M-ORTHOGONALITY AND HOLOMORPHIC RIGIDITY IN COMPLEX BANACH SPACES

C. MARTIN EDWARDS AND REMO V. HÜGLI

ABSTRACT. It is shown that the complex tangent space R_a at a point a on the surface of the unit ball A_1 in a complex Banach space A coincides with the complex linear span

$$\mathrm{lin}_{\mathbb{C}}(\{ia\}^{\square} \cap \{a\} \cap A_1),$$

of the set $\{ia\}^{\square} \cap \{a\} \cap A_1$, where, for a subset L of A,

$$L^{\square} = \{a \in A : \|a \pm b\| = \max\{\|a\|, \|b\|\}, \forall b \in L\}$$

is the M-orthogonal complement of L. It is also shown that if B is a holomorphically rigid closed subspace of A then $B^{\square}$ is equal to $\{0\}$. In the special case in which A is a JBW*-triple and B is a weak*-closed subtriple of A, it is shown that the M-orthogonal complement $B^{\square}$ of B coincides with the algebraic annihilator $B^{\perp}$ of B, that the complex tangent space $R_{L_B}(A)$ at the set L_B of elements of B of unit norm is weak*-closed and also coincides with $B^{\square}$, that a second tangent space $T^n_{L_B}(A)$ at L_B is weak*-closed and coincides with the algebraic kernel $\mathrm{Ker}(B)$ of B, and that B is holomorphically rigid in A if and only if $B^{\square}$ is equal to $\{0\}$.

1. INTRODUCTION

This paper is concerned with the geometric and holomorphic properties of a complex Banach space A. The study of the M-structure of Banach spaces originated in the late sixties and has proved fruitful in investigations into their structure, particularly when A is a dual space [1], [2], [7], [11], [12]. Whilst M-ideals and M-summands of Banach spaces have been studied in depth, little attention has been given to M-complementation of subsets of a Banach space. It is to this subject that much of this paper is devoted. For a subset L of the complex Banach space A, the M-orthogonal complement $L^{\square}$ of L is defined by

$$L^{\square} = \{a \in A : \|a \pm b\| = \max\{\|a\|, \|b\|\}, \forall b \in L\}. \tag{1.1}$$

The M-orthogonal complement is not, in general, a subspace of A, but it does enjoy some homogeneity properties. In a recent paper [4], Arazy and Kaup studied the properties of different "tangent spaces" to a subset L of the surface of the unit ball A_1 in the complex Banach space A and related them to the concept of holomorphic rigidity of a subspace B in A. The first part of this paper is devoted to a study of the connections between M-orthogonal complements, tangent spaces, and holomorphic rigidity. In particular, it is shown that the linear span R_a of the tangent disc S_a at a point a of norm one in A coincides with the complex linear span of the set

1

Received April 8, 2003.

$\{ia\}^\square \cap \{a\}^\square \cap A_1$ and that the M-orthogonal complement $B^\square$ of a holomorphically rigid subspace B is necessarily equal to $\{0\}$.

A class of dual complex Banach spaces, the algebraic, geometric and holomorphic properties of which are delicately linked, is that consisting of spaces the open unit balls of which form symmetric domains, the so-called JBW*-triples. A dual complex Banach space A with symmetric open unit ball automatically possesses a triple product $\{\ldots\} : A \times A \times A \to A$ having various algebraic and topological properties. For a subset B of A, the algebraic annihilator $B^\perp$ of B is defined to be the set of elements a in A for which $\{B\,a\,A\}$ is equal to $\{0\}$ and the kernel Ker(B) of B is defined to be the larger set of elements a in A for which $\{B\,a\,B\}$ is equal to $\{0\}$. It follows that, for such a complex Banach space A, it is possible to ask questions about the relationships between M-orthogonal complements, tangent spaces, algebraic annihilators and algebraic kernels of subsets of A. As is often the case in this area, it is the equivalence of purely algebraic properties on the one hand, with purely geometric or holomorphic properties on the other, that provides the most intriguing results. It is shown that, for any weak*-closed subtriple B of the JBW*-triple A, the algebraic annihilator $B^\perp$ of B coincides with one of the tangent spaces to the unit ball A_1 of A at the set L_B of elements of norm one in B, and that the algebraic kernel of B coincides with the other tangent space. A consequence of this is that B is rigid in A if and only if its algebraic annihilator is zero. All of these results fall into the category mentioned above of proving the equivalence of algebraic properties of the JBW*-triple A with its geometric or holomorphic properties. Examples of JBW*-triples are JBW*-algebras, and, in particular W*-algebras or von Neumann algebras.

This paper owes much to the work of Arazy and Kaup [4], and, in particular, the proof of the final result is a straightforward extension of their proof for the considerably more restrictive situation in which B is a weak*-closed inner ideal in A. However, in other cases, it is by giving proofs that are algebraic and geometric, rather than holomorphic, that allows the extension from inner ideals to subtriples to be made. The paper is organised as follows. In §2 the relationship between M-orthogonal complementation, tangent spaces and holomorphic rigidity for an arbitrary complex Banach space is considered, and §3 is concerned with the definition and properties of a JBW*-triple A. In particular, the various tangent spaces to the unit ball at elements on its surface are identified in terms of the algebraic structure of A. In §4, the main results are proved, and, in the final section these are applied to obtain some new information about W*-algebras.

2. M-ORTHOGONAL COMPLEMENTS AND TANGENT SPACES

In this section, the connections between tangent spaces and the concepts of holomorpic rigidity, introduced in [4], and M-orthogonal complementation are investigated.

Recall that a partially ordered set $\mathcal{P}$ is said to be a *lattice* if, for each pair (e, f) of elements of $\mathcal{P}$, the supremum $e \vee f$ and the infimum $e \wedge f$ exist with respect to the partial ordering of $\mathcal{P}$. The partially ordered set $\mathcal{P}$ is said to be a *complete lattice* if, for any subset M of $\mathcal{P}$, the supremum $\vee M$ and the infimum $\wedge M$ exist. A complete lattice has a greatest element and a least element, denoted by 1 and 0 respectively.

Let E be a complex vector space and let C be a convex subset of E. A convex subset F of C is said to be a *face* of C provided that, if $tx_1 + (1-t)x_2$ is an element

2

of F, where x_1 and x_2 lie in C and $0 < t < 1$, then x_1 and x_2 lie in F. A face F of C is said to be *proper* if it differs from C and from the the empty set. Since the intersection of a family of faces of C is also a face of C, for each subset L of C there exists a smallest face $\mathrm{face}(L)$ of C containing L. In this case $\mathrm{face}(L)$ is said to be the face of C *generated* by L. An element x in C for which $\{x\}$ is a face is said to be an *extreme point* of C. Let τ be a locally convex Hausdorff topology on E and let C be τ-closed. Let $\mathcal{F}_\tau(C)$ denote the set of τ-closed faces of C. Both $\emptyset$ and C are elements of $\mathcal{F}_\tau(C)$ and the intersection of an arbitrary family of elements of $\mathcal{F}_\tau(C)$ again lies in $\mathcal{F}_\tau(C)$. Hence, with respect to ordering by set inclusion, $\mathcal{F}_\tau(C)$ forms a complete lattice. A subset F of C is said to be a τ-*exposed face* of C if there exists a τ-continuous linear functional a on E and a real number t such that, for all elements x in $C \setminus F$,

$$\mathrm{Re}(a(x)) < t$$

and, for all elements x in F,

$$\mathrm{Re}(a(x)) = t.$$

Let $\mathcal{E}_\tau(C)$ denote the set of τ-exposed faces of C. Clearly $\mathcal{E}_\tau(C)$ is contained in $\mathcal{F}_\tau(C)$ and the intersection of a finite number of elements of $\mathcal{E}_\tau(C)$ again lies in $\mathcal{E}_\tau(C)$. The intersection of an arbitrary family of elements of $\mathcal{E}_\tau(C)$ is said to be a τ-*semi-exposed face* of C. Let $\mathcal{S}_\tau(C)$ denote the set of τ-semi-exposed faces of C. Clearly $\mathcal{E}_\tau(C)$ is contained in $\mathcal{S}_\tau(C)$, and the intersection of an arbitrary family of elements of $\mathcal{S}_\tau(C)$ again lies in $\mathcal{S}_\tau(C)$. Hence, with respect to the ordering by set inclusion $\mathcal{S}_\tau(C)$ forms a complete lattice, and the infimum of a family of elements of $\mathcal{S}_\tau(C)$ coincides with its infimum when taken in $\mathcal{F}_\tau(C)$.

