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A semigroup approach to the geometry of domains in complex Banach spaces

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Abstract

We study some geometric properties of holomorphic mappings on a Banach space by using the theory of linear and nonlinear semigroups in the spirit of Hille-Yosida. In addition, we obtain some new results on the asymptotic behavior of one-parameter semigroups of holomorphic self-mappings. This enables us to study those mappings on the Hilbert ball which are star-like with respect to a boundary point.

Key words: Asymptotic behavior; Generator; Semigroup; Star-like mapping

1 Vector fields and semigroups of holomorphic mappings

Let $\mathcal{D}$ be a domain (that is, a nonempty open connected subset) in a complex Banach space $(X, |\cdot|)$. We denote by $\mathrm{Hol}(\mathcal{D}, X)$ the vector space of all holomorphic mappings from $\mathcal{D}$ into X and by $\widetilde{\mathrm{Hol}}(\mathcal{D}, X)$ the subspace of $\mathrm{Hol}(\mathcal{D}, X)$ consisting of all those mappings in $\mathrm{Hol}(\mathcal{D}, X)$ which are bounded on each ball strictly inside $\mathcal{D}$.

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The space $\widetilde{\mathrm{Hol}}(\mathcal{D}, X)$ is a locally convex space under the family of seminorms

$$p_B(g) := \sup\{|g(x)| : x \in B\},$$

where B is a ball strictly inside $\mathcal{D}$.

Definition 1.1 *To each $g \in \mathrm{Hol}(\mathcal{D}, X)$ there corresponds a holomorphic vector field V_g which is the linear operator on $\mathrm{Hol}(\mathcal{D}, X)$ defined by*

$$V_g(f)(x) := f'(x)g(x), \tag{1.1}$$

where $f \in \mathrm{Hol}(\mathcal{D}, X)$ and $x \in \mathcal{D}$.

Each vector field V_g is locally integrable in the following sense: For each $x \in \mathcal{D}$ there exist a neighborhood U of x and a number $\delta > 0$ such that the Cauchy problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + g(u(t,x)) = 0 \\ u(0,x) = x \end{cases} \tag{1.2}$$

has a unique solution $\{u(t,x)\} \subset \mathcal{D}$ defined on the set $\{|t| < \delta\} \times U \subset \mathbb{R} \times \mathcal{D}$.

Definition 1.2 *A holomorphic vector field V_g is said to be right semi-complete (respectively, complete) on $\mathcal{D}$ if the solution of the Cauchy problem (1.2) is well defined on all of $\mathbb{R}^+ \times \mathcal{D}$ (respectively, $\mathbb{R} \times \mathcal{D}$), where $\mathbb{R}^+ = [0, \infty)$ (respectively, $\mathbb{R} = (-\infty, \infty)$).*

Definition 1.3 *A family $\{\mathcal{S}(t) : t \in \mathbb{R}^+\} \subset \mathrm{Hol}(\mathcal{D}, \mathcal{D})$ of holomorphic self-mappings of $\mathcal{D}$ is called a (one-parameter) semigroup if*

$$\mathcal{S}(s+t) = \mathcal{S}(s) \circ \mathcal{S}(t), \;\; s, t \in \mathbb{R}^+, \tag{1.3}$$

and

$$\mathcal{S}(0) = I,$$

where I denotes the identity operator.

A semigroup $\{\mathcal{S}(t)\}$ is said to be (strongly) continuous if

$$\lim_{t \to 0^+} \mathcal{S}(t)(x) = x \tag{1.4}$$

for each $x \in \mathcal{D}$.

Definition 1.4 *Let $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$, be a strongly continuous semigroup of holomorphic mappings of $\mathcal{D}$. If the strong limit*

$$g(x) = \lim_{t\to 0^+} \frac{1}{t}\left(x - \mathcal{S}(t)(x)\right) \tag{1.5}$$

exists for each $x \in \mathcal{D}$, then g is called the (infinitesimal) generator of the semigroup $\{\mathcal{S}(t)\}$. In this case we will say that $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$, is a differentiable semigroup. The set of all generators on $\mathcal{D}$ will be denoted by $\mathcal{G}_+(\mathcal{D})$.

