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WRONSKIAN OF FUNDAMENTAL SYSTEM OF DELAY DIFFERENTIAL EQUATIONS *

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Dedicated to Lina Fazylovna Rakhmatullina and
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on the occasion of their jubilees

Abstract. Wronskian is one of the classical objects in the theory of ordinary differential equations. For delay differential equations this notion did not exist because of unnatural traditional definition of homogeneous equations, leading to an infinite-dimensional fundamental system. Azbelev's definition of homogeneous equations allowed to obtain a finite-dimensional space of solutions of delay equations and to construct on this base the theory of these equations similar to the classical one for ordinary differential equations. Properties of Wronskian lead to important conclusions on behavior of solutions of delay equations. For instance, nonvanishing of Wronskian ensures Sturm's separation theorem (between two adjoint zeros of a solution there is a zero of each other nontrivial solution) for delay equations. An important use of Wronskian is the asymptotic behavior of delay equations. Thus a growth of Wronskian implies existence of unbounded solutions of delay differential equations. One of the results based on estimates of Wronskian's growth is the following: all solutions of the equation

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty),$$

with positive nondecreasing and bounded on $[0, +\infty)$ coefficient $p(t)$ are bounded if and only if $\int\limits_0^{+\infty} \tau(t)\, dt < \infty$. If p and τ are nonzero ω−periodic functions the following assertion based on Sturm's separation theorem and estimates of Wronskian's growth, is obtained: if distance between each two zeros of nontrivial solutions is different from 2ω, then all solutions of this equation are unbounded.

Key Words. Delay equation, fundamental system, Wronskian

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$$x(\xi) = 0,\ x'(\xi) = 0 \quad for\ \xi < 0,$$

where

$$\begin{aligned} g(t) &= f(t) - q(t)\psi(t-\theta(t))A(t-\theta(t)) - \\ &= -p(t)\varphi(t-\tau(t))A(t-\tau(t)). \end{aligned} \tag{1.10}$$

$$A(t) = \begin{cases} 1, & t < 0, \\ 0, & 0 \leq t, \end{cases}$$

is equivalent to equation (1.4), (1.5). It is clear now that "traditional" homogeneous equation (1.6), (1.5) is nonhomogeneous with a special right hand side (see, (1.10) with $f = 0$) by Azbelev's definition and properties of homogeneous equation (1.6), (1.8) allow to conclude about behavior of solutions of delayed equation (1.6), (1.5).

2. Notes about Wronskian, Nonoscillation and Positivity of Green Function. Various connections between nonoscillation properties of homogeneous equations and sign properties of Green functions for boundary value problems (BVP) with ordinary differential equations (ODE) are well known. Let us refer to A.Ju.Levin's paper [39] which is the classical survey in this topic. One of the main results was firstly obtained by E.S.Chichkin [9] and later independently by A.Ju.Levin [39] and by E.F.Beckenbach and R.Bellman [5] and can be formulated as follows.

THEOREM 2.1. *[39] If the equation*

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t)x^{(i)}(t) = 0, \quad t \in [0,b],$$

is nonoscillatory on $[0,b]$ (each nontrivial solution has at most $n-1$ zeros with zeros counted the number of times equal to their multiplicity on this interval), then the Green functions of de La Vallee-Poussin problems

$$(Mx)(t) = f(t),\ x^{(i)}(t_j) = 0, \quad t \in [0,b],$$

$$0 = t_1 < t_2 < ... < t_m = b,\ i = 0,...,k_j - 1,\ j = 1,...,m,\ k_1 + ... + k_m = n,$$

behave regularly, i.e.

$$G(t,s)(t-t_1)^{k_1}...(t-t_m)^{k_m} > 0,\ t_j < t < t_{j+1},\ t,s \in [0,b],\ j = 1,...,m-1.$$

The approach of the works [5,9,39] is based on a fact that for each fixed s the function $G(\cdot,s)$ is a solution of the homogeneous ODE $Mx = 0$. That is not true for delayed equation. It is therefore not possible to develop this approach and to transfer the results concerning regular behavior of Green function for ODE, to more general classes of differential equations. An analog of such the theorem for sufficiently wide class of $n-$th order FDE $Lx = 0$ was obtained in [3,4] on quite a different base. A mathematical formulation of this result for FDE requires many special notations and definitions and will not be done here. In this paper we focus our attention on properties of Wronskian of the fundamental system and their influence on oscillation and asymptotic behavior of second order delay equations. Let us only note that the main restrictions determining this wide class is nonvanishing of Wronskian $W(t)$ of the fundamental system of FDE $Lx = 0$ on $[0,b]$. In this case there exists a corresponding ODE $Mx = 0$ with the same fundamental system. It can be presupposed that nonvanishing of Wronskian is a natural bound of a "similar" oscillation behavior of corresponding functional and ordinary differential equations.