When E is a complex Banach space with dual space E^* the abbreviations n and w^* will be used for the norm topology of E and the weak* topology of E^*. For each subset F of the unit ball E_1 in E and G of the unit ball E_1^* of E^*, let the subsets F' and $G_{\prime}$ be defined by

$$F' = \{a \in E_1^* : a(x) = 1 \; \forall x \in F\}, \qquad G_{\prime} = \{x \in E_1 : a(x) = 1 \; \forall a \in G\}.$$

Notice that F lies in $\mathcal{S}_n(E_1)$ if and only if

$$(F')_{\prime} = F,$$

G lies in $\mathcal{S}_{w^*}(E_1^*)$ if and only if

$$(G_{\prime})' = G,$$

and the mappings $F \mapsto F'$ and $G \mapsto G_{\prime}$ are anti-order isomorphisms between $\mathcal{S}_n(E_1)$ and $\mathcal{S}_{w^*}(E_1^*)$ and are inverses of each other. The reader is referred to [18] for details. Observe that every proper face of E_1 consists of elements of norm one.

Let A be a complex Banach space and let a and b be elements of A. Then, a and b are said to be *M-orthogonal* if

$$\|a + b\| = \|a - b\| = \max\{\|a\|, \|b\|\}.$$

More generally, the *M-orthogonal complement* $L^\square$ of a subset L of A is defined as in (1.1). The proofs of the following results can be found in [21].

3

Lemma 2.1. *Let A be a complex Banach space, let a and b be M-orthogonal elements of A, and let s and t elements of $\mathbb{R}$ such that*

$$\|a\| \le \|b\|, \qquad |s| \le |t|.$$

Then, the following results hold.

(i) *The elements sa and tb are M-orthogonal.*

(ii) *$\|sa + tb\| = \|sa - tb\| = \|tb\|$.*

Lemma 2.2. *Let A be a complex Banach space, with unit ball A_1, and let a and b be elements of A of norm one. Then, the following conditions are equivalent.*

(i) *The elements a and b are M-orthogonal.*

(ii) *The elements $a + b$ and $a - b$ are contained in the face $\mathrm{face}(a)$ of A_1 generated by a.*

(iii) *The elements $a + b$ and $a - b$ are contained in a proper face G of A_1.*

(iv) *There exist proper faces F and G of A_1 such that $a + b$ lies in F and $a - b$ lies in G.*

These results have the following corollary.

Corollary 2.3. *Let A be a complex Banach space and let a and b be elements of A. Then, the following conditions are equivalent.*

(i) *For all positive real numbers s, the elements sa and b are M-orthogonal.*

(ii) *For all real numbers s and t, the elements sa and tb are M-orthogonal.*

(iii) *The closed unit ball $(\mathbb{R}a \oplus \mathbb{R}b)_1$ in the subspace $\mathbb{R}a \oplus \mathbb{R}b$ of A coincides with $(\mathbb{R}a)_1 \oplus (\mathbb{R}b)_1$.*

Proof. (i)$\Rightarrow$(ii) This is immediate from Lemma 2.1.

(ii)$\Rightarrow$(iii) Since, for real numbers s and t,

$$\|sa + tb\| = \max\{\|sa\|, \|tb\|\},$$

the result is immediate.

(iii)$\Rightarrow$(i) Let s be a real number such that

$$\|sa\| \le \|b\|,$$

and let C be the convex set $\|b\|(\mathbb{R}a \oplus \mathbb{R}b)_1$. Since b lies in $\|b\|(\mathbb{R}b)_1$ and both sa and $-sa$ lie in $\|b\|(\mathbb{R}a)_1$, the elements $b + sa$ and $b - sa$ lie in C, and, since

$$b = \tfrac{1}{2}(b + sa) + \tfrac{1}{2}(b - sa),$$

it can be seen that $b + sa$ and $b - sa$ lie in the face $\mathrm{face}(b)$ of C generated by b. It follows from Lemma 2.2 that b and sa are M-orthogonal. A similar argument can be used in the case in which

$$\|sa\| \ge \|b\|,$$

thereby completing the proof. $\qquad\qquad\square$

The results above give some information about the M-orthogonal complement $\{a\}^\square$ of the set $\{a\}$ in the complex Banach space A. In order to introduce tangent spaces to the unit ball at an element a of norm one in A, the complex structure must be considered.

Lemma 2.4. *Let A be a complex Banach space, with closed unit ball A_1, and let a be an element of A_1 of norm one. Then:*

(i) $\{a\}^\square \cap A_1 = \{b \in A : \|a + tb\| = 1, \forall t \in [-1, 1]\}$;

(ii) $\{ia\}^\square \cap \{a\}^\square \cap A_1 \subseteq 2^{\frac{1}{2}}\{b \in A : \|a + tb\| = 1, \forall t \in \mathbb{C}, |t| \leq 1\}$;

(iii) $\{ia\}^\square \cap A_1 = i\{a\}^\square \cap A_1$;

(iv) *the real linear span $\mathrm{lin}_\mathbb{R}(\{ia\}^\square \cap \{a\}^\square \cap A_1)$ of the set $\{ia\}^\square \cap \{a\}^\square \cap A_1$ coincides with its complex linear span $\mathrm{lin}_\mathbb{C}(\{ia\}^\square \cap \{a\}^\square \cap A_1)$.*

Proof. (i) Let b be an element of $\{a\}^\square \cap A_1$. Then, by M-orthogonality, both $a + b$ and $a - b$ lie in A_1, and, since

$$a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b)$$

it follows that $a + b$ and $a - b$ lie in face(a). However, for all elements t of $[-1, 1]$,

$$a + tb = \tfrac{1}{2}(1 + t)(a + b) + \tfrac{1}{2}(1 - t)(a - b), \tag{2.1}$$

and it follows that $a + tb$, and, similarly, $a - tb$ lie in face(a). Since a is of norm one, face(a) is a proper face of A_1, from which it follows that $a + tb$ is of norm one. Consequently,

$$\{a\}^\square \cap A_1 \subseteq \{b \in A : \|a + tb\| = 1, \forall t \in [-1, 1]\}.$$

Now, let b be an element of A such that, for all elements t in $[-1, 1]$, $a + tb$ is of norm one. Then, in particular,

$$\|a + b\| = \|a - b\| = 1, \tag{2.2}$$

and it follows that

$$2\|b\| = \|(a + b) - (a - b)\| \leq \|a + b\| + \|a - b\| = 2,$$

and, hence that b is of norm not greater than one. It follows from (2.2) that

$$1 = \|a + b\| = \|a - b\| = \max\{\|a\|, \|b\|\}.$$

Therefore, a and b are M-orthogonal, and b lies in $\{a\}^\square \cap A_1$, as required.

(ii) Let b be an element of $\{ia\}^\square \cap \{a\}^\square \cap A_1$. Then, by (i), for all t in $[-1, 1]$,

$$\|a + itb\| = \|a + tb\| = 1,$$

and, using (2.1), the elements $a \pm b$ and $a \pm ib$ lie in face(a). Observe that, for each complex number s of modulus not greater than one, there exist real numbers s_1, s_2, s_3 and s_4 in $[-1, 1]$ of sum one such that

$$sb = 2^{\frac{1}{2}}(s_1 b - s_2 b + is_3 b - is_4 b),$$

and, hence that

$$a + 2^{-\frac{1}{2}} sb = s_1(a + b) + s_2(a - b) + s_3(a + ib) + s_4(a - ib) \in \mathrm{face}(a).$$

Arguing as in the proof of (i), it follows that $a + 2^{-\frac{1}{2}} sb$ is of norm one, as required.

5

(iii) Observe that, by (i), an element b in A lies in the set $\{ia\}^\square \cap A_1$ if and only if, for all real numbers t in $[-1, 1]$,

$$1 = \|ia + tb\| = \|a - itb\|$$

which occurs if and only if $-ib$ lies in $\{a\}^\square \cap A_1$, as required.

(iv) Since, by (iii), the set $\{ia\}^\square \cap \{a\}^\square \cap A_1$ is invariant under multiplication by i, this follows immediately.

$\square$

For each element a of norm one in the complex Banach space A, the *tangent disc* S_a at a is defined by

$$S_a = \{b \in A : \|a + sb\| = 1, \forall s \in \mathbb{C}, |s| \le 1\},$$

and the complex linear span of S_a is denoted by R_a. In [4] the notation Θ_a is used instead of R_a. It is now possible to relate the M-orthogonal complements of elements a of A of norm one to the tangent space R_a.

Lemma 2.5. *Let A be a complex Banach space, with unit ball A_1, and let a be an element of norm one in A_1. Then, the subspace R_a coincides with the complex linear span* $\text{lin}_{\mathbb{C}}(\{ia\}^\square \cap \{a\}^\square \cap A_1)$ *of the set* $\{ia\}^\square \cap \{a\}^\square \cap A_1$.

Proof. This follows immediately from Lemma 2.4. $\square$

Let A be the dual space of a complex Banach space E. For any subset L of elements of norm one in A, the spaces $R_L(A)$, $R_L^{w^*}(A)$, $T_L^{w^*}(A)$, and $T_L^n(A)$ are defined by

$$R_L(A) = \bigcap_{a \in L} R_a, \quad R_L^{w^*}(A) = \bigcap_{a \in L} \overline{R_a}^{w^*}, \quad T_L^{w^*}(A) = \bigcap_{a \in L} (\{a\}_{\prime})^{\circ}, \quad T_L^n(A) = \bigcap_{a \in L} (\{a\}')_{\circ},$$

respectively, where, for subsets V of E and W of A^*, V° and $W_{\circ}$ respectively denote the topological annihilators of V and W in A. These represent various different tangent spaces to the unit ball A_1 at L. It is clear that $R_L(A)$ is contained in $R_L^{w^*}(A)$, and that $T_L^n(A)$ is contained in $T_L^{w^*}(A)$.

