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Central kernels of subspaces of JB*-triples ☆

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Abstract

An investigation of the norm central kernel $k_n(L)$ of an arbitrary norm-closed subspace L of a JB*-triple and the central kernel $k(L)$ of a weak*-closed subspace L of a JBW*-triple is carried out. It is shown that these geometrically defined objects have purely algebraic characterizations, the results providing new information about C*-algebras and W*-algebras.

Keywords: M-ideal; M-summand; JB*-triple; JBW*-triple

1. Introduction

This paper represents a further investigation into the central structure of Banach spaces. In the late sixties and early seventies, in ground-breaking work, Alfsen, Cunningham, Effros, and Roy [1,2,9,10] introduced the concepts of M-ideals, M-summands, and L-summands in real Banach spaces. In the following years their results were extended to complex Banach spaces, a full description being given in Behrends' treatise [6].

For a complex Banach space A and any closed subspace L of A, there exists a greatest M-ideal $k_n(L)$ of A contained in L, known as the norm central kernel of L in A. In the case in which A is a dual space and L is weak*-closed, there exists a greatest M-summand $k(L)$ of A

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contained in A, known as the central kernel $k(L)$ of L in A. It is the investigation of these two central kernels that is the subject of this paper.

A complex Banach space A having the property that its open unit ball is a bounded symmetric domain possesses a canonical triple product $\{\cdots\} : A \times A \times A \to A$ with respect to which A forms a JB*-triple. In the case in which A is a dual space, A is said to be a JBW*-triple, and its predual A_* is unique up to isometric isomorphism. The second dual of a JB*-triple is a JBW*-triple. The predual of a JBW*-triple has been proposed as a model for the state space of a physical system [26–29]. Such a space has the highly desirable property that its image under a contractive linear projection is of the same category [34,40]. In this case central properties of the JBW*-triple correspond to classical properties of the physical system. Examples of JB*-triples are C*-algebras, JB*-algebras, Hilbert C*-modules and spin triples. It is the interplay between the geometric, holomorphic, and algebraic structure of JB*-triples that has fascinated many authors over recent years.

Whilst much is known about the central structure of JB*-triples [13,23–25], no attention has yet been given to an investigation into the properties of the norm central kernel $k_n(L)$ of an arbitrary norm-closed subspace L of a JB*-triple or the central kernel $k(L)$ of an arbitrary weak*-closed subspace L of a JBW*-triple. The main results of the paper show that these objects, which are defined purely in geometrical terms can be described purely algebraically. The support space of a subset of the predual of a JBW*-triple plays an important part in the construction of contractive projections [15,16,20,31]. In the course of the investigations into the central structure of a weak*-closed subspace L of a JBW*-triple A, a new algebraic characterization of the algebraic annihilator $s(L_\circ)^\perp$ of the support space $s(L_\circ)$ of the topological annihilator $L_\circ$ of a weak*-closed subspace L is discovered.

The paper is organised as follows. In Section 2, definitions are given, notation is established, and certain preliminary results are described. In Section 3, the norm central kernel of a norm-closed subspace of a JB*-triple is investigated, and, in Section 4, the results of Section 3 are applied to study the central kernel of a weak*-closed subspace of a JBW*-triple. The final section considers the applications of the main results to C*-algebras and W*-algebras.

2. Preliminaries

Let A be a complex Banach space. A linear projection S on A is said to be an *M-projection* if, for each element a in A,

$$\|a\| = \max\{\|Sa\|, \|a - Sa\|\}.$$

A closed subspace which is the range of an M-projection is said to be an *M-summand* of A, and A is said to be the *M-sum*

$$A = SA \oplus_\infty (\mathrm{id}_A - S)A$$

of the M-summands SA and $(\mathrm{id}_A - S)A$. A linear projection T on a complex Banach space E is said to be an *L-projection* if, for each element x of E,

$$\|x\| = \|Tx\| + \|x - Tx\|.$$

A closed subspace which is the range of an L-projection is said to be an *L-summand* of E, and E is said to be the *L-sum*

$$E = TE \oplus_1 (\mathrm{id}_E - T)E$$

of the L-summands TE and $(\mathrm{id}_E - T)E$.

For a subset M of the complex Banach space E, having dual space E^*, let

$$M^\circ = \{x \in E^*\colon\ x(a) = 0,\ \forall a \in M\},$$

and, for a subset L of E^*, let

$$L_\circ = \{a \in E\colon\ x(a) = 0,\ \forall x \in L\},$$

be the *topological annihilators* of M and L, respectively. The mapping $M \mapsto M^\circ$ is a bijection from the family of L-summands of E onto the family of weak*-closed M-summands of E^*. When ordered by set inclusion, the family of L-summands of E forms a complete Boolean lattice, the lattice operations being defined for a family $\{M_j\colon\ j \in \Lambda\}$ of L-summands in E, by

$$\bigwedge_{j\in\Lambda} M_j = \bigcap_{j\in\Lambda} M_j, \qquad \bigvee_{j\in\Lambda} M_j = \overline{\operatorname{lin}\Big(\bigcup_{j\in\Lambda} M_j\Big)},$$

the closure being in the norm topology. It follows that for any family $\{L_j\colon\ j \in \Lambda\}$ of weak*-closed M-summands of the dual space E^* of E, the weak*-closure of their linear span is also an M-summand. A norm-closed subspace L of the complex Banach space A is said to be an *M-ideal* if its topological annihilator L° is an L-summand of its dual space. It follows from the remarks above that for any family $\{L_j\colon\ j \in \Lambda\}$ of M-ideals in A, the norm-closure of their linear span is also an M-ideal in A. For details, the reader is referred to [1,2,9,10].

It can now be seen that, for each closed subspace L of a complex Banach space A, there exists a greatest M-ideal $k_n(L)$ of A contained in L. The M-ideal $k_n(L)$ is said to be the *norm central kernel* of L in A. Similarly, for each complex Banach space E^* which is a dual space and each weak*-closed subspace L of E^*, there exists a greatest weak*-closed M-summand $k(L)$ of E^* contained in L. The M-summand $k(L)$ is said to be the *central kernel* of L in E^*.

