(k_\delta; \delta) \subset K\,, \quad \text{for } i \geqslant i_0\,,$$

from which we derive

$$\varphi(y_i) \leqslant \lambda + k_\delta\,, \quad \text{for } i \geqslant i_0\,,$$

and so, by the lower semicontinuity,

$$\varphi(y_0) \leqslant \lambda + k_\delta\,.$$

Now we observe that k_δ may be choosen in such a way that $\lim_{\delta\to 0} k_\delta = 0$ (in fact if $B(k_1; 1) \subset K$ then $B(\delta k_1; \delta) \subset K$) and so $\varphi(y_0) \leqslant \lambda$. □

Now we may prove a Weierstrass type theorem for lower semicontinuous n-valued functions.

THEOREM 4.5. – Let Y be a compact Hausdorff space and $\varphi\colon Y \to \boldsymbol{R}^n$ be lower semicontinuous. Then

$$\operatorname*{Inf}_{y\in Y} \varphi(y) = \operatorname*{Min}_{y\in Y} \varphi(y)\,.$$

PROOF. – Let $\lambda \in \boldsymbol{R}^n \cap \operatorname*{Inf}_{y\in Y} \varphi(y)$; then there exists a sequence $\{y_i\} \subset Y$ such that $\lim_{i\to+\infty} \varphi(y_i) = \lambda$.

Since Y is compact there exist a generalized subsequence $\{\eta_j\}$ of $\{y_i\}$ and $y_0 \in Y$ such that $\lim_j \eta_j = y_0$; we have also $\lim_j \varphi(\eta_j) = \lambda$.

By Proposition 4.4 we derive $\varphi(y_0) \leqslant \lambda$ and so $\varphi(y_0) = \lambda$.

Now we must prove that $-\infty \notin \operatorname*{Inf}_{y\in Y} \varphi(y)$.

By contradiction, let $\{\lambda_m\} \subset \operatorname{cl}\{\varphi(y)\colon y \in Y\}$ be a non-increasing sequence which has no lower bound and let $\{y_m\} \subset Y$ be a sequence such that $\|\varphi(y_m) - \lambda_m\| \leqslant 1/m$. Now we fix m_0; if $m \geqslant m_0$ we have $\lambda_{m_0} - \lambda_m \in K$ and given $k_0 \in \operatorname{int} K$ such that $k_0 + B(0, 1) \subset K$ we have

$$\lambda_m + B(0; 1/m) = \lambda_m - \lambda_{m_0} + \lambda_{m_0} + k_0 - k_0 - B(0; 1/m) \subset \lambda_{m_0} + k_0 - K;$$

so $\varphi(y_m) \in \lambda_{m_0} + k_0 - K$, that is $\varphi(y_m) \leqslant \lambda_{m_0} + k_0$.

Now we may choose a generalized subsequence $\{\eta_j\} = \{y_{m(j)}\}$ of $\{y_m\}_{m\geqslant m_0}$ which converges to a point $y_0 \in Y$ and we have

$$\varphi(y_0) \leqslant \lambda_{m_0} + k_0\,.$$

The same may be done starting from λ_{m_0+1} (we shall use in this case the generalized subsequence $\{y_{m(j)}\}_{m(j)\geqslant m_0+1}$ which converges to the same point y_0).

Finally we obtain $\varphi(y_0) \leqslant \lambda_m + k_0$ for every m and so $\{\lambda_m\}$ is lower bounded, which is a contradiction. □

We remark that in the hypotheses of Theorem 4.5 φ may have no lower bound; for example let $\varphi(t) = (-1/t, 1/t)$, $t \in (0, 1]$, $\varphi(0) = (-1, 0)$.

The following results will be used in the next section and clarify the meaning of Definition 4.1.