For complex Banach spaces A and B, the set of holomorphic mappings from an open subset U of B into A is denoted by $\text{Hol}(U, A)$. When M and L are arbitrary subsets of B and A, respectively, a mapping h from M to L is said to lie in $\text{Hol}(M, L)$ if there exists an open set U in B containing M and h is the restriction $f|_M$ of an element f of $\text{Hol}(U, A)$ to M. For the definition and properties of holomorphic mappings the reader is referred to [3], [14], [15], [23], [29], and [30]. Let A_1 and B_1 be the closed unit balls in A and B, respectively. A bounded linear operator T from B to A is said to be *holomorphically rigid* if it is the only element h in $\text{Hol}(B_1, A)$ such that

$$h(B_1) \subseteq \|T\|A_1, \quad h(0) = 0, \quad h'(0) = T,$$

where h' denotes the Fréchet derivative of h. A closed subspace B of A is said to be *rigid* in A if the natural embedding of B into A is holomorphically rigid. The next result relates holomorphic rigidity to M-orthogonality.

Lemma 2.6. *Let A be a complex Banach space and let B be a rigid subspace of A. Then, the M-orthogonal complement $B^\square$ of B in A is equal to $\{0\}$.*

6

Proof. Let a be an element of norm one in $B^{\square}$, and let y be an element in the unit ball B_1^* of the dual space of B. Define the mapping h from B to A, for elements b in B, by

$$h(b) = b + y(b)^2 a.$$

Then, h is holomorphic and, by M-orthogonality, for each b in B,

$$\|h(b)\| = \|b + y(b)^2 a\| = \max\{\|b\|, |y(b)^2|\|a\|\} = \|b\|.$$

It follows that $h(B_1)$ is a subset of A_1. Moreover,

$$h(0) = 0, \qquad h'(0) = \mathrm{id}_B.$$

But, since y and a are non-zero, h is not the identity on B, and, hence, B is not rigid in A. It follows that $B^{\square}$ is equal to $\{0\}$. $\qquad\square$

3. JBW*-ALGEBRAS AND JBW*-TRIPLES

A Jordan *-algebra A which is also a complex Banach space such that, for all elements a and b in A,

$$\|a^*\| = \|a\|, \quad \|a \circ b\| \le \|a\|\|b\|, \quad \|\{a\,a\,a\}\| = \|a\|^3,$$

where

$$\{a\,b\,c\} = a \circ (b^* \circ c) + (a \circ b^*) \circ c - b^* \circ (a \circ c)$$

is the Jordan triple product on A, is said to be a *Jordan C*-algebra* [42] or *JB*-algebra* [43]. A Jordan C*-algebra which is the dual of a Banach space is said to be a *Jordan W*-algebra* [16] or a *JBW*-algebra* [43]. Examples of JB*-algebras are C*-algebras and examples of JBW*-algebras are W*-algebras, in both cases equipped with the Jordan product

$$a \circ b = \tfrac{1}{2}(ab + ba).$$

The self-adjoint parts of JB*-algebras and JBW*-algebras are said to be *JB-algebras* and *JBW-algebras* respectively. For the properties of C*-algebras and W*-algebras the reader is referred to [36] and [38], and for the algebraic properties of Jordan algebras to [25], [31], and [35]. The set $\mathcal{P}(A)$ of self-adjoint idempotents, the *projections*, in a JBW*-algebra A forms a complete orthomodular lattice with respect to the partial ordering defined, for elements e and f in $\mathcal{P}(A)$, by $e \le f$ if

$$e \circ f = e,$$

and the orthocomplementation $e \mapsto 1 - e$, where 1 is the unit in A. The *centre* $Z(A)$ of A consists of elements c of A such that, for all elements a and b in A,

$$c \circ (a \circ b) = a \circ (c \circ b).$$

The centre $Z(A)$ of A forms a commutative W*-algebra, the complete Boolean lattice $\mathcal{P}(Z(A))$ coinciding with the centre $\mathcal{ZP}(A)$ of the complete orthomodular lattice $\mathcal{P}(A)$. Furthermore, there exists an ortho-order isomorphism $z \mapsto z \circ A$ from $\mathcal{P}(Z(A))$ onto the complete Boolean lattice of weak*-closed ideals in A [16], [25].

Recall that a complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\,b\,c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies the identity

$$[D(a, b), D(c, d)] = D(\{a\,b\,c\}, d) - D(c, \{d\,a\,b\}), \tag{3.1}$$

7

where [,] denotes the commutator and D is the mapping from $A \times A$ to A defined by

$$D(a,b)c = \{a\ b\ c\},$$

is said to be a *Jordan*-triple*. The Jordan*-triple A is said to be *anisotropic* if an element a is equal to zero if and only if $\{a\ a\ a\}$ is equal to zero. A subspace B of a Jordan*-triple A is said to be a *subtriple* if $\{B\ B\ B\}$ is contained in B, is said to be an *inner ideal* if $\{B\ A\ B\}$ is contained in B, and is said to be and *ideal* if both $\{B\ A\ A\}$ and $\{A\ B\ A\}$ are contained in B. When A is also a Banach space such that D is continuous from $A \times A$ to the Banach space $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of [8], Definition 5.1, with non-negative spectrum and satisfies

$$\|D(a,a)\| = \|a\|^2, \tag{3.2}$$

then A is said to be a *JB*-triple*. Observe that a JB*-triple is anisotropic. A JB*-triple which is the dual of a Banach space is said to be a *JBW*-triple*. In this case A has a unique predual denoted by A_*. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [5], [6], [13], [28], [32], [33], [39], [40], and [41].

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal* when $D(a,b)$ is equal to zero. For a subset B of A, denote by $B^\perp$ the subset of A which consists of all elements in A which are orthogonal to all elements in B. The subset $B^\perp$ is said to be the *annihilator* of B. Then, $B^\perp$ is a weak*-closed inner ideal in A. Moreover, for subsets B and C of A,

$$B^\perp \cap B \subseteq \{0\}, \qquad B \subseteq B^{\perp\perp}, \qquad B^\perp = B^{\perp\perp\perp},$$

and if B is contained in C then $C^\perp$ is contained in $B^\perp$. For each non-empty subset B of the JBW*-triple A, the *kernel* $\mathrm{Ker}(B)$ of B is the weak*-closed subspace of elements a in A for which $\{B\ a\ B\}$ is equal to $\{0\}$. It follows that the annihilator $B^\perp$ of B is contained in $\mathrm{Ker}(B)$ and that $B \cap \mathrm{Ker}(B)$ is contained in $\{0\}$.

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\ u\ u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. For each tripotent u in the JBW*-triple A, the weak*-continuous conjugate linear operator $Q(u)$ and, for j equal to 0, 1 or 2, the weak*-continuous linear operators $P_j(u)$, are defined by

$$Q(u)(a) = \{u\ a\ u\}, \qquad P_2(u)(a) = Q(u)^2(a),$$
$$P_1(u) = 2(D(u,u) - Q(u)^2), \quad P_0(u) = \mathrm{id}_A - 2D(u,u) + Q(u)^2.$$

The linear operators $P_j(u)$ are weak*-continuous projections onto the eigenspaces $A_j(u)$ of $D(u,u)$ corresponding to eigenvalues $j/2$. The corresponding decomposition

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is said to be the *Peirce decomposition* of A relative to u. For j, k, and l equal to 0, 1, or 2, $A_j(u)$ is a sub-JBW*-triple such that $\{A_j(u)\ A_k(u)\ A_l(u)\}$ is contained in $A_{j-k+l}(u)$ when $j - k + l$ is equal to 0, 1, or 2, and is equal to $\{0\}$ otherwise. Moreover,

$$\{A_2(u)\ A_0(u)\ A\} = \{A_0(u)\ A_2(u)\ A\} = \{0\}, \tag{3.3}$$

8

and $A_0(u)$ and $A_2(u)$ are inner ideals in A. Observe that,

$$\{u\}^\perp = A_2(u)^\perp = A_0(u) \tag{3.4}$$

and

$$\mathrm{Ker}(A_2(u)) = A_0(u) \oplus A_1(u), \quad \mathrm{Ker}(A_0(u)) = A_2(u) \oplus A_1(u). \tag{3.5}$$

For more details the reader is referred to [17] and [19]. With respect to the product $(a, b) \mapsto a \circ b$ and involution $a \mapsto a^\dagger$ defined by

$$a \circ b = \{a\ u\ b\}, \quad a^\dagger = \{u\ a\ u\}$$

$A_2(u)$ is a JBW*-algebra with unit u. For two elements u and v of $\mathcal{U}(A)$, write $v \leq u$ if $\{v\ u\ v\}$ is equal to v. The set of tripotents v in $\mathcal{U}(A)$ such that $v \leq u$ coincides with the set $\mathcal{P}(A_2(u))$ of projections in the JBW*-algebra $A_2(u)$. It follows from the spectral theorem that the linear span of $\mathcal{U}(A)$ is weak*-dense in A. Let $\mathcal{U}(A)^-$ be the union of the set $\mathcal{U}(A)$ and a point set $\{\omega\}$ and, for all elements u in $\mathcal{U}(A)^-$ write $u \leq \omega$. It is clear that this defines a partial ordering on $\mathcal{U}(A)^-$. Observe that, when A is a W*-algebra, $\mathcal{U}(A)$ is the set of partial isometries in A.

