Acta Math. Hungar.
86 (4) (2000), 291–303.

EXISTENCE OF ALMOST PERIODIC SOLUTIONS FOR JUMPING DISCONTINUOUS SYSTEMS

M. FEČKAN (Bratislava)

Abstract. An existence result for almost periodic sequences of ordinary differential equations with linear boundary value conditions is derived by using the Banach fixed point theorem together with a method of majorant functions. An application is given to a damped pendulum with a jumping length and external force.

1. Introduction

Let us consider the motion of the damped mathematical pendulum with changing length [5] $l = l(t)$ and external force $e = e(t)$ given by

$$l(t)\ddot{\phi} + c\dot{\phi} + \sin\phi = e(t). \tag{1.1}$$

We suppose that $l(t)$, $e(t)$ are step functions. Very recently, the stability of linear ordinary differential equations with step function coefficients has been studied in [1], [2], [3], [4], [5].

In this paper we assume that $l(t)$, $e(t)$ have certain almost periodicity in the following sense: there are sequences

$$\{t_n\}_{n\in\mathbf{Z}} \subset \mathbf{R}, \quad \{l_k\}_{k\in\mathbf{Z}} \subset \mathbf{C}, \quad \{e_k\}_{k\in\mathbf{Z}} \subset \mathbf{C},$$
$$\{T_k\}_{k\in\mathbf{Z}} \subset \mathbf{C}, \quad \{\omega_k\}_{k\in\mathbf{Z}} \subset \mathbf{R}$$

such that

$$t_n = nT + \sum_{k\in\mathbf{Z}} T_k e^{\imath\omega_k n} \quad \forall\, n \in \mathbf{Z}, \quad T > 0, \quad \sum_{k\in\mathbf{Z}} |T_k| < T/2,$$

and for any $t_n < t < t_{n+1}$, we have

$$e(t) = \sum_{k\in\mathbf{Z}} e_k e^{\imath\omega_k n}, \quad l(t) = \sum_{k\in\mathbf{Z}} l_k e^{\imath\omega_k n},$$

where

$$\sum_{k\in\mathbf{Z}} |e_k| < \infty, \quad \sum_{k\in\mathbf{Z}} |l_k| < \infty.$$

Consequently, we suppose that $l(t)$, $e(t)$ are step functions with almost periodic jumpings. We are interested in finding conditions on $l(t)$, $e(t)$, c that (1.1) has a bounded solution on $\mathbf{R}$ with the same almost periodic properties as $l(t)$, $e(t)$. To handle this problem, in Section 2, we study a sequence of ordinary differential equations with linear boundary value conditions. We rewrite this sequence into an ordinary differential equation on a certain Banach space with a linear boundary value condition. We solve this boundary value problem by using the Banach fixed point theorem together with a method of majorant functions. We are motivated in applying this approach by the books [6], [7], where continuous almost periodic ordinary differential equations are widely studied. In Section 3, we apply results of Section 2 to (1.1). We consider for simplicity a concrete form of (1.1).

Finally, we note that our approach can be directly modified for investigation of the existence of bounded solutions to almost periodic difference equations. For instance, let us consider the difference equation

$$x_{n+2} + ax_{n+1} + x_n = bx_n^3 + d_1 \cos n\sqrt{2} + d_2 \sin 3n, \quad n \in \mathbf{Z}, \tag{1.2}$$

where $a \in \mathbf{R}$, $|a| > 2$ and $b, d_1, d_2 \in \mathbf{R}$. It can be shown that if

$$27|b|\big(|d_1| + |d_2|\big)^2 < 4\big(|a| - 2\big)^3,$$

then (1.2) has a solution of the form

$$x_n = \sum_{k,p\in\mathbf{Z}} z_{kp} e^{\imath(k\sqrt{2}+3p)n}, \quad \sum_{k,p\in\mathbf{Z}} |z_{kp}| \leqq \frac{3}{2} \frac{|d_1| + |d_2|}{|a| - 2}.$$

