For every i and j we have $M_j^i \supset M_j^{i+1}$. Indeed, if $p \in M_j^{i+1}$, then $(p \cdot m_j^{i+1}) \otimes \chi \in V|_{G_{i+1}\times T}$. By (27), this implies $(p \cdot m_j^i) \otimes \chi \in V|_{G_i \times T}$; that is, $p \in M_j^i$. Now it follows from Lemma 8, that for every j there is an index $i(j)$ such that $M_j^i = M_j^{i(j)}$ for every $i \geqslant i(j)$. Let $i_1 = \max\{i_0, i(1), \ldots, i(k)\}$. Then we have $M_j^i = M_j^{i_1}$ for every $i \geqslant i_1$.

By the definition of Ξ we can find polynomials $p_j \in M_j^{i_1}$ $(j = 1, \ldots, k)$ such that (26) holds for every $x \in F$. If $i > i_1$ then $M_j^i = M_j^{i_1}$ for every j, and hence $(p_j \cdot m_j^i) \otimes \chi \in V|_{G_i \times T}$ for every $i > i_1$ and $j = 1, \ldots, k$.

It follows from (27) that for every $j = 1, \ldots, k$ there is a function $m_j : \mathbb{Q}^n \to \mathbb{C}$ such that $m_j|_{G_i} = m_j^i$ for every $i \geqslant i_1$. It is clear that m_j is an exponential on $\mathbb{Q}^n$. By $(p_j m_j^i) \otimes \chi \in V|_{G_i \times T}$ we can find a function $g_j^i \in V$ such that $g_j^i|_{G_i} = (p_j m_j^i) \otimes \chi$. It is easy to see that the sequence $(g_j^i)_{i \geqslant i_1}$ converges pointwise to the function $(p_j m_j) \otimes \chi$, and thus $(p_j m_j) \otimes \chi \in V$.

Now $\psi = \sum (p_j m_j) \otimes \chi$ is an exponential polynomial belonging to V. Taking into consideration that m_j is an extension of $m_j^{i_1}$ and that $F \subset G_{i_0} \times T \subset G_{i_1} \times T$, it follows from (26) that $|(p \cdot m)(x)\chi(t) - \psi(x, t)| < \varepsilon$ for every $x \in F$. This completes the proof.

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Orthogonal pairs of weak*-closed inner ideals in a JBW*-triple

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Abstract

Pre-symmetric complex Banach spaces have been proposed as models for state spaces of physical systems. A neutral GL-projection on a pre-symmetric space represents an operation on the corresponding system, and has as its range a further pre-symmetric space which represents the state space of the resulting system. Two neutral GL-projections S and T on the pre-symmetric space A_* are said to be L-orthogonal if for all elements x in SA_* and y in TA_*,

$$\|x \pm y\| = \|x\| + \|y\|.$$

By studying the algebraic properties of the dual space A of A_*, which is a JBW*-triple, it is shown that, provided that the orthogonal neutral GL-projections S and T satisfy a certain geometrical condition, there exists a smallest neutral GL-projection $S \vee T$ majorizing both S and T, and that S, T and $S \vee T$ form a compatible family.

1. *Introduction*

This paper presents a further investigation into the structure of JBW*-triples, examples of which include JBW*-algebras, Hilbert spaces, spin triples and W*-algebras. A complex Banach space A_* is said to be pre-symmetric if the open unit ball in its dual space A is a bounded symmetric domain. In this case the holomorphic structure of the open unit ball leads to the existence of a triple product on A with respect to which it is a JBW*-triple. Since the predual of a JBW*-triple is unique, there is a bijection $A_* \mapsto A$ from the set of pre-symmetric Banach spaces onto the set of JBW*-triples [**3**, **4**, **10**, **11**, **31**, **33–35**]. The motivation for using the apparently redundant notion of pre-symmetry is the fact that many purely geometric properties of the pre-symmetric space A_* are equivalent to purely algebraic properties of the JBW*-triple A. Furthermore, in one approach to the theory of statistical physical systems the state space of the system is represented by a pre-symmetric space A_*, the geometric properties of which represent physical properties of the system in question [**27–30**]. Although the results presented in this paper are mainly algebraic and analytic in nature, and present new information about the structure of JBW*-triples, it is their geometric and physical analogues that might be thought to be of greatest interest.

Operations on the physical system the state space of which is represented by the pre-symmetric space A_* are represented by contractive linear projections on A_* or, equivalently, by weak*-continuous contractive linear projections on A. Groundbreaking work of Kaup [**36**] and Stachó [**41**] may be applied to show that the range of a contractive linear projection on the pre-symmetric complex Banach space A_* is itself pre-symmetric, a highly desirable property for a model of a physical system. Properties of physical operations give rise to geometric properties of contractive projections. For example, a contractive linear projection S on A_* is said to be neutral if an element x in A_* for which $\|Sx\|$ and $\|x\|$ coincide necessarily lies in the range of S, and is said to be a GL-projection if the L-orthogonal complement

$$(SA_*)^\circ = \{x \in A_* : \|x \pm y\| = \|x\| + \|y\|, \forall y \in SA_*\}$$

of the range SA_* of S is contained in the kernel of S. Both of these geometrical properties may be interpreted physically. A linear projection R on the JBW*-triple A is said to be structural if, for all elements a, b, and c in A,

$$R\{a\ Rb\ c\} = \{Ra\ b\ Rc\}.$$

It was shown in [**17**, **19**, **20**] that structural projections are automatically contractive and weak*-continuous, and that the mapping $R \mapsto RA$ is a bijection from the complete lattice $\mathcal{S}(A)$ of structural projections on A onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. More recently, in [**15**], it was shown that the mapping $S \mapsto S^*$ is a bijection between the set of neutral GL-projections on A_* and the complete lattice $\mathcal{S}(A)$, thereby linking the purely physical and geometric properties of the pre-symmetric space A_* with the purely algebraic properties of A.

For each element J of $\mathcal{I}(A)$, the kernel Ker(J) of J is defined to be the set of elements a in A for which the triple product $\{J\ a\ J\}$ is equal to zero, and the annihilator $J^\perp$ of J is defined to be the set of elements a in A for which $\{J\ a\ A\}$ is equal to zero. For each element J in $\mathcal{I}(A)$, the annihilator $J^\perp$ also lies in $\mathcal{I}(A)$, and A enjoys the generalized Peirce decomposition

$$A = J_0 \oplus J_1 \oplus J_2, \tag{1·1}$$

where,

$$J_0 = J^\perp, \qquad J_2 = J, \qquad J_1 = \mathrm{Ker}(J) \cap \mathrm{Ker}(J^\perp). \tag{1·2}$$

The structural projections onto J and $J^\perp$ are denoted by $P_2(J)$ and $P_0(J)$, respectively, and the projection $\mathrm{id}_A - P_2(J) - P_0(J)$ onto J_1 is denoted by $P_1(J)$. Furthermore,

$$\{A\ J_0\ J_2\} = \{0\}, \quad \{A\ J_2\ J_0\} = \{0\}, \tag{1·3}$$

and, for j, k and l equal to 0, 1 or 2, the Peirce arithmetical relations,

$$\{J_j\ J_k\ J_l\} \subseteq J_{j+l-k}, \tag{1·4}$$

when $j + l - k$ is equal to 0, 1 or 2 and

$$\{J_j\ J_k\ J_l\} = \{0\}, \tag{1·5}$$

otherwise, hold, except in the cases when (j, k, l) is equal to $(0, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(2, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$ or $(1, 1, 1)$. For j equal to 0, 1 or 2, writing $P_j(J)_*$ for the pre-adjoint of $P_j(J)$ and J_{*j} for its range, it is clear that A_* also enjoys a Peirce decomposition

$$A_* = J_{*0} \oplus J_{*1} \oplus J_{*2},$$

and that $P_0(J)_*$ and $P_2(J)_*$ are neutral GL-projections. In general, however, J_1 is not a JBW*-triple and $P_1(J)$ and, hence $P_1(J)_*$, is not contractive. A remarkable result, proved in [**21**], shows that the Peirce-one projections $P_1(J)$ and $P_1(J)_*$ are contractive if and only if the Peirce arithmetical relations (1·4) and (1·5) hold in all cases. In this case J is said to be a Peirce inner ideal. In general, the physical operation corresponding to $P_2(J)_*$ has a natural complementary operation corresponding to $P_0(J)_*$, and, in some sense, the space J_{*1} represents the information lost in performing the operations. In the case in which J is Peirce, $P_1(J)_*$ not only is contractive but also is a GL-projection [**15**]. This could be interpreted as indicating that the information, apparently lost in the measurement process, can possibly be retrieved using the operation corresponding to $P_1(J)_*$.

Two weak*-closed inner ideals J and K in the JBW*-triple A are said to be compatible when, for j and k equal to 0, 1 or 2, the Peirce projections $P_j(J)$ and $P_k(K)$ commute [**16**]. The corresponding physical operations may be thought to be simultaneously performable. A weak*-closed inner ideal I is compatible with all weak*-closed inner ideals in A if and only if it is an ideal in A, or, equivalently, if and only if $P_2(I)_*$ is an L-projection on A_*, or, equivalently, if and only if the Peirce-one space I_1 is zero [**16**]. The sets $\mathcal{ZI}(A)$ of weak*-closed ideals in A and $\mathcal{ZS}(A)$ of corresponding central elements of $\mathcal{S}(A)$, or M-projections, form order isomorphic Boolean sub-complete lattices of $\mathcal{I}(A)$ and $\mathcal{S}(A)$, respectively, and both are order isomorphic to the complete Boolean lattice of L-projections on A_* [**1**, **2**, **5**, **8**, **9**, **16**]. It is clear that physical operations represented by L-projections on A_* may be considered to be classical.