Recall that, for each element u in $\mathcal{U}(A)$, the set $\{u\}_{\prime}$ is a norm-exposed face of $A_{*,1}$. Define $\{\omega\}_{\prime}$ to be the set $A_{*,1}$. The following result was proved in [18].

Lemma 3.1. *Let A be a JBW*-triple with predual A_*. Then, the following results hold.*

(i) *The mapping $u \mapsto \{u\}_{\prime}$ is an order isomorphism from the partially ordered set $\mathcal{U}(A)^-$ of tripotents in A, with a largest element adjoined, onto the complete lattice $\mathcal{F}_n(A_{*,1})$ of all norm-closed faces of the closed unit ball $A_{*,1}$ in A_*, and, hence, $\mathcal{U}(A)^-$ is a complete lattice.*

(ii) *The mapping $u \mapsto \{u\}_{\prime}'$ is an anti-order-isomorphism from $\mathcal{U}(A)^-$ onto the complete lattice $\mathcal{F}_{w^*}(A_1)$ of weak*-closed faces of the closed unit ball A_1 in A and*

$$\{u\}_{\prime}' = u + A_0(u)_1.$$

Let a be an arbitrary element of the JBW*-triple A and define the sequence (a^{2j+1}) inductively by

$$a^1 = a, \qquad a^{2j+1} = \{a\ a^{2j-1}\ a\}$$

Then, for positive integers j, k and l,

$$\{a^{2j-1}\ a^{2k-1}\ a^{2l-1}\} = a^{2(j+k+l)-3}. \tag{3.6}$$

The reader is referred to [34] for details. A proof of the next result can be found in [20].

Lemma 3.2. *Let a be an element of norm one in the JBW*-triple A. Then, the following results hold.*

(i) *The sequence (a^{2j-1}) converges in the weak* topology to an element $u(a)$ in $\mathcal{U}(A)$ such that*

$$\{u(a)\ a\ u(a)\} = u(a)$$

9

and the norm-exposed faces $\{u(a)\}_,$ *and* $\{a\}_,$ *of* $A_{*,1}$ *coincide. Further-more,* $u(a)$ *is contained in the set* $\overline{face(a)}^{w^*}$ *and the weak*-closed face* $u(a)+$ $A_0(u(a))_1$ *coincides with* $face(\overline{face(a)}^{w^*})$.

(ii) *There exists a smallest element* $r(a)$ *in* $\mathcal{U}(A)$ *such that* a *is positive in the JBW*-algebra* $A_2(r(a))$ *and, in* $A_2(r(a))$,

$$0 \leq u(a) \leq a^{2j-1} \leq a \leq r(a).$$

Furthermore, the smallest weak*-closed subtriple $W(a)$ *of* A *containing* a *is an associative JBW*-subalgebra of* $A_2(r(a))$, *with unit* $r(a)$, *and coincides with the weak*-closed linear span of the sequence* (a^{2j-1}).

The tripotent $r(a)$ described in Lemma 3.2 is said to be the *support* of a. The proof of the following result may be found in [21]

Lemma 3.3. *Let* a *and* b *be elements of norm one in the JBW*-triple* A. *Then,* a *and* b *are orthogonal if and only if their support tripotents* $r(a)$ *and* $r(b)$ *are orthogonal, in which case the element* $a+b$ *is of norm one and*

$$r(a+b) = r(a) + r(b).$$

The next two results produce a list of properties of the annihilators of single elements of the JBW*-triple A.

Lemma 3.4. *Let* a *be an element of a JBW*-triple* A. *Then, for* j *equal to* $1, 2, \ldots$, *the annihilator* $\{a\}^{\perp}$ *of* $\{a\}$ *is contained in the annihilator* $\{a^{2j-1}\}^{\perp}$ *of* $\{a^{2j-1}\}$.

Proof. The result is clear when j is equal to 1. Suppose that the result holds for j equal to k, and let b be an element of $\{a\}^{\perp}$. Then,

$$D(b, a) = D(b, a^{2k-1}) = 0,$$

and, applying (3.1),

$$\begin{aligned} D(b, a^{2k+1}) &= D(b, \{a\,a\,a^{2k-1}\}) \\ &= D(\{a\,a^{2k-1}\,b\}, a) + D(b,a)D(a, a^{2k-1}) - D(a, a^{2k-1})D(b,a) \\ &= D(D(b, a^{2k-1})a, a) = 0. \end{aligned}$$

The result follows by induction. $\square$

The corollary below summarises the properties of annihilators of elements of norm one.

Corollary 3.5. *Let* A *be a JBW*-triple, with predual* A_*, *let* A_1 *and* $A_{*,1}$ *be the unit balls in* A *and* A_*, *respectively, let* a *be an element of norm one in* A_1, *and let* $u(a)$ *and* $r(a)$ *be the tripotents in* A *associated with* a. *Then, the following results hold.*

(i) *The annihilator* $W(a)^{\perp}$ *of the smallest weak*-closed subtriple* $W(a)$ *of* A *containing* a *coincides with the annihilator* $\{a\}^{\perp}$ *of* $\{a\}$.

(ii) *The annihilator* $\{a\}^{\perp}$ *of* $\{a\}$ *is contained in the annihilator* $\{u(a)\}^{\perp}$ *of* $\{u(a)\}$, *which coincides with the weak*-closed inner ideal* $A_0(u(a))$.

(iii) *The annihilator $\{a\}^{\perp}$ of $\{a\}$ coincides with the annihilator $\{r(a)\}^{\perp}$ of $\{r(a)\}$, which also coincides with the weak*-closed inner ideal $A_0(r(a))$.*

(iv) *The weak*-closed subspace $(\{a\}_{\prime})^{\circ}$ of A coincides with the kernel $A_0(u(a)) \oplus A_1(u(a))$ of the weak*-closed inner ideal $A_2(u(a))$.*

(v) *The norm-exposed face $\{a\}_{\prime}$ of $A_{*,1}$ is contained in the topological annihilator $(\{u(a)\}^{\perp})_{\circ}$ of the weak*-closed inner ideal $\{u(a)\}^{\perp}$, which is itself contained in the topological annihilator $(\{a\}^{\perp})_{\circ}$ of $\{a\}^{\perp}$.*

(vi) *The annihilator $\{a\}^{\perp}$ of $\{a\}$ is contained in the weak*-closed subspace $(\{a\}_{\prime})^{\circ}$ of A.*

Proof. Since a is contained in $W(a)$, it follows that $W(a)^{\perp}$ is contained in $\{a\}^{\perp}$. The reverse inclusion follows from Lemma 3.2(ii), Lemma 3.4, and the weak*-continuity of the triple product, thereby completing the proof of (i). It follows from Lemma 3.2(i) that $u(a)$ lies in $W(a)$, and (ii) then follows from (i). Since, by Lemma 3.2(ii), $r(a)$ is also contained in $W(a)$, (i) again shows that $\{a\}^{\perp}$ is contained in $\{r(a)\}^{\perp}$. Since, by (3.4), $\{r(a)\}^{\perp}$ coincides with $A_0(r(a))$, and, by Lemma 3.2(ii), a lies in $A_2(r(a))$, it follows that $\{r(a)\}^{\perp}$ is contained in $\{a\}^{\perp}$, and (iii) holds. The norm-exposed face $\{a\}_{\prime}$, which, by Lemma 3.2(i), coincides with $\{u(a)\}_{\prime}$, is the normal state space of the JBW*-algebra $A_2(u(a))$, and, therefore, its linear span coincides with the predual $A_2(u(a))_*$ of $A_2(u(a))$. Hence,

$$(\{a\}_{\prime})^{\circ} = (\mathrm{lin}_{\mathbb{C}}(\{a\}_{\prime}))^{\circ} = (A_2(u(a))_*)^{\circ} = A_0(u(a)) \oplus A_1(u(a)),$$

thereby completing the proof of (iv). It is clear that $\{a\}_{\prime}$ is contained in $(\{u(a)\}^{\perp})_{\circ}$, which coincides with $A_2(u(a))_* \oplus A_1(u(a))_*$. The last inclusion of (v) follows by applying the topological annihilator to (ii). Since $\{a\}^{\perp}$ is a weak*-closed subspace, (vi) follows by applying the topological annihilator to (v). $\square$

4. MAIN RESULTS

Let A be a JBW*-triple and let B be a subset of A. The first part of this section is concerned with the connections between the M-orthogonal complement $B^{\square}$ of B and the algebraic annihilator $B^{\perp}$ of B. The first result along these lines, the proof of which relies upon that of [24], Lemma 1.3, follows.