2. Almost periodic solutions

Let $\{\omega_k\}_{k\in\mathbf{Z}}$ be a sequence of real numbers such that

$$p\omega_{k_1} + q\omega_{k_2} \in \{\omega_k\}_{k\in\mathbf{Z}} \quad \forall p, q, k_1, k_2 \in \mathbf{Z},$$

$$\omega_k \neq \omega_l \quad \text{whenever} \quad k \neq l.$$

Let us consider a sequence of ordinary differential equations

$$\begin{cases} \dot{x}_n = A(n)x_n + f(n, x_n, t) + h(n, t), \\ x_{n+1}(0) = E(n)x_n(1), \quad t \in [0,1], \quad n \in \mathbf{Z}, \end{cases} \tag{2.1}$$

where

$$A(n)=\sum_{k\in\mathbf{Z}}C_k e^{\imath\omega_k n},\quad h(n,t)=\sum_{k\in\mathbf{Z}}D_k(t)e^{\imath\omega_k n},$$

$$f(n,x,t)=\sum_{j\geqq 1}\sum_{k\in\mathbf{Z}}B_{jk}(t)e^{\imath\omega_k n}x^j,\quad C_k\in\mathcal{L}(\mathbf{C}^m),\quad C=\sum_{k\in\mathbf{Z}}\|C_k\|<\infty,$$

$$D_k\in C\big([0,1],\mathbf{C}^m\big),\quad D=\sum_{k\in\mathbf{Z}}|D_k|<\infty,\quad |D_k|=\max_{[0,1]}\big|D_k(t)\big|,$$

$$E(n)=\sum_{k\in\mathbf{Z}}E_k e^{\imath\omega_k n},\quad E_k\in\mathcal{L}(\mathbf{C}^m),\quad E=\sum_{k\in\mathbf{Z}}\|E_k\|<\infty,$$

$$B_{jk}\in C\big([0,1],\mathcal{L}^j(\mathbf{C}^m)\big),\quad \sum_{j\geqq 1}\Big(\sum_{k\in\mathbf{Z}}|B_{jk}|\Big)R^j<\infty,\quad |B_{jk}|=\max_{[0,1]}\big|B_{jk}(t)\big|$$

for some constant $R>0$. We put

$$\Omega(r)=\sum_{j\geqq 1}\Big(\sum_{k\in\mathbf{Z}}|B_{jk}|\Big)r^j$$

for $0\leqq r<R$. We are interested in almost periodic solutions of (2.1) of the form

$$\text{(F)}\qquad x_n(t)=\sum_{k\in\mathbf{Z}}a_k(t)e^{\imath\omega_k n}$$

such that

$$\max_{[0,1]}\sum_{k\in\mathbf{Z}}\big|a_k(t)\big|<\infty,\quad \max_{[0,1]}\sum_{k\in\mathbf{Z}}\big|\dot a_k(t)\big|<\infty.$$

Let

$$l_1=\Big\{\{a_k\}_{k\in\mathbf{Z}}\,\Big|\,a_k\in\mathbf{C}^m,\ |a|=\sum_{k\in\mathbf{Z}}|a_k|<\infty\Big\}.$$

We intend to rewrite (2.1) as an ordinary differential equation on l_1 with a linear boundary value condition. We have

$$A(n)\sum_{k\in\mathbf{Z}}a_k e^{\imath\omega_k n}=\Big(\sum_{j\in\mathbf{Z}}C_j e^{\imath\omega_j n}\Big)\sum_{k\in\mathbf{Z}}a_k e^{\imath\omega_k n}=\sum_{l\in\mathbf{Z}}\sum_{\omega_j+\omega_k=\omega_l}C_j a_k e^{\imath\omega_l n}.$$

Moreover, we get

$$\sum_{l\in\mathbf{Z}}\Big|\sum_{\omega_j+\omega_k=\omega_l} C_j a_k\Big| \leqq \sum_{l\in\mathbf{Z}}\sum_{\omega_j+\omega_k=\omega_l} \|C_j\|\cdot|a_k| \leqq C|a|.$$