It is now possible to describe the material that appears in this paper. Two weak*-closed inner ideals J and K are said to be orthogonal if J is contained in the annihilator $K^\perp$ of K. The relationship is clearly symmetric, and it is shown in [**14**] that orthogonality of J and K is equivalent to L-orthogonality of the of the pre-symmetric spaces J_{*2} and K_{*2}. It is the purpose of this paper to investigate the compatibility of two orthogonal weak*-closed inner ideals J and K with each other, and with various weak*-closed inner ideals which contain them. Since it is not known if, in general, two weak*-closed inner ideals, one of which is contained in the other, are compatible, it is hardly surprising that some additional condition is needed in order to make any progress. What can be shown is that, provided that the larger of the two inner ideals is Peirce, then the inner ideals are compatible. Using this fact, in the first main result it is proved that, if one of the orthogonal pair J and K is Peirce then J and K are compatible. In order to make further progress it appears to be necessary to consider the situation in which both J and K are Peirce. In this case, it is easily shown that

$$B = J \oplus K \oplus J_1 \cap K_1 \tag{1·6}$$

is a weak*-closed inner ideal in A. However, it may not be Peirce, and a deep analysis of B is required in order to show that B is compatible with J and K. Since B is a weak*-closed inner ideal containing J and K it is clear that the smallest weak*-closed inner ideal $J \vee K$ containing J and K is also contained in B. The final results of the paper describe $J \vee K$ and show that J and K are compatible with $J \vee K$.

The paper is organized as follows. In Section 2 definitions are given and notation is established. In Section 3 an analysis of the weak*-closed inner ideal B, described above, is carried out, and in Section 4, the properties of the smallest weak*-closed inner ideal containing an orthogonal pair of weak*-closed inner ideals is investigated. The final section is devoted to a consideration of examples.

2. *Preliminaries*

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\ b\ c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and, for elements a, b, c and d in A, satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\ b\ c\}, d) - D(c, \{d\ a\ b\}), \tag{2·1}$$

where [,] denotes the commutator, and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a,b)c = \{a\ b\ c\},$$

is said to be a *Jordan*-triple*. A Jordan*-triple A for which the vanishing of $\{a\ a\ a\}$ implies that a itself vanishes is said to be *anisotropic*. For each element a in A, the conjugate linear mapping $Q(a)$ from A to itself is defined, for each element b in A, by

$$Q(a)b = \{a\ b\ a\}.$$

For details about the properties of Jordan*-triples the reader is referred to [**37**].

A Jordan*-triple A which is also a Banach space such that D is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of [**6**, definition 5·1], with non-negative spectrum, and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

is said to be a *JB*-triple*. A subspace B of a JB*-triple A is said to be a *subtriple* if $\{B\ B\ B\}$ is contained in B. A subspace B is clearly a subtriple if and only if, for each element a in B, the element $\{a\ a\ a\}$ lies in B. Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. For each element a in a JB*-triple the smallest closed subtriple $A(a)$ containing a is triple isomorphic to a commutative C*-algebra, the Gelfand representation of $A(a)$ thereby giving rise to a functional calculus. A subspace J of a JB*-triple A is said to be an *inner ideal* if $\{J\ A\ J\}$ is contained in J and is said to be an *ideal* if $\{A\ A\ J\}$ and $\{A\ J\ A\}$ are contained in J. Every norm-closed subtriple of a JB*-triple A is a JB*-triple [**35**], and a norm-closed subspace J of A is an ideal if and only if $\{J\ J\ A\}$ is contained in J [**7**]. A JB*-triple A which is the dual of a Banach space A_* is said to be a *JBW*-triple*. In this case the *predual* A_* of A is unique and, for elements a and b in A, the operators $D(a,b)$ and $Q(a)$ are weak*-continuous. It follows that a weak*-closed subtriple B of a JBW*-triple A is a JBW*-triple. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [**3**, **4**, **10**, **11**, **31**, **34–36**, **42**, **43**]. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras, for the properties of which the reader is referred to [**12**, **32**, **44**, **45**].

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\ u\ u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. For each tripotent u in A, the weak*-continuous linear operators $P_0(u)$, $P_1(u)$ and $P_2(u)$, defined by

$$P_0(u) = \mathrm{id}_A - 2D(u,u) + Q(u)^2,\ P_1(u) = 2(D(u,u) - Q(u)^2),$$
$$P_2(u) = Q(u)^2, \tag{2·2}$$

are mutually orthogonal projection operators on A with sum id_A. For j equal to 0, 1 or 2, the range of $P_j(u)$ is the weak*-closed eigenspace $A_j(u)$ of $D(u,u)$ corresponding to the eigenvalue $(1/2)j$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u) \tag{2·3}$$

is the *Peirce decomposition* of A relative to u. Moreover, $A_0(u)$ and $A_2(u)$ are inner ideals in A, $A_1(u)$ is a subtriple of A and $A_j(u)$ is said to be the *Peirce j-space* corresponding to the tripotent u. Furthermore,

$$\{A\ A_2(u)\ A_0(u)\} = \{A\ A_0(u)\ A_2(u)\} = \{0\} \tag{2·4}$$

and, for j, k and l equal to 0, 1 or 2,

$$\{A_j(u)\ A_k(u)\ A_l(u)\} \subseteq A_{j+l-k}(u) \tag{2·5}$$

when $j + l - k$ is equal to 0, 1 or 2, and

$$\{A_j(u)\ A_k(u)\ A_l(u)\} = \{0\} \tag{2·6}$$

otherwise.

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal* when $D(a,b)$ is equal to zero. For a subset L of A, denote by $L^\perp$ the subset of A which consists of all elements in A which are orthogonal to all elements in L. The subset $L^\perp$ is said to be the *annihilator* of L. Then, $L^\perp$ is a weak*-closed inner ideal in A. Moreover, for subsets L, M of A, $L^\perp \cap L \subseteq \{0\}$, $L \subseteq L^{\perp\perp}$, $L \subseteq M$ implies that $M^\perp \subseteq L^\perp$, and $L^\perp$ and $L^{\perp\perp\perp}$ coincide.

For each non-empty subset B of the JBW*-triple A, the *kernel* Ker(B) of B is the weak*-closed subspace of elements a in A for which $\{B\ a\ B\}$ is equal to $\{0\}$. It follows that the annihilator $B^\perp$ of B is contained in Ker(B) and that $B \cap \mathrm{Ker}(B)$ is contained in $\{0\}$. A subtriple B of A is said to be *complemented* [**20**] if A coincides with $B \oplus \mathrm{Ker}(B)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal. A linear projection R on the JBW*-triple A is said to be a *structural projection* [**38**] if, for each element a in A,

$$RQ(a)R = Q(Ra). \tag{2·7}$$

The main results of [**17**], [**19**] and [**20**] show that the range RA of a structural projection R is a complemented subtriple, that the kernel ker$R$ of the map R coincides with Ker(RA), that every structural projection is contractive and weak*-continuous, and, most significantly, that every weak*-closed inner ideal is complemented.

Let $\mathcal{I}(A)$ denote the complete lattice of weak*-closed inner ideals in the JBW*-triple A and let $\mathcal{S}(A)$ denote the set of structural projections on A. The results of [**17**] can be used to show that the set $\mathcal{S}(A)$ of structural projections on A is a complete lattice with respect to the ordering defined, for elements Q and R, by $Q \preceq R$ if QR is equal to R, and the mapping $R \mapsto RA$ is an order isomorphism from $\mathcal{S}(A)$ onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A.

For each element J of $\mathcal{I}(A)$, the annihilator $J^\perp$ also lies in $\mathcal{I}(A)$ and A enjoys the *generalized Peirce decomposition* described in (1·1)–(1·3). The Peirce relations given in (1·4) and (1·5) hold, except in the cases when (j,k,l) is equal to $(0,1,1)$, $(1,1,0)$, $(1,0,1)$, $(2,1,1)$, $(1,1,2)$, $(1,2,1)$ or $(1,1,1)$. When the relations hold in all cases then J is said to be a *Peirce inner ideal*. When J is the Peirce-two space $A_2(u)$ corresponding to a tripotent u the generalized Peirce decomposition reduces to that described in (2·2)–(2·6). A pair J and K of elements of $\mathcal{I}(A)$ is said to be *compatible* if, for j and k equal to 0, 1 or 2,

$$[P_j(J), P_k(K)] = 0. \tag{2·8}$$

Let A be a complex Banach space. A linear projection R on A is said to be an *M-projection* if, for each element a in A,

$$\|a\| = \max\{\|Ra\|, \|a - Ra\|\}.$$

A closed subspace which is the range of an M-projection is said to be an *M-summand* of A, and A is said to be equal to the *M-sum*

$$A = RA \oplus_M (\mathrm{id}_A - R)A$$

of the M-summands RA and $(\mathrm{id}_A - R)A$. For details the reader is referred to [**1**, **2**, **8**, **9**]. The results of [**3**, **34**] show that the set of M-summands of a JBW*-triple A coincides with its weak*-closed ideals.

A structural projection $P_2(I)$ on the JBW*-triple A which commutes with every structural projection on A is said to be *central*. It is shown in [**16**] that a weak*-closed inner ideal I in A is an ideal if and only if one of the following equivalent conditions holds: $P_2(I)$ is central; I is compatible with every weak*-closed inner ideal in A; the Peirce one-space I_1 is equal to $\{0\}$. The set $\mathcal{ZI}(A)$ of weak*-closed ideals in A is a Boolean sub-complete lattice of $\mathcal{I}(A)$, such that the annihilator $I^\perp$ of each element I in $\mathcal{ZI}(A)$ also lies in $\mathcal{ZI}(A)$. The *central hull* $c(L)$ of a subspace L of A is defined by

$$c(L) = \bigwedge\{I \in \mathcal{ZI}(A) : L \subseteq I\},$$

the smallest weak*-closed ideal in A that contains L.

3. *Orthogonal pairs of weak*-closed inner ideals*

In this section some properties of orthogonal pairs of weak*-closed inner ideals in a JBW*-triple are investigated. Before attempting this, several preliminary results are required. The proof of the following result can be found in [**21**, theorem 4·8].