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\ b\ c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and, for elements a, b, c and d in A, satisfies the identity

$$\big[D(a, b), D(c, d)\big] = D\big(\{a\ b\ c\}, d\big) - D\big(c, \{d\ a\ b\}\big), \tag{2.1}$$

where [,] denotes the commutator, and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a, b)c = \{a\ b\ c\},$$

is said to be a *Jordan*-triple*. For an element a in the Jordan*-triple A and for n equal to $1, 2, \ldots,$ define

$$a^1 = a, \qquad a^{2n+1} = \{a\ a^{2n-1}\ a\}.$$

Observe that for non-negative integers l, m, and n,

$$\{a^{2l+1}\ a^{2m+1}\ a^{2n+1}\} = a^{2(l+m+n)+3}. \tag{2.2}$$

A Jordan*-triple for which the vanishing of a^3 implies that a itself vanishes is said to be *anisotropic*. For elements a and b in A, the conjugate linear mapping $Q(a, b)$ from A to itself is defined, for each element c in A, by

$$Q(a, b)c = \{a\ c\ b\}.$$

For details about the properties of Jordan*-triples the reader is referred to [35].

A Jordan*-triple A which is also a Banach space such that D is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of [7, Definition 5.1], with non-negative spectrum, and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

is said to be a *JB*-triple*. A subspace B of a JB*-triple A is said to be a *subtriple* if $\{B\ B\ B\}$ is contained in B. A subspace B is clearly a subtriple if and only if, for each element a in B, the element a^3 lies in B. Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. A subspace J of a JB*-triple A is said to be an *inner ideal* if $\{J\ A\ J\}$ is contained in J and is said to be an *ideal* if $\{A\ A\ J\}$ and $\{A\ J\ A\}$ are contained in J. Every norm-closed subtriple of a JB*-triple A is a JB*-triple [33], and a norm-closed subspace J of A is an ideal if and only if $\{J\ J\ A\}$ is contained in J [8]. For each element a in a JB*-triple A, the smallest norm-closed subtriple $A(a)$ of A containing a is isometrically triple isomorphic to the commutative C*-algebra $C_0(\sigma_A(a))$ of complex-valued continuous functions on the bounded, locally compact subset $\sigma_A(a)$ of $\mathbb{R}^+$ which have limit zero at zero. Under the isomorphism the element a^{2n+1} is mapped into the function ι^{2n+1} defined, for each element t in $\sigma_A(a)$, by

$$\iota^{2n+1}(t) = t^{2n+1}.$$

The isometric triple isomorphism from $C_0(\sigma_A(a))$ onto $A(a)$ is said to be the *functional calculus* corresponding to a. A JB*-triple A which is the dual of a Banach space A_* is said to be a *JBW*-triple*. In this case the *predual* A_* of A is unique up to isometric isomorphism and, for elements a and b in A, the operators $D(a,b)$ and $Q(a,b)$ are weak*-continuous. It follows that a weak*-closed subtriple B of a JBW*-triple A is a JBW*-triple. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [4,5,11,12,30,32–34,41,42].

When A is a JB*-triple the M-ideals of A coincide with its norm-closed ideals, and, when A is a JBW*-triple its M-summands coincide with its weak*-closed ideals [4,32]. Hence, the norm central kernel $k_n(L)$ of a norm-closed subspace L of the JB*-triple A is the greatest norm-closed ideal of A contained in L, and the central kernel $k(L)$ of a weak*-closed subspace L of a JBW*-triple A is the greatest weak*-closed ideal of A contained in L [24,25].

3. Subspaces of JB*-triples

This section is devoted to an investigation of the norm central kernel of a norm-closed subspace of a JB*-triple. The results are proved using a series of mainly algebraic lemmas.

Lemma 3.1. *Let A be a JB*-triple, let L be a norm-closed subspace of A, and let*

$$J_L = \{a \in A:\ D(b,c)a \in L,\ \forall b,c \in A\}.$$

Then, J_L is a norm-closed inner ideal of A contained in L.

Proof. Since L is a norm-closed subspace, by the linearity and separate norm-continuity of the triple product, it is clear that J_L is a norm-closed subspace of A. Furthermore, by polarization, it can be seen that an element a of A lies in J_L if and only if, for all elements b in A, the element $D(b,b)a$ lies in L. Let a be an element of J_L, and let b and c be elements of A. Then, by (2.1),

$$\begin{aligned} D(b,b)\{a\ c\ a\} &= D(b,b)D(a,c)a \\ &= D(a,c)D(b,b)a + D\big(D(b,b)a,c\big)a - D\big(a,D(b,b)c\big)a \\ &= 2D\big(D(b,b)a,c\big)a - D\big(a,D(b,b)c\big)a \end{aligned}$$

which lies in L. It follows that the element $\{a\ c\ a\}$ lies in J_L, and, again by polarization, J_L is an inner ideal in A.

For an element a in J_L, using the functional calculus, there exists a sequence (d_j) in the norm-closed subtriple $A(a)$ generated by a such that the sequence $(D(d_j,d_j)a)$ converges in norm to a. However, for j equal to $1,2,\ldots,$ the element $D(d_j,d_j)a$ lies in L, and, since L is closed, the element a therefore lies in L. This completes the proof of the lemma. □

The following lemmas, which are of a technical algebraic nature, aim to give an alternative algebraic description of the norm-closed inner ideal J_L.

Lemma 3.2. *Let A be a Jordan*-triple, let L be a subspace of A, and let a be an element of A such that, for all elements b in A, the element $D(a,a)b$ lies in L. Then, for all elements b in A, the elements $Q(a,a^3)b$ and $D(a,a^5)b$ lie in L.*

Proof. Observe that, by using [35, JP1], twice, for each element b in A,

$$\begin{aligned} Q\big(a,a^3\big)b &= \big\{a\ b\ \{a\ a\ a\}\big\} = \big\{a\ \{b\ a\ a\}\ a\big\} \\ &= \big\{a\ \{a\ a\ b\}\ a\big\} = \big\{a\ a\ \{a\ b\ a\}\big\} \\ &= D(a,a)\{a\ b\ a\}, \end{aligned} \tag{3.1}$$

which lies in L by hypothesis. Using (2.1) observe that

$$D\big(a,a^5\big)b = 2D(a,a)\big\{a^3\ a\ b\big\} - Q\big(a,a^3\big)\{a\ b\ a\}, \tag{3.2}$$

which, by hypothesis and (3.1), lies in L. □

Lemma 3.3. *Let A be a Jordan*-triple, let L be a subspace of A, and let a be an element of A such that, for all elements b in A, the element $D(a,a)b$ lies in L. Then, for j equal to $1,2,\ldots$ and all elements b in A, the elements $Q(a,a^{4j-1})b$ and $D(a,a^{4j+1})b$ lie in L.*

