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A geometric characterization of structural projections on a JBW*-triple ☆

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Abstract

A structural projection R on a Jordan*-triple A is a linear projection such that, for all elements a, b and c in A,

$$R\{a\ Rb\ c\} = \{Ra\ b\ Rc\}.$$

The L-orthogonal complement $G^{\diamond}$ of a subset G of a complex Banach space E is the set of elements x in E such that, for all elements y in G,

$$||x \pm y|| = ||x|| + ||y||.$$

A contractive projection P on E is said to be neutral if the condition that

$$||Px|| = ||x||$$

implies that the elements Px and x coincide, and is said to be a GL-projection if the L-orthogonal complement $(PE)^{\diamond}$ of the range PE of P is contained in the kernel $\ker(P)$ of P. It is shown that, for a JBW*-triple A, with predual A_*, a linear projection R on A is structural if and only if it is the adjoint of a neutral GL-projection on A_*, thereby giving a purely geometric characterization of structural projections.

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1. Introduction

The work of Kaup, Upmeier, Vigué and others showed that there is a natural partially defined Jordan triple product on any complex Banach space A, which, when the open unit ball in A is a bounded symmetric complex domain, is universally defined [32–34], [44–47]. In this case, A is said to be a JB*-triple. The part played by idempotents in the study of algebraic structures endowed with a binary product is played by tripotents in the theory of JB*-triples. A class of JB*-triples that are extremely well-endowed with tripotents is that of JBW*-triples, which are JB*-triples that are Banach dual spaces. Examples of JBW*-triples are JBW*-algebras and, in particular, W*-algebras, or von Neumann algebras, many properties of which may be discovered by investigating their triple structure.

In examining the properties of a JBW*-triple A it is found that an important part is played by the weak*-closed inner ideals in A [13,14,16–20,24,25]. A subspace J of A is said to be an inner ideal if the subspace $\{J\,A\,J\}$ is contained in J. A linear projection R on A is said to be structural if, for all elements a, b and c in A,

$$R\{a\,Rb\,c\} = \{Ra\,b\,Rc\}.$$

It is clear that the range of a structural projection is an inner ideal, and, in [14], it was shown that every weak*-closed inner ideal in A is the range of a unique structural projection which is automatically both contractive and weak*-continuous. Consequently, every structural projection is the adjoint of a contractive linear projection on the predual A_* of A. The contractive projections P on A_* that arise in this way also have the property that they are neutral, in that, if x is an element of A_* for which $||Px||$ and $||x||$ coincide then Px and x also coincide. It follows from the results of [19] that the mapping $P \mapsto P^*A$ is a bijection from the family $\mathcal{N}(A)$ of neutral projections on A_* for which P^*A is a subtriple of A onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A.

The question then arises of whether it is possible to discover a further geometric property of a neutral projection P on A_* which would automatically ensure that P^*A is a subtriple of A. Alternatively, the problem can be described as that of finding a geometric characterization of the pre-adjoints of structural projections. The solution of the problem would then, of course, lead to a purely geometric characterization of structural projections, the definition of which is purely algebraic. By introducing a property described as the GL-property, the solution to the problem is presented in this paper. The results rest heavily upon those of Friedman and Russo [27–29] and of two of the authors of this paper [20,23].

The predual A_* of a JBW*-triple A has been proposed as a model of the state space of a statistical physical system [26], in which contractive linear mappings on A_* represent operations or filters on the system. Contractive linear projections on A_* represent repeatable operations, and both neutrality and the GL-property can be interpreted physically. A contractive projection is neutral if and only if, when the transmission probability of a state under the corresponding operation is one, then the state is unchanged by the operation. A contractive projection has the

GL-property if and only if a state of the system 'orthogonal' to the set of states transmitted by the corresponding operation has zero probability of being transmitted by the operation.

The paper is organized as follows. In Section 2 definitions are given and notation is established, and in Section 3 certain preliminary results concerning contractive projections on the predual of a JBW*-triple, some of which are well known, are presented. The main results of the paper are proved in Section 4, where the properties of GL-projections are studied, and the geometric characterization of structural projections is given. The final section is devoted to the description of examples of GL-projections on the preduals of arbitrary JBW*-triples, and this is applied to the special cases of a W*-algebra and a complex Hilbert space.

2. Preliminaries

A complex vector space A equipped with a triple product $(a,b,c) \mapsto \{a\,b\,c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and, for elements a, b, c and d in A, satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\,b\,c\}, d) - D(c, \{d\,a\,b\}),$$

where $[\,.\,,\,.\,]$ denotes the commutator, and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a,b)c = \{a\,b\,c\}$$

is said to be a *Jordan*-triple*. A Jordan*-triple A for which the vanishing of $\{a\,a\,a\}$ implies that a itself vanishes is said to be *anisotropic*. For each element a in A, the conjugate linear mapping $Q(a)$ from A to itself is defined, for each element b in A, by

$$Q(a)b = \{a\,b\,a\}.$$

A subspace B of a Jordan*-triple A is said to be a *subtriple* if $\{B\,B\,B\}$ is contained in B. Clearly, a subspace B is a subtriple if and only if, for each element a in B, the element $\{a\,a\,a\}$ lies in B. For details about the properties of Jordan*-triples the reader is referred to [35,37].