LEMMA 3·1. *Let A be a JBW*-triple and let K be a weak*-closed inner ideal in A, with corresponding Peirce projections $P_0(K)$, $P_1(K)$, and $P_2(K)$. Then, K is a Peirce inner ideal if and only if the the linear mapping ϕ_K defined by*

$$\phi_K = 2P_2(K) + 2P_0(K) - \mathrm{id}_A = \mathrm{id}_A - 2P_1(K) \tag{3·1}$$

is an isometry from A onto itself.

Observe that by [**35**, proposition 5·5], the linear isometry ϕ_K appearing in Lemma 3·1 is a triple automorphism of A.

LEMMA 3·2. *Let A be a JBW*-triple, let J and K be weak*-closed inner ideals in A with K Peirce, let $P_0(J)$, $P_1(J)$ and $P_2(J)$, and $P_0(K)$, $P_1(K)$ and $P_2(K)$ be the Peirce projections corresponding to J and K respectively, and let ϕ_K be the triple automorphism of A given by*

$$\phi_K = 2P_2(K) + 2P_0(K) - \mathrm{id}_A.$$

Then, the following conditions are equivalent:

(i) $\phi_K(J) \subseteq J$;
(ii) $\phi_K(J) = J$;
(iii) $J \subseteq \phi_K(J)$;
(iv) $\phi_K P_2(J) = P_2(J)\phi_K$.

Proof. Observe that ϕ_K^2 coincides with id_A. If (i) holds then

$$J = \phi_K^2(J) \subseteq \phi_K(J),$$

and (ii) and (iii) hold. Similarly, if (iii) holds so also do (ii) and (i). Hence, (i),(ii) and (iii) are equivalent.

If (ii) holds it follows that

$$P_2(J)\phi_K P_2(J) = \phi_K P_2(J). \tag{3·2}$$

Let a be an element of the kernel $\ker(P_2(J))$ of the structural projection $P_2(J)$, which, by [**20**, lemma 4·4], coincides with the kernel $\mathrm{Ker}(J)$ of J. Then, since ϕ_K is a triple homomorphism,

$$\{J\ \phi_K(a)\ J\} = \{\phi_K(J)\ \phi_K(a)\ \phi_K(J)\} = \phi_K(\{J\ a\ J\}) = \{0\}.$$

Therefore, $\phi_K(\ker(P_2(J)))$ is contained in $\ker(P_2(J))$. Arguing as before,

$$\ker(P_2(J)) = \phi_K^2(\ker(P_2(J))) \subseteq \phi_K(\ker(P_2(J))),$$

and it follows that $\phi_K(\ker(P_2(J)))$ and $\ker(P_2(J))$ coincide. Hence,

$$(\mathrm{id}_A - P_2(J))\phi_K(\mathrm{id}_A - P_2(J)) = \phi_K(\mathrm{id}_A - P_2(J)). \tag{3·3}$$

Combining (3·2) and (3·3), it can be seen that (iv) holds. Conversely, if (iv) holds it is clear that (i) holds. This completes the proof of the lemma.

It is now possible to prove the following important lemma.

LEMMA 3·3. *Let A be a JBW*-triple and let J and K be weak*-closed inner ideals in A, with K Peirce and J contained in K. Then, J and K form a compatible pair.*

Proof. Since J is contained in K it is clear that

$$P_2(J)P_2(K) = P_2(J). \tag{3·4}$$

Moreover, the annihilator $K^\perp$ of K is contained in the annihilator $J^\perp$ of J, and, therefore

$$P_0(K)P_0(J) = P_0(K). \tag{3·5}$$

Furthermore, by [**17**, theorem 5·3],

$$J_* \subseteq K_*, \quad (K^\perp)_* \subseteq (J^\perp)_*,$$

and it follows that

$$P_2(J)_* P_2(K)_* = P_2(J)_*, \quad P_0(K)_* P_0(J)_* = P_0(K)_*.$$

Taking adjoints,

$$P_2(K)P_2(J) = P_2(J), \quad P_0(J)P_0(K) = P_0(K). \tag{3·6}$$

From (3·4)–(3·6),

$$[P_2(J), P_2(K)] = [P_0(J), P_0(K)] = 0. \tag{3·7}$$

Moreover, by (3·6),

$$P_2(J)P_0(K) = P_2(J)P_0(J)P_0(K) = 0, \quad P_0(K)P_2(J) = P_0(K)P_2(K)P_2(J) = 0, \tag{3·8}$$

from which it follows that

$$[P_2(J), P_0(K)] = 0. \tag{3·9}$$

Observe that, by (3·6) and (3·8),

$$\begin{aligned}\phi_K(J) &= (2P_2(K) + 2P_0(K) - \mathrm{id}_A)J \\ &= (2P_2(K) + 2P_0(K) - \mathrm{id}_A)P_2(J)A \subseteq J. \qquad (3\cdot10)\end{aligned}$$

Let a be an element of $J^\perp$. Then, using (3·10) and the fact that ϕ_K is a triple homomorphism,

$$\{\phi_K(a)\ J\ A\} \subseteq \{\phi_K(a)\ \phi_K(J)\ \phi_K(A)\} = \phi_K(\{a\ J\ A\}) = \{0\},$$

and it follows that $\phi_K(J^\perp)$ is contained in $J^\perp$. Using Lemma 3·2, it can be seen that

$$[P_0(J), \phi_K] = 0. \qquad (3\cdot11)$$

By (3·7) and (3·11),

$$[P_0(J), P_2(K)] = 0. \qquad (3\cdot12)$$

Since

$$P_0(J) + P_1(J) + P_2(J) = P_0(K) + P_1(K) + P_2(K) = \mathrm{id}_A,$$

equations (3·7), (3·9), and (3·12) are sufficient to show that, for j and k equal to 0, 1 and 2,

$$[P_j(J), P_k(K)] = 0,$$

as required.

It is now possible to prove the first important result.

THEOREM 3·4. *Let J and K be orthogonal weak*-closed inner ideals in a JBW*-triple A one of which is a Peirce inner ideal. Then, J and K form a compatible pair.*

Proof. Without loss of generality suppose that K is a Peirce inner ideal. Then, by [**13**, theorem 4·2], $K^\perp$ and $K^{\perp\perp}$ are also Peirce inner ideals. Since J is contained in $K^\perp$, by Lemma 3·3, J and $K^\perp$ form a compatible pair, and, hence, by [**13**, theorem 4·4], J and $K^{\perp\perp}$ form a compatible pair. Using [**13**, lemma 4·1(ii)], for j equal to 0, 1 and 2,

$$[P_j(J), P_0(K)] = [P_j(J), P_0(K^{\perp\perp})] = 0. \qquad (3\cdot13)$$

Moreover, by [**25**, lemma 3·12],

$$P_2(J)P_2(K) = P_2(K)P_2(J) = 0. \qquad (3\cdot14)$$

Since J is contained in $K^\perp$,

$$P_2(J)P_0(K) = P_2(J) \qquad (3\cdot15)$$

and, using [**17**, theorem 5·3],

$$J_* \subseteq K_*^\perp.$$

Therefore,

$$P_2(J)_*P_0(K)_* = P_2(J)_*$$

and, taking adjoints,

$$P_0(K)P_2(J) = P_2(J). \qquad (3\cdot16)$$

It follows from (3·15) and (3·16) that

$$[P_2(J), P_0(K)] = 0. \qquad (3\cdot17)$$

Since

$$P_0(J) + P_1(J) + P_2(J) = P_0(K) + P_1(K) + P_2(K) = \mathrm{id}_A,$$

equations (3·13), (3·14) and (3·17) are sufficient to show that, for j and k equal to 0, 1 and 2,

$$[P_j(J), P_k(K)] = 0,$$

as required.

This theorem has the following corollary, that is an immediate consequence of [**13**, corollary 4·5].

COROLLARY 3·5. *Let J and K be orthogonal Peirce weak*-closed inner ideals in a JBW*-triple A. Then,*

$$\{J, K, J^\perp, K^\perp, J^{\perp\perp}, K^{\perp\perp}, J^{\perp\perp}\cap J_1, K^{\perp\perp}\cap K_1\}$$

forms a family of pairwise compatible Peirce weak-closed inner ideals in A.*

The next result displays the existence of a weak*-closed inner ideal containing a pair of orthogonal Peirce weak*-closed inner ideals that is not, in general, equal to A.

THEOREM 3·6. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A, with corresponding Peirce spaces J_0, J_1 and J_2, and K_0, K_1 and K_2. Then, the subspace B of A defined by*

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1$$

is a weak-closed inner ideal in A with kernel* Ker(B) *given by*

$$\mathrm{Ker}(B) = J_0 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0.$$

Proof. Observe that, since J and K are inner ideals in A

$$\{J_2\ A\ J_2\} \subseteq J_2, \quad \{K_2\ A\ K_2\} \subseteq K_2. \qquad (3\cdot18)$$

By Theorem 3·4, J and K form a compatible pair, and, therefore, by [**16**, theorem 3·4], the intersection table of A relative to the pair J and K is given by:

$\cap$	J_2	J_1	J_0
K_2	$\{0\}$	$\{0\}$	K_2
K_1	$\{0\}$	$J_1\cap K_1$	$J_0\cap K_1$
K_0	J_2	$J_1\cap K_0$	$J_0\cap K_0$

Since J_2 is contained in K_0 and K_2 is contained in J_0, using (1·3)–(1·5) and the intersection table above,

$$\begin{aligned}\{J_2\ A\ K_2\} &= \{J_2\ J_0 \oplus J_1 \oplus J_2\ K_2\} \\ &= \{J_2\ J_0\ K_2\} + \{J_2\ J_1\ K_2\} + \{J_2\ J_2\ K_2\} \\ &\subseteq \{0\} + \{J_2\ J_1\ J_0\} \cap \{K_0\ J_1 \cap K_1 + J_1 \cap K_0\ K_2\} + \{J_2\ J_2\ J_0\} \\ &\subseteq \{0\} + J_1 \cap (K_1 + \{0\}) + \{0\} = J_1 \cap K_1. \qquad (3\cdot19)\end{aligned}$$