Lemma 4.1. *Let A be a JBW*-triple and let a be an element in A. Then, the annihilator $\{a\}^{\perp}$ of $\{a\}$ is contained in the M-orthogonal complement $\{a\}^{\square}$ of $\{a\}$.*

Proof. Let b be a non-zero element of $\{a\}^{\perp}$. It follows from Corollary 2.3 that, in order to show that b is M-orthogonal to a, it is sufficient to prove it for the case in which a is of norm one and b is of norm less than one. In this case, using (3.3), (3.6) and the fact that $D(a, b)$ is equal to zero, for j equal to $1, 2, \dots$,

$$\|a \pm b\| = \|(a \pm b)^{3^j}\|^{3^{-j}} = \|a^{3^j} \pm b^{3^j}\|^{3^{-j}} \le (2)^{3^{-j}}.$$

Allowing j to increase indefinitely, it can be seen that

$$\|a \pm b\| \le 1.$$

On the other hand,

$$2 = 2\|a\| = \|(a + b) + (a - b)\| \le \|a + b\| + \|a - b\| \le 2,$$

and it follows that

$$\|a \pm b\| = 1 = \max\{\|a\|, \|b\|\},$$

as required. □

The next step on the way to the proof of the first main theorem follows.

Lemma 4.2. *Let A be a JBW*-triple, with closed unit ball A_1, and let a be an element in A of norm one. Then, the set $\{a\}^{\square} \cap A_1$ is contained in the annihilator $\{u(a)\}^{\perp}$ of $\{u(a)\}$.*

Proof. For each element b of $\{a\}^{\square} \cap A_1$,

$$\|a \pm b\| = \max\{\|a\|, \|b\|\} = 1.$$

Since

$$a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b),$$

it follows that the elements $a + b$ and $a - b$ lie in the face $\mathrm{face}(a)$ of A_1 generated by a. By Lemma 3.2(i),

$$a \in \mathrm{face}(a) \subseteq u(a) + A_0(u(a))_1.$$

Therefore,

$$a + b \in u(a) + A_0(u(a))_1 \subseteq a + A_0(u(a)),$$

and it can be seen that b lies in $A_0(u(a))$, as required. □

The next result is immediate from Lemma 4.1 and Lemma 4.2.

Corollary 4.3. *Let A be a JBW*-triple and let $\mathcal{V}$ be a non-empty subset of the set $\mathcal{U}(A)$ of tripotents in A. Then, the set $\mathcal{V}^{\square} \cap A_1$ coincides with the set $\mathcal{V}^{\perp} \cap A_1$.*

It is now possible to prove the first main result.

Theorem 4.4. *Let A be a JBW*-triple, and let B be a weak*-closed subtriple of A. Then, the annihilator $B^{\perp}$ of B coincides with the M-orthogonal complement $B^{\square}$ of B.*

Proof. That $B^{\perp}$ is contained in $B^{\square}$ is immediate from Lemma 4.1. Let a be an element of $B^{\square}$. Then, for each tripotent u in B and each positive real number s, the element a is M-orthogonal to su. It follows from Corollary 4.3 that, for all real numbers s, the element $a/\|a\|$ is M-orthogonal to su, and, hence that the elements $u + (a/\|a\|)$ and $u - (a/\|a\|)$ are of norm one. An argument similar to that used in the proof of Lemma 4.2 shows that the elements $u \pm (a/\|a\|)$ are contained in the face $\mathrm{face}(u)$ of the unit ball A_1 in A generated by u, which itself is contained in the weak*-semi-exposed face $u + A_0(u)_1$ of A_1. Hence, a lies in $A_0(u)$. It follows that, for all elements u in $\mathcal{U}(B)$,

$$D(u, a) = 0.$$

Since the linear span of $\mathcal{U}(B)$ is weak*-dense in B, the weak*-continuity of the triple product implies that, for all elements b in B,

$$D(b, a) = 0,$$

and a lies in $B^{\perp}$, as required. □

12

Before proving the second main result the following lemma is needed.

Lemma 4.5. *Let A be a JBW^*-triple, let a be an element of norm one in A having support $r(a)$, and let R_a be the linear span of the tangent disc S_a at a. Then, the annihilator $\{a\}^\perp$ of $\{a\}$, which coincides with the Peirce-zero space $A_0(r(a))$, is contained in R_a.*

Proof. By Lemma 4.1,

$$\{a\}^\perp \subseteq \{a\}^\square, \qquad \{a\}^\perp = \{ia\}^\perp \subseteq \{ia\}^\square.$$

Therefore, by Lemma 2.5,

$$
\begin{aligned}
\{a\}^\perp = \{ia\}^\perp &= \mathrm{lin}_{\mathbb{C}}(\{ia\}^\perp \cap \{a\}^\perp \cap A_1) \\
&\subseteq \mathrm{lin}_{\mathbb{C}}(\{ia\}^\square \cap \{a\}^\square \cap A_1) \\
&= R_a,
\end{aligned}
$$

as required. $\qquad\square$

The main result given below identifies the various tangent spaces to the unit ball A_1 in the JBW^*-triple A at the set of elements of norm one in a weak*-closed subtriple B of A.

Theorem 4.6. *Let A be a JBW^*-triple, with closed unit ball A_1, let B be a weak*-closed subtriple of A, and let L_B be the set of elements of B of norm one. Then, the following results hold.*

(i) *The subspace $R_{L_B}(A)$ and the weak*-closed subspace $R_{L_B}^{w^*}(A)$ both coincide with the M-orthogonal complement $B^\square$ and annihilator $B^\perp$ of B.*

(ii) *The weak*-closed subspace $T_{L_B}^{w^*}(A)$ and the norm-closed subspace $T_{L_B}^n(A)$ both coincide with the kernel $\mathrm{Ker}(B)$ of B.*

Proof. (i) Observe that the equality of $B^\square$ and $B^\perp$ follows from Theorem 4.4. For each element a of norm one in A, since the inner ideal $\{u(a)\}^\perp$ is weak*-closed, it can be seen from Lemma 2.4(iii) and Lemma 4.2 that

$$\overline{R_a}^{w^*} = \overline{\mathrm{lin}_{\mathbb{C}}(\{ia\}^\square \cap \{a\}^\square \cap A_1)}^{w^*} \subseteq \{u(a)\}^\perp.$$

Since B is a weak*-closed subtriple, for each element a in B of norm one, the smallest weak*-closed subtriple $W(a)$ of A containing a is contained in B, and, in particular, by Lemma 3.2(i), the tripotent $u(a)$ is contained in B. Since the linear span of $\mathcal{U}(B)$ is weak*-dense in B, it follows that

$$R_{L_B}^{w^*}(A) = \bigcap_{a \in L_B} \overline{R_a}^{w^*} \subseteq \bigcap_{a \in L_B} \{u(a)\}^\perp = \bigcap_{u \in \mathcal{U}(B)} \{u\}^\perp = B^\perp. \qquad (4.1)$$

By Lemma 4.5, it can be seen that

$$B^\perp = \bigcap_{a \in L_B} \{a\}^\perp \subseteq \bigcap_{a \in L_B} R_a = R_{L_B}(A). \qquad (4.2)$$

It follows from (4.1) and (4.2) that

$$R_{L_B}^{w^*}(A) = R_{L_B}(A) = B^\square = B^\perp,$$

as required.

(ii) Observe that, by Corollary 3.5(iv) and Lemma 3.2,

$$T_{L_B}^{w^*}(A) = \bigcap_{a \in L_B} (\{a\}_{\prime})^\circ = \bigcap_{a \in L_B} \mathrm{Ker}(A_2(u(a))) = \bigcap_{u \in \mathcal{U}(B)} \mathrm{Ker}(A_2(u)). \qquad (4.3)$$

Let b be an element of $T_{L_B}^{w^*}(A)$. Then, by (4.3), for all elements u in $\mathcal{U}(B)$,

$$\{A_2(u)\, b\, A_2(u)\} = \{0\}, \qquad (4.4)$$

and, in particular,

$$\{u\, b\, u\} = 0. \qquad (4.5)$$

For non-zero elements u and v of $\mathcal{U}(B)$, the element $(u+v)/\|u+v\|$ lies in B, and it follows from Lemma 3.2(ii) that $r((u+v)/\|u+v\|)$ lies in $\mathcal{U}(B)$ and that $u+v$ is contained in $A_2(r((u+v)/\|u+v\|))$. Therefore, by (4.4),

$$\{(u+v)\, b\, (u+v)\} = 0, \qquad (4.6)$$

and, combining (4.5) and (4.6),

$$\{u\, b\, v\} = 0.$$

Since the linear span of $\mathcal{U}(B)$ is weak*-dense in B, it can be seen that

$$\{B\, b\, B\} = \{0\},$$

and b lies in $\mathrm{Ker}(B)$. It follows from (4.3) that $T_{L_B}^{w^*}(A)$ is contained in $\mathrm{Ker}(B)$. On the other hand, for every element b in $\mathrm{Ker}(B)$ and u in $\mathcal{U}(B)$,

$$\{u\, b\, u\} = 0,$$

and, therefore, b lies in $\mathrm{Ker}(A_2(u))$. It follows from (4.3) that b lies in $T_{L_B}^{w^*}(A)$, and the reverse inclusion is proved.