Consequently, $\{A(n)\}_{n\in\mathbf{Z}}$ generates a bounded linear mapping $L : l_1 \to l_1$ given by

$$L : \{a_k\}_{k\in\mathbf{Z}} \to \Big\{\sum_{\omega_j+\omega_k=\omega_l} C_j a_k\Big\}_{l\in\mathbf{Z}}.$$

We note that $\|L\| \leqq C$. Similarly, we can check that the sequences

$$\{f(n,x,t)\}_{n\in\mathbf{Z}} \quad \text{and} \quad \{h(n,t)\}_{n\in\mathbf{Z}}$$

generate mappings

$$F : B_R\times[0,1]\to l_1, \quad H : [0,1]\to l_1,$$

respectively, where $B_R = \{a\in l_1 \mid |a| < R\}$. Moreover, F is analytic in $a\in B_R$ and F, H are continuous. Indeed, we compute

$$\sum_{j\geqq 1}\sum_{k\in\mathbf{Z}} B_{jk}(t)\mathrm{e}^{\imath\omega_k n}\Big(\sum_{p\in\mathbf{Z}} a_p \mathrm{e}^{\imath\omega_p n}\Big)^j$$

$$= \sum_{l\in\mathbf{Z}}\sum_{\omega_{p_1}+\omega_{p_2}+\cdots+\omega_{p_j}+\omega_k=\omega_l} B_{jk}(t)a_{p_1}a_{p_2}\cdots a_{p_j}\mathrm{e}^{\imath\omega_l n}.$$

Moreover, for $a\in l_1$, $|a| < R$, we get

$$\sum_{l\in\mathbf{Z}}\Big|\sum_{\omega_{p_1}+\omega_{p_2}+\cdots+\omega_{p_j}+\omega_k=\omega_l} B_{jk}(t)a_{p_1}a_{p_2}\cdots a_{p_j}\Big| \leqq \Omega(|a|).$$

Consequently, we arrive at the formulas

$$F(a,t)_l = \sum_{\omega_{p_1}+\omega_{p_2}+\cdots+\omega_{p_j}+\omega_k=\omega_l} B_{jk}(t)a_{p_1}a_{p_2}\cdots a_{p_j}, \quad \forall\, l\in\mathbf{Z},$$

$$H(t)_l = D_l(t), \quad \forall\, l\in\mathbf{Z}.$$

The above computations also give

$$|F(a,t)| \leqq \Omega(|a|), \quad |H(t)| \leqq D.$$

Furthermore, for $v = \{v_k\}_{k\in\mathbf{Z}} \in l_1$, we have

$$\left(D_a F(a,t)v\right)_l = \sum_{\omega_{p_1}+\omega_{p_2}+\dots+\omega_{p_j}+\omega_k=\omega_l} B_{jk}(t)\big(v_{p_1}a_{p_2}\cdots a_{p_j}$$

$$+a_{p_1}v_{p_2}a_{p_3}\cdots a_{p_j}+\dots+a_{p_1}\cdots a_{p_{j-1}}v_{p_j}\big).$$

Hence

$$\left|D_a F(a,t)v\right| \leqq \sum_{l\in\mathbf{Z}} \sum_{\omega_{p_1}+\omega_{p_2}+\dots+\omega_{p_j}+\omega_k=\omega_l} |B_{jk}|\big(|v_{p_1}|\cdot|a_{p_2}|\cdots|a_{p_j}|$$

$$+|a_{p_1}|\cdot|v_{p_2}|\cdot|a_{p_3}|\cdots|a_{p_j}|+\dots+|a_{p_1}|\cdots|a_{p_{j-1}}|\cdot|v_{p_j}|\big) \leqq \Omega'\big(|a|\big)\,|v|.$$