PROPOSITION 4.6. – Let Y be a convex set in a vector space and $\varphi\colon Y \to \boldsymbol{R}^n$ be properly quasi-convex and lower semicontinuous on the intersection of Y with any finite dimensional space. Let V be a one dimensional space and $\{y_m\} \subset Y \cap V$ be a sequence such that $\lim_{m\to+\infty} y_m = y_0$ and $\{\varphi(y_m)\}$ be lower bounded. Then there is a subsequence $\{y_{m_i}\}$ of $\{y_m\}$ such that $\{\varphi(y_{m_i})\}$ is convergent and $\varphi(y_0) \leqslant \lim_{i\to+\infty} \varphi(y_{m_i})$.

PROOF. – There is a subsequence of $\{y_m\}$, which we shall note also $\{y_m\}$, such that $y_{m+1} \in \operatorname{co}\{y_m, y_0\}$ for every m. Since φ is properly quasi-convex we have $\varphi(y_{m+1}) \leqslant \varphi(y_0)$ or $\varphi(y_{m+1}) \leqslant \varphi(y_m)$. If $\varphi(y_m) \leqslant \varphi(y_0)$ for a countably set of the indices m then we obtain a subsequence of $\{\varphi(y_m)\}$ which is also upper bounded and so it is bounded in norm by Theorem 2.1.

Finally we have a subsequence $\{y_{m_i}\}$ of $\{y_m\}$ such that $\{\varphi(y_{m_i})\}$ is convergent to $\sigma \in \boldsymbol{R}^n$.

Given $k_0 \in \operatorname{int} K$ we have $\sigma \in \operatorname{int}(\sigma + k_0 - K)$ and so $\varphi(y_{m_i}) \in \sigma + k_0 - K$ for i large enough, that is $\varphi(y_{m_i}) \leqslant \sigma + k_0$; from this by the semicontinuity we have $\varphi(y_0) \leqslant \sigma + k_0$ for every $k_0 \in \operatorname{int} K$; so $\varphi(y_0) \leqslant \sigma$.

If $\varphi(y_m) \leqslant \varphi(y_0)$ holds only for a finite set of the indices m the sequence $\{\varphi(y_m)\}$ is non-increasing for $m \geqslant m_0$, where m_0 is a suitable index.

Then $\{\varphi(y_m)\}$ is also upper bounded and we may argue as in the previous case. □

THEOREM 4.7. – Let Y be a convex set in a Hausdorff topological vector space and $\varphi\colon Y \to R^n$ be properly quasi-convex, lower bounded and lower semicontinuous on the intersection of Y with any finite dimensional space. Then there exists $\lambda_0 \in \boldsymbol{R}^n$ such that $\{\lambda_0\} = \operatorname*{Inf}_{y\in Y} \varphi(y)$.

PROOF. – We have $\operatorname*{Inf}_{y\in Y} \varphi(y) \subset \boldsymbol{R}^n$ since φ is lower bounded. Now let $\lambda_1, \lambda_2 \in \operatorname*{Inf}_{y\in Y} \varphi(y)$ and $\{y'_i\}$, $\{y''_i\}$ be two sequences in Y such that $\lim_{i\to+\infty} \varphi(y'_i) = \lambda_1$ and $\lim_{i\to+\infty} \varphi(y''_i) = \lambda_2$.

Given $t\in[0,1]$ we put $y_i(t)=ty_i'+(1-t)y_i''$; since φ is properly quasi-convex for every i and t we have $\varphi(y_i(t))\leqslant\varphi(y_i')$ or $\varphi(y_i(t))\leqslant\varphi(y_i'')$.

Let i be fixed; if $\varphi(y_i(t))\leqslant\varphi(y_i')$ then $\varphi(y_i(\tau))\leqslant\varphi(y_i')$ for $\tau\in[t,1]$. In fact for such a value of τ there exists θ, $0\leqslant\theta\leqslant1-t$, such that $t+\theta=\tau$; so

$$y_i(\tau)=\tau y_i'+(1-\tau)y_i''=\frac{\theta}{1-t}y_i'+\left(1-\frac{\theta}{1-t}\right)y_i(t)$$