In fact, it can be shown (see, for example, [10]) that if $\mathcal{D}$ is bounded, then a continuous semigroup is differentiable if and only if the convergence in (1.4) is uniform on each ball strictly inside $\mathcal{D}$, i.e.,

$$\text{T-}\lim_{t\to 0^+} \mathcal{S}(t) = I. \tag{1.6}$$

Here $\text{T-}\lim_{t\to 0^+} \mathcal{S}(t)$ refers to the limit in the T-topology of locally uniform convergence [6,8].

Moreover, in this case the mapping u defined by

$$u(t,x) = \mathcal{S}(t)(x), \quad (t,x) \in \mathbb{R}^+ \times \mathcal{D}, \tag{1.7}$$

is the solution of the Cauchy problem (1.2) (see, for example, [10]).

If a semigroup $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$, has a continuous extension to all of $\mathbb{R}$, then $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}$, is actually a (one-parameter) group of automorphisms of $\mathcal{D}$. The converse is also true: If an element $\mathcal{S}(t_0)$, $t_0 > 0$, of a semigroup $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$, is an automorphism of $\mathcal{D}$, then so is each $\mathcal{S}(t)$ and the semigroup can be continuously extended to a (one-parameter) group. Thus, a holomorphic vector field V_g, defined by (1.1), is semi-complete (respectively, complete) on $\mathcal{D}$ if and only if g is a generator of a one-parameter semigroup (respectively, group) of holomorphic self-mappings on $\mathcal{D}$.

If $\mathcal{D}$ is bounded, then a semigroup (group) $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$ (respectively, $t \in \mathbb{R}$), induces a linear semigroup (group) $\{\mathcal{L}(t)\}$ of linear mappings $\mathcal{L}(t) : \widetilde{\text{Hol}}(\mathcal{D}, X) \mapsto \widetilde{\text{Hol}}(\mathcal{D}, X)$ defined by

$$\left(\mathcal{L}(t)f\right)(x) := f(\mathcal{S}(t)(x)), \tag{1.8}$$

where $t \in \mathbb{R}^+$ ($t \in \mathbb{R}$) and $x \in \mathcal{D}$.

This semigroup is called the semigroup of composition operators on $\widetilde{\text{Hol}}(\mathcal{D}, X)$. If $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$ ($t \in \mathbb{R}$), is T-continuous, (that is, differentiable), then $\{\mathcal{L}(t)\}$, $t \in \mathbb{R}^+$ $(t \in \mathbb{R})$, is also differentiable and

$$\begin{cases} \frac{\partial \mathcal{L}(t)f}{\partial t} + V_g(\mathcal{L}(t)f) = 0 \\ \mathcal{L}(0)f = f \end{cases} \tag{1.9}$$

for all $f \in \widetilde{\mathrm{Hol}}(\mathcal{D}, X)$, where $g = -\left.\frac{d\mathcal{S}(t)}{dt}\right|_{t=0^+}$.

In other words, the holomorphic vector field V_g, defined by (1.1) and considered a linear operator on $\widetilde{\mathrm{Hol}}(\mathcal{D}, X)$, is the infinitesimal generator of the semigroup $\{\mathcal{L}(t)\}$. It is sometimes called a Lie generator [2]. Thus a holomorphic vector-field V_g is semi-complete (respectively, complete) if and only if it is the generator of a linear semigroup (respectively, group) of composition operators on $\widetilde{\mathrm{Hol}}(\mathcal{D}, X)$.

Moreover, using the exponential formula representation for the linear semigroup,

$$\mathcal{L}(t)f = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} V_g^k f = \exp\left[-tV_g\right] f, \tag{1.10}$$

(see, for example, [9,11]), we also have

$$\mathcal{S}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} V_g^k I = \exp\left[-tV_g\right] I. \tag{1.11}$$

In other words, a T-continuous semigroup of holomorphic self-mappings on a bounded domain can be represented in exponential form by the holomorphic vector field corresponding to its generator.

Other exponential and product formulas for semigroups of holomorphic mappings can be found, for example, in [9,10].