It remains to prove that the subspaces $T_{L_B}^{w^*}(A)$ and $T_{L_B}^{n}(A)$ coincide. It is clear that $T_{L_B}^{n}(A)$ is contained in $T_{L_B}^{w^*}(A)$. Let b be an element of $T_{L_B}^{w^*}(A)$, which, from above, coincides with $\mathrm{Ker}(B)$. Then,

$$\{B\, b\, B\} = 0,$$

and, using the results of [13] and identifying A with its canonical image in A^{**},

$$\{\overline{B}^{w^*}\, b\, \overline{B}^{w^*}\} = 0.$$

It follows that b lies in the kernel $\mathrm{Ker}(\overline{B}^{w^*})$ of the weak*-closed subtriple $\overline{B}^{w^*}$ of A^{**}, which, from above, coincides with $T_{L_{\overline{B}^{w^*}}}^{w^*}(A^{**})$. By definition, it follows that, for all elements a in $L_{\overline{B}^{w^*}}$,

$$b(\{a\}_{\prime}) = \{0\}.$$

In particular, this holds for all elements a in L_B, and, therefore, making the canonical identification,

$$b(\{a\}') = \{0\}.$$

It follows that, for all elements a in L_B, b lies in the subspace $(\{a\}')_\circ$ and, hence, in $T_{L_B}^{n}(A)$, as required. $\qquad \square$

14

Observe that it is a consequence of this result that the subspaces $R_{L_B}(A)$ and $T_{L_B}^n(A)$ are automatically weak*-closed. The proof of the last main result follows quite closely the proof of the less general result in [4]. The proof of the following two lemmas may also be found in [4].

Lemma 4.7. *Let A be a JBW^*-triple, with closed unit ball A_1, let B be a weak*-closed subtriple of A, let L_B be the set of elements in B of norm one, and let g be a holomorphic mapping from B to A such that, for each element a in L_B, the element $g(a)$ lies in the subspace R_a. Then g maps B into the subspace $R_{L_B}(A)$.*

Proof. Since $R_{L_B}(A)$ is weak*-closed, it is also norm-closed. The result then follows from [4], Corollary 7.3. □

A subset V of A is said to be a *set of determinancy* in A if, for every open connected set U containing V, the restriction mapping $h \mapsto h|_V$ from $\mathrm{Hol}(U, \mathbb{C})$ to $\mathrm{Hol}(V, \mathbb{C})$ is injective.

Lemma 4.8. *Let A and B be complex Banach spaces, let U be an open subset of A, let the subset V of U be a set of determinancy in A, and let h be a holomorphic mapping from U to B mapping V into a closed subspace L of B. Then, h maps U into L.*

The next lemma, a consequence of the maximum principle for holomorphic mappings, is related to Lemma III.1.2 in [23].

Lemma 4.9. *Let A be a complex Banach space, with unit ball A_1, and let h be a holomorphic function from an open subset of $\mathbb{C}$ containing the closed unit disc $\mathbb{C}_1$ in $\mathbb{C}$ into A_1 such that*

$$\|h(0)\| = 1,$$

and for each element s of $\mathbb{C}$ of modulus one,

$$\|h(s)\| \leq 1.$$

Then, the set $h(\mathbb{C}_1)$ is contained in the smallest norm-semi-exposed face $\{h(0)\}'$, of A_1 containing $h(0)$.

Proof. For each element x of $\{h(0)\}'$,

$$(x \circ h)(0) = x(h(0)) = 1,$$

and for each element s of $\mathbb{C}$ of modulus one,

$$|(x \circ h)(s)| \leq 1.$$

The maximum principle shows that the holomorphic function $(x \circ h)$ is identically equal to one on $\mathbb{C}_1$, and, therefore, $h(\mathbb{C}_1)$ is contained in $\{h(0)\}'_{\prime}$, as required. □

It is now possible to prove the final result.

Theorem 4.10. *Let A be a JBW^*-triple and and let B be a weak*-closed subtriple of A. Then B is rigid in A if and only if the annihilator $B^{\perp}$ of B is equal to $\{0\}$.*

15

Proof. If B is rigid in A then it follows from Theorem 4.4 and Lemma 2.6 that $B^\perp$ is equal to $\{0\}$. Conversely, suppose that this is the case and let h be a holomorphic mapping from B to A such that

$$h(B_1) \subseteq A_1, \quad h(0) = 0, \quad h'(0) = \mathrm{id}_B. \tag{4.7}$$

In order to complete the proof it is necessary to show that h is equal to id_B. By Theorem 4.6, it is sufficient to show that the holomorphic mapping

$$g = h - \mathrm{id}_B$$

maps B into R_{L_B}. By Lemma 4.7, it is therefore only required to show that, for each element a in L_B, $g(a)$ lies in R_a. From (4.7) it can be seen that the Taylor series of g and h at 0 are given by

$$g = \sum_{j=2}^{\infty} g_j, \qquad h = \mathrm{id}_B + \sum_{j=2}^{\infty} g_j \tag{4.8}$$

where g_j is a homogeneous polynomial of degree j. For each element a of L_B, and for each non-zero element s of $\mathbb{C}_1$. let

$$r_a(s) = s^{-1} h(sa) - a.$$

If the modulus $|s|$ of s is sufficiently small then, using (4.8),

$$r_a(s) = \sum_{j=2}^{\infty} s^{j-1} g_j(a).$$

By defining $r_a(0)$ to be equal to 0, r_a can be regarded as a holomorphic function on $\mathbb{C}_1$. By (4.7), for each element a in L_B and each element s of $\mathbb{C}_1$, ———————— $|s| = 1$

$$1 \geq \|h(sa)\| = \|s(a + r_a(s))\| = |s|\|a + r_a(s)\| = \|a + r_a(s)\|.$$

Therefore, the holomorphic function $s \mapsto a + r_a(s)$ defined on some open set containing $\mathbb{C}_1$ sends the set of complex numbers of unit modulus into the unit ball A_1 of A. When a is a tripotent u in B, by Lemma 4.9 and Lemma 3.1(ii),

$$u + r_u(1) = h(u) = u + g(u) \in u + A_0(u)_1. \tag{4.9}$$

It follows that, for every tripotent u in B, $g(u)$ lies in $A_0(u)$. For a general element a in L_B, having support $r(a)$ in A, which, by Lemma 3.2, lies in B, the weak*-closed subspace

$$B_2(r(a)) = A_2(r(a)) \cap B,$$

is a JBW*-algebra, the self-adjoint part $B_2(r(a))_{\mathrm{sa}}$ of which consists of elements b in B for which

$$\{r(a)\, b\, r(a)\} = b.$$

As in the proof of [4], Proposition 7.11, the set $\exp(iA_2(r(a))_{\mathrm{sa}})$ is a set of determinancy in $A_2(r(a))$ and, hence, $\exp(iA_2(r(a))_{\mathrm{sa}}) \cap B$ is a set of determinancy in $B_2(r(a))$. The set $\exp(iA_2(r(a))_{\mathrm{sa}}) \cap B$ consists of elements of $\mathcal{U}(B)$ all having the same Peirce spaces as $r(a)$. From (4.9) it follows that g maps every element of $\exp(iA_2(r(a))_{\mathrm{sa}}) \cap B$ into $A_0(r(a))$. Therefore, by Lemma 4.8, Corollary 3.5, and Lemma 4.5,

$$g(a) \in g(B_2(r(a))) \subseteq A_0(r(a)) = \{r(a)\}^\perp = \{a\}^\perp \subseteq R_a,$$

and the proof is complete. □

16

The above proof provides information about a holomorphic function of the type described in 4.7. The following corollary is now immediate from Theorem 4.4 and the proof of Theorem 4.10.

Corollary 4.11. *Under the conditions described in Theorem 4.10, let h be a holomorphic function from B to A, such that*

$$h(B_1) \subseteq A_1, \quad h(0) = 0, \quad h'(0) = \mathrm{id}_B.$$

Then, h is equal to $\mathrm{id}_B + g$, where g is a holomorphic function mapping B to its M-complement $B^\square$.