Let us consider the following equation

$$(\mathcal{L}x)(t) \equiv x''(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) = f(t), \tag{2.1}$$

$$p_i(t) \geq 0, \quad t \in [0,+\infty),$$

$$x(\xi) = 0 \text{ for } \xi < 0, \tag{2.2}$$

where p_i and f are locally summable functions and τ_i are nonnegative measurable functions ($i = 1,...,n$). The fundamental system of this equation is shown to be two- dimensional and Wronskian $W(t)$ of a certain fundamental system can be considered

It is known [1] that a general solution of equation (2.1), (2.2) has the following representation

$$x(t) = \int_0^t C(t,s)f(s)\,ds + x_1(t)x(0) + x_2(t)x'(0), \tag{2.3}$$

Here $C(t,s)$ is the Cauchy function of equation (2.1),(2.2). Note that for every fixed $s \in [0,+\infty)$ the function $C(\cdot,s)$ is a solution of "s-truncated" equation

$$(\mathcal{L}_s x)(t) \equiv x''(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) = 0, \quad t \in [s,+\infty), \tag{2.4}$$

(2.5) $$x(\xi) = 0 \text{ for } \xi < s,$$

and $C(s,s) = 0$, $C_t'(s,s) = 1$. Functions x_1 and x_2 are solutions of homogeneous equation $\mathcal{L}x = 0$ satisfying the conditions $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 0$, $x_2'(0) = 1$. If BVP

(2.6) $$(\mathcal{L}x)(t) = f(t), \quad t \in [0,b], \quad x(0) = 0, \; x(b) = 0,$$

has a unique solution for each summable f, then this solution has the following representation

(2.7) $$x(t) = \int_0^b G(t,s) f(s)\, ds,$$

where $G(t,s)$ is called the Green function of problem (2.6). Let us introduce the following compact operator $K_{\nu\mu}: \mathcal{C}_{[\nu,\mu]} \mapsto \mathcal{C}_{[\nu,\mu]}$ by the following equality

(2.8) $$(K_{\nu\mu}x)(t) = -\int_\nu^\mu G_{\nu\mu}(t,s) \sum_{j=1}^n p_j(s) x(s - \tau_j(s))\, ds,$$
where $x(\xi) = 0$ for $\xi < \nu$,

$G_{\nu\mu}(t,s)$ is the Green function of boundary value problem

(2.9) $$x''(t) = f(t), \; t \in [\nu,\mu], \quad x(\nu) = 0, \quad x(\mu) = 0.$$

Denote $r_{_{\nu\mu}}$ the spectral radius of $K_{\nu\mu}$.

DEFINITION 2.1. *[1] We say that $[\nu,\mu]$ is non-oscillation interval of homogeneous equation*

(2.10) $$(\mathcal{L}x)(t) \equiv x''(t) + \sum_{i=1}^n p_i(t) x(t - \tau_i(t)) = 0,$$

$$p_i(t) \geq 0, \; t \in [0,+\infty), \; x(\xi) = 0 \text{ for } \xi < 0,$$

if each of its solutions has at most one zero on $[\nu,\mu]$.

Denote by $D_{[0,b]}$ the space of functions $x : [0,b] \to R$, with absolutely continuous derivative x'.

THEOREM 2.2. *The following assertions are equivalent:*

a) $[0,b]$ is non-oscillation interval of equation (2.10).

b) $C(t,s) > 0$ for $0 \leq s < t \leq b$,

c) there exists the Green function of BVP (2.6) and $G(t,s) < 0$ for $t,s \in (0,b)$,

d) the spectral radius $r_{_{0b}}$ of the operator K_{0b} is less than one.

e) there exists a positive function $v \in D_{[0,b]}$ such that $(\mathcal{L}v)(t) \leq 0$ for $t \in [0,b]$ and $v(0) + v(b) - \int_0^b (\mathcal{L}v)(t)dt > 0$.