A Jordan*-triple A which is also a Banach space such that D is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of [6], Definition 5.1, with non-negative spectrum, and satisfies

$$||D(a,a)|| = ||a||^2$$

is said to be a *JB***-triple*. Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. A subspace J of a JB*-triple A is said to be an *inner ideal* if $\{J\,A\,J\}$ is contained in J and is said to be an *ideal* if $\{A\,A\,J\}$ and $\{A\,J\,A\}$ are contained in J. Every norm-closed subtriple of a JB*-triple A is a JB*-triple [32], and a norm-closed subspace J of A is an ideal if and only if $\{J\,A\,A\}$ is contained in J [7]. A JB*-triple A which is the dual of a Banach space A_* is said to be a *JBW***-triple*. In this case the *predual* A_* of A is unique and, for elements a and b in A, the operators $D(a,b)$ and $Q(a)$ are weak*-continuous. It follows that a weak*-closed subtriple B of a JBW*-triple A is a JBW*-triple. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [3,4,10,11,27,31–33,42,43]. Examples of JB*-triples are JB*-algebras, and examples of JBW*-triples are JBW*-algebras, for the properties of which the reader is referred to [12,30,48,49].

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal*, when $D(a,b)$ is equal to zero. For a subset L of A, denote by $L^\perp$ the subset of A which consists of all elements in A which are orthogonal to all elements in L. The subset $L^\perp$ is said to be the *annihilator* of L. Then, $L^\perp$ is a weak*-closed inner ideal in A. Moreover, for subsets L and M of A,

$$L^\perp \cap L \subseteq \{0\}, \quad L \subseteq L^{\perp\perp}, \quad L^\perp = L^{\perp\perp\perp},$$

and if L is contained in M then $M^\perp$ is contained in $L^\perp$.

For each non-empty subset B of the JBW*-triple A, the *kernel* $\mathrm{Ker}(B)$ of B is the weak*-closed subspace of elements a in A for which $\{B\,a\,B\}$ is equal to $\{0\}$. It follows that the annihilator $B^\perp$ of B is contained in $\mathrm{Ker}(B)$ and that $B \cap \mathrm{Ker}(B)$ is contained in $\{0\}$. A subtriple B of A is said to be *complemented* [20] if A coincides with $B \oplus \mathrm{Ker}(B)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal.

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\,u\,u\}$ is equal to u. The set of tripotents in A is denoted by $\mathscr{U}(A)$. Notice that the weak*-closure of the linear hull of $\mathscr{U}(A)$ coincides with A. For each tripotent u in the JBW*-triple A, the weak*-continuous conjugate linear operator $Q(u)$ and, for j equal to 0, 1 or 2, the weak*-continuous linear operators $P_j(u)$ are defined by

$$Q(u)a = \{u\,a\,u\}, \quad P_2(u)a = Q(u)^2, P_1(u) = 2(D(u,u) - Q(u)^2), \quad P_0(u) = I - 2D(u,u) + Q(u)^2.$$

The linear operators $P_j(u)$ are weak*-continuous projections onto the eigenspaces $A_j(u)$ of $D(u,u)$ corresponding to eigenvalues $j/2$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is the *Peirce decomposition* of A relative to u. For j, k and l equal to 0, 1 or 2, $A_j(u)$ is a weak*-closed subtriple of A such that

$$\{A_j(u)\,A_k(u)\,A_l(u)\} \subseteq A_{j-k+l}(u),$$

when $i-j+k$ is equal to 0, 1 or 2, and is equal to $\{0\}$ otherwise, and

$$\{A_2(u)\, A_0(u)\, A\} = \{A_0(u)\, A_2(u)\, A\} = \{0\}.$$

Notice that $A_0(u)$ and $A_2(u)$ are inner ideals in A. Observe that two elements u and v in $\mathcal{U}(A)$ are orthogonal if and only if v is contained in $A_0(u)$. For two elements u and v in $\mathcal{U}(A)$, write $u \leqslant v$ if $\{u\, v\, u\}$ coincides with u or, equivalently, if $v-u$ is a tripotent orthogonal to u. This defines a partial ordering on $\mathcal{U}(A)$.

For each non-zero element x in the predual A_* of the JBW*-triple A there exists a non-zero element $e^A(x)$ in $\mathcal{U}(A)$ that is the smallest element of the set of elements u of $\mathcal{U}(A)$ such that

$$u(x) = \|x\|.$$

The tripotent $e^A(x)$ is said to be the *support tripotent* of x [15,27]. For each non-empty subset G of A_*, the *support space* $s(G)$ of G is the smallest weak*-closed subspace of A such that, for all x in G, the support tripotent $e^A(x)$ of x lies in $s(G)$. It can easily be seen that

$$s(G)^{\perp} = \left(\overline{\operatorname{lin}\{e^A(x) : x \in G\}}^{\mathrm{w}^*}\right)^{\perp} = \bigcap_{x \in G} A_0(e^A(x)), \tag{2.1}$$

a weak*-closed inner ideal in A.

Let E be a complex Banach space. A linear projection P on E is said to be an *L-projection* if, for each element x in E,

$$\|x\| = \|Px\| + \|x - Px\|.$$

A closed subspace which is the range of an L-projection is said to be an *L-summand* of E, and E is said to be the *L-sum*

$$E = PE \oplus_{\mathrm{L}} (\mathrm{id}_E - P)E$$

of PE and $(\mathrm{id}_E - P)E$. A linear projection S on E is said to be an *M-projection* if, for each element a in E,

$$\|a\| = \max\{\|Sa\|, \|a - Sa\|\}.$$

A closed subspace which is the range of an M-projection is said to be an *M-summand* of E, and E is said to be equal to the *M-sum*

$$E = SE \oplus_{\mathrm{M}} (\mathrm{id}_E - S)E$$

of the M-summands SE and $(\mathrm{id}_A - S)E$. The families of L-projections on E and M-projections on its dual space E^* form complete Boolean lattices and the mapping $P \mapsto P^*$ is an order isomorphism between them. For details, the reader is referred to [1,2,5,8,9]. The results of [3,31] show that the set of M-summands of a JBW*-triple A coincides with the set of its weak*-closed ideals.