and the result follows by the proper quasi convexity of φ. We define $t_i=\inf\{t:\varphi(y_i(t))\leqslant\varphi(y_i')\}$; by Proposition 4.6 we obtain $\varphi(y_i(t_i))\leqslant\varphi(y_i')$ and since $t_i=\sup\{t:\varphi(y_i(t))\leqslant\varphi(y_i'')\}$ we have also $\varphi(y_i(t_i))\leqslant\varphi(y_i'')$. The sequence $\{\varphi(y_i(t_i))\}$ is lower and upper bounded; then it is bounded in norm and it has a subsequence which converges to $\lambda\in\boldsymbol{R}^n$. By $\varphi(y_i(t_i))\leqslant\varphi(y_i')$ and $\varphi(y_i(t_i))\leqslant\varphi(y_i'')$ we obtain $\lambda\leqslant\lambda_1$, $\lambda\leqslant\lambda_2$ and so $\lambda=\lambda_1=\lambda_2$. □

THEOREM 4.8. – Let Y be a compact and convex set in a Hausdorff topological vector space and $\varphi\colon Y\to\boldsymbol{R}^n$ be properly quasi-convex and lower semicontinuous. Then there exists $y_0\in Y$ such that $\{\varphi(y_0)\}=\operatorname{Inf}_{y\in Y}\varphi(y)=\operatorname{Min}_{y\in Y}\varphi(y)$.

PROOF. – By Theorem 4.5 $\operatorname{Inf}_{y\in Y}\varphi(y)=\operatorname{Min}_{y\in Y}\varphi(y)\in\boldsymbol{R}^n$.

Then let $\varphi(y_1)\in\operatorname{Min}_{y\in Y}\varphi(y)$ and $\varphi(y_2)\in\operatorname{Min}_{y\in Y}\varphi(y)$.

Since φ is properly quasi-convex we have $\varphi(y(t))=\varphi(y_1)$ or $\varphi(y(t))=\varphi(y_2)$, where $y(t)=ty_1+(1-t)y_2$. We define $t_0=\inf\{t:\varphi(y(t))=\varphi(y_1)\}=\sup\{t:\varphi(y(t))=\varphi(y_2)\}$; by the lower semicontinuity, since $\varphi(y(t))$ is constant in $(t_0,1]$ and in $[0,t_0)$ we obtain $\varphi(y(t_0))=\varphi(y_1)=\varphi(y_2)$. □

We remark that if φ is not lower semicontinuous then the conclusion of Theorem 4.7 may not hold: for example let $\varphi(y)=(0,0)$ if $-1\leqslant y<0$ and $\varphi(y)=(1,-1)$ if $0\leqslant y\leqslant1$.

In what follows whenever we are in the situation of Theorem 4.7 or Theorem 4.8 we shall use the symbol $\operatorname{Inf}_{y\in Y}\varphi(y)$ also to denote the single element which belongs to the set $\operatorname{Inf}_{y\in Y}\varphi(y)$.

THEOREM 4.9. – Let X be a convex set in a vector space and Y be a compact and convex set in a Hausdorff topological vector space; let $g\colon X\times Y\to\boldsymbol{R}^n$ such that $g(x,\cdot)$ is properly quasi convex and lower semicontinuous.

Then the following statements hold:

(i) if $-g(\cdot,y)$ is properly quasi-convex for every $y\in Y$ then the n-valued function $x\to-\operatorname{Inf}_{y\in Y}g(x,y)\equiv\psi(x)$ is properly quasi-convex;

(ii) if $-g(\cdot,y)$ is lower semicontinuous for every $y\in Y$ on the intersection of X with any finite dimensional space then also the n-valued function $x\to-\operatorname{Inf}_{y\in Y}g(x,y)\equiv\psi(x)$ is so;

(iii) if g is upper bounded and $-g(\cdot,y)$ is properly quasi-convex and lower semicontinuous on the intersection of X with any finite dimensional space then the n-valued function $y\to\operatorname{Sup}_{x\in X}g(x,y)\equiv\varphi(y)$ is properly quasi convex and lower semicontinuous.

PROOF. – First of all we remark that by Theorems 4.7 and 4.8 the functions φ, ψ considered in (i), (ii), (iii) are well-defined.