The similar assertion for a first order FDE was obtained in [22], where related stability properties were also studied.

If we set $v(t) = t(b-t)$ in the assertion *e)* the following is obtained.

COROLLARY 2.1. *[1] If $\operatorname{vrai\,sup}_{s \in [h(t),t]} \sum_{i=1}^n p_i(t) \leq \frac{8}{b^2}$ for $t \in [0,b]$, then $[0,b]$ is non- oscillation interval of equation (2.10).*

If we use the theorem on integral inequality [1,35] for the operator K_{0b} the following is obtained.

COROLLARY 2.2. *[1] If $\int_0^b \sum_{i=1}^n p_i(s)\, ds \leq \frac{4}{b}$ then $[0,b]$ is non- oscillation interval of equation (2.10).*

3. Sturm's Separation Theorem.

For ordinary homogeneous differential equation (1.3) the classical Sturm's separation theorem is valid. It can be formulated as follows: between two adjoint zeros of each nontrivial solution there is one and only one zero of linearly independent solution. Validity of the Sturm's theorem follows from non-vanishing of Wronskian $W(t)$ of a fundamental system of ODE (1.3). Really, let us suppose existence of two zeros t_1 and t_2 of nontrivial solution x_2 between adjoint zeros of x_1. Consider the following function $y(t) = \frac{x_2(t)}{x_1(t)}$. From the form of $y(t)$ it follows that $y(t_1) = y(t_2) = 0$, but this contradicts to the fact that the derivative $y'(t) = \frac{W(t)}{[x_1(t)]^2}$ preserves sign for $t \in [t_1,t_2]$. Generally speaking, Sturm's separation theorem is not valid for delayed differential equations. It is even possible that a second order delay equation has both oscillating and nonoscillating solutions. Wronskian can vanish and its zeros do not depend on the chosen fundamental system [1]. The first results about non-vanishing of Wronskian were obtained by N.V.Azbelev in [1] due to the "smallness" of delays. Let us denote $h_i(t) = t - \tau_i(t)$ and $h(t) = \min_{1 \leq i \leq n} h_i(t)$.

DEFINITION 3.1. *[1] We say that for equation (2.10) the h-condition is fulfilled if*

(3.1) $$r_{h(t)t} < 1 \text{ for almost all } t \in (0,+\infty).$$

THEOREM 3.1. *[1] If for equation (2.10) the h-condition is fulfilled, then $W(t) \neq 0$ and $|W(t)|$ does not decrease for $t \in [0,+\infty)$.*

In [1] is demonstrated that h-condition is essential for nonvanishing of Wronskian in [1].

COROLLARY 3.1. *[1] If at least one of the conditions*

$$a)\ (t-h(t))\int_{h(t)}^{t}\sum_{i=1}^{n}p_i(s)\,ds \leq 4 \text{ for } t\in(0,+\infty),$$

or

$$b)\ (t-h(t))^2 \operatorname{vraisup}_{s\in[h(t),t]}\sum_{i=1}^{n}p_i(s)\leq 8 \text{ for } t\in(0,+\infty),$$

for equation (2.10) is fulfilled, then $W(t)\neq 0$ and $|W(t)|$ does not decrease for $t\in[0,+\infty)$.

Note that each of the conditions *a)* and *b)* ensures that h-condition is fulfilled for equation (2.10). S.M.Labovskii [36] proved the following analog of Sturm's theorem.

THEOREM 3.2. *[36] If in equation (2.10) $n=1$ and h is a nondecreasing function, then $W(t)\neq 0$ and $|W(t)|$ does not decrease for $t\in[0,+\infty)$.*

In fact, the conditions of this theorem ensure that the h-condition for equation (2.10) is fulfilled. In the paper [15] of 1993 nonvanishing of Wronskian was obtained through several other conditions, basic of them being the "smallness" of difference of delays $\tau_i-\tau_j$, where $i,j=1,...,n$.