Similarly,

$$\begin{aligned}\{J_2\ A\ J_1 \cap K_1\} &= \{J_2\ J_2 \oplus J_1 \oplus J_0\ J_1 \cap K_1\}\\ &= \{J_2\ J_2\ J_1 \cap K_1\} + \{J_2\ J_1\ J_1 \cap K_1\} + \{J_2\ J_0\ J_1 \cap K_1\}\\ &\subseteq \{J_2\ J_2\ J_1\} \cap \{K_0\ K_0\ K_1\} + \{J_2\ J_1\ J_1\} + \{0\}\\ &\subseteq J_1 \cap K_1 \oplus J_2 \oplus \{0\} = J_1 \cap K_1 \oplus J_2 \qquad (3{\cdot}20)\end{aligned}$$

and

$$\{K_2\ A\ J_1 \cap K_1\} \subseteq J_1 \cap K_1 \oplus K_2. \qquad (3{\cdot}21)$$

Finally,

$$\begin{aligned}\{J_1 \cap K_1\ A\ J_1 \cap K_1\} &= \{J_1 \cap K_1\ J_2 \oplus J_1 \oplus J_0\ J_1 \cap K_1\}\\ &= \{J_1 \cap K_1\ J_2\ J_1 \cap K_1\} + \{J_1 \cap K_1\ J_1\ J_1 \cap K_1\}\\ &\quad + \{J_1 \cap K_1\ J_0\ J_1 \cap K_1\}\\ &\subseteq \{J_1\ J_2\ J_1\} \cap \{K_1\ K_0\ K_1\}\\ &\quad + \{J_1 \cap K_1\ J_1 \cap K_1 + J_1 \cap K_0\ J_1 \cap K_1\} + \{J_1\ J_0\ J_1\}\\ &\subseteq J_0 \cap K_2 + J_1 \cap K_1 + J_1 \cap K_2 + J_2\\ &= K_2 \oplus J_1 \cap K_1 \oplus J_2. \qquad (3{\cdot}22)\end{aligned}$$

It follows from (3·18)–(3·22) that B is an inner ideal in A.

Using the orthogonality of J and K and (1·3)–(1·5),

$$\{J_2\ J_0 \cap K_1\ J_2\} = \{J_2\ J_1 \cap K_0\ J_2\} = \{J_2\ J_0 \cap K_0\ J_2\} = \{0\}, \qquad (3{\cdot}23)$$
$$\{K_2\ J_0 \cap K_1\ K_2\} = \{K_2\ J_1 \cap K_0\ K_2\} = \{K_2\ J_0 \cap K_0\ K_2\} = \{0\}, \qquad (3{\cdot}24)$$
$$\{J_2\ J_0 \cap K_1\ K_2\} = \{J_2\ J_1 \cap K_0\ K_2\} = \{J_2\ J_0 \cap K_0\ K_2\} = \{0\}. \qquad (3{\cdot}25)$$

Similarly,

$$\{J_2\ J_0 \cap K_1\ J_1 \cap K_1\} = \{J_2\ J_0 \cap K_0\ J_1 \cap K_1\} = \{0\}, \qquad (3{\cdot}26)$$
$$\{K_2\ J_0 \cap K_1\ J_1 \cap K_1\} = \{K_2\ J_0 \cap K_0\ J_1 \cap K_1\} = \{0\}. \qquad (3{\cdot}27)$$

Moreover,

$$\{J_2\ J_1 \cap K_0\ J_1 \cap K_1\} \subseteq \{J_2\ J_1\ J_1\} \cap \{K_0\ K_0\ K_1\} \subseteq J_2 \cap K_1 = \{0\} \qquad (3{\cdot}28)$$

and, similarly,

$$\{K_2\ J_0 \cap K_1\ J_1 \cap K_1\} = \{0\}. \qquad (3{\cdot}29)$$

Furthermore,

$$\{J_1 \cap K_1\ J_0 \cap K_1\ J_1 \cap K_1\} \subseteq J_2 \cap K_1 = \{0\}, \qquad (3{\cdot}30)$$
$$\{J_1 \cap K_1\ J_1 \cap K_0\ J_1 \cap K_1\} \subseteq J_1 \cap K_2 = \{0\}, \qquad (3{\cdot}31)$$
$$\{J_1 \cap K_1\ J_0 \cap K_0\ J_1 \cap K_1\} \subseteq J_2 \cap K_2 = \{0\}. \qquad (3{\cdot}32)$$

It follows from (3·23)–(3·32) that

$$J_0 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0 \subseteq \mathrm{Ker}(B).$$

Since J and K is a compatible pair, using the intersection table above, it follows that

$$A = B \oplus J_0 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0 \subseteq B \oplus \mathrm{Ker}(B) \subseteq A, \qquad (3{\cdot}33)$$

which implies that B is a complemented inner ideal in A. Therefore, by [**17**, lemma 3·2], B is weak*-closed and, by (3·33),

$$\mathrm{Ker}(B) = J_0 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0,$$

as required.

The theorem above has two important corollaries.

COROLLARY 3·7. *Under the conditions of Theorem* 3·6,

$$\{J^{\perp}, K^{\perp}, B^{\perp}, J^{\perp\perp}, K^{\perp\perp}\}$$

forms a family of pairwise compatible weak-closed inner ideals in A.*

Proof. Recall that, by [**13**, theorem 4·2], $J^{\perp}$, $K^{\perp}$, $J^{\perp\perp}$ and $K^{\perp\perp}$ are Peirce weak*-closed inner ideals. Since J is contained in B, it follows that $B^{\perp}$ is contained in the Peirce weak*-closed inner ideal $J^{\perp}$. Therefore, by Lemma 3·3, $(B^{\perp}, J^{\perp})$ is a compatible pair. Using [**13**, theorem 4·4], it can be seen that $(B^{\perp}, J^{\perp\perp})$ forms a compatible pair. The same applies when J is replaced by K, and the result follows.

COROLLARY 3·8. *Under the conditions of Theorem* 3·6, *the stuctural projection with range B is given by*

$$P_2(B) = P_2(J) + P_2(K) + P_1(J)P_1(K).$$

Proof. Again using the compatibility of the pair J and K and the intersection diagram given above, it can be seen that the linear mapping R given by

$$R = P_2(J) + P_2(K) + P_1(J)P_1(K)$$

is a projection onto B with kernel equal to the kernel $\mathrm{Ker}(B)$ of B. The result follows from [**17**, theorem 3·4].

In order to study the compatibility of the weak*-closed inner ideals J, K and B, a detailed analysis of the structure of B is required. As a first step, the Peirce decomposition of A relative to the weak*-closed inner ideal $J_0 \cap K_0$ is considered.

LEMMA 3·9. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A, with corresponding Peirce spaces* J_0, J_1 *and* J_2, *and* K_0, K_1 *and* K_2. *Then, the kernel* $\mathrm{Ker}(J_0 \cap K_0)$ *of the weak*-closed inner ideal* $J_0 \cap K_0$ *in A is given by*

$$\mathrm{Ker}(J_0 \cap K_0) = J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1.$$

Proof. Observe that, by (1·3)–(1·5), since both J and K are Peirce inner ideals,

$$\{J_0 \cap K_0\ J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1\ J_0 \cap K_0\} = \{0\}$$

and, using the intersection table of A corresponding to the compatible pair J and K given above, it follows that

$$\begin{aligned}A &= (J_0 \cap K_0) \oplus (J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1)\\ &\subseteq (J_0 \cap K_0) \oplus \mathrm{Ker}(J_0 \cap K_0) = A,\end{aligned}$$

from which it follows that

$$\mathrm{Ker}(J_0 \cap K_0) = J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1,$$

as required.

LEMMA 3·10. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A, with corresponding Peirce spaces J_0, J_1 and J_2, and K_0, K_1 and K_2, and let*

$$M = (J_0 \cap K_0) \cap (J_1 \cap K_1)^{\perp}.$$

Then, M is a weak-closed ideal in the weak*-closed inner ideal $J_0 \cap K_0$.*

Proof. Let a_{00} lie in M, and let b_{00} and c_{00} lie in $J_0 \cap K_0$. Since $J_0 \cap K_0$ is a subtriple of A, the element $\{a_{00}\ b_{00}\ c_{00}\}$ lies in $J_0 \cap K_0$, and, using (1·3)–(1·5), (2·1) and the intersection table corresponding to the compatible pair J and K,

$$\begin{aligned}\{\{a_{00}\ b_{00}\ c_{00}\}\ J_1 \cap K_1\ A\} &= \{\{a_{00}\ b_{00}\ c_{00}\}\ J_1 \cap K_1\ J_2 \oplus K_2 \oplus J_1 \cap K_1 \\ &\quad \oplus J_1 \cap K_0 \oplus J_0 \cap K_1 \oplus J_0 \cap K_0\} \\ &= \{0\} \oplus \{0\} \oplus \{\{a_{00}\ b_{00}\ c_{00}\}\ J_1 \cap K_1\ J_1 \cap K_1\} \\ &\quad \oplus \{0\} \oplus \{0\} \oplus \{0\} \\ &= D(J_1 \cap K_1, J_1 \cap K_1)D(c_{00}, b_{00})a_{00} \\ &= D(c_{00}, b_{00})D(J_1 \cap K_1, J_1 \cap K_1)a_{00} \\ &\quad + D(J_1 \cap K_1, \{J_1 \cap K_1\ c_{00}\ b_{00}\})a_{00} \\ &\quad - D(\{c_{00}\ b_{00}\ J_1 \cap K_1\}, J_1 \cap K_1)a_{00}. \end{aligned} \quad (3·34)$$

However, since a_{00} lies in $(J_1 \cap K_1)^{\perp}$,

$$\begin{aligned} D(J_1 \cap K_1, J_1 \cap K_1)a_{00} &= \{J_1 \cap K_1\ J_1 \cap K_1\ a_{00}\} = \{0\}, \\ D(J_1 \cap K_1, \{J_1 \cap K_1\ c_{00}\ b_{00}\})a_{00} &\subseteq \{a_{00}\ J_1 \cap K_1\ J_1 \cap K_1\} = \{0\}, \\ D(\{c_{00}\ b_{00}\ J_1 \cap K_1\}, J_1 \cap K_1)a_{00} &\subseteq \{J_1 \cap K_1\ J_1 \cap K_1\ a_{00}\} = \{0\}, \end{aligned}$$

and it follows from (3·34) that the element $\{a_{00}\ b_{00}\ c_{00}\}$ lies in $(J_1 \cap K_1)^{\perp}$. Hence, $\{M\ J_0 \cap K_0\ J_0 \cap K_0\}$ is contained in M, and the result follows from [**7**, proposition 1·3].