The boundary value condition of (2.1) gives

$$\sum_{k\in\mathbf{Z}} E_k e^{\imath\omega_k n}\left(\sum_{p\in\mathbf{Z}} a_p(1)e^{\imath\omega_p n}\right) = \sum_{l\in\mathbf{Z}} a_l(0)e^{\imath\omega_l}e^{\imath\omega_l n},$$

$$e^{-\imath\omega_l}\sum_{\omega_k+\omega_p=\omega_l} E_k a_p(1) = a_l(0) \quad \forall\, l\in\mathbf{Z}.$$

We introduce a linear continuous mapping $M : l_1 \to l_1$ given by

$$(Ma)_k = e^{-\imath\omega_k}\sum_{\omega_p+\omega_q=\omega_k} E_p a_q.$$

We have $\|M\| \leqq E$. Clearly the boundary value condition of (2.1) is expressed by $a(0) = Ma(1)$. Summarizing, we arrive at the following result.

THEOREM 2.1. *Problem* (2.1) *generates a boundary value problem on* l_1 *given by*

$$\dot a = La + F(a,t) + H(t), \quad a(0) = Ma(1), \tag{2.2}$$

where L, F, H, M *are defined above.*

Let us solve (2.2). The variation of constants formula for (2.2) gives

$$a(t) = e^{Lt}a(0) + \int_0^t e^{L(t-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds.$$

The boundary value condition of (2.2) implies

$$Me^L a(0) + M \int_0^1 e^{L(1-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds = a(0).$$

By assuming

$$\text{(2.3)} \qquad I - Me^L : l_1 \to l_1 \quad \text{is continuously invertible,}$$

we get

$$a(0) = KM \int_0^1 e^{L(1-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds, \quad K = \big(I - e^L\big)^{-1}.$$

Hence (2.2) is rewritten in the form

$$\text{(2.4)} \qquad a(t) = e^{Lt}KM \int_0^1 e^{L(1-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds + \int_0^t e^{L(t-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds.$$

We can prove the main result of this section.

THEOREM 2.2. *Assume* (2.3). *Let us put*

$$\Phi(r) = \big(e^C\|K\|E + 1\big)\frac{e^C - 1}{C}\big(\Omega(r) + D\big)$$

for r, $0 \leqq r < R$. *If there is an* r_0, $0 < r_0 < R$, *such that* $\Phi(r_0) \leqq r_0$ *and* $\Phi'(r_0) < 1$, *then* (2.1) *has an almost periodic solution which can be obtained by an iterative method.*

PROOF. We solve (2.4) on $X = C\big([0,1], l_1\big)$ by using the Banach fixed point theorem to a mapping $G : X \to X$ given by

$$G(x)(t) = e^{Lt}KM \int_0^1 e^{L(1-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds + \int_0^t e^{L(t-s)}\big(F\big(a(s),s\big) + H(s)\big)\,ds$$

for $x \in X$, $|x| < R$. The above computations imply

$$\left|G(x)\right| \leqq \left(\mathrm{e}^C\|K\|E+1\right)\frac{\mathrm{e}^C-1}{C}\left(\Omega\left(|x|\right)+D\right),$$

$$\left\|DG(x)\right\| \leqq \left(\mathrm{e}^C\|K\|E+1\right)\frac{\mathrm{e}^C-1}{C}\Omega'\left(|x|\right).$$

Hence, we arrive at

$$\left|G(x)\right| \leqq \Phi\left(|x|\right), \quad \left\|DG(x)\right\| \leqq \Phi'\left(|x|\right).$$

On the ball $B_{r_0} = \left\{x \in X \mid |x| \leqq r_0\right\}$, we have

$$\left|G(x)\right| \leqq \Phi(r_0) \leqq r_0, \quad \left\|DG(x)\right\| \leqq \Phi'(r_0) < 1.$$

The Banach fixed point theorem gives a unique solution of $x = G(x)$ on B_{r_0}. □

The main difficulty in applying Theorem 2.1 is the verification of (2.3), which is equivalent to $1 \notin \sigma(M\mathrm{e}^L)$. Here σ means the spectrum.