THEOREM 3.3. *[15] Let*

1) the functions $h_i(t)=t-\tau_i(t)$ be nondecreasing and the inequalities $\tau_{i+1}(t)\leq\tau_i(t)$ hold for almost all $t\in[0,+\infty)$,

2) the functions p_{i+1}/p_i be nondecreasing for $i=1,...,n-1$,

3) at least one of the following inequalities a) or b) be fulfilled:

$$a)\ [\tau_1(t)-\tau_n(t)]\int_{h_1(t)}^{h_n(t)}\sum_{i=1}^{n}p_i(s)\,ds\leq 1 \quad \text{for almost all } t\in(0,+\infty),$$

$$b)\ [\tau_1(t)-\tau_n(t)]^2 \operatorname{vraisup}_{s\in[h(t),t]}\sum_{i=1}^{n}p_i(s)\leq 2 \text{ for almost all } t\in(0,+\infty).$$

Then $W(t)\neq 0$ for $t\in[0,+\infty)$.

Note that in this assertion h-condition is not generally speaking fulfilled [15]. In the paper [15] it is demonstrated that each of the conditions 1), 2) and 3) is essential. An analog of Sturm's separation theorem for neutral equation was obtained in [12]. Note that another formulation of Sturm's separation theorem was proposed by Yu.I. Domshlak [17] in terms of big semi-cycles introduced in the Myshkis's monograph [41]. Analogs of Sturm's theorem for solutions of Dirichlet and Neumann problems for partial differential equations

were obtained in [16].

4. Distance between Zeros of Solutions. For ordinary second order differential equations distance between zeros is one of the classical topics. This distance for solutions of delay equations was estimated in works by N.V.Azbelev [1] and his group [12,15, 36], by Yu.I. Domshlak [17], S.V.Eliason [20], A.D.Myshkis [41] and S.B.Norkin [42].

A basic assertion estimating nonoscillation interval from below was proposed by N.V.Azbelev in [1].

THEOREM 4.1. *[1] Let $h-$condition be fulfilled for equation (2.10). If $r_{\nu\mu}<1$ then each nontrivial solution has at most one zero on $[\nu,\mu]$.*

Other lower estimates of nonoscillation interval one can find in [15], where instead of $h-$condition the smallness of the difference of delays $\tau_i-\tau_j$ is required.

Remark 4.1. Now it is clear that h-condition means that each nontrivial solution has at most one zero of nontrivial solution x of homogeneous equation (2.10) on $[h(t),t]$ for every $t\in(0,+\infty)$. Note that the condition 3) in Theorem 3.3 ensures that there is no zero of derivative x' of each nontrivial solution having zero on $[h_1(t),h_n(t)]$. Lower estimates of the distance between adjoint zeros for neutral second order equations were obtained in [12]. Conditions of Corollaries 2.1 and 2.2 ensure the inequality $r_{\nu\mu}<1$.

Let us denote $\nu^*=vrai\inf_{t\in[\nu,\mu]}h(t)$ and formulate an assertion about upper estimate of nonoscillation interval.

THEOREM 4.2. *[16] If $r_{\nu\mu}\geq 1$, then each solution of equation (2.1) has zero on the interval $[\nu^*,\mu]$.*

The following assertion about differential inequality ensures a lower estimate of the spectral radius $r_{\nu\mu}$.

THEOREM 4.3. *[1] If there exists a positive on (ν,μ) function $v\in D_{\nu,\mu}$ and such that $v(\nu)=v(\mu)=0$ and $v''(t)+\sum_{i=1}^{n}p_i(t)v(t-\tau_i(t))\geq 0$, where $v(s)=0$ for $s<\nu$, then $r_{\nu\mu}\geq 1$.*

Many questions of the qualitative theory of functional- differential equations are reduced to estimates of nonoscillation intervals. Consider one of the classical problems about existence of periodic solution of the equation

$$x''(t)+\sum_{i=1}^{n}p_i(t)x(t-\tau_i(t))=f(t),\quad t\in(-\infty,+\infty), \tag{4.1}$$

where p_i,τ_i and f are $\omega-$periodic functions, such that

$p_i\geq 0, ess\sup_{t\in[0,\omega]}\sum_{i=1}^{n}p_i(t)>0,\ \tau_i\geq 0$ for $i=1,...,n$.

It is known that there exists a unique $\omega-$periodic solution of equation (4.1) for each summable f if and only if the corresponding homogeneous equation

$$x''(t)+\sum_{i=1}^{n}p_i(t)x(t-\tau_i(t))=0,\quad t\in(-\infty,+\infty), \tag{4.2}$$

has only the trivial periodic solution (see, the paper by Yu.V.Komlenko [33]). Note the following result based on a lower estimate of nonoscillation interval.