Proof. That the result holds when j is equal to 1 follows from Lemma 3.2. Suppose, inductively, that the result holds when j is equal to n. Then, using [35, JP1], twice, for each element b in A,

$$\begin{aligned} Q\big(a,a^{4(n+1)-1}\big)b &= \big\{a\ b\ a^{4n+3}\big\} = \big\{a\ b\ \big\{a\ a^{4n+1}\ a\big\}\big\} \\ &= \big\{a\ \big\{b\ a\ a^{4n+1}\big\}\ a\big\} = \big\{a\ \big\{a^{4n+1}\ a\ b\big\}\ a\big\} \\ &= D\big(a,a^{4n+1}\big)\{a\ b\ a\} \end{aligned} \tag{3.3}$$

which, by hypothesis, lies in L. Using [35, JP9],

$$D\big(a,a^{4(n+1)+1}\big)b = 2D(a,a)\big\{a^{4n+3}\ a\ b\big\} - Q\big(a,a^{4n+3}\big)\{a\ b\ a\}, \tag{3.4}$$

which, by hypothesis and (3.3), lies in L. This completes the proof of the lemma. □

The next result requires the use of the functional calculus and is therefore not necessarily valid for a Jordan*-triple.

Lemma 3.4. *Let A be a JB*-triple, let L be a norm-closed subspace of A, and let a be an element of A such that, for all elements b of A, the element $D(a,a)b$ lies in L. Then, the following results hold.*

(i) *For j equal to $0,1,2,\ldots$ and all elements b of A, the elements $Q(a,a^{2j+3})b$ and $D(a,a^{2j+1})b$ lie in L.*
(ii) *For all elements b in A, the element $Q(a,a)b$ lies in L.*

Proof. Let $C_0(\sigma_A(a))$ be the commutative C*-algebra of continuous functions on the bounded locally compact subset $\sigma_A(a)$ of $\mathbb{R}^+$ that have limit zero at zero. Then, the norm-closed *-subalgebra of $C_0(\sigma_A(a))$ generated by the set of functions $\{\iota^{4j}\colon j=1,2,\ldots\}$ satisfies the conditions of the Stone–Weierstrass theorem for locally compact Hausdorff spaces, and, hence, coincides with $C_0(\sigma_A(a))$. It follows that, given a positive real number ϵ, there exist a positive integer n and complex numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$, such that

$$\left\|\iota^2-\sum_{j=1}^{n}\alpha_j\iota^{4j}\right\|<\frac{\epsilon}{\|a\|}.$$

Using the functional calculus, it follows that,

$$\left\|a^3-\sum_{j=1}^{n}\alpha_j a^{4j+1}\right\|\leqslant\|\iota\|\left\|\iota^2-\sum_{j=1}^{n}\alpha_j\iota^{4j}\right\|<\epsilon.$$

By Lemma 3.3, for j equal to $1,2,\ldots,n$ and any element b in A, the element $D(a,a^{4j+1})b$ lies in L, and, since

$$\left\|D\big(a,a^3\big)b-\sum_{j=1}\overline{\alpha_j}D\big(a,a^{4j+1}\big)b\right\|<\|a\|\|b\|\epsilon,$$

it can be seen that the element $D(a,a^3)b$ lies in L.

Observe that, as in the proof of Lemma 3.3, for each element b in A,

$$Q\big(a,a^5\big)b=\big\{a\ b\ \big\{a\ a^3\ a\big\}\big\}=D\big(a,a^3\big)\{a\ b\ a\},\tag{3.5}$$

which, from above, lies in L. An induction argument similar to that used in the proof of Lemma 3.3 now shows that, for j equal to $1,2,\ldots,$ and all elements b in A, the elements $Q(a,a^{4j+1})b$ and $D(a,a^{4j-1})b$ lie in L. Combining these facts with the results of Lemma 3.3 completes the proof of (i).

Observe that, using (2.2), for j equal to $0,1,2,\ldots,$

$$\big(a^3\big)^{2j+1}=a^{6j+3}=a^{2(3j)+3}.$$

Therefore, using (i), for j equal to $0,1,2,\ldots,$ and all elements b in A, the element $Q(a,(a^3)^{2j+1})b$ lies in L. Since the family of finite linear combinations of elements of the form $(a^3)^{2j+1}$ is dense in the JB*-triple $A(a^3)$ generated by the element a^3, it follows from the linearity and separate norm-continuity of the triple product that, for all elements c in $A(a^3)$ and b in A, the element $Q(a,c)b$ lies in L. However, the function $\iota^{1/3}$ is continuous on the bounded locally compact set $\sigma_A(a^3)$, and has limit zero at zero. Therefore, the functional calculus shows that the element a lies in $A(a^3)$. Consequently, the element $Q(a,a)b$ lies in L, as required. □

The final lemma paves the way for the main result of this section.

Lemma 3.5. *Let A be a JB*-triple, let L be a norm-closed subspace of A, and let*

$$J_L = \{a \in A\colon\ D(b,c)a \in L,\ \forall b, c \in A\}.$$

Then,

$$J_L = \{a \in A\colon\ D(a,a)b \in L,\ \forall b \in A\}.$$

Proof. Observe that if a is an element of J_L then, for all elements b in A,

$$D(a,a)b = D(b,a)a,$$

which is contained in L. Conversely, suppose that a is an element of A such that, for all elements b in A, the element $D(a,a)b$ lies in L. Then, by (2.1),

$$D(b,b)a^3 = D(b,b)D(a,a)a = 2D(a,a)\{b\ b\ a\} - Q(a)\{b\ b\ a\}$$

which, by hypothesis and Lemma 3.4(ii), lies in L. By polarization it follows that, for all elements b and c in A, the element $D(b,c)a^3$ lies in L, and, hence, that the element a^3 lies in J_L. However, by Lemma 3.1, J_L is a norm-closed inner ideal in A, and, therefore, the JB*-triple $A(a^3)$ is contained in J_L. Using the functional calculus as in the proof of Lemma 3.4, the cube root a of a^3 lies in $A(a^3)$ and, hence, in J_L. This completes the proof of the lemma. □

It is now possible to present the main result concerning JB*-triples which gives the required algebraic characterization of the norm central kernel of an arbitrary norm-closed subspace of a JB*-triple.