Now we prove (i). Let $x_1,x_2\in X$, $t\in[0,1]$, $x(t)=tx_1+(1-t)x_2$.

By Theorem 4.8 we may write, for a suitable $y_0\in Y$:

$$\psi(x(t))=-g(x(t),y)\leqslant-g(x_1,y_0)\quad\text{or}\quad\leqslant-g(x_2,y_0)$$

and so

$$\psi(x(t))\leqslant-\operatorname{Inf}_{y\in Y}g(x_1,y)=\psi(x_1)\quad\text{or}\quad\psi(x(t))\leqslant-\operatorname{Inf}_{y\in Y}g(x_2,y)=\psi(x_2)\,.$$

As to (ii), given a finite dimensional space V it is easily seen that

$$\{x\colon\psi(x)\leqslant\alpha,\ x\in V\cap X\}=\bigcap_{y\in Y}\{x\colon-g(x,y)\leqslant\alpha,\ x\in V\cap X\}$$

and so the left hand side set is closed.

In an analogous way we may prove (iii). □

5. – Minimax theorems.

We may state the minimax theorem relative to properly quasi-convex functions in a very general fashion.

THEOREM 5.1. – Let X be a convex set in a vector space and Y be a compact and convex set in a Hausdorff topological vector space. Let $f\colon X\times Y\to\boldsymbol{R}^n$ satisfy:

(i) for every y the function $x\to-f(x,y)$ is lower bounded, properly quasi-convex and lower semicontinuous on the intersection of X with any finite dimensional space;

(ii) for every x the function $y\to f(x,y)$ is properly quasi-convex and lower semicontinuous.

Then

$$\operatorname{Sup}_{x\in X}\operatorname{Min}_{y\in Y}f(x,y)=\operatorname{Min}_{y\in Y}\operatorname{Sup}_{x\in X}f(x,y).$$

PROOF. – By Theorems 4.7, 4.8, 4.9 and the hypotheses we have $\operatorname{Sup}_{x\in X}\operatorname{Min}_{y\in Y} f(x,y)=\alpha$ and $\operatorname{Min}_{y\in Y}\operatorname{Sup}_{x\in X} f(x,y)=\beta$ for suitable $\alpha,\beta\in\boldsymbol{R}^n$.

We have also

$$\operatorname{Min}_{y\in Y} f(x,y)\leqslant f(x,y)\,,\quad \text{for every } (x,y)\in X\times Y\,,$$

and so:

$$\operatorname{Sup}_{x\in X}\operatorname{Min}_{y\in Y} f(x,y)\leqslant \operatorname{Sup}_{x\in X} f(x,y)\,,\quad \text{for every } y\in Y\,,$$

from which we derive $\alpha\leqslant\beta$.

Now, given $k_0\in\operatorname{int}K$, we consider the family of set

$$\Gamma=\{\{y\colon f(x,y)\leqslant\alpha+\varepsilon k_0\}\colon x\in X,\ \varepsilon\in\boldsymbol{R},\ \varepsilon>0\}\,.$$

Since for every $x\in X$ we have $\operatorname{Min}_{y\in Y} f(x,y)\leqslant\alpha$, that is for every $x\in X$ there exists $y(x)\in Y$ such that $f(x,y(x))\leqslant\alpha$, every set in Γ is non-empty; moreover if $F\in\Gamma$, F is compact since F is a closed set, by (ii), in the compact space Y.

Hence it is well known that $\bigcap_{F\in\Gamma} F\neq\emptyset$ if and only if the family Γ has the finite intersection property.

We prove this fact by induction. Let $h\in\boldsymbol{N}\setminus\{0\}$ and suppose that the intersection of ν sets $F_i\in\Gamma$ be non-empty for every $\nu\leqslant h$ and every choice of the sets F_i; by contradiction let us assume that there exist $h+1$ sets $F_i^0\in\Gamma$ such that $\bigcap_{i=1}^{h+1}F_i^0=\emptyset$.

We may suppose

$$F_i^0=\{y\colon f(x_i^0,y)\leqslant\alpha+\varepsilon_i^0k_0\}\,,$$

where $x_i^0\in X$ and $0<\varepsilon_1^0<\varepsilon_2^0<\ldots<\varepsilon_{h+1}^0$.