3. Projections on the predual of a JBW*-triple

Recall that a linear projection P on a complex Banach space E is said to be *contractive* if, for all elements x in E,

$$\|Px\| \leqslant \|x\|.$$

Observe that the adjoint P^* of a contractive projection P on E is a weak*-continuous contractive projection on the dual space E^* of E, that the dual $(PE)^*$ of the range PE of P is canonically isometrically isomorphic to the range P^*E^* of P^*, and that the Banach space of all weak*-continuous linear functionals on P^*E^* is isometrically isomorphic to PE. The first part of this section is concerned with the properties of contractive projections on the predual of a JBW*-triple.

Let A be a JBW*-triple with predual A_*. The following result is an immediate consequence of the main results of [33,41].

Lemma 3.1. *Let P be a contractive projection on the predual A_* of the JBW*-triple A, and let P^* be the adjoint of P. Then, with respect to the triple product $\{\ldots\}_{P^*A}$ from $P^*A \times P^*A \times P^*A$ to P^*A, defined, for elements a, b, and c in P^*A, by*

$$\{a\,b\,c\}_{P^*A} = P^*\{a\,b\,c\},$$

the range P^*A of P^* is a JBW*-triple.

Further properties of contractive projections can be determined from the results of [20,28].

Lemma 3.2. *Under the conditions of Lemma* 3.1, *the following results hold*:

(i) *The support space $s(PA_*)$ of the norm-closed subspace PA_* of A_* is a weak*-closed subtriple of A.*
(ii) *The space $s(PA_*) \oplus s(PA_*)^\perp$ is a weak*-closed subtriple of A in which $s(PA_*)$ and $s(PA_*)^\perp$ are weak*-closed ideals.*
(iii) *The weak*-closed subspace P^*A of A is contained in $s(PA_*) \oplus s(PA_*)^\perp$, and the restriction ϕ of the M-projection from $s(PA_*) \oplus_M s(PA_*)^\perp$ onto $s(PA_*)$ is a weak*-continuous isometric triple isomorphism from the JBW*-triple P^*A endowed with the triple product $\{\ldots\}_{P^*A}$ onto the sub-JBW* -triple $s(PA_*)$ of A.*
(iv) *The inverse ϕ^{-1} of ϕ is the restriction of P^* to $s(PA_*)$, the predual of which can be identified with PA_*, and the pre-adjoints ϕ_* and $(\phi^{-1})_*$ are the identity mappings on PA_*.*

Proof. For the proof of (i)–(iii), see [20,28]. Observe that, by [28, Lemma 2.1], for elements x and y in PA_*, the elements $P^*e^A(x) - e^A(x)$ and $e^A(y)$ are orthogonal. Allowing y to run through PA_*, it follows that $P^*e^A(x) - e^A(x)$ lies in $s(PA_*)^\perp$.

Hence

$$P^*e^A(x) = e^A(x) + (P^*e^A(x) - e^A(x))$$

is the M-decomposition of the element $P^*e^A(x)$. Therefore,

$$\phi(P^*e^A(x)) = e^A(x).$$

It follows from (iii) that ϕ^{-1} and P^* agree on a weak*-dense subspace of $s(PA_*)$ and, therefore, by weak*-continuity, coincide.

Observe that, for each element x in the predual PA_* of the JBW*-triple P^*A and each element y in PA_*,

$$e^A(y)((\phi^{-1})^*x) = \phi^{-1}(e^A(y))(x) = P^*e^A(y)(x) = e^A(y)(Px) = e^A(y)(x).$$

Therefore, the predual $s(PA_*)_*$ of the sub-JBW*-triple $s(PA_*)$ of A can be identified with PA_*, which completes the proof of (iv). $\square$

From this result two corollaries can be deduced.

Corollary 3.3. *Under the conditions of Lemma* 3.2, *if either P^*A is contained in $s(PA_*)$ or $s(PA_*)$ is contained in P^*A then P^*A and $s(PA_*)$ coincide.*

Proof. This is immediate from Lemma 3.2(iii) and (iv).

A tripotent u in the JBW*-triple A is said to be *σ-finite* if, in the partially ordered set $\mathcal{U}(A)$, it does not majorize an uncountable orthogonal subset of $\mathcal{U}(A)$. Let $\mathcal{U}_\sigma(A)$ denote the set of σ-finite tripotents in A. It follows from [22, Theorem 3.2] that a tripotent u lies in $\mathcal{U}_\sigma(A)$ if and only if there exists an element x in A_* such that u coincides with the support tripotent $e^A(x)$.

Corollary 3.4. *Under the conditions of Lemma* 3.2, *the set $\mathcal{U}_\sigma(s(PA_*))$ of σ-finite tripotents in the sub-JBW*-triple $s(PA_*)$ of A coincides with the set $\mathcal{U}_\sigma(A) \cap s(PA_*)$.*