5. APPLICATIONS TO W*-ALGEBRAS

Let A be a W*-algebra with unit 1. Recall that, with respect to the triple product defined, for elements a, b, and c of A, by

$$\{a \, b \, c\} = \tfrac{1}{2}(ab^*c + cb^*a),$$

A is a JBW*-triple. Let α be a *-antiautomorphism of A of order two, and let

$$H(A, \alpha) = \{a \in A : \alpha(a) = a\}, \quad S(A, \alpha) = \{a \in A : \alpha(a) = -a\}.$$

Then $H(A, \alpha)$ and $S(A, \alpha)$ are weak*-closed subtriples of A, such that

$$A = H(A, \alpha) \oplus S(A, \alpha), \tag{5.1}$$

and $H(A, \alpha)$ is, in fact, a JBW*-subalgebra of A with unit 1. Furthermore,

$$\begin{aligned} \{H(A, \alpha) \, S(A, \alpha) \, H(A, \alpha)\} &\subseteq S(A, \alpha), \\ \{S(A, \alpha) \, H(A, \alpha) \, S(A, \alpha)\} &\subseteq H(A, \alpha), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \{H(A, \alpha) \, H(A, \alpha) \, S(A, \alpha)\} &\subseteq S(A, \alpha), \\ \{S(A, \alpha) \, S(A, \alpha) \, H(A, \alpha)\} &\subseteq H(A, \alpha). \end{aligned} \tag{5.3}$$

It is possible to identify the "tangent spaces" to the closed unit ball A_1 in A at the intersection of $H(A, \alpha)$ and $S(A, \alpha)$ with elements of A_1 of norm one, and, hence, to determine whether $H(A, \alpha)$ and $S(A, \alpha)$ are holomorphically rigid in A. Recall that, by Theorem 4.6,

$$R_{L_{H(A,\alpha)}}(A) = H(A, \alpha)^\perp = H(A, \alpha)^\square, \tag{5.4}$$

$$T^n_{L_{H(A,\alpha)}}(A) = \mathrm{Ker}(H(A, \alpha)), \tag{5.5}$$

the same equalities holding when $H(A, \alpha)$ is replaced by $S(A, \alpha)$.

Theorem 5.1. *Let A be a W*-algebra, let α be a *-antiautomorphism of A of order two, and let $H(A, \alpha)$ be the weak*-closed subtriple and JBW*-subalgebra of A consisting of elements of A invariant under α. Then, the tangent spaces $R_{L_{H(A,\alpha)}}(A)$ and $T^n_{L_{H(A,\alpha)}}(A)$ are both equal to $\{0\}$, and $H(A, \alpha)$ is holomorphically rigid in A.*

Proof. From (5.4) and (5.5), it is clearly sufficient to show that $\mathrm{Ker}(H(A, \alpha))$ is equal to $\{0\}$. Let a be an element of $\mathrm{Ker}(H(A, \alpha))$. Then, for all elements b and c in $H(A, \alpha)$,

$$\{b \, a \, c\} = 0,$$

and, choosing b and c equal to 1, it follows that a is equal to 0. $\square$

17

The situation regarding $H(A, \alpha)$ is therefore very straightforward. However, since $S(A, \alpha)$ is not a JBW*-subalgebra of A, in this case the situation is more complicated.

Theorem 5.2. *Let A be a W^*-algebra, let α be a $*$-antiautomorphism of A of order two, let $H(A, \alpha)$ be the weak*-closed subtriple and JBW*-subalgebra of A consisting of elements of A invariant under α, and let $S(A, \alpha)$ be the weak*-closed subtriple of A consisting of elements a of A for which $\alpha(a)$ is equal to $-a$. Then, the tangent space $R_{L_{S(A,\alpha)}}(A)$ to A_1 at $L_{H(A,\alpha)}$ coincides with the largest weak*-closed ideal of A contained in $H(A, \alpha)$ and is a commutative subalgebra of A.*

Proof. By (5.4), $R_{L_{S(A,\alpha)}}(A)$ coincides with $S(A, \alpha)^\perp$. Let a be an element of $S(A, \alpha)^\perp$, and, using (5.1), let a_h and a_s be the unique elements of $H(A, \alpha)$ and $S(A, \alpha)$, respectively, such that

$$a = a_h + a_s.$$

Then,

$$0 = \{a_s \ a \ a_s\} = \{a_s \ a_h \ a_s\} + \{a_s \ a_s \ a_s\},$$

and, using (5.1), (5.2), and the anisotropy of A, a_s is equal to zero, a is equal to a_h, and $S(A, \alpha)^\perp$ is contained in $H(A, \alpha)$. Now, suppose that a lies in A, b_h in $H(A, \alpha)$, c_s in $S(A, \alpha)$, and d_h and e_h in $S(A, \alpha)^\perp$. Then, using (3.1), and the fact that $D(c_s, d_h)$ is equal to zero,

$$
\begin{aligned}
\{c_s \ \{b_h \ e_h \ d_h\} \ a\} &= D(c_s, \{d_h \ e_h \ b_h\})a \\
&= D(\{e_h \ b_h \ c_s\}, d_h)a + [D(c_s, d_h), D(e_h, b_h)]a \\
&= D(\{e_h \ b_h \ c_s\}, d_h)a = 0,
\end{aligned}
$$

since, by (5.3), $\{e_h \ b_h \ c_s\}$ lies in $S(A, \alpha)$. It follows that

$$\{H(A, \alpha) \ S(A, \alpha)^\perp \ S(A, \alpha)^\perp\} \subseteq S(A, \alpha)^\perp. \tag{5.6}$$

Therefore, using (5.6),

$$
\begin{aligned}
\{A \ S(A, \alpha)^\perp \ S(A, \alpha)^\perp\} &= \{H(A, \alpha) \oplus S(A, \alpha) \ S(A, \alpha)^\perp \ S(A, \alpha)^\perp\} \\
&= \{H(A, \alpha) \ S(A, \alpha)^\perp \ S(A, \alpha)^\perp\} \\
&\subseteq S(A, \alpha)^\perp.
\end{aligned}
$$

It follows from [9], Proposition 1.3, that the weak*-closed inner ideal $S(A, \alpha)^\perp$ is an ideal in A that is also contained in $H(A, \alpha)$.

Suppose that I is a further weak*-closed ideal in A that is contained in $H(A, \alpha)$, and let a lie in its annihilator $I^\perp$. Then,

$$\alpha\{\alpha(a) \ I \ A\} = \{a \ \alpha(I) \ A\} = \{a \ I \ A\} = 0,$$

and it can be seen that $\alpha(a)$ lies in $I^\perp$. Since α is of order two, it follows that $\alpha(I^\perp)$ and $I^\perp$ coincide. From [28],

$$A = I \oplus I^\perp \tag{5.7}$$

and, for each element a_s in $S(A, \alpha)$, there exist unique elements b in I and c in $I^\perp$ such that

$$a_s = b + c.$$

18

Therefore,

$$b + c = a_s = -\alpha(a_s) = -\alpha(b) - \alpha(c),$$

and, using (5.7), both b and c lie in $S(A, \alpha)$. However, b also lies in $H(A, \alpha)$, and is, therefore, equal to zero. It follows that $S(A, \alpha)$ is contained in $I^\perp$, and, taking annihilators, that I is contained in $S(A, \alpha)^\perp$. Hence $S(A, \alpha)^\perp$ is the largest weak*-closed ideal in A that is contained in $H(A, \alpha)$. Therefore, there exists a unique central projection z in A such that $S(A, \alpha)^\perp$ coincides with zA. Observe that, since zA is contained in $H(A, \alpha)$, for each element a in A,

$$za = \alpha(za) = \alpha(a)\alpha(z),$$

and, choosing a equal to 1, it follows that z lies in the intersection of $\mathcal{P}(Z(A))$ and $H(A, \alpha)$, which, from [22], Lemma 3.2, coincides with $\mathcal{P}(Z(H(A, \alpha)))$, the complete Boolean lattice of central projections in the JBW*-algebra $H(A, \alpha)$. Observe that, for elements a and b in zA,

$$ab = (za)(zb) = z(ab) = \alpha(zab) = z\alpha(b)\alpha(a) = zba = ba,$$

and $S(A, \alpha)^\perp$ is commutative, as required. $\qquad\square$

The theorem has the following immediate corollary.