The result above leads immediately to the following lemma.

LEMMA 3·11. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A, with corresponding Peirce spaces J_0, J_1 and J_2, and K_0, K_1 and K_2, and let B be the weak*-closed inner ideal in A given by*

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1.$$

Then, there exists a unique weak-closed ideal I contained in the central hull $c(J_0 \cap K_0)$ of the weak*-closed inner ideal $J_0 \cap K_0$ of A such that,*

$$B^{\perp} \cap I = (J_0 \cap K_0) \cap I, \quad B^{\perp} \cap I^{\perp} = \{0\}.$$

Proof. By Lemma 3·10, using [**24**, corollary 3·8], there exists a unique weak*-closed ideal I in $c(J_0 \cap K_0)$ such that

$$(J_0 \cap K_0) \cap (J_1 \cap K_1)^{\perp} = (J_0 \cap K_0) \cap I.$$

It follows that

$$(J_0 \cap K_0) \cap ((J_0 \cap K_0) \cap (J_1 \cap K_1)^{\perp})^{\perp} = (J_0 \cap K_0) \cap I^{\perp}.$$

However,

$$\begin{aligned} B^{\perp} &= (J_2 \oplus K_2 \oplus J_1 \cap K_1)^{\perp} \\ &= J_0 \cap K_0 \cap (J_1 \cap K_1)^{\perp}, \end{aligned}$$

from which the result follows.

In order to study the Peirce decomposition of A corresponding to B it is necessary to prove the following fairly general lemma.

LEMMA 3·12. *Let A be a JBW*-triple, let M be a weak*-closed inner ideal in A and let I be a weak*-closed ideal in A. Then, the annihilator $M^{\perp}$ and kernel* Ker(M) *of M have the following properties:*

(i) $M^{\perp} \cap I = (M \cap I)^{\perp} \cap I$;

(ii) $\mathrm{Ker}(M) \cap I = \mathrm{Ker}(M \cap I) \cap I$.

Proof. (i) Since M and I are compatible,

$$M = (M \cap I) \oplus (M \cap I^{\perp})$$

and, hence,

$$M^{\perp} = (M \cap I)^{\perp} \cap (M \cap I^{\perp})^{\perp}.$$

Therefore,

$$M^{\perp} \cap I = (M \cap I)^{\perp} \cap (M \cap I^{\perp})^{\perp} \cap I. \quad (3·35)$$

However, since $M \cap I^{\perp}$ is contained in $I^{\perp}$,

$$I = I^{\perp\perp} \subseteq (M \cap I^{\perp})^{\perp}$$

and it follows from (3·35) that

$$M^{\perp} \cap I = (M \cap I)^{\perp} \cap I,$$

as required.

(ii) Since $M \cap I$ is contained in M, it follows that Ker(M) is contained in Ker$(M \cap I)$ and, hence, that

$$\mathrm{Ker}(M) \cap I \subseteq \mathrm{Ker}(M \cap I) \cap I. \quad (3·36)$$

Suppose that a lies in Ker$(M \cap I) \cap I$. Then, since M is complemented, there exist elements b in M and c in Ker(M) such that

$$a = b + c.$$

Then,

$$\{b\ a\ b\} = \{b\ b\ b\} + \{b\ c\ b\} = \{b\ b\ b\},$$

from which it follows that $\{b\ b\ b\}$ lies in I. Using the functional calculus it follows that b lies in I and, by linearity, that c also lies in I. It follows that c lies in Ker$(M) \cap I$ which, by (3·36), is contained in Ker$(M \cap I) \cap I$. Again using linearity, b is contained in Ker$(M \cap I)$ $\cap$ $(M \cap I)$ and is, therefore, equal to zero. Hence, a is contained in Ker$(M) \cap I$ and the proof is complete.

It is now possible to prove the main result of this section.

THEOREM 3·13. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A, with corresponding Peirce spaces J_0, J_1 and J_2, and K_0, K_1 and K_2, and let B be the weak*-closed inner ideal in A given by*

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1.$$

Then,

$$\{J, K, B, J^{\perp}, K^{\perp}, J^{\perp\perp}, K^{\perp\perp}, J^{\perp\perp}\cap J_1, K^{\perp\perp}\cap K_1\}$$

is a family of pairwise compatible weak-closed inner ideals in A.*

Proof. Let I be the weak*-closed ideal in A defined in Lemma 3·11. Then, since I is compatible with every weak*-closed inner ideal, using Theorem 3·6 and Lemmas 3·9, 3·11 and 3·12,

$$(B\cap I)_2\cap I = B\cap I = (J_2\cap I)\oplus(K_2\cap I)\oplus((J_1\cap K_1)\cap I), \quad (3\cdot 37)$$
$$(B\cap I)_0\cap I = B^{\perp}\cap I = (J_0\cap K_0)\cap I, \quad (3\cdot 38)$$
$$\begin{aligned}(B\cap I)_1\cap I &= \mathrm{Ker}(B\cap I)\cap\mathrm{Ker}(B^{\perp}\cap I)\cap I\\ &= \mathrm{Ker}(B)\cap\mathrm{Ker}(J_0\cap K_0)\cap I\\ &= (J_1\cap K_0\oplus J_0\cap K_1\oplus J_0\cap K_0)\\ &\quad\cap(J_2\oplus K_2\oplus J_1\cap K_1\oplus J_1\cap K_0\oplus J_0\cap K_1)\cap I\\ &= (J_1\cap K_0\oplus J_0\cap K_1)\cap I\\ &= ((J_1\cap K_0)\cap I)\oplus((J_0\cap K_1)\cap I).\end{aligned} \quad (3\cdot 39)$$

Since there is a unique structural projection onto a weak*-closed inner ideal, it follows from Corollary 3·8 that

$$P_2(B\cap I)P_2(I) = (P_2(J)+P_2(K)+P_1(J)P_1(K))P_2(I), \quad (3\cdot 40)$$

$$P_0(B\cap I)P_2(I) = P_0(J)P_0(K)P_2(I) \quad (3\cdot 41)$$

and, hence, that

$$P_1(B\cap I)P_2(I) = (P_1(J)P_0(K)+P_0(J)P_1(K))P_2(I). \quad (3\cdot 42)$$

Applying similar arguments with $I^{\perp}$ replacing I, it can be seen that

$$(B\cap I^{\perp})_2\cap I^{\perp} = (J_2\cap I^{\perp})\oplus(K_2\cap I^{\perp})\oplus((J_1\cap K_1)\cap I^{\perp}),$$
$$(B\cap I^{\perp})_0\cap I^{\perp} = B^{\perp}\cap I^{\perp} = \{0\},$$
$$\begin{aligned}(B\cap I^{\perp})_1\cap I^{\perp} &= \mathrm{Ker}(B\cap I^{\perp})\cap I^{\perp}\\ &= (J_1\cap K_0\oplus J_0\cap K_1\oplus J_0\cap K_0)\cap I^{\perp}\\ &= ((J_1\cap K_0)\cap I^{\perp})\oplus((J_0\cap K_1)\cap I^{\perp})\oplus((J_0\cap K_0)\cap I^{\perp}).\end{aligned}$$

It follows from Corollary 3·8 that

$$\begin{aligned}P_2(B\cap I^{\perp})P_0(I) &= (P_2(J)+P_2(K)+P_1(J)P_1(K))P_0(I),\\ P_0(B\cap I^{\perp})P_0(I) &= 0,\\ P_1(B\cap I^{\perp})P_0(I) &= (P_1(J)P_0(K)+P_0(J)P_1(K)+P_0(J)P_0(K))P_0(I).\end{aligned} \quad (3\cdot 43)$$

Using (3·40)–(3·43), it follows that

$$\begin{aligned}P_2(B) &= P_2(J)+P_2(K)+P_1(J)P_1(K),\\ P_0(B) &= P_0(J)P_0(K)P_2(I),\\ P_1(B) &= P_1(J)P_0(K)+P_0(J)P_1(K)+P_0(J)P_0(K)P_0(I).\end{aligned}$$

Hence, for j and k equal 0, 1 or 2, $P_j(B)$ commutes with $P_k(J)$ and $P_k(K)$, and B is compatible with both J and K. That B is compatible with $J^{\perp}$, $K^{\perp}$, $J^{\perp\perp}$, $K^{\perp\perp}$ $J^{\perp\perp}\cap J_1$ and $K^{\perp\perp}\cap K_1$ follows from [**13**, theorem 4·4].

4. *The supremum of an orthogonal pair of weak*-closed inner ideals*

In the previous section the properties of a weak*-closed inner ideal B containing two orthogonal Peirce weak*-closed inner ideals J and K were investigated. Before going on to discuss the smallest weak*-closed inner ideal containing J and K one further property of B is required.