We put $G=Y$ if $h=1$, $G=\bigcap_{i=3}^{h+1}\{y\colon f(x_i^0,y)\leqslant\alpha+\varepsilon_1^0k_0\}$ otherwise and $K(x)=\{y\in G\colon f(x,y)<\alpha+\varepsilon_1^0k_0\}$.

For every $x\in X$, by (ii) $K(x)$ is a convex set; moreover $K(x)\neq\emptyset$ for every $x\in X$ since

$$K(x)\supset\{y\colon f(x,y)\leqslant\alpha+\varepsilon_1^0k_0/2\}\neq\emptyset\,,\quad \text{if } h=1\,,$$

and, by the inductive hypothesis, if $h\geqslant2$

$$K(x)\supset\bigcap_{i=3}^{h+1}\{y\colon f(x_i^0,y)\leqslant\alpha+\varepsilon_1^0k_0/2\}\cap\{y\colon f(x,y)\leqslant\alpha+\varepsilon_1^0k_0/2\}\neq\emptyset\,.$$

Now let $x_1,x_2\in X$ and $[x_1,x_2]=\{tx_1+(1-t)x_2\colon t\in[0,1]\}$; we have $K(z)\subset K(x_1)\cup K(x_2)$ for every $z\in[x_1,x_2]$.

In fact by (i) we may write

$$K(z)=\{y\in G\colon f(z,y)<\alpha+\varepsilon_1^0k_0\}\subset\{y\in G\colon f(x_1,y)<\alpha+\varepsilon_1^0k_0\}\cup$$
$$\cup\{y\in G\colon f(x_2,y)<\alpha+\varepsilon_1^0k_0\}=K(x_1)\cup K(x_2)\,.$$

Now we prove that for every $z\in[x_1^0,x_2^0]$ we have

$$(*)\qquad K(z)\subset K(x_1^0)\setminus K(x_2^0)\quad\text{or}\quad K(z)\subset K(x_2^0)\setminus K(x_1^0)\,.$$

Indeed let $y_1\in K(z)\cap K(x_1^0)$ and $y_2\in K(z)\cap K(x_2^0)$; we put

$$T_j=[y_1,y_2]\cap F_j^0\,,\qquad j=1,2,\ \text{if } h=1\,,$$
$$T_j=[y_1,y_2]\cap F_j^0\cap\bigcap_{i=3}^{h+1}F_i^0\,,\qquad j=1,2,\ \text{if } h\geqslant2\,.$$

We have $T_j\neq\emptyset$, $j=1,2$, since

$$y_j\in K(x_j^0)\subset F_j^0\,,\qquad j=1,2,\ \text{if } h=1\,,$$

and

$$y_j\in K(x_j^0)\subset F_j^0\cap G\subset F_j^0\cap\bigcap_{i=3}^{h+1}F_i^0\,,\qquad \text{if } h\geqslant2\,.$$

Moreover we obtain $T_1\cap T_2\subset\bigcap_{i=1}^{h+1}F_i^0=\emptyset$; but we have also

$$[y_1,y_2]\supset T_1\cup T_2\supset[y_1,y_2]\cap(F_1^0\cup F_2^0)\cap G\supset(\text{since } \varepsilon_1^0<\varepsilon_2^0)\supset$$
$$\supset[y_1,y_2]\cap\big(K(x_1^0)\cup K(x_2^0)\big)\supset[y_1,y_2]\cap K(z)=[y_1,y_2]$$

and so $T_1\cup T_2=[y_1,y_2]$; this is a contradiction since T_1, T_2 are closed sets by the definition; so (*) is proved.

Now let $H_j^0=\{z\in[x_1^0,x_2^0]\colon K(z)\subset K(x_j^0)\}$, $j=1,2$.

It is easily seen by (i) that the sets H_j^0, $j=1,2$, are convex; moreover we have $x_j^0\in H_j^0$, $j=1,2$, and, by (*), $[x_1^0,x_2^0]\subset H_1^0\cup H_2^0$.