Theorem 3.6. *Let A be a JB*-triple, let L be a norm-closed subspace of A, having norm central kernel $k_n(L)$, and let*

$$J_L = \{a \in A\colon\ D(b,c)a \in L,\ \forall b, c \in A\},$$

and

$$I_L = \{a \in A\colon\ Q(b,c)a \in L,\ \forall b, c \in A\}.$$

Then, J_L is a norm-closed inner ideal of A, such that

$$I_L = k_n(L) \subseteq J_L \subseteq L.$$

Proof. That J_L is a norm-closed inner ideal of A contained in L was proved in Lemma 3.1.

By the linearity and separate norm-continuity of the triple product it is clear that I_L is a norm-closed subspace of A. Let a be an element of I_L. Then, for each element b in A, it can be seen that the element

$$D(a,a)b = Q(b,a)a$$

lies in L. It follows from Lemma 3.5 that the element a lies in J_L, and, hence, I_L is contained in J_L.

Now let a be an element of I_L and let b and c be elements of A. Then, the element a lies in J_L, and, using (2.1), the element

$$Q(b,b)\{c\ a\ a\} = 2Q\big(b, \{a\ c\ b\}\big)a - D\big(\{b\ a\ b\}, c\big)a,$$

lies in L. By polarization it can be seen that the element $\{c\ a\ a\}$ lies in I_L. It follows from [8, Proposition 1.3], that I_L is a norm-closed ideal in A. Therefore, I_L is contained in the norm central kernel $k_n(L)$ of L.

Finally, let J be a norm-closed ideal of A contained in L. Then, for each element a in J and b in A, the element $\{b\ a\ b\}$ lies in J and, hence, in L. It follows by polarization that the element a lies in I_L. Therefore, J is contained in I_L, from which it follows that $k_n(L)$ is contained in I_L as required. □

When the norm-closed subspace L of the JB*-triple A discussed above is a subtriple of A, rather more can be said about its norm central kernel.

Theorem 3.7. *Let A be a JB*-triple, let L be a norm-closed subtriple of A, having norm central kernel $k_n(L)$, and let*

$$J_L = \big\{a \in A\colon\ D(b,c)a \in L,\ \forall b,c \in A\big\},$$

and

$$I_L = \big\{a \in A\colon\ Q(b,c)a \in L,\ \forall b,c \in A\big\}.$$

Then,

$$I_L = k_n(L) = J_L \subseteq L.$$

Proof. By Theorem 3.6, the set J_L is a norm-closed inner ideal of A contained in L. It will first be shown that J_L is an ideal in the JB*-triple L. Let a be an element of J_L and let c be an element of L. Then, using (2.1), for all elements b in A,

$$\begin{aligned} D(b,b)\{a\ a\ c\} &= D(b,b)D(a,a)c \\ &= D(a,a)D(b,b)c + D\big(\{b\ b\ a\},a\big)c - D\big(a,\{a\ b\ b\}\big)c \\ &= D(a,a)D(b,b)c + \big\{D(b,b)a\ a\ c\big\} - \big\{a\ D(b,b)a\ c\big\}. \end{aligned}$$

Since a lies in J_L it follows from Lemma 3.5 that the element $D(a,a)D(b,b)c$ lies in L, and, from the definition of J_L, the element $D(b,b)a$ lies in L. Since a, c and $D(b,b)a$ are elements of the subtriple L, it can be concluded that the element $D(b,b)\{a\ a\ c\}$ lies in L. By polarization, it can be seen that the element $\{a\ a\ c\}$ lies in J_L. Again, by polarization, it follows from [8, Proposition 1.3], that J_L is an ideal in L.

In order to show that J_L is an ideal in A, let a be an element of J_L and let b be an element of A. Since J_L is an inner ideal the element a^3 lies in J_L. By Lemma 3.5, the element $D(a,a)b$, and, by Lemma 3.4(ii), the element $Q(a,a)b$ lie in L. Since J_L is an ideal in L, the elements $\{a^3\ a\ D(a,a)b\}$ and $\{a^3\ Q(a,a)b\ a\}$ lie in J_L. Therefore, using [35, JP9],

$$\begin{aligned} \big\{a^3\ a^3\ b\big\} = D\big(a^3,a^3\big)b &= 2D\big(a^3,a\big)D(a,a)b - Q\big(a^3,a\big)Q(a,a)b \\ &= 2\big\{a^3\ a\ D(a,a)b\big\} - \big\{a^3\ Q(a,a)b\ a\big\}, \end{aligned}$$

which implies that the element $\{a^3\ a^3\ b\}$ lies in J_L. Using the functional calculus, an arbitrary element c in J_L possesses a cube root a in J_L. Applying the argument above, it follows that for each element c in J_L and each element b in A, the element $\{c\ c\ b\}$ lies in J_L. Therefore, by polarization and [8, Proposition 1.3], it can be seen that J_L is a norm-closed ideal in A.

However, by Theorem 3.6, the greatest norm-closed ideal $k_n(L)$ of A that is contained in L is contained in J_L. Therefore, $k_n(L)$ and J_L coincide and the proof is complete. □

4. Subspaces of JBW*-triples

Recall that a JBW*-triple A is a JB*-triple that is the dual of a complex Banach space A_* and that the predual A_* is unique up to isometric isomorphism. The techniques used to study the norm central kernel of a norm-closed subspace of a JB*-triple can easily be adapted to the study of the central kernel of a weak*-closed subspace of a JBW*-triple, and yield rather more detailed information.

Theorem 4.1. *Let A be a JBW*-triple, with predual A_*, let L be a weak*-closed subspace of A, having central kernel $k(L)$, let*

$$J_L = \{a \in A\colon\ D(b,c)a \in L,\ \forall b, c \in A\},$$

and

$$I_L = \{a \in A\colon\ Q(b,c)a \in L,\ \forall b, c \in A\}.$$

Then, J_L is a weak-closed inner ideal of A, such that*

$$J_L = \{a \in A\colon\ D(a,a)b \in L,\ \forall b \in A\},$$

and

$$I_L = k(L) \subseteq J_L \subseteq L.$$

Proof. Since the triple product is separately weak*-continuous and since L is weak*-closed, it is clear that J_L and I_L are weak*-closed subspaces of L. Moreover, since the central kernel $k(L)$ of L is a norm-closed ideal in A, $k(L)$ is contained in $k_n(L)$. On the other hand, the weak*-closure of $k_n(L)$ is a weak*-closed ideal of A contained in L, and, hence, $k_n(L)$ is contained in $k(L)$. Therefore, the central kernel $k(L)$ and the norm central kernel $k_n(L)$ of L coincide, and the result follows from Theorem 3.6. □