Proof. By Lemma 3.2, the JBW*-triple $s(PA_*)$ has predual PA_*. Let u be a σ-finite tripotent in $s(PA_*)$. Since the partial ordering of tripotents is preserved under triple isomorphisms, it follows from Lemma 3.2(iii) that P^*u is a σ-finite tripotent in the JBW*-triple P^*A endowed with the triple product $\{\ldots\}_{P^*A}$. It follows from Lemma 3.2(iv) that there exists an element x in PA_* such that

$$P^*u = e^{P^*A}(x). \tag{3.1}$$

Observe that

$$u(x) = u(Px) = P^* u(x) = e^{P^* A}(x)(x) = ||x||,$$

and it follows that, in $\mathscr{U}(A)$,

$$e^A(x) \leqslant u. \tag{3.2}$$

Furthermore,

$$P^* e^A(x)(x) = e^A(x)(Px) = e^A(x)(x) = ||x||,$$

and, since $e^A(x)$ lies in $s(PA_*)$, using Lemma 3.2(iii) it can be seen that, in $\mathscr{U}(P^*A)$,

$$e^{P^* A}(x) \leqslant P^* e^A(x). \tag{3.3}$$

Combining (3.1)–(3.3), it can be seen that

$$P^* u = e^{P^* A}(x) \leqslant P^* e^A(x) \leqslant P^* u.$$

It follows that $P^* u$ and $P^* e^A(x)$ coincide, and, again using Lemma 3.2(iii), that u and $e^A(x)$ coincide. Hence u lies in $\mathscr{U}_\sigma(A) \cap s(PA_*)$.

Conversely, let u be an element of $\mathscr{U}_\sigma(A)$ contained in $s(PA_*)$ and let $\{u_j : j \in \Lambda\}$ be an orthogonal family of elements of $\mathscr{U}(s(PA_*))$ majorized by u. Since $s(PA_*)$ is a sub-JBW*-triple of A, it follows that the family $\{u_j : j \in \Lambda\}$ is countable. Hence u is contained in $\mathscr{U}_\sigma(s(PA_*))$, as required. $\square$

Let E be a complex Banach space with dual E^*, and let E_1 and E_1^* be the closed unit balls in E and E^*, respectively. For subsets G of E and H of E^*, let G° and $H_\circ$ denote the topological annihilators of G in E^* and H in E, respectively. When G and H are $\mathbb{R}$-homogeneous let $G^\sharp$ and $H_\sharp$ be the subsets of E^* and E consisting of elements that attain their norms on G and H, respectively. To be precise,

$$G^\sharp = \{a \in E^* : ||a|| = \sup\{|a(x)| : x \in G \cap E_1\}\},$$

$$H_\sharp = \{x \in E : ||x|| = \sup\{|a(x)| : a \in H \cap E_1^*\}\}.$$

A contractive projection P on E is said to be *neutral* if, whenever an element x of E is such that $||Px||$ and $||x||$ coincide then Px and x also coincide. Observe that an L-projection provides an example of a neutral projection. The next result describes some properties of neutral projections, some parts of which may be found in [20,29,39].

Lemma 3.5. *Let E be a complex Banach space and let P be a contractive projection on E. Then, the following conditions on P are equivalent:*

(i) *The projection P is neutral.*
(ii) *Every weak*-continuous linear functional on the range P^*E^* of the adjoint P^* of P has a unique weak*-continuous Hahn-Banach extension to E^*.*
(iii) *Every contractive projection S on E having the property that the range S^*E^* of its adjoint S^* coincides with P^*E^* is neutral.*
(iv) *The set $(P^*E^*)_\sharp$ coincides with the range PE of P.*

Furthermore, if the contractive projection P is neutral and S is a further contractive projection such that P^*E^* and S^*E^* coincide then P and S coincide.

Proof. (i) $\Rightarrow$ (ii): Let z be a weak*-continuous linear functional on P^*E^*. Then, there exists an element x in E such that,

$$Px = x$$

for all elements a in E^*,

$$a(x) = P^*a(x) = P^*a(z) \tag{3.4}$$

and

$$||x|| = ||z||. \tag{3.5}$$

Let y be a further element of E such that, for all elements a in E^*,

$$P^*a(y) = P^*a(z) \tag{3.6}$$

and

$$||y|| = ||z||. \tag{3.7}$$

It follows from (3.4) and (3.6) that, for all elements a of E^*,

$$a(Py) = P^*a(y) = P^*a(z) = a(x),$$

and hence that Py and x coincide. It follows from (3.5) and (3.7) that

$$||y|| = ||x|| = ||Py||,$$

and, since P is neutral, Py and y coincide. Therefore, x and y coincide, as required.

(ii) $\Rightarrow$ (iii): Let x be an element of E such that

$$||Sx|| = ||x||.$$

Then, for all elements a of P^*E^*,

$$a(x) = S^*a(x) = a(Sx),$$

and both Sx and x are weak*-continuous extensions of the restriction of x to P^*E^*. It follows that Sx and x coincide.

(iii) $\Rightarrow$ (i): This is immediate.

(i) $\Rightarrow$ (iv): Since P is contractive, P^* is contractive, and, therefore, $P^*E_1^*$ coincides with $P^*E^* \cap E_1^*$. Hence, for each element x in E,

$$\begin{aligned} ||Px|| &= \sup\{|a(Px)| : a \in E_1^*\} = \sup\{|P^*a(x)| : a \in E_1^*\} \\ &= \sup\{|b(x)| : b \in P^*E^* \cap E_1^*\}. \end{aligned}$$

Therefore, x is contained in $(P^*E^*)_\sharp$ if and only if $||Px||$ and $||x||$ coincide. But, since P is neutral, this occurs if and only if x lies in PE.

(iv) $\Rightarrow$ (ii): Let x lie in PE and let y be an element of E that is a Hahn–Banach extension of the restriction of x to P^*E^*. It follows that y lies in $(P^*E^*)_\sharp$ and, therefore, in PE. Hence, x and y coincide, as required.