Corollary 5.3. *Under the conditions of Theorem 5.2, the weak*-closed subtriple $S(A, \alpha)$ is holomorphically rigid in the W^*-algebra A if and only if the only central projection z in A for which the ideal zA is contained in $H(A, \alpha)$ is equal to zero.*

Theorem 5.4. *Let A be a W^*-algebra, let α be a *-antiautomorphism of A of order two, let $H(A, \alpha)$ be the weak*-closed subtriple and JBW*-subalgebra of A consisting of elements of A invariant under α, and let $S(A, \alpha)$ be the weak*-closed subtriple of A consisting of elements a of A for which $\alpha(a)$ is equal to $-a$. Then, there exists a central projection w in $H(A, \alpha)$ such that the tangent space $T^r_{L_{S(A,\alpha)}}(A)$ to A_1 at $L_{S(A,\alpha)}$ coincides with the weak*-closed ideal $wH(A, \alpha)$ in $H(A, \alpha)$.*

Proof. Using (5.5), from [25], Proposition 4.3.6, it must be shown that the weak*-closed subspace $\mathrm{Ker}(S(A, \alpha))$ of A is a weak*-closed ideal in $H(A, \alpha)$. Precisely the same argument as that used in the proof of Theorem 5.2, shows that $\mathrm{Ker}(S(A, \alpha))$ is contained in $H(A, \alpha)$. Let a_h and b_h lie in $\mathrm{Ker}(S(A, \alpha))$, let c_h lie in $H(A, \alpha)$ and let d_s lie in $S(A, \alpha)$. Then, using (3.1), (5.3), and the fact that

$$\{\{b_h\ c_h\ d_s\}\ a_h\ d_s\} \in \{S(A, \alpha)\ \mathrm{Ker}(S(A, \alpha))\ S(A, \alpha)\} = \{0\},$$
$$\{d_s\ a_h\ \{b_h\ c_h\ d_s\}\} \in \{S(A, \alpha)\ \mathrm{Ker}(S(A, \alpha))\ S(A, \alpha)\} = \{0\},$$
$$\{d_s\ a_h\ d_s\} \in \{S(A, \alpha)\ \mathrm{Ker}(S(A, \alpha))\ S(A, \alpha)\} = \{0\},$$

it can be seen that

$$\begin{aligned}
\{d_s\ \{a_h\ b_h\ c_h\}\ d_s\} &= D(d_s, \{a_h\ b_h\ c_h\})d_s \\
&= D(\{b_h\ c_h\ d_s\}, a_h)d_s + [D(d_s, a_h)D(b_h, c_h)]d_s \\
&= 0.
\end{aligned}$$

By polarization, it follows that $\{a_h\ b_h\ c_h\}$ lies in $\mathrm{Ker}(S(A, \alpha))$ and, hence, that

$$\{\mathrm{Ker}(S(A, \alpha))\ \mathrm{Ker}(S(A, \alpha))\ H(A, \alpha)\} \subseteq \mathrm{Ker}(S(A, \alpha)).$$

19

Again using [9], Proposition 1.6, it follows that $\mathrm{Ker}(S(A, \alpha))$ is an ideal in the JBW*-algebra $H(A, \alpha)$, as required. $\qquad\square$

A simple example of the kind described above is that in which A is the W*-algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices, and α is the mapping sending an element a to its transpose. In this case $H(A, \alpha)$ is the JBW*-algebra of $n \times n$ symmetric complex matrices and $S(A, \alpha)$ is the JBW*-triple of anti-symmetric complex matrices. In this case, both A and $H(A, \alpha)$ have trivial centres. In particular, it follows that both $H(A, \alpha)$ and $S(A, \alpha)$ are holomorphically rigid in A.

REFERENCES

[1] E.M. Alfsen and E.G. Effros, Structure in real Banach spaces I, *Ann. Math.* **96** (1972), 98-128.

[2] E.M. Alfsen and E.G. Effros, Structure in real Banach spaces II, *Ann. Math.* **96** (1972), 129-174.

[3] J. Arazy, An application of infinite dimensional holomorphy to the geometry of Banach spaces, in *Geometrical Aspects of Functional Analysis*, Lecture Notes in Mathematics, Vol. 1267, Springer-Verlag, Berlin/Heidelberg/New York, 1980.

[4] J. Arazy and W. Kaup, On the holomorphic rigidity of linear operators on complex Banach spaces, *Quart. J. of Math. Oxford* **50** (1999), 249-277.

[5] T.J. Barton and R.M. Timoney, Weak*-continuity of Jordan triple products and its applications, *Math. Scand.* **59** (1986), 177-191.

[6] T.J. Barton, T. Dang, and G. Horn, Normal representations of Banach Jordan triple systems, *Proc. Amer. Math. Soc.* **102** (1987), 551-555.

[7] E. Behrends, *M-structure and the Banach-Stone Theorem*, Lecture Notes in Mathematics, Vol. 736, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

[8] F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge Univ. Press, Cambridge, 1971.

[9] L.J. Bunce and C-H. Chu, Compact operations, multipliers and the Radon Nikodym property in JB*-triples, *Pacific J. Math.* **153** (1992), 249-265.

[10] E. Cartan, Sur les domaines bornés homogènes de l'espace de n variables complexes, *Abh. Math. Sem. Univ. Hamburg* **11** (1935), 116-162.

[11] F. Cunningham Jr., M-structure in Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **63** (1967), 613-629.

[12] F. Cunningham Jr., E.G. Effros, and N.M. Roy, M-structure in dual Banach spaces, *Israel J. Math.* **14** (1973), 304-309.

[13] S. Dineen, The second dual of a JB*-triple system, in *Complex analysis, Functional Analysis and Approximation Theory* (J. Mujica, Ed.), North-Holland, Amsterdam, 1986.

[14] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.

[15] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, Berlin/Heidelberg/New York, 1999.

[16] C.M. Edwards, On Jordan W*-algebras, *Bull. Sc. Math.* **104** (1980), 393-403.

[17] C.M. Edwards, K. McCrimmon, and G.T. Rüttimann, The range of a structural projection, *J. Funct. Anal.* **139** (1996), 196-224.

[18] C.M. Edwards and G.T. Rüttimann, On the facial structure of the unit balls in a JBW*-triple and its predual, *J. London Math. Soc.* **38** (1988), 317-322.

[19] C.M. Edwards and G.T. Rüttimann, Structural projections on JBW*-triples, *J. London Math. Soc.* **53** (1996), 354-368.

[20] C.M. Edwards and G.T. Rüttimann, Compact tripotents in bi-dual JB*-triples, *Math. Proc. Cambridge Philos. Soc.* **120** (1996), 155-173.

[21] C.M. Edwards and G.T. Rüttimann, Orthogonal faces of the unit ball in a Banach space. *Atti Sem. Mat. Fis. Univ. Modena* **49** (2001), 473-493.

[22] C.M. Edwards, G.T. Rüttimann, and S.Yu. Vasilovsky, Invariant inner ideals in W*-algebras, *Math. Nachr.* **172** (1995), 95-108.

[23] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North Holland, Amsterdam, 1980.

[24] Y. Friedman and B. Russo, Structure of the predual of a JBW*-triple, *J. Reine Angew. Math.* **356** (1985), 67-89.

[25] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Pitman, London, 1984.

[26] L. A. Harris, Schwarz's Lemma in normed linear spaces, *Proc. Nat. Acad. Sci., U.S.A.*, **62** (1969), 1014-1017.

[27] L. A. Harris, Bounded symmetric domains in infinite dimensional spaces, in *Proceedings on Infinite Dimensional Holomorphy* (T.L. Hayden and T.J. Suffridge Eds.), Lecture Notes in Mathematics, Vol. 364, Spronger-Verlag, Berlin/Heidelberg/New York, 1974.

[28] G. Horn, Characterization of the predual and the ideal structure of a JBW*-triple, *Math. Scand.* **61** (1987), 117-133.

[29] J.M. Isidro and L.L. Stachó, *Holomorphic Automorphism Groups in Banach Spaces: An Elementary Introduction*, North-Holland, Amsterdam, 1985.

[30] V. I. Istrăţescu, *Strict convexity and complex strict convexity, theory and applications*, Lecture Notes in Pure Mathematics, Vol. 89, Marcel Dekker, Basel/New York, 1984.

[31] N. Jacobson, *Structure and Representation of Jordan Algebras*, Amer. Math. Soc. Colloquium Publications, Vol. 39, American Mathematical Society, Providence, 1968.

[32] W. Kaup, Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183** (1983), 503-529.

[33] W. Kaup, Contractive projections on Jordan C*-algebras and generalizations, *Math. Scand.* **54** (1984), 95-100.

[34] O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, University of California, Irvine, 1977.

[35] E. Neher, *Jordan triple systems by the grid approach*, Lecture Notes in Mathematics, Vol. 1280, Springer-Verlag, Berlin/Heidelberg/New York, 1987.

[36] G.K. Pedersen, *C*-algebras and their automorphism groups*, London Mathematical Society Monographs, Vol. 14, Academic Press, London, 1979.

[37] R.S. Phillips, On symplectic mappings of contraction operators, *Studia Math.*, **31**, 15-27 (1969).

[38] S. Sakai, *C*-algebras and W*-algebras*, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[39] L.L. Stachó, A projection principle concerning biholomorphic automorphisms, *Acta Sci. Math.* **44**, 99-124 (1982).

[40] H. Upmeier, *Symmetric Banach manifolds and Jordan C*-algebras*, North-Holland, Amsterdam, 1985.

[41] H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics*, Regional Conference Series in Mathematics, Vol. 67, Amer. Math. Soc., Providence, 1986.

[42] J.D.M. Wright, Jordan C*-algebras, *Michigan Math. J.* **24** (1977), 291-302.

[43] M.A. Youngson, A Vidav theorem for Banach Jordan algebras, *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 263-272.

THE QUEEN'S COLLEGE, OXFORD, UNITED KINGDOM
E-mail address: martin.edwards@queens.ox.ac.uk

UNIVERSITY OF CALIFORNIA, IRVINE, U.S.A.
E-mail address: huegli@stat.unibe.ch