Recall that an element u in a JBW*-triple A is said to be a *tripotent* if u^3 is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. For each tripotent u in A, the linear operators $P_0(u)$, $P_1(u)$, and $P_2(u)$, defined by

$$P_0(u) = \mathrm{id}_A - 2D(u,u) + Q(u)^2,$$
$$P_1(u) = 2\big(D(u,u) - Q(u)^2\big),$$
$$P_2(u) = Q(u)^2,$$

are mutually orthogonal weak*-continuous projection operators on A with sum id_A. For j equal to 0, 1, or 2, the range of $P_j(u)$ is the eigenspace $A_j(u)$ of $D(u,u)$ corresponding to the eigenvalue $\frac{1}{2}j$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is the *Peirce decomposition* of A relative to u. In particular, $A_2(u)$ and $A_0(u)$ are a weak*-closed inner ideals in A. Observe that there exist mutually orthogonal contractive linear projections $P_0(u)_*$, $P_1(u)_*$, and $P_2(u)_*$ on the predual A_* of A with sum id_{A_*}, the ranges of which are the preduals $A_0(u)_*$, $A_1(u)_*$, and $A_2(u)_*$ of $A_0(u)$, $A_1(u)$, and $A_2(u)$, respectively [30].

For two tripotents u and v in the JBW*-triple A, write $u \leqslant v$ if $\{u\ v\ u\}$ is equal to u. This relation is a partial ordering on the set $\mathcal{U}(A)$ of tripotents in A, and the set $\mathcal{U}(A)\tilde{}$, consisting of

$\mathcal{U}(A)$ with a largest element adjoined, forms a complete lattice. Two tripotents u and v are said to be *compatible* if their Peirce projections commute, or, equivalently, if

$$A = \bigoplus_{j,k=0}^{2} \big(A_j(u) \cap A_k(v)\big).$$

If u lies in a Peirce space of v, then u and v are compatible [37]. Two tripotents u and v are said to be orthogonal if u lies in the Peirce-zero space $A_0(v)$ of v, and two such tripotents are, therefore, compatible. For each element x of the predual A_* of the JBW*-triple A there exists a smallest element $e(x)$ of $\mathcal{U}(A)\tilde{}$ for which

$$x\big(e(x)\big) = \|x\|,$$

and $e(x)$ is said to be *support tripotent* of x [30]. More generally, for each subset M of the predual A_* of A, the weak*-closed linear span $s(M)$ of the set $\{e(x)\colon\ x \in M\}$ is said to be the *support space* of M [16]. Observe that the annihilator $s(M)^\perp$ of the support space $s(M)$ of M, is given by

$$s(M)^\perp = \bigcap_{x\in M} A_0\big(e(x)\big), \tag{4.1}$$

which, being the intersection of weak*-closed inner ideals in A, is itself a weak*-closed inner ideal in A [20,31].

Recall that for any subset L of the JBW*-triple A, the *kernel* $\mathrm{Ker}(L)$ is the weak*-closed subspace of A consisting of elements a in A for which $\{L\ a\ L\}$ is equal to zero, and the (algebraic) *annihilator* $L^\perp$ is the weak*-closed inner ideal of A contained in $\mathrm{Ker}(L)$ consisting of elements of A for which $\{L\ a\ A\}$ is equal to zero. A weak*-closed subtriple L of A is said to be *complemented* if

$$A = L \oplus \mathrm{Ker}(L).$$

Such a subtriple is an inner ideal in A and every weak*-closed inner ideal arises in this way. A linear projection R on the JBW*-triple A is said to be a *structural projection* [36] if, for each element a in A,

$$RQ(a,a)R = Q(Ra, Ra).$$

The range of a structural projection is a weak*-closed inner ideal, and every weak*-closed inner ideal arises in this manner. For each weak*-closed inner ideal L of A, the annihilator $L^\perp$ is a weak*-closed inner ideal and A enjoys the *generalized Peirce decomposition*

$$A = L_2 \oplus L_1 \oplus L_0,$$

relative to L, where

$$L_2 = L, \qquad L_0 = L^\perp, \qquad L_1 = \mathrm{Ker}(L) \cap \mathrm{Ker}\big(L^\perp\big).$$

The structural projections the ranges of which are L_2 and L_0 are denoted by $P_2(L)$ and $P_0(L)$, respectively, and the projection

$$P_1(L) = \mathrm{id}_A - P_2(L) - P_0(L)$$

denotes the projection onto L_1. Then $P_0(L)$, $P_1(L)$, and $P_2(L)$ are mutually orthogonal weak*-continuous linear projections on A with sum id_A. Observe that the pre-adjoints $P_0(L)_*$, $P_1(L)_*$, and $P_2(L)_*$ are mutually orthogonal projections onto the preduals $L_{0,*}$, $L_{1,*}$, and $L_{2,*}$ of L_0, L_1, and L_2, respectively. For more details the reader is referred to [17,19–21].

Before proving the main result connecting the theory of support spaces to the earlier results, one more lemma is required.

Lemma 4.2. *Let A be a JBW*-triple, with predual A_*, let M be a subset of A_* having support space $s(M)$ and topological annihilator M°, and let $k(s(M)^\perp)$, $k(\mathrm{Ker}(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel* $\mathrm{Ker}(s(M))$ *of $s(M)$, and M°, respectively. Then,*

$$s(M)^\perp \subseteq \mathrm{Ker}\big(s(M)\big) \subseteq M^\circ,$$

and

$$k\big(s(M)^\perp\big) = k\big(\mathrm{Ker}\big(s(M)\big)\big) = k\big(M^\circ\big).$$

Proof. It is clear that $s(M)^\perp$ is contained in $\mathrm{Ker}(s(M))$. If a is an element of $\mathrm{Ker}(s(M))$ then, since $e(x)$ lies in $s(M)$, for all elements x in M,

$$Q\big(e(x), e(x)\big)a = \big\{e(x)\ a\ e(x)\big\} = 0.$$

Therefore,

$$P_2\big(e(x)\big)a = Q\big(e(x), e(x)\big)^2 a = 0,$$

and

$$x(a) = P_2\big(e(x)\big)_* x(a) = x\big(P_2\big(e(x)a\big)\big) = 0,$$

and a is contained in M°. This completes the first part of the proof. A proof of the second part can be found in [14]. □

It is now possible to present the most interesting result of this section of the paper.