2 A general dynamic approach to star-like mappings in Banach spaces

Definition 2.1 *A subset M of a Banach space X is said to be star-shaped if for each $w \in M$ and $t \geq 0$ the point $e^{-t}w$ also belongs to M.*

Definition 2.2 *If $\mathcal{D}$ is a domain in X, then a biholomorphic mapping $f \in$ $\mathrm{Hol}(\mathcal{D}, X)$ is said to be a star-shaped mapping on $\mathcal{D}$ if the closure $Cl(\Omega)$ of its image $\Omega = f(\mathcal{D})$ is a star-shaped set.*

In addition, if $0 \in \Omega$, then we will say that f is star-like with respect to an interior point; if $0 \in \partial\Omega$, the boundary of Ω, then f is said to be star-like with respect to a boundary point.

Our first result is a consequence of (1.10).

Theorem 2.1 *Let $\mathcal{D}$ be a domain in a complex Banach space X, and let g belong to $\mathrm{Hol}(\mathcal{D}, X)$. If the vector field V_g corresponding to g is semi-complete, then for each element f of $\mathrm{Ker}(I - V_g) \subset \mathrm{Hol}(\mathcal{D}, X)$, the set $f(\mathcal{D})$ is star-shaped.*

If $f \in \mathrm{Hol}(\mathcal{D}, X)$ is biholomorphic on $\mathcal{D}$, then the converse assertion is also true.

Theorem 2.2 *Let f be a biholomorphic mapping on a domain $\mathcal{D}$ in X such that $f(\mathcal{D})$ is star-shaped. Then there is a semi-complete holomorphic vector field V_g such that $f \in \mathrm{Ker}(I - V_g)$.*

Corollary 2.1 *A biholomorphic mapping f on a domain $\mathcal{D} \subset X$ is star-like if and only if it satisfies the differential equation*

$$f(x) = f'(x)g(x), \tag{2.1}$$

where g is the generator of a one-parameter semigroup of holomorphic self-mappings of $\mathcal{D}$.

Corollary 2.2 *Let $\mathcal{S}(t)$ be a one-parameter T-continuous semigroup of holomorphic self-mappings of a domain $\mathcal{D}$. If for some $F \in \mathrm{Hol}(\mathcal{D}, X)$ there exists the strong limit*

$$f = \lim_{t\to\infty} \exp(tI)F(\mathcal{S}(t)), \tag{2.2}$$

then $f(\mathcal{D})$ is star-shaped.

Remark. It can be shown (see, for example, [4]) that if $g \in \mathcal{G}_+(\mathcal{D})$ is bounded and satisfies the conditions

$$g(a) = 0, \qquad a \in \mathcal{D}, \tag{2.3}$$

and

$$g'(a) = I, \tag{2.4}$$

and F is a translation by a, then the limit f in (2.2) exists and $f(\mathcal{D})$ is a star-shaped domain (with respect to $0 = f(a)$). In other words, under conditions

(2.3) and (2.4), the equation

$$f(x) = f'(x)g(x)$$

can be solved and for each $x \in \mathcal{D}$,

$$f(x) = \lim_{t\to\infty} e^t\Big(\mathcal{S}(t)(x) - a\Big),$$

where $\mathcal{S}(t)$ is the semigroup generated by g.

These connections between semigroups of holomorphic mappings and the geometry of domains in Banach spaces show the importance of the study of the asymptotic behavior of such semigroups. For example, an exponential rate of convergence of $\mathcal{S}$ yields a distortion theorem for f.

3 Asymptotic behavior

In this section we announce new results regarding the asymptotic behavior and rates of convergence of semigroups (flows) of holomorphic mappings on the open unit Hilbert ball $\mathbb{B}$. We begin with the following definition.

Definition 3.1 *Let $\mathcal{S} = \{F(t)\}_{t\geq 0}$ be a flow on $\mathbb{B}$ which is generated by f. We will say that a point $\tau \in \overline{\mathbb{B}}$, the closure of $\mathbb{B}$, is a globally attractive point for $\mathcal{S}$ if for each $x \in \mathbb{B}$, the strong limit*

$$\lim_{t\to\infty} F(t)x = \tau,$$

uniformly on each ball strictly inside $\mathbb{B}$.

If $\tau \in \mathbb{B}$, then τ is the unique asymptotically stable stationary point of $\mathcal{S}$. If $\tau \in \partial\mathbb{B}$, the boundary of $\mathbb{B}$, then we will call it the attractive sink point of $\mathcal{S}$.

For the case of holomorphic generators, the attractivity of a stationary point can be completely described in terms of their derivatives.