Finally, suppose that P is neutral and that S is a contractive projection on E such that P^*E^* and S^*E^* coincide. Then, for each element x in E, the weak*-continuous linear functionals Px and Sx are Hahn–Banach extensions of the restriction of Px to P^*E^*. It follows from (ii) that Px and Sx coincide. $\square$

Recall that a linear projection R on the JBW*-triple A is said to be a *structural projection* [36] if, for each element a in A,

$$RQ(a)R = Q(Ra). \tag{3.8}$$

The following results, the proofs of which can be found in [14,19,20], relate neutral projections on the predual A_* of A with structural projections on A and with weak*-closed inner ideals in A.

Lemma 3.6. *Let A be a JBW*-triple with predual A_*. Let $\mathscr{I}(A)$ denote the family of* weak*-*closed inner ideals in A, let $\mathscr{S}(A)$ denote the family of structural projections on A, and let $\mathscr{N}(A_*)$ denote the family of neutral projections on A_* with the property that P^*A is a subtriple of A. Then the following results hold*:

(i) *The mapping $P \mapsto P^*$ is a bijection from $\mathscr{N}(A_*)$ onto $\mathscr{S}(A)$.*
(ii) *The mapping $R \mapsto RA$ is a bijection from $\mathscr{S}(A)$ onto $\mathscr{I}(A)$.*

In proving the result above the proofs of the following two lemmas become evident.

Lemma 3.7. *Let A be a JBW*-triple, with predual A_*, and let J be a subtriple of A. Then, the following conditions are equivalent*:

(i) *J is a weak*-closed inner ideal in A.*
(ii) *The set $J_\sharp$ of elements of A_* attaining their norm on J is a subspace of A_*.*
(iii) *For every tripotent u in J, the Peirce-two space $A_2(u)$ is contained in J.*

Lemma 3.8. *Let A be a JBW*-triple, with predual A_*, let R be a structural projection on A, and let J be the weak*-closed inner ideal RA in A. Then, the following results hold*:

(i) *R is the unique structural projection with range J.*
(ii) *R is contractive and weak*-continuous.*
(iii) *The kernel* $\mathrm{Ker}(J)$ *of J coincides with the kernel* $\ker(R)$ *of R.*
(iv) *The predual J_* of J coincides with $J_\sharp$ which consists of the set of elements x in A_* the support tripotent $e^A(x)$ of which lies in J.*

Observe that an L-projection P on the predual A_* of the JBW*-triple A provides an example of an element of $\mathcal{N}(A_*)$. In Section 4 a more general class of projections on a complex Banach space E will be introduced. In the special case in which E is the predual of a JBW*-triple A this class contains $\mathcal{N}(A_*)$.

4. GL-projections

Recall that elements x and y of the complex Banach space E are said to be *L-orthogonal* [29] provided that

$$\|x \pm y\| = \|x\| + \|y\|.$$

Observe that if x and y are L-orthogonal, then so also are x and $-y$, $-x$ and y, and $-x$ and $-y$. The set of elements L-orthogonal to the set of all elements in a subset G of E is said to be the *L-orthogonal complement* of G and is denoted by $G^\diamond$. The proof of the following characterization of L-orthogonality for elements in the predual A_* of a JBW*-triple A may be found in [23, Theorem 5.4].

Lemma 4.1. *Let A be a JBW*-triple, and let x and y be elements of the predual A_* of A. Then x and y are L-orthogonal if and only if their support tripotents $e^A(x)$ and $e^A(y)$ are orthogonal.*

The relationship between L-orthogonality and support spaces is summarised in the following result.

Lemma 4.2. *Let A be a JBW*-triple, with predual A_*, and let G be a non-empty subset of A_*, having L-orthogonal complement $G^\diamond$ and support space $s(G)$. Then, the following results hold*:

(i) *The support space* $s(G^\diamond)$ *of* $G^\diamond$ *coincides with weak***-closed inner ideal* $s(G)^\perp$, *and the predual* $(s(G)^\perp)_\sharp$ *of* $s(G)^\perp$ *coincides with* $G^\diamond$.
(ii) *The kernel* $\mathrm{Ker}(s(G)^\perp)$ *of* $s(G)^\perp$ *coincides with the topological annihilator* $(G^\diamond)^\circ$ *of* $G^\diamond$.
(iii) *The L-orthogonal complement* $G^\diamond$ *of* G *is contained in the topological annihilator* $s(G)_\circ$ *of* $s(G)$.

Proof. (i) By Lemma 4.1, an element x of A_* lies in $G^\diamond$ if and only if, for all elements y in G, the tripotent $e^A(x)$ lies in the Peirce-zero space $A_0(e^A(y))$. By (2.1), it follows that x lies in $G^\diamond$ if and only if $e^A(x)$ lies in the weak*-closed inner ideal $s(G)^\perp$. Lemma 3.8(iv) shows that $e^A(x)$ lies in $s(G)^\perp$ if and only if x lies in $(s(G)^\perp)_\sharp$, thereby completing the proof.

(ii) By Lemma 3.8(iii), the kernel $\mathrm{Ker}(s(G)^\perp)$ of the weak*-closed inner ideal $s(G)^\perp$ coincides with the kernel of the structural projection onto $s(G)^\perp$, which, by Lemma 3.8(iv), itself coincides with $((s(G)^\perp)_\sharp)^\circ$. The result follows from (i).

(iii) Observe that $s(G)$ is contained in $\mathrm{Ker}(s(G)^\perp)$. Using (i) and Lemma 3.8(iv),

$$G^\diamond = (s(G)^\perp)_\sharp = \mathrm{Ker}(s(G)^\perp)_\circ \subseteq s(G)_\circ,$$

as required. □

In general, the L-orthogonal complement $G^\diamond$ of a non-empty subset G of a Banach space E is not a subspace of E. Lemma 4.2 shows that, when E is the predual of a JBW*-triple, $G^\diamond$ is always a subspace of E.