LEMMA 4·1. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A, with corresponding Peirce spaces J_0, J_1 and J_2, and K_0, K_1 and K_2, and let B be the weak*-closed inner ideal in A given by*

$$B = J_2\oplus K_2\oplus J_1\cap K_1.$$

Then, the Peirce decomposition of B associated with J is given by

$$B = (J)_{B,2}\oplus(J)_{B,1}\oplus(J)_{B,0} = J_2\oplus J_1\cap K_1\oplus K_2$$

and J and K form a compatible pair of Peirce weak-closed inner ideals in B such that*

$$J^{\perp}\cap B = K,\quad K^{\perp}\cap B = J.$$

Proof. By Theorems 3·6 and 3·13, using the compatibility of J, K and B,

$$J^{\perp}\cap B = K,\quad K^{\perp}\cap B = J$$

and

$$\mathrm{Ker}_B(J) = \mathrm{Ker}(J)\cap B = K_2\oplus J_1\cap K_1,$$

$$\mathrm{Ker}_B(J^{\perp}\cap B) = \mathrm{Ker}_B(K) = J_2\oplus J_1\cap K_1.$$

Hence, again using Theorem 3·6,

$$(J)_{B,1} = \mathrm{Ker}_B(J)\cap\mathrm{Ker}_B(J^{\perp}\cap B) = J_1\cap K_1$$

and the relative Peirce decomposition of B associated with J is as stated above. Furthermore, using Corollary 3·8, since $P_2(J)$ is a projection on B with range equal to J and kernel equal to $\mathrm{Ker}_B(J)$, it follows from [**17**, theorem 3·4], that the Peirce projection $P_{B,2}(J)$ on B corresponding to J is given by $P_2(J)$. By symmetry the same applies to K and it follows that

$$P_{B,2}(J) = P_2(J),\quad P_{B,1}(J) = P_1(J)P_1(K),\quad P_{B,0}(J) = P_2(K).$$

In particular $P_{B,1}(J)$ is contractive and, by [**21**, theorem 4·8], J is a Peirce weak*-closed inner ideal in B. By symmetry the same applies to K, and by the compatibility of J and K in A it can be seen that, for j and k equal to 0, 1 or 2, the relative Peirce projections $P_{B,j}(J)$ and $P_{B,k}(K)$ commute and J and K form a compatible pair in B.

The key result allowing the supremum of the orthogonal Peirce weak*-closed inner ideals J and K to be defined is the following lemma.

LEMMA 4·2. *Let J be a Peirce weak*-closed inner ideal in a JBW*-triple A and let J_0, J_1 and J_2 be the Peirce spaces corresponding to J. Then, the smallest weak*-closed inner ideal $J_2 \vee J_0$ containing J_2 and J_0 is given by*

$$J_2 \vee J_0 = J_2 \oplus J_0 \oplus \overline{\operatorname{lin}\{J_0\ J_1\ J_2\}}^{w^*},$$

where $\overline{\operatorname{lin}\{J_0\ J_1\ J_2\}}^{w^*}$ is the weak-closure of the linear span of the set*

$$\{J_0\ J_1\ J_2\} = \{\{a_0\ a_1\ a_2\} : a_j \in J_j,\ j = 0, 1, 2\}.$$

Proof. Observe that, by (1·4), $\{J_0\ J_1\ J_2\}$ is contained in J_1, and, therefore, $\overline{\operatorname{lin}\{J_0\ J_1\ J_2\}}^{w^*}$ is also contained in the weak*-closed subtriple J_1 of A. Let M be a weak*-closed inner ideal in A containing J_2 and J_0. Then,

$$\{J_0\ J_1\ J_2\} \subseteq \{M\ A\ M\} \subseteq M$$

and it can be seen that

$$J_2 \oplus J_0 \oplus \overline{\operatorname{lin}\{J_0\ J_1\ J_2\}}^{w^*} \subseteq M.$$

Since the weak*-closure of an inner ideal is an inner ideal, it remains to show that

$$N = J_2 \oplus J_0 \oplus \operatorname{lin}\{J_0\ J_1\ J_2\}$$

is an inner ideal in A. Since J_2 and J_0 are inner ideals in A, to complete the proof it must be shown that

$$\{J_0\ A\ J_2\} \subseteq N, \tag{4·1}$$

$$\{J_0\ A\ \{J_0\ J_1\ J_2\}\} \subseteq N, \tag{4·2}$$

$$\{J_2\ A\ \{J_0\ J_1\ J_2\}\} \subseteq N, \tag{4·3}$$

$$\{\{J_0\ J_1\ J_2\}\ A\ \{J_0\ J_1\ J_2\}\} \subseteq N. \tag{4·4}$$

Observe that

$$\{J_0\ A\ J_2\} = \{J_0\ J_0 \oplus J_1 \oplus J_2\ J_2\} = \{J_0\ J_1\ J_2\} \subseteq N$$

and (4·1) holds. Using (1·3)–(1·5),

$$\begin{aligned}\{J_0\ A\ \{J_0\ J_1\ J_2\}\} &= \{J_0\ J_0 \oplus J_1\ \{J_0\ J_1\ J_2\}\} \\ &\subseteq \{J_0\ J_0\ \{J_0\ J_1\ J_2\}\} + \{J_0\ J_1\ \{J_0\ J_1\ J_2\}\} \\ &\subseteq \{J_0\ J_0\ \{J_0\ J_1\ J_2\}\} + J_0.\end{aligned} \tag{4·5}$$

Moreover, by (2·1), using the fact that $D(J_2, J_0)$ is equal to zero,

$$\begin{aligned}D(\{J_0\ J_1\ J_2\}, J_0)J_0 &= D(J_2, \{J_0\ J_0\ J_1\})J_0 + [(D(J_0, J_1), D(J_2, J_0)]J_0 \\ &= \{J_2\ \{J_0\ J_0\ J_1\}\ J_0\} \subseteq \{J_0\ J_1\ J_2\}.\end{aligned} \tag{4·6}$$

It follows from (4·5)–(4·6) that (4·2) holds, and (4·3) is proved in a similar manner. Observe that, by (1·3)–(1·5),

$$\{\{J_0\ J_1\ J_2\}\ J_2\ \{J_0\ J_1\ J_2\}\} \subseteq \{J_1\ J_2\ J_1\} \subseteq J_0 \subseteq N,$$

$$\{\{J_0\ J_1\ J_2\}\ J_0\ \{J_0\ J_1\ J_2\}\} \subseteq \{J_1\ J_0\ J_1\} \subseteq J_2 \subseteq N$$

and in order to complete the proof of (4·4) and, hence, that of the lemma, it remains to prove that

$$\{\{J_0\ J_1\ J_2\}\ J_1\ \{J_0\ J_1\ J_2\}\} \subseteq N. \tag{4·7}$$

For j equal to 0, 1 and 2, let a_j, b_j and c_j be elements of J_j. Using (2·1),

$$\begin{aligned}\{\{a_0\ a_1\ a_2\}\ b_1\ \{c_0\ c_1\ c_2\}\} &= D(\{a_0\ a_1\ a_2\}, b_1)D(c_0, c_1)c_2 \\ &= D(c_0, c_1)D(\{a_0\ a_1\ a_2\}, b_1)c_2 \\ &\quad + D(\{\{a_0\ a_1\ a_2\}\ b_1\ c_0\}, c_1)c_2 \\ &\quad - D(c_0, \{c_1\ \{a_0\ a_1\ a_2\}\ b_1\})c_2.\end{aligned} \tag{4·8}$$

Since

$$D(\{a_0\ a_1\ a_2\}, b_1)c_2 = \{\{a_0\ a_1\ a_2\}\ b_1\ c_2\} \subseteq \{J_1\ J_1\ J_2\} \subseteq J_2$$

it follows that

$$D(c_0, c_1)D(\{a_0\ a_1\ a_2\}, b_1)c_2 \subseteq \{c_0\ c_1\ J_2\} \subseteq \{J_0\ J_1\ J_2\} \subseteq N. \tag{4·9}$$

Moreover, since

$$\{\{a_0\ a_1\ a_2\}\ b_1\ c_0\} \subseteq \{J_1\ J_1\ J_0\} \subseteq J_0,$$

it can be seen that

$$D(\{\{a_0\ a_1\ a_2\}\ b_1\ c_0\}, c_1)c_2 \in \{J_0\ c_1\ c_2\} \subseteq \{J_0\ J_1\ J_2\} \subseteq N \tag{4·10}$$

and, since

$$\{c_1\ \{a_0\ a_1\ a_2\}\ b_1\} \in \{c_1\ J_1\ b_1\} \subseteq J_1,$$

it can be seen that

$$D(c_0, \{c_1\ \{a_0\ a_1\ a_2\}\ b_1\})c_2 \in \{c_0\ J_1\ c_2\} \subseteq \{J_0\ J_1\ J_2\} \subseteq N. \tag{4·11}$$

Therefore, (4·7) follows from (4·8)–(4·11) and the proof is complete.

It is now possible to prove the first important result of this section.

THEOREM 4·3. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A with corresponding Peirce spaces J_0, J_1 and J_2, and K_0, K_1 and K_2. Then, the smallest weak*-closed inner ideal $J \vee K$ containing J and K is given by*

$$J \vee K = J_2 \oplus K_2 \oplus \overline{\operatorname{lin}\{J_2\ J_1 \cap K_1\ K_2\}}^{w^*},$$

where $\overline{\operatorname{lin}\{J_2\ J_1 \cap K_1\ K_2\}}^{w^*}$ is the weak-closure of the linear span of the set*

$$\{J_2\ J_1 \cap K_1\ K_2\} = \{\{a_{20}\ a_{11}\ a_{02}\} : a_{jk} \in J_j \cap K_k,\ j, k = 0, 1, 2\}.$$

Proof. By Lemma 4·1, $J \vee K$ is a weak*-closed inner ideal in the weak*-closed inner ideal B in A given by

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1,$$

in which J is Peirce with relative Peirce decomposition given by

$$(J)_{B,2} = J_2, \quad (J)_{B,1} = J_1 \cap K_1, \quad (J)_{B,0} = K_2.$$

The result follows immediately from Lemma 4·2.

The result above has several important consequences. However, before embarking upon a discussion of them, the following lemma is required.