Theorem 4.3. *Let A be a JBW*-triple, with predual A_*, let M be a subset of A_* having support space $s(M)$ and topological annihilator M°, let $k(s(M)^\perp)$, $k(\mathrm{Ker}(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel* $\mathrm{Ker}(s(M))$ *of $s(M)$, and M°, respectively, and let*

$$J_{M^\circ} = \big\{a \in A\colon\ D(b, c)a \in M^\circ,\ \forall b, c \in A\big\},$$

and

$$I_{M^\circ} = \big\{a \in A\colon\ Q(b, c)a \in M^\circ,\ \forall b, c \in A\big\}.$$

Then, the following results hold.

(i) $J_{M^\circ} = s(M)^\perp \subseteq \mathrm{Ker}(s(M)) \subseteq M^\circ$.
(ii) $I_{M^\circ} = k(s(M)^\perp) = k(\mathrm{Ker}(s(M))) = k(M^\circ)$.

Proof. (i) To show that J_{M° is contained in $s(M)^\perp$, let a be an element of J_{M° and let x be an element of M. Since M° is a weak*-closed subspace of A, it follows from Theorem 4.1 that the element $D(a, a)e(x)$ lies in M°, and, hence,

$$x\big(\big\{a\ a\ e(x)\big\}\big) = 0.$$

By [3, Proposition 1.2], it follows that, for all elements x of M, the element a lies in $A_0(e(x))$, and, hence, a lies in the weak*-closed inner ideal $s(M)^\perp$.

Now suppose that u is a tripotent in $s(M)^\perp$ and let x be an element of M. Then the tripotent u lies in $A_0(e(x))$. The tripotents u and $e(x)$ are orthogonal, and, hence, compatible, and, therefore the pre-adjoint $D(u,u)_*$ of the weak*-continuous linear operator $D(u,u)$ satisfies

$$D(u,u)_* x = P_2(u)_* P_2\bigl(e(x)\bigr)_* x + \frac{1}{2} P_1(u)_* P_2\bigl(e(x)\bigr)_* x = 0,$$

since, by compatibility, $P_2(u)_* P_2(e(x))_*$ and $P_1(u)_* P_2(e(x))_*$ are projections onto the zero subspace. Hence, for each element b in A and each element x in M,

$$x\bigl(\{u\ u\ b\}\bigr) = x\bigl(D(u,u)b\bigr) = D(u,u)_* x(b) = 0,$$

and the element $D(u,u)b$ is contained in $M_\circ$. Therefore, by Theorem 4.1, the element u lies in J_{M°. Since $s(M)^\perp$ and J_{M° are both inner ideals in A, it follows from [19, Lemma 2.3], that

$$s(M)^\perp = \bigcup_{u \in \mathcal{U}(s(M)^\perp)} A_2(u) \subseteq \bigcup_{u \in \mathcal{U}(J_{M^\circ})} A_2(u) = J_{M^\circ},$$

and the proof of (i) is complete.

(ii) This follows immediately from Theorem 4.1 and Lemma 4.2. □

Similar to the situation that pertains in the case of JB*-triples, when the weak*-closed subspace M° of the JBW*-triple A is a subtriple of A, rather more can be said.

Theorem 4.4. *Let A be a JBW*-triple, with predual A_*, let M be a subset of A_*, having support space $s(M)$ and topological annihilator M°, let $k(s(M)^\perp)$, $k(\mathrm{Ker}(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel $\mathrm{Ker}(s(M))$ of $s(M)$, and M°, respectively, and let*

$$J_{M^\circ} = \bigl\{a \in A\colon\ D(b,c)a \in M^\circ,\ \forall b, c \in A\bigr\},$$

and

$$I_{M^\circ} = \bigl\{a \in A\colon\ Q(b,c)a \in M^\circ,\ \forall b, c \in A\bigr\}.$$

If M° is a subtriple of A then the weak-closed inner ideal J_{M° is an ideal in A, and*

$$s(M)^\perp = J_{M^\circ} = I_{M^\circ} = k\bigl(s(M)^\perp\bigr) = k\bigl(\mathrm{Ker}\bigl(s(M)\bigr)\bigr) = k\bigl(M^\circ\bigr).$$

Proof. Since M° is a weak*-closed subtriple of A such that $k_n(M^\circ)$ and $k(M^\circ)$ coincide, it follows from Theorem 3.7 that J_{M° is a weak*-closed ideal of A contained in M° and containing the central kernel $k(M^\circ)$ of M°. Hence, the weak*-closed ideal J_{M° coincides with $k(M^\circ)$, and the result follows from Theorem 4.3. □

Since, for any weak*-closed subspace L of the JBW*-triple A, the double topological annihilator $(L_\circ)^\circ$ coincides with L, Theorems 4.3 and 4.4 apply when M° is replaced by any weak*-closed subspace L, in which case M is replaced by the topological annihilator $L_\circ$.

The central kernel $k(L)$ of a weak*-closed inner ideal in the JBW*-triple A has been extensively studied [24,25]. Applying Theorem 4.4 yields a new algebraic characterization of $k(L)$.

Corollary 4.5. *Let A be a JBW*-triple, with predual A_*, let L be a weak*-closed inner ideal in A, let L_0, L_1, and L_2, and $L_{0,*}$, $L_{1,*}$, and $L_{2,*}$ be the Peirce spaces corresponding to L in A and A_*, respectively, let $k(L)$ be the central kernel of L, and let*

$$J_L = \{a \in A\colon\ D(b,c)a \in L,\ \forall b,c \in A\},$$

and

$$I_L = \{a \in A\colon\ Q(b,c)a \in L,\ \forall b,c \in A\}.$$

Then,

$$s(L_{1,*} \oplus L_{0,*})^{\perp} = J_L = I_L = k\big(s(L_{1,*} \oplus L_{0,*})^{\perp}\big) = k\big(\mathrm{Ker}\big(s(L_{1,*} \oplus L_{0,*})\big)\big) = k(L).$$

Proof. Observing that the topological annihilator $L_\circ$ of L coincides with $L_{1,*} \oplus L_{0,*}$, the result is immediate from Theorem 4.4. □

It is worth remarking that the case in which the weak*-closed subspace L is a subtriple far from exhausts the interesting situations. For example, if the subset M of the predual A_* of the JBW*-triple A, consists of the single point $\{x\}$, then

$$s(M) = \mathbb{C}e(x), \qquad s(M)^{\perp} = A_0\big(e(x)\big),$$
$$\mathrm{Ker}\big(s(M)\big) = A_0\big(e(x)\big) \oplus A_1\big(e(x)\big), \qquad M^{\circ} = \ker(x),$$

and the following corollary holds.