If $f \in \mathcal{G}_+(\mathbb{B})$ and $\tau \in \mathrm{Null}(f)$, the null point set of f, then τ is (globally) attractive if and only if the spectrum of the linear operator $f'(\tau)$, the Fréchet derivative of f at τ, lies strictly in the right-half plane (see, for example, [9]). But, as far as we know, for generators with no null points, the situation has been described only for the one-dimensional case, that is, when $\mathbb{B} = \Delta$, the open unit disk in the complex plane $\mathbb{C}$ [5].

Even in this case the usual approach treats the interior and boundary cases separately. The new approach which we will sketch below provides a unified description in a general Hilbert space both for the case of an interior stationary point and for that of a boundary sink point.

Let $\mathbb{B}$ be the open unit ball in a complex Hilbert space $\mathcal{H}$. For a fixed $\tau \in \overline{\mathbb{B}}$, the closure of $\mathbb{B}$, and an arbitrary $x \in \mathbb{B}$, we define a non-Euclidean "distance" between x to τ by the formula

$$d_\tau(x) = \frac{|1 - \langle x, \tau\rangle|^2}{1 - \|x\|^2}\left(1 - \sigma(x,\tau)\right), \tag{3.1}$$

where

$$\sigma(x,\tau) = \frac{(1-\|x\|^2)(1-\|\tau\|^2)}{|1-\langle x,\tau\rangle|^2}, \quad x \in \mathbb{B},\ \tau \in \overline{\mathbb{B}}.$$

Geometrically, the sets

$$E(\tau, s) = \left\{x \in \mathbb{B} : d_\tau(x) < s\right\}, \quad s > 0,$$

are ellipsoids. If $\tau \in \mathbb{B}$, then these sets are exactly the ρ-balls

$$E(\tau, s) = \left\{x \in \mathbb{B} : \rho(x,\tau) < r\right\}$$

centered at $\tau \in \mathbb{B}$ and of radius $r = \tanh^{-1}\sqrt{\frac{s}{s+1-\|\tau\|^2}}$. If $\tau \in \partial\mathbb{B}$, the boundary of $\mathbb{B}$, then the sets

$$E(\tau, s) = \left\{x \in \mathbb{B} : d_\tau(x) = \frac{|1-\langle x,\tau\rangle|^2}{1-\|x\|^2} < s\right\}, \quad s > 0,$$

are ellipsoids which are internally tangent to the unit sphere $\partial\mathbb{B}$ at τ.

For fixed $\tau \in \overline{\mathbb{B}}$ and $x \in \partial E(\tau, s), x \neq \tau$, consider now the non-zero vector

$$x^* = \frac{1}{1-\sigma(x,\tau)}\left(\frac{1}{1-\|x\|^2}x - \frac{1}{1-\langle \tau, x\rangle}\tau\right). \tag{3.2}$$

As in [1], it can be shown that x^* is a support functional of the smooth convex set $E(\tau, s)$ at x, normalized by the condition

$$\lim_{x\to\tau}\langle x-\tau, x^*\rangle = 1.$$

Then for a mapping $f : \mathbb{B} \to \mathcal{H}$, the so-called "flow-invariance condition",

$$\operatorname{Re}\langle f(x), x^{*}\rangle \geq 0, \tag{3.3}$$

is necessary for f to be a generator of a continuous flow for which the sets $E(\tau, s)$ are invariant.

Conversely, if condition (3.3) holds for some $\tau \in \mathbb{B}$ and all $x \in \mathbb{B}$, then τ must be a stationary point of $\mathcal{S} = \{F(t)\}_{t \geq 0}$, hence a null point of f.

If f has no null point, then it can be shown exactly as in Theorem 3.1 in [1] that there is a unique boundary point $\tau \in \partial\mathbb{B}$ such that (3.3) holds. This point τ is the sink point for the flow generated by f.

In order to classify the asymptotic behavior of flows we will introduce a condition which is finer than (3.3). More precisely, for a point $\tau \in \overline{\mathbb{B}}$ and $f \in \mathcal{G}_{+}(\mathbb{B})$ we consider the nonnegative function on $(0, \infty)$ defined by

$$\omega_{\flat}(s) := \inf_{d_{\tau}(x) \leq s} 2 \operatorname{Re}\langle f(x), x^{*}\rangle, \quad s > 0, \tag{3.4}$$

where x^{*} is defined by (3.2).