LEMMA 4·4. *Let J be a Peirce weak*-closed inner ideal in a JBW*-triple A, let J_0, J_1 and J_2 be the Peirce spaces corresponding to J and let $J_2 \vee J_0$ be the smallest weak*-closed inner ideal in A containing J_2 and J_0. Then, the weak*-closed subspace $\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}$ of the weak*-closed subtriple J_1 of A is an ideal in J_1, and the Peirce decomposition of A associated with $J_2 \vee J_0$ is given by*

$$A = (J_2 \vee J_0)_2 \oplus (J_2 \vee J_0)_1 \oplus (J_2 \vee J_0)_0 = (J_2 \vee J_0) \oplus \{J_0\ J_1\ J_2\}^{\perp} \cap J_1 \oplus \{0\}.$$

Proof. Observe that, by (1·3)–(1·5) and (2·1),

$$\begin{aligned}\{J_1\ J_1\ \{J_0\ J_1\ J_2\}\} &= D(J_1, J_1)D(J_0, J_1)J_2\\ &= D(J_0, J_1)D(J_1, J_1)J_2 + D(\{J_1\ J_1\ J_0\}, J_1)J_2\\ &\quad - D(J_0, \{J_1\ J_1\ J_1\})J_2\\ &\subseteq \{J_0\ J_1\ J_2\} + \{J_0\ J_1\ J_2\} + \{J_0\ J_1\ J_2\}\\ &\subseteq \mathrm{lin}\{J_0\ J_1\ J_2\}.\end{aligned}$$

It follows that

$$\{J_1\ J_1\ \mathrm{lin}\{J_0\ J_1\ J_2\}\} \subseteq \mathrm{lin}\{J_0\ J_1\ J_2\}$$

and, by the weak*-continuity of the triple product,

$$\{J_1\ J_1\ \overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}\} \subseteq \overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}.$$

Therefore, by [7, proposition 1·3], $\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}$ is a weak*-closed ideal in the JBW*-triple J_1.

Notice that

$$\begin{aligned}(J_2 \vee J_0)^{\perp} &= (J_2 \oplus J_0 \oplus \overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*})^{\perp}\\ &= J^{\perp} \cap J^{\perp\perp} \cap (\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*})^{\perp} = \{0\}.\end{aligned} \tag{4·12}$$

Moreover, since J_2, J_0 and $\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}$ are contained in $J_2 \vee J_0$, it is clear that $\mathrm{Ker}(J_2 \vee J_0)$ is contained in $\mathrm{Ker}(J_2) \cap \mathrm{Ker}(J_0) \cap \mathrm{Ker}(\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*})$, which coincides with $\mathrm{Ker}(\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}) \cap J_1$. However, since $\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}$ is a weak*-closed ideal in J_1, it follows that

$$\mathrm{Ker}(J_2 \vee J_0) \subseteq \mathrm{Ker}(\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}) \cap J_1 = (\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*})^{\perp} \cap J_1. \tag{4·13}$$

Furthermore, since $\{J_0\ J_1\ J_2\}$ lies in $\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*}$, it follows that

$$(\overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*})^{\perp} \subseteq \{J_0\ J_1\ J_2\}^{\perp}$$

and, by the weak*-continuity and linearity of the triple product, it follows that the reverse inclusion holds. Therefore, using (4·13),

$$\mathrm{Ker}(J_2 \vee J_0) \subseteq \{J_0\ J_1\ J_2\}^{\perp} \cap J_1$$

and, since $J_2 \vee J_0$ is a weak*-closed inner ideal in A and, therefore, complemented,

$$\begin{aligned}A &= (J_2 \vee J_0) \oplus \mathrm{Ker}(J_2 \vee J_0)\\ &\subseteq J_2 \oplus J_0 \oplus \overline{\mathrm{lin}\{J_0\ J_1\ J_2\}}^{w^*} \oplus \{J_0\ J_1\ J_2\}^{\perp} \cap J_1\\ &= A.\end{aligned}$$

Hence,

$$\mathrm{Ker}(J_2 \vee J_0) = \{J_0\ J_1\ J_2\}^{\perp} \cap J_1$$

and, using (4·12), the proof is complete.

Using Lemma 4·4, an immediate corollary of Theorem 4·3 can now be stated.

COROLLARY 4·5. *Under the conditions of Theorem 4·3 the following are equivalent:*

(i) $J \vee K = J_2 \oplus K_2 \oplus J_1 \cap K_1$;

(ii) $\overline{\mathrm{lin}\{J_2\ J_1 \cap K_1\ K_2\}}^{w^*} = J_1 \cap K_1$;

(iii) $\{J_2\ J_1 \cap K_1\ K_2\}^{\perp} \cap J_1 \cap K_1 = \{0\}$.

In the special case in which J and K coincide with the Peirce two-spaces of two orthogonal tripotents in the JBW*-triple A a little more can be said.

COROLLARY 4·6. *Let u and v be orthogonal tripotents in the JBW*-triple A having Peirce spaces $A_j(u)$ and $A_j(v)$ for j equal to 0, 1 and 2. Then*

$$\overline{\mathrm{lin}\{A_2(u)\ A_1(u) \cap A_1(v)\ A_2(v)\}}^{w^*} = A_1(u) \cap A_1(v).$$

Proof. Since v is contained in the weak*-closed inner ideal $A_0(u)$ in A, it follows that $A_2(v)$ is contained in $A_0(u)$ and the Peirce weak*-closed inner ideals $A_2(u)$ and $A_2(v)$ are orthogonal. Since u and v lie in $A_2(u) \vee A_2(v)$ it follows that the tripotent $u + v$ lies in $A_2(u) \vee A_2(v)$, and, hence, that $A_2(u+v)$ is contained in $A_2(u) \vee A_2(v)$. By [**26**, lemma 5·3],

$$\begin{aligned}A_2(u) \oplus A_2(v) \oplus A_1(u) \cap A_1(v) &= A_2(u + v) \subseteq A_2(u) \vee A_2(v)\\ &\subseteq A_2(u) \oplus A_2(v) \oplus A_1(u) \oplus A_1(v)\end{aligned}$$

and the result follows from Corollary 4·5.

The second main result of the section is now proved.

THEOREM 4·7. *Let J and K be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple A and let $J \vee K$ be the smallest weak*-closed inner ideal in A containing J and K. Then,*

$$\{J, K, J \vee K, J^{\perp}, K^{\perp}, J^{\perp\perp}, K^{\perp\perp}, J^{\perp\perp} \cap J_1, K^{\perp\perp} \cap K_1\}$$

is a family of pairwise compatible weak-closed inner ideals in A.*

Proof. Let ϕ_J be the triple automorphism of A defined, as in (3·1), by

$$\phi_J = 2P_2(J) + 2P_0(J) - \mathrm{id}_A = \mathrm{id}_A - 2P_1(J) \tag{4·14}$$

and observe that, for an element a of the subtriple $J_2 \oplus J_0$,

$$\phi_J(a) = a.$$

Since K_2 is contained in J_0, the subspace $J + K$ is contained in $J_2 \oplus J_0$ and it follows that ϕ_J is the identity on $J + K$. Since $J + K$ is contained in $J \vee K$ it can be seen that $J \vee K$ is the smallest weak*-closed inner ideal in A containing $J+K$. Since ϕ_J is a triple automorphism of A, $\phi_J(J \vee K)$ is a weak*-closed inner ideal in A such that

$$J+K=\phi_J(J+K)\subseteq \phi_J(J\vee K).$$

It follows that

$$J\vee K\subseteq \phi_J(J\vee K)$$

and, hence, by Lemma 3·2, that $\phi_J(J\vee K)$ and $J\vee K$ coincide, with

$$\phi_J P_2(J\vee K)=P_2(J\vee K)\phi_J. \tag{4·15}$$

Let a be an element of $(J\vee K)^{\perp}$. Then

$$\{\phi_J a\ J\vee K\ A\}=\{\phi_J a\ \phi_J(J\vee K)\ \phi_J(A)\}=\phi_J\{a\ J\vee K\ A\}=\{0\}$$

and $\phi_J a$ lies in $(J\vee K)^{\perp}$. Hence, $\phi_J((J\vee K)^{\perp})$ is contained in $(J\vee K)^{\perp}$ and Lemma 3·2 again applies to show that $\phi_J((J\vee K)^{\perp})$ and $(J\vee K)^{\perp}$ coincide, with

$$\phi_J P_0(J\vee K)=P_0(J\vee K)\phi_J. \tag{4·16}$$

Since,

$$P_0(J\vee K)+P_1(J\vee K)+P_2(J\vee K)=\mathrm{id}_A, \tag{4·17}$$

it follows from (4·15)–(4·16) that, for j equal to 0, 1 or 2,

$$\phi_J P_j(J\vee K)=P_j(J\vee K)\phi_J. \tag{4·18}$$

Since J is contained in $J\vee K$ and, as a consequence of [**17**, theorem 5·3], J_* is contained in $(J\vee K)_*$,

$$P_2(J\vee K)P_2(J)=P_2(J),\quad P_2(J\vee K)_*P_2(J)_*=P_2(J)_*$$

and, taking adjoints,

$$P_2(J\vee K)P_2(J)=P_2(J)P_2(J\vee K)=P_2(J). \tag{4·19}$$

Similarly, since the weak*-closed inner ideal $(J\vee K)^{\perp}$ is contained in $J^{\perp}$,

$$P_0(J)P_0(J\vee K)=P_0(J\vee K)P_0(J)=P_0(J\vee K). \tag{4·20}$$

By (4·14), (4·18) and (4·19), $P_2(J\vee K)$ commutes with $P_2(J)$ and $P_1(J)$ and, since

$$P_0(J)+P_1(J)+P_0(J)=\mathrm{id}_A,$$

also with $P_0(J)$. Similarly, by (4·14), (4·18) and (4·20), $P_0(J\vee K)$ commutes with $P_0(J)$ and $P_1(J)$ and hence also with $P_2(J)$. Therefore, for j equal to 0, 1 and 2, $P_j(J)$ commutes with $P_0(J\vee K)$ and $P_2(J\vee K)$ and hence, by (4·17), also with $P_1(J\vee K)$. It follows that J and $J\vee K$ form a compatible pair and similarly so also do K and $J\vee K$. The remainder of the proof follows immediately from [**13**, theorem 4·4].