Corollary 4.6. *Let A be a JBW*-triple with predual A_*, let x be an element of A_* having support tripotent $e(x)$ and kernel* $\ker(x)$*, for j equal to* 0, 1*, and* 2*, let $A_j(e(x))$ be the Peirce j-space corresponding to $e(x)$, and let*

$$I = \{a \in A\colon\ x\big(Q(b,c)a\big) = 0,\ \forall b,c \in A\}.$$

Then, the central kernels satisfy

$$I = k\big(A_0\big(e(x)\big)\big) = k\big(A_0\big(e(x)\big) \oplus A_1\big(e(x)\big)\big) = k\big(\ker(x)\big).$$

Since it is very rarely the case that the weak*-closed inner ideal $A_0(e(x))$ is an ideal in the JBW*-triple A, this result confirms the strength of the condition in Theorem 4.4 that M° is a subtriple.

5. C*-algebras and W*-algebras

The results above are now able to throw some new light upon the theory of C*-algebras and W*-algebras for the properties of which the reader is referred to [38,39]. Recall that a norm-closed subspace of a C*-algebra A is an M-ideal if and only if it is an algebraic ideal, and a weak*-closed subspace of a W*-algebra is an M-summand if and only if it is an algebraic ideal. Furthermore, with respect to the multiplication defined, for elements a, b, and c of A by

$$\{a\ b\ c\} = \frac{1}{2}\big(ab^*c + cb^*a\big),$$

the C*-algebra A is a JB*-triple. Similarly, a W*-algebra is a JBW*-triple [41].

Theorem 5.1. *Let A be a C^*-algebra, let L be a norm-closed subspace of A, having norm central kernel $k_n(L)$, and let*

$$J_L = \{a \in A\colon\ bc^*a + ac^*b \in L,\ \forall b, c \in A\},$$
$$I_L = \{a \in A\colon\ ba^*c + ca^*b \in L,\ \forall b, c \in A\},$$

and

$$\tilde{I}_L = \{a \in A\colon\ ba^*c \in L,\ \forall b, c \in A\}.$$

Then, J_L is a norm-closed inner ideal of A, such that

$$J_L = \{a \in A\colon\ aa^*b + ba^*a \in L,\ \forall b \in A\},$$

and

$$I_L = \tilde{I}_L = k_n(L) \subseteq J_L \subseteq L.$$

Proof. Most of the proposition is immediate from Lemma 3.5 and Theorem 3.6. Observe that it is clear that $\tilde{I}_L$ is contained in I_L. On the other hand, it can be seen that $\tilde{I}_L$ is a norm-closed algebraic ideal in A. Suppose that J is any norm-closed algebraic ideal in A contained in L. Since norm-closed algebraic ideals are *-subalgebras, for elements b and c in A and a in J, the element ba^*c lies in J and, hence, in L. It follows that the element a lies in $\tilde{I}_L$. Hence, J is contained in $\tilde{I}_L$, which is, therefore, the greatest norm-closed ideal in A contained in L. Since the sets of norm-closed ideals and M-ideals coincide, it follows that $\tilde{I}_L$ is equal to the norm central kernel $k_n(L)$ of L as required. □

Observe that Theorem 3.7 leads to the following result, which applies, for example, when L is a norm-closed *-subalgebra of the C*-algebra A.

Theorem 5.2. *Let A be a C*-algebra, let L be a norm-closed subtriple of A, having norm central kernel $k_n(L)$, and let*

$$J_L = \{a \in A\colon\ bc^*a + ac^*b \in L,\ \forall b, c \in A\},$$
$$I_L = \{a \in A\colon\ ba^*c + ca^*b \in L,\ \forall b, c \in A\},$$

and

$$\tilde{I}_L = \{a \in A\colon\ ba^*c \in L,\ \forall b, c \in A\}.$$

Then,

$$I_L = \tilde{I}_L = k_n(L) = J_L \subseteq L.$$

Let A be a W*-algebra with unit 1_A, and let $\mathcal{P}(A)$ be the complete orthomodular lattice of self-adjoint idempotents in A, the ordering being given by $e \leqslant f$ if and only if ef is equal to e, and the orthocomplementation being given by

$$e \mapsto e' = 1_A - e.$$

Let $Z(A)$ be the commutative W*-algebra that is the algebraic centre of A. Then $\mathcal{P}(Z(A))$ coincides with the complete Boolean lattice that is the orthomodular lattice centre $\mathcal{ZP}(A)$ of $\mathcal{P}(A)$. For each element e in $\mathcal{P}(A)$, the *central support* $c(e)$ of e is defined by

$$c(e) = \bigwedge\{z \in \mathcal{ZP}(A)\colon\ e \leqslant z\}.$$

A pair (e, f) of elements of $\mathcal{P}(A)$ is said to be *centrally equivalent* if $c(e)$ and $c(f)$ coincide. The common central support is denoted by $c(e, f)$. When endowed with the product ordering, the set $\mathcal{CP}(A)$ of centrally equivalent pairs of elements of $\mathcal{P}(A)$ forms a complete lattice in which the lattice supremum coincides with the supremum in the product lattice, but, in general, the lattice infimum does not. The results of [18] show that the mapping $(e, f) \mapsto eAf$ is an order isomorphism from $\mathcal{CP}(A)$ onto the complete lattice of weak*-closed inner ideals in A. The restriction of the mapping to the complete Boolean sublattice $\mathcal{ZCP}(A)$ of pairs (z, z), where z lies in $\mathcal{ZP}(A)$, is an order isomorphism onto the complete Boolean lattice of weak*-closed ideals in A.