LEMMA 2.3. *Assume*

$$A(n) = A, \quad E(n) = S \quad \forall n \in \mathbf{Z}.$$

If $\mathrm{e}^{\imath\omega_k} \notin \sigma(S\mathrm{e}^A)$ $\forall k \in \mathbf{Z}$ *and*

$$m_0 = \sup_{k\in\mathbf{Z}}\left\|\left(S\mathrm{e}^A - \mathrm{e}^{\imath\omega_k}I\right)^{-1}\right\| < \infty, \tag{2.6}$$

then (2.3) *is satisfied along with* $\|K\| \leqq m_0$.

PROOF. We have

$$\left((I - M\mathrm{e}^L)a\right)_k = \left(I - \mathrm{e}^{-\imath\omega_k}S\mathrm{e}^A\right)a_k.$$

If $\mathrm{e}^{\imath\omega_k} \notin \sigma(S\mathrm{e}^A)$ then $\left(I - \mathrm{e}^{-\imath\omega_k}S\mathrm{e}^A\right)^{-1}$ exists and we can put

$$(Ka)_k = \left(I - \mathrm{e}^{-\imath\omega_k}S\mathrm{e}^A\right)^{-1}a_k \quad \forall k \in \mathbf{Z}.$$

Hence

$$|Ku| \leqq \sum_{k\in\mathbf{Z}}\left\|\left(\mathrm{e}^{\imath\omega_k}I - S\mathrm{e}^A\right)^{-1}\right\| \cdot |a_k| \leqq m_0|a|. \qquad \square$$

LEMMA 2.4. *Assume*

$$A(n) = A + A_1(n), \quad E(n) = S + E_1(n) \quad \forall n \in \mathbf{Z},$$

where

$$A_1(n) = \sum_{k\in\mathbf{Z}} C_{k1} e^{\imath\omega_k n}, \quad C_{k1} \in \mathcal{L}(\mathbf{C}^m), \quad C_1 = \sum_{k\in\mathbf{Z}} \|C_{k1}\| < \infty,$$

$$E_1(n) = \sum_{k\in\mathbf{Z}} E_{k1} e^{\imath\omega_k n}, \quad E_{k1} \in \mathcal{L}(\mathbf{C}^m), \quad E_1 = \sum_{k\in\mathbf{Z}} \|E_{k1}\| < \infty.$$

If (2.6) *holds along with*

$$\big(\|S\|C_1 + E_1\big) e^{\|A\|+C_1} m_0 < 1, \tag{2.7}$$

then (2.3) *holds with*

$$\|K\| \leqq m_0 \big/ \Big(1 - \big(\|S\|C_1 + E_1\big) e^{\|A\|+C_1} m_0\Big). \tag{2.8}$$

PROOF. Now we have $M = M_0 + M_1$, $L = L_0 + L_1$, where like above M_0, M_1, L_0, L_1 correspond to S, $E_1(\cdot)$, A, $A_1(\cdot)$, respectively. Moreover, we get

$$\|M_0\| \leqq \|S\|, \quad \|M_1\| \leqq E_1, \quad \|L_0\| \leqq \|A\|, \quad \|L_1\| \leqq C_1.$$

We compute

$$I - Me^L = I - M_0 e^{L_0} + M_0(e^{L_0} - e^{L_0+L_1}) - M_1 e^{L_0+L_1}.$$

Since

$$\big\| M_0(e^{L_0} - e^{L_0+L_1})\big\| \leqq \|S\| e^{\|A\|+C_1} C_1, \quad \|M_1 e^{L_0+L_1}\| \leqq E_1 e^{\|A\|+C_1},$$

the Neumann theorem implies the assertion of Lemma 2.4. $\square$

3. A damped pendulum with a jumping length and external force

Let us consider a pendulum given by

$$l_n\ddot{\phi} + c\dot{\phi} + \sin\phi = \xi + \gamma\sin\omega_3 n, \tag{3.1}$$