Let us define

$$t^0=\sup\{t\colon tx_1^0+(1-t)x_2^0\in H_2^0,\ t\in[0,1]\}$$

and let $z^0=t^0x_1^0+(1-t^0)x_2^0$. By (*) $z^0\notin H_1^0\cap H_2^0$.

If $z^0 \in H_1^0 \setminus H_2^0$, that is $K(z^0) \subset K(x_1^0)$ and $K(z^0) \cap K(x_2^0) = \emptyset$, we have $z_0 \neq x_2^0$ and since the sets H_j^0, $j = 1, 2$, are convex, $K(z) \subset K(x_2^0)$ for every $z \in [x_2^0, z^0)$; hence if $y^0 \in K(z^0)$ then $y^0 \notin K(z)$ for every $z \in [x_2^0, z^0)$.

Now we choose $y^0 \in K(z^0)$ such that $f(z^0, y^0) < \alpha + \varepsilon_1^0 k_0/2$: this is possible by the inductive hypothesis. We have

$$(**) \qquad \alpha + \varepsilon_1^0 k_0 - f(z, y^0) \notin K \setminus \{0\}, \quad \text{for every } z \in [x_2^0, z^0).$$

By Proposition 4.6 and (i) we may find a sequence $\{z_m\} \subset [x_2^0, z^0)$ such that $\lim_{m\to+\infty} z_m = z^0$ and $\{f(z_m, y^0)\}$ converges to a point $\sigma \in \boldsymbol{R}^n$.

By (i) we have $f(z^0, y^0) \geqslant \sigma$ and by (**)

$$\alpha + \varepsilon_1^0 k_0 - f(z^0, y^0) \leqslant \alpha + \varepsilon_1^0 k_0 - \sigma \notin \operatorname{int}(K \setminus \{0\}) = \operatorname{int} K;$$

so we obtain

$$\alpha + \varepsilon_1^0 k_0 - f(z^0, y^0) \notin \operatorname{int} K.$$

On the other hand, by the choice of y^0 we have

$$\alpha + \varepsilon_1^0 k_0 - f(z^0, y^0) \in (K \setminus \{0\}) + \varepsilon_1^0 k_0/2 \subset \operatorname{int} K,$$

which is a contradiction.

If $z^0 \in H_2^0 \setminus H_1^0$ we may argue in an analogous way and so the family Γ has the finite intersection property.

Hence there exists $y_0 \in Y$ such that $y_0 \in F$ for every $F \in \Gamma$, that is

$$f(x, y_0) \leqslant \alpha + \varepsilon k_0, \quad \text{for every } x \in X \text{ and } \varepsilon > 0.$$

Then we have $f(x, y_0) \leqslant \alpha$ for every $x \in X$ and so $\beta \leqslant \alpha$. □

The problem is more difficult if we use convex functions instead of properly quasi-convex ones.

In this case we have only the following weaker result:

THEOREM 5.2. – Let $X \subset \boldsymbol{R}^m$, $Y \subset \boldsymbol{R}^k$, be two compact and convex sets. Let $f\colon X \times Y \to \boldsymbol{R}^n$ such that

(i) f is continuous in $X \times Y$;

(ii) $-f(\cdot, y)$ and $f(x, \cdot)$ are convex in X and Y respectively.

Then there exist $\beta_0 \in \operatorname{co} \operatorname{Inf}_0_{y\in Y} \operatorname{Max}_{x\in X} f(x, y)$ and $\alpha_0 \in \operatorname{co} \operatorname{Sup}_0_{x\in X} \operatorname{Min}_{y\in Y} f(x, y)$ such that $\beta_0 \leqslant \alpha_0$.

PROOF. – We remark that by Theorem 4.5 $\operatorname{Min}_{y\in Y} f(x, y) = \operatorname{Inf}_{y\in Y} f(x, y) \neq \emptyset$ for every $x \in X$ and $\operatorname{Max}_{x\in X} f(x, y) = \operatorname{Sup}_{x\in X} f(x, y) \neq \emptyset$ for every $y \in Y$.