It is clear that the function $\omega_{\flat}$ is decreasing on $(0, \infty)$.

Theorem 3.1 (Theorem on universal rates of convergence) *If $f \in \mathcal{G}_{+}(\mathbb{B})$ and $\mathcal{S} = \{F(t)\}_{t \geq 0}$ is the flow generated by f, then the following are equivalent.*

(a) For some point $\tau \in \overline{\mathbb{B}}$, the number $\omega_{\flat}(0)$ $(= \lim_{s \to 0^{+}} \omega_{\flat}(s))$ is positive;

(b) for some point $\tau \in \overline{\mathbb{B}}$, there is a decreasing function $\omega : (0, \infty) \mapsto (0, \infty)$ such that

$$d_{\tau}(F(t)x) \leq e^{-\omega(d_{\tau}(x))t} d_{\tau}(x), \quad x \in \mathbb{B},\ t \geq 0; \tag{3.5}$$

(c) for some point $\tau \in \overline{\mathbb{B}}$, there exists a number $\mu > 0$ such that

$$d_{\tau}(F(t)x) \leq e^{-\mu t} d_{\tau}(x), \quad x \in \mathbb{B},\ t \geq 0. \tag{3.6}$$

Moreover,

(ii) if $\tau \in \mathbb{B}$, then μ can be chosen as $\mu = \frac{\omega_\flat(0)}{4}$, but μ cannot be larger than $\omega_\flat(0)$ $(= \lim_{s\to 0^+} \omega_\flat(s))$;

(iii) if $\tau \in \partial\mathbb{B}$, then the maximal μ for which (3.6) holds is exactly $\omega_\flat(0)$, that is, $0 < \mu \leq \omega_\flat(0)$.

Thus we see that if the flow $\mathcal{S} = \{F(t)\}_{t\geq 0}$ converges to $\tau \in \overline{\mathbb{B}}$ with a rate of convergence of exponential type, then the rate of convergence is, in fact, uniform in terms of the "distance" d_τ. A key tool in the proof of Theorem 3.1 is the following surprising fact:

If $\omega_\flat(0)$ is positive, then $\omega_\flat$ is bounded away from zero. Moreover, if $\tau \in \partial\mathbb{B}$, then $\omega_\flat$ is simply a positive constant: $\omega_\flat(s) = \omega_\flat(0) = \beta$ for all $s \in (0,\infty)$.

It can be shown that this constant β is equal to the so-called angular derivative of f (if it exists) at the point $\tau \in \partial\mathbb{B}$. In view of Theorem 3.1(iii), this number β gives the best rate of exponential convergence of $\mathcal{S} = \{F(t)\}_{t\geq 0}$.

These facts can be considered a continuous analog of the classical Julia-Wolff-Carathéodory Theorem.

Remark. As we mentioned at the end of Section 2, the universal estimate (3.6) can be used to obtain distortion theorems for star-like mappings.

Returning to the theme of Section 2, the following natural invariance problem arises (cf. [7], pp. 112 and 156):

If $f : \mathbb{B} \mapsto \mathcal{H}$ is star-like, on which subsets of $\mathbb{B}$ does it continue to be star-like?

This problem, in turn, is closely related to the following approximation problem:

If $f : \mathbb{B} \mapsto \mathcal{H}$ is star-like on $\mathbb{B}$, how does one find a sequence of nice domains $\{\mathcal{D}_n\}$ such that $\bigcup_{n=1}^{\infty} \mathcal{D}_n = \mathbb{B}$ and $f(\mathcal{D}_n)$ is star-like for each n?

If $f(\tau) = 0$ with $\tau \in \mathbb{B}$, then an answer can be obtained by combining the Schwarz Lemma with an appropriate Möbius transformation (cf. the one-dimensional argument in [3], p. 41). This approach is no longer applicable when τ belongs to the boundary $\partial\mathbb{B}$ of $\mathbb{B}$. However, in this case the approximation problem can be solved by applying Theorems 2.1 and 3.1.

Corollary 3.1 *Let $f : \mathbb{B} \mapsto \mathcal{H}$ be star-like with $f(\tau)$ $\left(= \lim_{r\to 1^-} f(r\tau)\right) = 0$. Then for each $s \in (0,\infty)$, the sets $f(E(\tau,s))$ are star-shaped.*

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