5. *Examples and remarks*

Let C be a W*-algebra and let $\mathcal{P}(C)$ be the complete orthomodular lattice of self-adjoint idempotents in C. Let $Z(C)$ be the commutative W*-algebra which is the algebraic centre of C. Then, $\mathcal{P}(Z(C))$ coincides with the complete Boolean lattice that is the orthomodular

lattice centre $\mathcal{ZP}(C)$ of $\mathcal{P}(C)$. Moreover, with respect to the Jordan triple product defined, for elements a, b and c in C, by

$$\{a\ b\ c\}=\frac{1}{2}(ab^*c+cb^*a),$$

C is a JBW*-triple. For details, the reader is referred to [**39**, **40**, **42**].

For each element e in $\mathcal{P}(C)$, the *central support* $c(e)$ of e is defined by

$$c(e)=\bigwedge\{z\in\mathcal{ZP}(C): e\leqslant z\}.$$

A pair (e, f) of elements of $\mathcal{P}(C)$ is said to be *centrally equivalent* if $c(e)$ and $c(f)$ coincide. The common central support is denoted by $c(e, f)$. When endowed with the product ordering, the set $\mathcal{CP}(C)$ of centrally equivalent pairs of elements of $\mathcal{P}(C)$ forms a complete lattice in which the lattice supremum coincides with the supremum in the product lattice, but, in general, the lattice infimum does not. The results of [**18**] show that the mapping $(e, f)\mapsto eCf$ is an order isomorphism from $\mathcal{CP}(C)$ onto $\mathcal{I}(C)$.

A JBW*-triple A is said to be *rectangular* if there exists a W*-algebra C and an element (p, q) of $\mathcal{CP}(C)$ such that A is isomorphic to the JBW*-triple pCq. In what follows the rectangular JBW*-triples A and pCq will be identified. Let $\mathcal{CP}(C)_{(p,q)}$ denote the principal order ideal in $\mathcal{CP}(C)$ consisting of elements (e, f) such that

$$(e, f)\leqslant(p, q).$$

Then, the mapping $(e, f)\mapsto eAf$ is an order isomorphism from $\mathcal{CP}(C)_{(p,q)}$ onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. Therefore, there exists a corresponding order isomorphism from $\mathcal{CP}(C)_{(p,q)}$ onto $\mathcal{S}(A)$.

The mapping $z\mapsto pz$ is a *-isomorphism from the commutative W*-algebra $c(p, q)Z(C)$ onto the centre $Z(pCp)$ of the hereditary sub-W*-algebra pCp of C. It follows that the same mapping determines an order isomorphism from the complete Boolean lattice $\mathcal{ZP}(C)_{c(p,q)}$ onto $\mathcal{ZP}(pCp)$ or, equivalently, $\mathcal{Z}(\mathcal{P}(C)_p)$. In order to simplify notation, for e in the principal order ideal $\mathcal{P}(C)_p$ of $\mathcal{P}(C)$, let

$$c^p(e)=\bigwedge\{zp: z\in\mathcal{ZP}(C)_{c(p,q)}, e\leqslant z\}.$$

It is clear that $c^p(e)$ coincides with $c(e)p$. The results of [**23**] show that the mapping μ, defined, for each element z of the complete Boolean lattice $\mathcal{ZP}(C)_{c(p,q)}$, and each element a in A, by

$$\mu(z)(a)=za,$$

is an order isomorphism onto the complete Boolean lattice of M-projections on A. It follows that the mapping $z\mapsto zA$ is an order isomorphism from $\mathcal{ZP}(C)_{c(p,q)}$ onto the complete Boolean lattice $\mathcal{ZI}(A)$ of weak*-closed ideals in A.

For each element (e, f) in $\mathcal{CP}(C)_{(p,q)}$ and each element z in $\mathcal{ZP}(C)_{c(p,q)}$, write

$$e'^p=p-e,\quad f'^q=q-f,\quad z'^{c(p,q)}=c(p,q)-z.$$

For an element (e, f) in $\mathcal{CP}(A)_{(p,q)}$, let

$$(e, f)'^{(p,q)}=(c^q(e'^q)e'^p, c^p(e'^p)f'^q).$$

Then, the mapping $(e, f)\mapsto(e, f)'^{(p,q)}$ is order reversing, and if J is the weak*-closed inner ideal eAf in A, then the annihilator $J^{\perp}$ coincides with $c^q(e'^q)e'^pAc^p(e'^p)f'^q$. It follows that the generalized Peirce decomposition of A corresponding to the weak*-closed inner ideal J is given by

$$J = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_2 = eAf, \qquad J_0 = c^q(e'^q)e'^p Ac^p(e'^p)f'^q$$

and

$$J_1 = ec^q(e'^q)Ac(e, f)f'^q + c(e, f)e'^p Ac^p(e'^p)f.$$

Furthermore, every weak*-closed inner ideal J is a Peirce inner ideal.

The results of [**22**, **23**] show that for two elements (e, f) and (g, h) of $\mathcal{CP}(C)_{(p,q)}$ the corresponding weak*-closed inner ideals

$$J = eAf, \quad K = gAh,$$

are orthogonal if and only if $(e, f) \leqslant (g, h)'^{(p,q)}$, or, equivalently, if and only if, in $\mathcal{P}(C)$,

$$e + g \leqslant p, \quad f + h \leqslant q.$$

In this case the general results take on a fairly straightforward form.

THEOREM 5·1. *Let C be a W*-algebra, let (p, q) be an element of the complete lattice $\mathcal{CP}(C)$ of pairs of centrally equivalent projections in C and let A be the rectangular JBW*-triple pCq. Let (e, f) and (g, h) be orthogonal elements of the complete lattice $\mathcal{CP}(C)_{(p,q)}$, let J and K be the weak*-closed inner ideals eAf and gAh and let J_0, J_1 and J_2, and K_0, K_1 and K_2 be the corresponding generalized Peirce spaces defined above. Let B be the weak*-closed inner ideal in A given by*

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1$$

and let $J \vee K$ be the smallest weak-closed inner ideal in A containing J and K. Then, the weak*-closed inner ideals B and $J \vee K$ are equal and both coincide with the weak*-closed inner ideal $(e + g)A(f + h)$ in A.*

Proof. Observe that the projections e and g commute, as do f and h. Notice that eAg'^q and $e'^p Ah$ are ideals in the JBW*-triple J_1, as are gAh'^q and $g'^p Ah$ in K_1. Therefore, using the orthogonality of the pairs e and g, and f and h, it can be seen that

$$\begin{aligned} J_1 \cap K_1 &= eAf'^q \cap gAh'^q \oplus eAf'^q \cap g'^p Ah \\ &\oplus e'^p Af \cap gAh'^q \oplus e'^p Af \cap g'^p Ah \\ &= eAh \oplus gAf. \end{aligned}$$

It follows that

$$B = eAf \oplus gAh \oplus eAh \oplus gAf = (e + g)A(f + h).$$

Let (r, s) be the element of $\mathcal{CP}(C)_{(p,q)}$ corresponding to the weak*-closed inner ideal $J \vee K$. Then,

$$(e, f) \leqslant (r, s), \quad (g, h) \leqslant (r, s)$$

and both e and g are majorised by r, and f and h are majorised by s. Hence,

$$e + g = e \vee g \leqslant r, \quad f + h = f \vee h \leqslant s$$

and it follows that

$$J \vee K \subseteq B = (e + g)A(f + h) \subseteq rAs = J \vee K,$$

as required.

When A is a W*-algebra the situation is described by choosing both p and q equal to the unit in the theorem above.

In conclusion, it is worth repeating that the main results of the paper could equally well be stated about L-orthogonal subspaces of the pre-symmetric space A_* that is the predual of the JBW*-triple A. The existence of a smallest subspace that is the range of a neutral GL-projection containing two L-orthogonal such spaces having the Peirce property, and the fact that all three such subspaces are pairwise compatible clearly has deep physical significance in any theory that uses pre-symmetric spaces as models for state spaces.

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The Hausdorff dimension of pulse-sum graphs

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Abstract

We consider random functions formed as sums of pulses

$$F(t) = \sum_{n=1}^{\infty} n^{-\alpha/D} G\big(n^{1/D}(t - X_n)\big) \quad (t \in \mathbb{R}^D)$$

where X_n are independent random vectors, $0 < \alpha < 1$, and G is an elementary "pulse" or "bump". Typically such functions have fractal graphs and we find the Hausdorff dimension of these graphs using a novel variant on the potential theoretic method.

1. *Introduction*

Many types of random fractal function have been proposed to model a wide range of phenomena from internet traffic to stock prices. One class of construction, studied in [**1**] and [**5**], depends on the superposition of randomly located "pulses" or "bumps" with width and amplitude decreasing in a self-similar manner. Here we investigate the Hausdorff dimension of the graph of such pulse-sum functions, which provides a measure of the irregularity or volatility of the process.

Let $g : \mathbb{R} \to \mathbb{R}$ be an even continuous function, decreasing on $[0, 1]$, equal to 0 on $[1, \infty)$ and such that $g(0) = 1$. We define the *elementary pulse* or *elementary bump* $G : \mathbb{R}^D \to \mathbb{R}$ to be the symmetrical function

$$G(t) = g(\|t\|)$$

where $\|t\| = \max\{|t_i|\}$ for $t = (t_1, \ldots, t_D) \in \mathbb{R}^D$. (The simplest instance to bear in mind is the "triangular bump" on $\mathbb{R}$, where $G(t) = g(t) = \max\{1 - |t|, 0\}$.) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we study the random function $F : \mathbb{R}^D \to \mathbb{R}$ given by a sum of randomly centred pulses

$$F(t) = \sum_{n=1}^{\infty} n^{-\alpha/D} G\big(n^{1/D}(t - X_n)\big), \tag{1·1}$$

where $0 < \alpha < 1$ and $(X_n)_{n \geqslant 1}$ is a sequence of independent random variables uniformly