Let

$$U = \{(x_0, y_0) \in X \times Y\colon f(x_0, y_0) \in \operatorname{Min}_{y\in Y} f(x_0, y)\}$$

and

$$V = \{(x_0, y_0) \in X \times Y\colon f(x_0, y_0) \in \operatorname{Max}_{x\in X} f(x, y_0)\};$$

we define also

$$(\operatorname{cl} U)_{x_0} = \{y\colon (x_0, y) \in \operatorname{cl} U\}, \quad x_0 \in X,$$
$$(\operatorname{cl} V)_{y_0} = \{x\colon (x, y_0) \in \operatorname{cl} V\}, \quad y_0 \in Y.$$

Since $X \times Y$ is compact and $\operatorname{cl} U$ is obviously closed, then $(\operatorname{cl} U)_{x_0}$ is compact and so, by a well-known consequence of Carathéodory's theorem (see [10, Theorem 17.2]), $U'_{x_0} = \operatorname{co}((\operatorname{cl} U)_{x_0})$ is compact; the same is true for $V'_{y_0} = \operatorname{co}((\operatorname{cl} V)_{y_0})$.

We define

$$U' = \bigcup_{x\in X} \{(x, y)\colon y \in \operatorname{co}((\operatorname{cl} U)_x)\},$$
$$V' = \bigcup_{y\in Y} \{(x, y)\colon x \in \operatorname{co}((\operatorname{cl} V)_y)\};$$

now we prove that U', V' are closed.

Let $\{(x_r, y_r)\} \subset U'$ be a sequence such that

$$y_r \in \operatorname{co}((\operatorname{cl} U)_{x_r}) \quad \text{and} \quad \lim_{r\to+\infty} (x_r, y_r) = (x_0, y_0).$$

By Carathéodory's theorem (see [10, Theorem 17.1]) we have $y_r = \sum_{i=0}^{k} \lambda_r^{(i)} y_r^{(i)}$, for suitable $y_r^{(i)} \in (\operatorname{cl} U)_{x_r}$ and $\lambda_r^{(i)} \geqslant 0$ such that $\sum_{i=0}^{k} \lambda_r^{(i)} = 1$. We may choose suitable convergent subsequences $\{y_{r_h}^{(i)}\}$, $\{\lambda_{r_h}^{(i)}\}$ such that $(x_{r_h}, y_{r_h}^{(i)}) \to (x_0, y_0^{(i)})$, $\lambda_{r_h}^{(i)} \to \lambda_0^{(i)}$.

Since $(x_0, y_0^{(i)}) \in \operatorname{cl} U$ we have $y_0^{(i)} \in (\operatorname{cl} U)_{x_0}$; moreover $\sum_{i=0}^{k} \lambda_0^{(i)} = 1$ and so $y_0 = \sum_{i=0}^{k} \lambda_0^{(i)} y_0^{(i)} \in \operatorname{co}((\operatorname{cl} U)_{x_0})$ and $(x_0, y_0) \in U'$. In an analogous way we may prove that V' is closed.

Since U'_x, V'_y are non-empty, closed and convex we may use [7, Theorem 2] and so $U' \cap V' \neq \emptyset$, that is there is (x_0, y_0) such that $y_0 \in \operatorname{co}((\operatorname{cl} U)_{x_0})$ and $x_0 \in \operatorname{co}((\operatorname{cl} V)_{y_0})$.

Hence $y_0 = \sum_{i=0}^{k} \lambda^{(i)} y^{(i)}$ for suitable $\lambda^{(i)} \geqslant 0$ such that $\sum_{i=0}^{k} \lambda^{(i)} = 1$ and $y^{(i)} \in (\operatorname{cl} U)_{x_0}$ such that $(x_0, y^{(i)}) = \lim_{r\to+\infty} (x_r, y_r^{(i)})$ for suitable sequences $\{(x_r, y_r^{(i)})\} \subset U$.