$$t_n = n + \alpha\sin\omega_1 n < t < n+1+\alpha\sin\omega_1(n+1) = t_{n+1}, \quad l_n = l + \beta\sin\omega_2 n,$$

$$\lim_{t\to t_n-}\phi(t) = \lim_{t\to t_n+}\phi(t), \quad l_{n-1}\lim_{t\to t_n-}\dot{\phi}(t) = l_n\lim_{t\to t_n+}\dot{\phi}(t), \quad \forall n\in\mathbf{Z},$$

where $c>0$, $l>0$, $\omega_1,\omega_2,\omega_3\in\mathbf{R}\setminus\{\mathbf{Z}\pi\}$, $\alpha,\beta,\gamma,\xi\in\mathbf{R}$, and

$$\left|\alpha\sin\frac{\omega_1}{2}\right| < 1/2, \quad |\beta| < l.$$

Here ϕ is the angle between the axis directed vertically downward and the thread. Equation (3.1) describes the motion of the pendulum with a sudden almost periodic change of the length together with an external force at any $t=t_n$, $n\in\mathbf{Z}$. For any t, $t_n \leqq t \leqq t_{n+1}$, we put

$$t = st_{n+1} + (1-s)t_n, \quad s\in[0,1], \quad \phi_n(s) = \phi(t).$$

Then (3.1) becomes

$$\dot{\phi}_n = -\delta\psi_n - \beta_0\phi_n, \tag{3.2}$$

$$\dot{\psi}_n = \delta\phi_n - \alpha_0\psi_n + c_n\psi_n + \frac{d_n}{\delta}\sin\phi_n - \frac{1}{l\delta}(\phi_n - \sin\phi_n) + \frac{c_n\beta_0}{\delta}\phi_n - \frac{h_n}{\delta},$$

$$\phi_{n+1}(0) = \phi_n(1), \quad \psi_{n+1}(0) = \psi_n(1) + g_n\psi_n(1) + \frac{\beta_0}{\delta}g_n\phi_n(1),$$

where

$$\delta^2 = \max\left\{\frac{1}{l+1}, \frac{1}{l} - \frac{c^2}{4l^2}\right\},$$

$$\alpha_0 = \frac{c-\sqrt{c^2+4\delta^2l^2-4l}}{2l}, \quad \beta_0 = \frac{c+\sqrt{c^2+4\delta^2l^2-4l}}{2l},$$

$$c_n = c\frac{\beta\sin\omega_2 n - 2\alpha l\cos\omega_1\frac{2n+1}{2}\sin\frac{\omega_1}{2}}{l(l+\beta\sin\omega_2 n)},$$

$$d_n = \frac{4\alpha l\cos\omega_1\frac{2n+1}{2}\sin\frac{\omega_1}{2} + 4\alpha^2 l\cos^2\omega_1\frac{2n+1}{2}\sin^2\frac{\omega_1}{2} - \beta\sin\omega_2 n}{l(l+\beta\sin\omega_2 n)},$$

$$h_n = \frac{\left(1 + 2\alpha\cos\omega_1\frac{2n+1}{2}\sin\frac{\omega_1}{2}\right)^2(\xi + \gamma\sin\omega_3 n)}{l + \beta\sin\omega_2 n},$$

$$g_n = \frac{\left(1 + 2\alpha\sin\frac{\omega_1}{2}\cos\omega_1\frac{2n+3}{2}\right)(l + \beta\sin\omega_2 n)}{\left(1 + 2\alpha\sin\frac{\omega_1}{2}\cos\omega_1\frac{2n+1}{2}\right)\left(l + \beta\sin\omega_2(n+1)\right)} - 1.$$

Assume that ω_1, ω_2, ω_3 are incommensurable, i.e.

$$p\omega_1 + q\omega_2 + r\omega_3 \neq 0 \quad \forall (p,q,r) \in \mathbf{Z}^3 \setminus \{(0,0,0)\}. \tag{3.3}$$