Corollary 4.3. *Let* A *be a* JBW**-triple with predual* A_*. *Then, the following results hold*:

(i) *The L-orthogonal complement* $G^\diamond$ *of a non-empty subset* G *of* A_* *is a norm-closed subspace of* A_*.
(ii) *Let* F, G *and* H *be mutually L-orthogonal subspaces of* A_*. *Then, the subspace* $F \oplus G$ *is L-orthogonal to* H.

Proof. The proof of (i) follows from Lemma 4.2(i). To prove (ii), observe that both F and G are contained in the closed subspace $H^\diamond$, from which it follows that the subspace $F \oplus G$ is contained in $H^\diamond$. □

It is now possible to give the central definition of this paper. A contractive projection P on the complex Banach space E is said to be a *GL-projection* if the L-orthogonal complement $(PE)^\diamond$ of its range is contained in its kernel $\ker(P)$.

Lemma 4.4. *Let E be a complex Banach space and let P be an L-projection on E. Then, P is a GL-projection on E.*

Proof. In this case $(PE)^{\diamond}$ coincides with $\ker(P)$.

Attention is now focused on GL-projections on the predual A_* of a JBW*-triple A. The first result shows that contractive projections on A_* give rise naturally to GL-projections.

Theorem 4.5. *Let A be a JBW*-triple with predual A_*, and let P be a contractive projection on A_*. Then, there exists a GL-projection S on A_* such that the range SA_* of S coincides with the range PA_* of P, and the range S^*A of its adjoint S^* coincides with the support space $s(PA_*)$ of PA_*.*

Proof. Let ϕ be the isometric triple isomorphism from P^*A onto $s(PA_*)$ defined in Lemma 3.2. Then, the mapping ϕP^* is a weak*-continuous contractive linear mapping from A onto $s(PA_*)$. Observe that, by Lemma 3.2(iv), for all elements a in A,

$$(\phi P^*)^2 a = \phi(P^*\phi)P^*a = (\phi P^*)a,$$

and ϕP^* is a contractive projection onto P^*A. Let S be the contractive projection on A_* such that S^* coincides with ϕP^*. Then, again using Lemma 3.2(iv),

$$SA_* = (\ker(S^*))_\circ = (\ker(\phi P^*))_\circ = (\ker(P^*))_\circ = PA_*,$$

as required. It remains to show that S is a GL-projection. However,

$$S^*A = s(PA_*) = s(SA_*) \subseteq \mathrm{Ker}(s(SA_*)^{\perp}),$$

and it follows from Lemma 4.2(ii) that

$$(SA_*)^{\diamond} = (\mathrm{Ker}(s(SA_*)^{\perp}))_\circ \subseteq (S^*A)_\circ = \ker(S),$$

as required. □

The next result shows that GL-projections on the predual of a JBW*-triple may be characterized in many different ways.

Theorem 4.6. *Let A be a JBW*-triple, with predual A_*, let P be a contractive projection on A_*, with adjoint P^*, and let $s(PA_*)$ be the support space of the range PA_* of P. Then, the following conditions are equivalent.*

(i) *P is a GL-projection.*
(ii) *The range P^*A of P^* is contained in the kernel* $\mathrm{Ker}(s(PA_*)^{\perp})$ *of the weak*-closed inner ideal $s(PA_*)^{\perp}$.*

(iii) $s(PA_*)$ *is contained in* P^*A.
(iv) $s(PA_*)$ *coincides with* P^*A.
(v) $s(PA_*)$ *contains* P^*A.
(vi) *The topological annihilator* $s(PA_*)_\circ$ *of* $s(PA_*)$ *is contained in the kernel* $\ker(P)$ *of* P.
(vii) $s(PA_*)^\perp$ *is contained in the weak***-closed inner ideal* $(P^*A)^\perp$.
(viii) $s(PA_*)^\perp$ *coincides with* $(P^*A)^\perp$.

Proof. (i)$\Leftrightarrow$(ii): This follows from Lemma 4.2.

(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v): This follows from Corollary 3.3.

(ii) $\Rightarrow$ (v): By Lemma 3.2(iii), for each element a in P^*A, there exist elements b in $s(PA_*)$ and c in $s(PA_*)^\perp$ such that

$$a = b + c.$$

It follows from (ii) that

$$0 = \{c\,a\,c\} = \{c\,b + c\,c\} = \{c\,b\,c\} + \{c\,c\,c\} = \{c\,c\,c\},$$

and, by the anisotropy of A, c is equal to 0. It follows that a lies in $s(PA_*)$, as required.

(v) $\Rightarrow$ (vi): By taking topological annihilators this is immediate.

(vi) $\Rightarrow$ (i): By Lemma 4.2(iii),

$$(PA_*)^\diamond \subseteq s(PA_*)_\circ \subseteq \ker(P),$$

as required.

(v) $\Rightarrow$ (vii): By taking algebraic annihilators this is immediate.

(vii) $\Rightarrow$ (ii): Observing that $s(PA_*)^{\perp\perp}$ is contained in $\mathrm{Ker}(s(PA_*)^\perp)$,

$$P^*A \subseteq (P^*A)^{\perp\perp} \subseteq s(PA_*)^{\perp\perp} \subseteq \mathrm{Ker}(s(PA_*)^\perp),$$

as required.

(iv) $\Rightarrow$ (viii) $\Rightarrow$ (vii): These trivially hold. $\square$

It can now be shown that there exists a unique GL-projection onto a particular norm-closed subspace of the predual of a JBW*-triple.