Since $f(x_r, y_r^{(i)}) \in \operatorname{Min}_{y\in Y} f(x_r, y)$, $i = 0, \ldots, k$, by (i) we have $f(x_0, y^{(i)}) \in \operatorname{cl}\left(\bigcup_{x\in X} \operatorname{Min}_{y\in Y} f(x, y)\right)$ and so there exists $\alpha_i \in \operatorname{Sup}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y)$ such that $f(x_0, y^{(i)}) \leqslant \alpha_i$; moreover $\alpha_i \in \operatorname{Sup}_0{}_{x\in X} \operatorname{Min}_{y\in Y} f(x, y)$; in fact there exists $y_0^{(i)} \in Y$ such that $f(x_0, y_0^{(i)}) \in \operatorname{Min}_{y\in Y} f(x_0, y)$ and $f(x_0, y_0^{(i)}) \leqslant f(x_0, y^{(i)}) \leqslant \alpha_i$. Now by (ii) we obtain $f(x_0, y_0) \leqslant \sum_{i=0}^{k} \lambda^{(i)}\alpha_i \equiv \alpha_0$.

In an analogous way we find $\beta_0 \in \operatorname{co}\left(\operatorname{Inf}_0{}_{y\in Y} \operatorname{Max}_{x\in X} f(x, y)\right)$ such that $\beta_0 \leqslant f(x_0, y_0)$ and the proof is complete. □

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Structural Assignment of Neumann Boundary Feedback Parabolic Equations: the unbounded Case in the Feedback Loop (*) (**).

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Summary. – *A parabolic equation defined on a bounded domain is considered, with input acting in* the Neumann (or mixed) boundary conditions, *and expressed as a* specified feedback *of the solution x of the form: $\langle\gamma x, w\rangle g$, where $w \in L_2(\Omega)$, $g \in L_2(\Gamma)$ and γ is a continuous operator for $\sigma < \frac{3}{4}$: $H^{2\sigma}(\Omega) \to L_2(\Omega)$. The* free *system is assumed unstable. In this case, the boundary feedback stabilization problem (in space dimension larger or equal to two) follows from an essentially more general result recently established by the authors in* [L8]: *under algebraic (full rank), verifiable conditions at the unstable eigenvalues, one can select* boundary *vectors, so that the corresponding feedback solutions decay* in the uniform operator norm *exponentially at $t \to \infty$. Here, this stabilization peoblem is pushed further and made more precise, under the additional assumption that the original* free *system be self-adjoint: we show, in fact, that one can further restrict the* boundary *vectors, so that the corresponding feedback solutions have the following more precise desirable* structural property (*the same enjoyed by* free stable *systems*): *they can be expressed as an infinite linear combination of decaying exponentials. A semigroup approach is employed. Since* structure *of feedback solutions is sought, the analysis here is much more technical and vastly different from* [L8], *where only* norm upper bound *was the goal.*

1. – Introduction and statement of main result.

Let Ω be a bounded open domain in R^ν with boundary Γ, assumed to be an $(\nu-1)$-dimensional variety with Ω locally on one side of Γ (¹). Let $A(\xi, \partial)$ be a uniformly strongly elliptic operator of order two in Ω of the form $A(\xi, \partial) = \sum_{|\alpha|\leqslant 2} a_\alpha(\xi)\partial^\alpha$, with smooth real coefficients a_α, where the symbol ∂ denotes differentiation. We begin by considering a diffusion open-loop system based on Ω with input applied on Γ through mixed (elastic) boundary conditions; that is

$$\frac{\partial x}{\partial t}(t, \xi) = -A(\xi, \partial)\, x(t, \xi) \qquad \text{in } (0, T]\times\Omega \tag{1.1}$$

$$x(0, \xi) = x_0(\xi)\,, \qquad \xi \in \Omega \tag{1.2}$$

$$\frac{\partial x(t, \zeta)}{\partial \eta} + b(\zeta)\, x(t, \zeta) = f(t, \zeta) \qquad \text{in } (0, T]\times\Gamma \ (\textit{Mixed B.C.})\,. \tag{1.3}$$

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(¹) Assumptions on Γ will be imposed as needed; see the statement of Theorem 1.2.