Take $\{\omega_k\}_{k\in\mathbf{Z}} = \{p\omega_1 + q\omega_2 + r\omega_3\}_{(p,q,r)\in\mathbf{Z}^3}$. (3.2) has the form of (2.1) with

$$A(n) = A = \begin{pmatrix} -\beta_0 & -\delta \\ \delta & -\alpha_0 \end{pmatrix}, \quad h(n,t) = (0, -h_n/\delta),$$

$$f(n,\phi,\psi,t) = \left(0, c_n\psi + \frac{c_n\beta_0}{\delta}\phi + \frac{d_n}{\delta}\sin\phi - \frac{1}{l\delta}(\phi - \sin\phi)\right),$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E(n) = S + E_1(n), \quad E_1(n) = \begin{pmatrix} 0 & 0 \\ \beta_0 g_n/\delta & g_n \end{pmatrix}.$$

Consider the norm $|(\phi,\psi)| = \sqrt{|\phi|^2 + |\psi|^2}$ on $\mathbf{C}^2$. We have

$$\sin\omega n = \frac{1}{2\imath}e^{\imath\omega n} - \frac{1}{2\imath}e^{-\imath\omega n},$$

$$\cos\omega\frac{2n+j}{2} = \frac{1}{2}e^{\imath\omega j/2}e^{\imath\omega n} + \frac{1}{2}e^{-\imath\omega j/2}e^{-\imath\omega n}, \quad j = 1,3.$$

Hence

$$|c_n| \leqq c\frac{|\beta| + 2l\left|\alpha\sin\frac{\omega_1}{2}\right|}{l(l - |\beta|)} = \Gamma_1,$$

$$|d_n| \leqq \frac{4l\left|\alpha\sin\frac{\omega_1}{2}\right| + 4\alpha^2 l\sin^2\frac{\omega_1}{2} + |\beta|}{l(l - |\beta|)} = \Gamma_2,$$

$$|h_n| \leqq (|\xi| + |\gamma|)\frac{\left(1 + 2\left|\alpha\sin\frac{\omega_1}{2}\right|\right)^2}{l - |\beta|} = \Gamma_3,$$

$$|g_n| \leqq \frac{4l\left|\alpha\sin\frac{\omega_1}{2}\right| + 4\left|\alpha\beta\sin\frac{\omega_1}{2}\right| + 2|\beta|}{\left(1 - 2\left|\alpha\sin\frac{\omega_1}{2}\right|\right)(l - |\beta|)} = \Gamma_4.$$

As a majorant function for $\Omega(r)$ from Section 2, in this case we take

$$\theta(r)=\left(1+\frac{\beta_0}{\delta}\right)\Gamma_1 r+\frac{1}{\delta}\Gamma_2\sinh r+\frac{1}{l\delta}(\sinh r-r).$$

In context of Lemma 2.4, it is not hard to see that

$$\|S\|=1,\quad E_1\leqq\left(\frac{\beta_0}{\delta}+1\right)\Gamma_4=\Gamma_5,$$

$$C_1=0,\quad \|A\|\leqq\sqrt{\alpha_0^2+\beta_0^2+2\delta^2}=\Gamma_6.$$

Furthermore, since $\|e^A\|\leqq e^{-2\alpha_0}$, we get for (2.6)

$$m_0\leqq 1/(1-e^{-2\alpha_0}).$$

Condition (2.7) is satisfied when

$$\Gamma_5 e^{\Gamma_6}<1-e^{-2\alpha_0}. \tag{3.4}$$

According to (2.8) of Lemma 2.4, we get

$$\|K\|\leqq 1/(1-e^{-2\alpha_0}-\Gamma_5 e^{\Gamma_6}).$$

Summarizing, we obtain that we can take

$$\begin{aligned}\Phi(r)&=\left(e^{\Gamma_6}\frac{1}{1-e^{-2\alpha_0}-\Gamma_5e^{\Gamma_6}}(1+\Gamma_5)+1\right)\frac{e^{\Gamma_6}-1}{\Gamma_6}\left(\theta(r)+\frac{\Gamma_3}{\delta}\right)\\&=\tau_1 r+\tau_2\sinh r+\tau_3(\sinh r-r)+\tau_4\end{aligned}$$