Corollary 4.7. *Under the conditions of Theorem* 4.6, *let* P *and* S *be GL-projections on* A_* *such that* PA_* *and* SA_* *coincide. Then* P *and* S *coincide.*

Proof. It follows from Theorem 4.6 that

$$P^*A = s(PA_*) = s(SA_*) = S^*A,$$

and, hence, that

$$\ker(P) = (P^*A)_\circ = (S^*A)_\circ = \ker(S).$$

Since P and S have the same range and kernel, P and S coincide. □

Using Corollary 3.4, it is also possible to give characterizations of GL-projections in terms of σ-finite tripotents.

Corollary 4.8. *Let A be a JBW*-triple, with predual A_*, let P be a contractive projection on A_*, with adjoint P^*, let $s(PA_*)$ be the support space of the range PA_* of P, and let $\mathcal{U}_\sigma(s(PA_*))$ and $\mathcal{U}_\sigma(P^*A)$, respectively, be the families of σ-finite tripotents in the sub-JBW*-triple $s(PA_*)$ of A and in the JBW*-triple P^*A endowed with the triple product $\{\ldots\}_{P^*A}$. Then, the following conditions are equivalent*:

(i) *P is a GL-projection.*
(ii) *$\mathcal{U}_\sigma(P^*A)$ coincides with the set $\{e^A(x) : x \in PA_*\}$.*
(iii) *$\mathcal{U}_\sigma(P^*A)$ coincides with the set $\mathcal{U}_\sigma(A) \cap s(PA_*)$.*
(iv) *$\mathcal{U}_\sigma(P^*A)$ is contained in $\mathcal{U}_\sigma(s(PA_*))$.*
(v) *$\mathcal{U}_\sigma(s(PA_*))$ is contained in $\mathcal{U}_\sigma(P^*A)$.*
(vi) *$\mathcal{U}_\sigma(P^*A)$ coincides with $\mathcal{U}_\sigma(s(PA_*))$.*

Proof. If (i) holds, then, by Theorem 4.6, P^*A and $s(PA_*)$ coincide, and (ii) and (iii) are equivalent consequences of Corollary 3.4. By [22, Corollary 3.5], the JBW*-triples P^*A and $s(PA_*)$ coincide with the weak*-closed linear spans of $\mathcal{U}_\sigma(P^*A)$ and $\mathcal{U}_\sigma(s(PA_*))$, respectively. Hence, by Corollary 3.4 and Theorem 4.6, conditions (i), (iv)–(vi) are all equivalent. Since condition (iii) clearly implies condition (iv), the proof is complete. □

It is now possible to prove the main result of the paper.

Theorem 4.9. *Let A be a JBW*-triple, with predual A_*, and let R be a linear projection on A. Then R is a structural projection if and only if there exists a neutral GL-projection P on A_* with adjoint equal to R.*

Proof. Let R be a structural projection on A. Then, by Lemma 3.6(i) there exists a unique neutral projection P on A_*, with the property that P^*A is a subtriple of A, such that P^* and R coincide. Furthermore, by Lemma 3.8(iv), PA_* is the predual of the weak*-closed inner ideal P^*A of A and, for all elements x of PA_*, the support tripotent $e^A(x)$ of x lies in P^*A. It follows from Corollary 4.8 that P is a GL-projection.

Conversely, let P be a neutral GL-projection on A_*. From Theorem 4.6(iv), P^*A coincides with $s(PA_*)$, which is a subtriple of A. The result now follows from Lemma 3.6(i). □

Observe that, by giving a purely geometrical characterization of structural projections, this result resolves the main problem attacked in this paper.

5. Examples

Having resolved the main problem, the question arises of whether the two properties of a contractive projection on the predual A_* of a JBW*-triple A of being neutral and being a GL-projection are both necessary. This is quickly resolved in the simple example of the JBW*-triple $\mathbb{C}^2$ endowed with the L-norm and pointwise operations, in which it is easy to find GL-projections that are not neutral and neutral projections that do not have the GL-property. The results given below show that non-neutral GL-projections can also be found on the predual of a general JBW*-triple.

Let A be a JBW*-triple with predual A_*. For each element J of the family $\mathscr{I}(A)$ of weak*-closed inner ideals in A, the annihilator $J^\perp$ also lies in $\mathscr{I}(A)$ and A enjoys the *generalized Peirce decomposition*

$$A = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_0 = J^\perp, \qquad J_2 = J, \qquad J_1 = \mathrm{Ker}(J) \cap \mathrm{Ker}(J^\perp).$$

The structural projections onto J and $J^\perp$ are denoted by $P_2(J)$ and $P_0(J)$, respectively, and the projection $\mathrm{id}_A - P_2(J) - P_0(J)$ onto J_1 is denoted by $P_1(J)$. Furthermore,

$$\{A\, J_0\, J_2\} = \{0\}, \quad \{A\, J_2\, J_0\} = \{0\},$$

and, for j, k and l equal to 0, 1 or 2, the Peirce arithmetical relations,

$$\{J_j\, J_k\, J_l\} \subseteq J_{j+l-k}, \tag{5.1}$$

when $j + l - k$ is equal to 0, 1 or 2, and

$$\{J_j\, J_k\, J_l\} = \{0\}, \tag{5.2}$$

otherwise, hold, except in the cases when (j, k, l) is equal to $(0, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(2, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, or $(1, 1, 1)$. Observe that, for j equal to 0, 1 or 2, the projections $P_j(J)$ are weak*-continuous, and it follows that there are projections $P_j(J)_*$ on A_* with adjoints $P_j(J)$. The ranges J_{j*} of these mutually orthogonal projections are the preduals of J_j.