Observe that an element u in the W*-algebra A is a tripotent if and only if

$$uu^*u = u,$$

or, equivalently, if and only if u is a partial isometry with initial projection $h(u)$ equal to u^*u and final projection $g(u)$ equal to uu^*. Observe that the Peirce spaces corresponding to u are given by

$$A_0(u) = g(u)'Ah(u)',$$
$$A_1(u) = g(u)Ah(u)' + g(u)'Ah(u),$$
$$A_2(u) = g(u)Ah(u).$$

For a subset M of A_*, the annihilator $s(M)^\perp$ of the support space $s(M)$ is a weak*-closed inner ideal in A, which, using [22, Lemma 2.4 and Corollary 2.5] is given by

$$s(M)^\perp = \bigcap_{x\in M} A_0\big(e(x)\big) = \bigcap_{x\in M} g\big(e(x)\big)'Af\big(e(x)\big)' = g'Ah', \tag{5.1}$$

where $e(x)$ is the support partial isometry of x and

$$g' = \bigwedge_{x\in M} g\big(e(x)\big)' = \Big(\bigvee_{x\in M} g\big(e(x)\big)\Big)', \tag{5.2}$$

$$h' = \bigwedge_{x\in M} h\big(e(x)\big)' = \Big(\bigvee_{x\in M} h\big(e(x)\big)\Big)'. \tag{5.3}$$

Observe that the central supports $c(g)$ and $c(h)$ are equal and are given by

$$c(g, h) = c\bigg(\Big(\bigvee_{x\in M} g\big(e(x)\big)\Big)\bigg) = \bigvee_{x\in M} c\big(e(x)e(x)^*\big) = \bigvee_{x\in M} c\big(e(x)^*e(x)\big). \tag{5.4}$$

It is now possible to apply the results of Section 4 to W*-algebras.

Theorem 5.3. *Let A be a W*-algebra, with predual A_*, let M be a subset of A_*, having support space $s(M)$ and topological annihilator M°, let $k(s(M)^\perp)$, $k(\mathrm{Ker}(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel $\mathrm{Ker}(s(M))$ of $s(M)$, and M°, respectively, let*

$$J_{M^\circ} = \big\{a \in A\colon\ bc^*a + ac^*b \in M^\circ,\ \forall b, c \in A\big\},$$
$$I_{M^\circ} = \big\{a \in A\colon\ ba^*c + ca^*b \in M^\circ,\ \forall b, c \in A\big\},$$

and

$$\tilde{I}_{M^\circ} = \big\{a \in A\colon\ ba^*c \in M^\circ,\ \forall b, c \in A\big\},$$

and let g and h be the projections in A defined in (5.1)–(5.3)*, having common central support* $c(g,h)$ *given by* (5.4)*. Then, the following results hold.*

(i) $g'Ah' = J_{M^\circ} = s(M)^\perp \subseteq \mathrm{Ker}(s(M)) \subseteq M^\circ$.
(ii) $c(g,h)'A = I_{M^\circ} = \tilde{I}_{M^\circ} = k(s(M)^\perp) = k(\mathrm{Ker}(s(M))) = k(M^\circ)$.

Proof. Much of the proof follows immediately from Theorem 4.3. Observe that it is clear that $\tilde{I}_{M^\circ}$ is contained in I_{M°. On the other hand, it can be seen that $\tilde{I}_{M^\circ}$ is a weak*-closed ideal of A. Suppose that J is any weak*-closed ideal in A contained in M°. Since weak*-closed ideals of A are *-subalgebras of A, for elements b and c in A and a in J, the element ba^*c lies in J and hence M°. It follows that the element a lies in $\tilde{I}_{M^\circ}$. Hence, J is contained in $\tilde{I}_{M^\circ}$, which is, therefore, the greatest weak*-closed ideal in A contained in M°. Since the sets of weak*-closed ideals and M-summands coincide, it follows that $\tilde{I}_{M^\circ}$ is equal to the central kernel $k(M^\circ)$ of M° as required.

Observe that, from (5.2) and (5.3), it can be seen that the weak*-closed inner ideals $g'Ah'$ and $s(M)^\perp$ coincide, thereby completing the proof of (i). However, it is not necessarily true that the projections g' and h' have a common central support. Therefore, the element of $\mathcal{CP}(A)$ corresponding to the weak*-closed inner ideal $s(M)^\perp$ is $(c(h')g', c(g')h')$. By [24, Theorem 4.1], it follows that the central kernel $k(s(M)^\perp)$ is given by

$$\begin{aligned} k\big(s(M)^\perp\big) &= c\big(\big(c(h')g'\big)'\big)'c\big(\big(c(g')h'\big)'\big)'A = c\big(\big(c(h')' \vee g\big)\big)'c\big(\big(c(g')' \vee h\big)\big)'A \\ &= \big(c(h')' \vee c(g)\big)'\big(c(g')' \vee c(h)\big)'A = \big(c(h') \wedge c(g,h)'\big)\big(c(g') \wedge c(g,h)'\big)A \\ &= c(g,h)'A, \end{aligned}$$

since $c(g,h)'$ is majorized by both $c(g')$ and $c(h')$. This completes the proof of (ii). □

The restrictive situation in which the topological annihilator M° of the subset M of the predual A_* of the W*-algebra A is a subtriple can be considered. This, of course, occurs if, for example, M° is a *-subalgebra of A.

Theorem 5.4. *Let A be a W*-algebra with predual A_*, let M be a subset of A_* having support space $s(M)$ and topological annihilator M°, let $k(s(M)^\perp)$, $k(\mathrm{Ker}(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel* $\mathrm{Ker}(s(M))$ *of $s(M)$, and M°, respectively, let*

$$\begin{aligned} J_{M^\circ} &= \big\{a \in A\colon\ bc^*a + ac^*b \in M^\circ,\ \forall b,c \in A\big\}, \\ I_{M^\circ} &= \big\{a \in A\colon\ ba^*c + ca^*b \in M^\circ,\ \forall b,c \in A\big\}, \end{aligned}$$

and

$$\tilde{I}_{M^\circ} = \big\{a \in A\colon\ ba^*c \in L,\ \forall b,c \in A\big\},$$

and let g and h be the projections in A defined in (5.1)–(5.3)*. If M° is a subtriple of A then, the projections $c(h')g$ and $c(g')h$ are central and satisfy*

$$c(h')g + c(h')' = c(g')h + c(g')' = c(g,h), \tag{5.5}$$

and

$$c(g,h)'A = s(M)^\perp = J_{M^\circ} = I_{M^\circ} = \tilde{I}_{M^\circ} = k\big(s(M)^\perp\big) = k\big(\mathrm{Ker}\big(s(M)\big)\big) = k\big(M^\circ\big).$$

Proof. Since, by Theorem 4.4, the weak*-closed inner ideal $g'Ah'$ is an ideal in A, the elements $(c(h')g', c(g')h')$ and $(c(g,h)', c(g,h)')$ of $\mathcal{CP}(A)$ coincide, and a calculation shows that (5.5) holds. The rest of the result follows from Theorem 5.3. □

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