as the function Φ in Theorem 2.2 where

$$\tau_1=\tau\left(1+\frac{\beta_0}{\delta}\right)\Gamma_1,\quad \tau_2=\tau\Gamma_2/\delta,\quad \tau_3=\tau/(l\delta),\quad \tau_4=\tau\Gamma_3/\delta,$$

$$\tau=\left(e^{\Gamma_6}\frac{1}{1-e^{-2\alpha_0}-\Gamma_5e^{\Gamma_6}}(1+\Gamma_5)+1\right)\frac{e^{\Gamma_6}-1}{\Gamma_6}.$$

If

$$\tau_1+\tau_2<1, \tag{3.5}$$

then there is a unique $r_0 > 0$ such that

$$\tau_1 + \tau_2 \cosh r_0 + \tau_3(\cosh r_0 - 1) = 1, \tag{3.6}$$

and for any τ_4 satisfying

$$\tau_4 < r_0 - \tau_1 r_0 - \tau_2 \sinh r_0 - \tau_3(\sinh r_0 - r_0), \tag{3.7}$$

there is a unique r_1, $0 < r_1 < r_0$, such that $\Phi(r_1) = r_1$ and $\Phi'(r_1) < 1$. We note that such r_1 is determined by the equation

$$\tau_1 r_1 + \tau_2 \sinh r_1 + \tau_3(\sinh r_1 - r_1) + \tau_4 = r_1. \tag{3.8}$$

Consequently, for such r_1, the conditions of Theorem 2.2 are satisfied. Summarizing, we arrive at the following result.

THEOREM 3.1. *Let* (3.3), (3.4) *and* (3.5) *be satisfied. Let* $r_0 > 0$ *be given by* (3.6). *If* (3.7) *holds, then* (3.1) *possesses an almost periodic solution* x *of the form* (F) *such that* $\sup_{\mathbf{R}} |x(\cdot)| \leqq r_1$, *where* r_1, $0 < r_1 < r_0$, *is given by* (3.8).

We note that $\tau_1 = \tau_2 = 0$, $\Gamma_5 = 0$ when $\alpha = \beta = 0$, and then (3.4), (3.5) are trivially satisfied.

References

[1] Á. Elbert, Stability of some difference equations, in *Advances in Difference Equations* (Proc. Second Int. Conf. Difference Eqns., Veszprém, Hungary, August 7–11, 1995), Gordon & Breach Science Publ. (London, 1997), pp. 165–187.

[2] Á. Elbert, On asymptotic stability of some Sturm–Liouville differential equations, *General Seminar of Mathematics, Univ. Patras*, **22–23** (1997/98) (to appear).

[3] J. R. Graef and J. Karsai, On irregular growth and impulses in oscillator equations, in *Advances in Difference Equations* (Proc. Second Int. Conf. Difference Eqns., Veszprém, Hungary, August 7–11, 1995), Gordon & Breach Science Publ. (London, 1997), pp. 253–262.

[4] L. Hatvani, On the existence of a small solution to linear second order differential equations with step function coefficients, *Diff. Int. Eqns.* (to appear).

[5] L. Hatvani and L. Stachó, On small solutions of second order differential equations with random coefficients, *Arch. Math. EQUADIFF 9* (Brno, 1997), **34** (1998), 119–126.

[6] J. J. Stoker, *Nonlinear Vibrations in Mechanical and Electrical Systems* (New York, 1950).

[7] T. G. Stryzhak, *Asymptotic Method of Normalization* (Kiev, 1984) (in Russian).

(Received May 18, 1998)

DEPARTMENT OF MATHEMATICAL ANALYSIS
COMENIUS UNIVERSITY
MLYNSKÁ DOLINA
842 15 BRATISLAVA
SLOVAKIA
E-MAIL: MICHAL.FECKAN@FMPH.UNIBA.SK