Theorem 5.1. *Let A be a JBW*-triple, with predual A_*, let J be a weak*-closed inner ideal in A, and let $P_2(J)_*$ and $P_0(J)_*$ be the pre-adjoints of the Peirce projections $P_2(J)$ and $P_0(J)$. Then, $P_2(J)_* + P_0(J)_*$ is a GL-projection on A_*.*

Proof. Observe that by [21, Lemma 4.4], $P_2(J)_* + P_0(J)_*$ is a contractive projection on A_*. By Edwards and Rüttimann [23, Theorem 5.4], the ranges J_{2*} and J_{0*} of $P_2(J)_*$ and $P_0(J)_*$ are L-orthogonal. Therefore, for each element x in $(P_2(J)_* + P_0(J)_*)A_*$, there exist uniquely y in J_{2*} and z in J_{0*} such that

$$x = y + z.$$

Since J_2 and J_0 are inner ideals, by Lemma 3.8(iv) and [23, Theorem 5.4],

$$e^A(x) = e^A(y) + e^A(z) \in J_2 + J_0 = (P_2(J) + P_0(J))A.$$

It follows that $s((P_2(J)_* + P_0(J)_*)A_*)$ is contained in $(P_2(J) + P_0(J))A$ and the result follows from Theorem 4.6(iii). $\square$

Recall that, for a weak*-closed inner ideal J in A, if relations (5.1) and (5.2) hold in all cases, then J is said to be a *Peirce inner ideal*.

Theorem 5.2. *Let A be a JBW*-triple, with predual A_*, let J be a Peirce inner ideal in A, and let $P_1(J)_*$ be the pre-adjoint of the Peirce-one projection $P_1(J)$. Then $P_1(J)_*$ is a GL-projection on A_*.*

Proof. It follows from [21, Theorem 4.8], that $P_1(J)$ and, hence, $P_1(J)_*$ is contractive. Furthermore, by (5.1), J_1 is a subtriple of A. Therefore, for x in J_{1*}, the support tripotent $e^{J_1}(x)$ is a tripotent in A such that

$$e^{J_1}(x)(x) = ||x||.$$

It follows that

$$e^A(x) \leqslant e^{J_1}(x). \tag{5.3}$$

Observe that, by Lemma 3.2, $P_1(J)$ is a triple isomorphism from $s(J_{1*})$ onto J_1. It follows that $P_1(J)e^A(x)$ is a tripotent in the subtriple J_1 such that

$$P_1(J)e^A(x)(x) = e^A(x)(P_1(J)_*x) = e^A(x)(x) = ||x||.$$

Therefore,

$$e^{J_1}(x) \leqslant P_1(J)e^A(x). \tag{5.4}$$

Using (5.3) and (5.4) and the triple isomorphism property of $P_1(J)$,

$$e^{J_1}(x) \leqslant P_1(J)e^A(x) \leqslant P_1(J)e^{J_1}(x) = e^{J_1}(x),$$

and it follows that

$$e^{J_1}(x) = P_1(J)e^A(x). \tag{5.5}$$

From (5.3), there exists a tripotent v orthogonal to $e^A(x)$ such that

$$e^{J_1}(x) = e^A(x) + v. \tag{5.6}$$

Applying $P_1(J)$, it follows from (5.5) that

$$P_1(J)v = 0,$$

and, hence, that v lies in the kernel $J_0 + J_2$ of $P_1(J)$. Using (5.1), (5.2), and (5.6),

$$\begin{aligned} v &= \{v\, e^A(x) + v\, v\} = \{v\, e^{J_1}(x)\, v\} \\ &\subseteq \{J_0 + J_2\, J_1\, J_0 + J_2\} \\ &= \{J_0\, J_1\, J_0\} + \{J_2\, J_1\, J_2\} + \{J_2\, J_1\, J_0\} \\ &\subseteq \{0\} + \{0\} + J_1. \end{aligned}$$

It follows that

$$v \in (J_0 + J_2) \cap J_1 = \{0\}.$$

Therefore, $e^A(x)$ coincides with $e^{J_1}(x)$ and, by Corollary 4.8, $P_1(J)_*$ is a GL-projection. $\square$

Observe that, by Theorem 4.9, the GL-projections $P_2(J)_* + P_0(J)_*$ and $P_1(J)_*$, described in Theorems 5.1 and 5.2, are not, in general, neutral.

When A is a W^*-algebra, for the properties of which the reader is referred to [38,40], with triple product defined, for elements a, b and c of A, by

$$\{a\, b\, c\} = \tfrac{1}{2}(ab^*c + cb^*a),$$

it follows from the results of [16] and those above that, for each pair e and f of projections in A with common central support, the mapping P on the predual A_* defined, for x in A_* and a in A, by

$$a(Px) = (ea(1_A - f) + (1_A - e)af)(x),$$

where 1_A is the unit in A, is a GL-projection on A_*, which is not, in general, neutral.

When A is a complex Hilbert space endowed with triple product defined, for elements a, b and c in A, by

$$\{a\,b\,c\} = \tfrac{1}{2}(\langle a, b\rangle c + \langle c, b\rangle a),$$

the set of closed subspaces of A coincides with the set of weak*-closed inner ideals in A, and, for every such subspace J,

$$J_2 = J, \qquad J_1 = J^{\mathrm{perp}}, \qquad J_0 = \{0\},$$

where J^{perp} denotes the Hilbert space orthogonal complement of J. In this case, it is clear that every GL-projection on the predual of A is neutral.

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