THEOREM 4.4. *[33] If*

$$\int_0^{\omega}\sum_{i=1}^{n}p_i(s)\,ds\le\frac{16}{\omega}, \tag{4.3}$$

then equation (4.2) has only the trivial periodic solution.

Inequality (4.3) ensures that there are no 3 zeros of nontrivial solution of (4.2) on each interval of path ω. Without loss of generality we assume that $\tau_i\ge\tau_n$ for $i=1,2,...,n-1$.

Let us introduce the following notations:

$$\begin{aligned} R &= ess\sup_{t\ge0}\sum_{i=1}^{n}p_i(t),\ P=ess\inf_{t\ge0}p_n(t),\\ \tau &= ess\sup_{t\ge0}(t-h(t)),\tau_-=ess\sup_{t\ge0}\tau_n(t),\\ g &= 0 \text{ for } \tau_-=0 \text{ and } g=a \text{ for } \tau_->0,\end{aligned} \tag{4.4}$$

where a satisfies the inequalities $a\sqrt{p}\ge1, a\ge\tau_-$.

The following assertion was obtained through combining of lower and upper estimates of nonoscillation interval.

THEOREM 4.5. *[13] Let h−condition be fulfilled for equation (2.10). If for a natural k the following inequality is fulfilled:*

$$(k-1)(\frac{\pi}{\sqrt{P}}+\tau+2g)<\frac{2\sqrt{2}}{\sqrt{R}}k, \tag{4.5}$$

and

$$\omega\in(0,\frac{4\sqrt{2}}{\sqrt{R}}]\cup[2(\frac{\pi}{\sqrt{P}}+\tau+2g),\frac{8\sqrt{2}}{\sqrt{R}}]\cup \tag{4.6}$$

$$...\cup[2(k-1)(\frac{\pi}{\sqrt{P}}+\tau+2g),\frac{4\sqrt{2}}{\sqrt{R}}k].$$

then equation (4.2) has only the trivial periodic solution.

Example 4.1 [13]. Consider the equation

$$x''(t)+100x(t)+x(t-\tau_1(t))=0, \tag{4.7}$$

If $\tau_1\le0.24$ there are 2 intervals of type (4.6), if $\tau_1\le0.1$ then $k=3$, if $\tau_1\le0.05$, then $k=4$, if $\tau_1\le0.002$ then $k=9$.

5. Growth of Wronskian and Existence of Unbounded Solutions. For ordinary differential equation (1.3) relations between growth of Wronskian and existence of unbounded solutions were obtained by P.Hartman in the classical monograph [25] and in the paper by P.Hartman and A.Winter [26]. In this part we consider this problem for delay equation (2.10). In the paper [14] it was demonstrated that the existence of unbounded solutions of delay equation (2.10) implies its Lyapunov's instability. Consider the Wronskian of the fundamental system of equation (2.10)

$$W(t)=\begin{vmatrix}x_1(t) & x_2(t)\\ x_1'(t) & x_2'(t)\end{vmatrix}.$$

For simplicity let us assume that $W(0)>0$.

THEOREM 5.1. *[14] If h-condition be fulfilled,*

$$\lim_{t\to+\infty}W(t)=+\infty \tag{5.1}$$

and there exists positive ε so that $\tau_i(t)\ge\varepsilon$ for $i=1,\dots,n$ and almost all $t\ge\nu$, then there exist unbounded solutions of equation (2.10).

Denote

$$R(t)=\sum_{i=1}^{n}p_i(t).$$

THEOREM 5.2. *[14] Let h-condition be fulfilled and*

$$\operatorname{vrailim}_{t\to+\infty}\frac{W(t)}{\sqrt{R(t)}}=\infty. \tag{5.2}$$

Then there exists unbounded solution of equation (2.10).

COROLLARY 5.1. *[14] If h-condition be fulfilled and*

$$\operatorname{vraisup}_{t\in[0,+\infty)}\sum_{i=1}^{n}p_i(t)<\infty,\ \lim_{t\to+\infty}W(t)=\infty,$$

then there exist unbounded solutions of equation (2.10).

A growth of Wronskian implies the following assertion.

COROLLARY 5.2. *If the delays $\tau_i, i=1,...,n)$ are bounded and*

$$\operatorname*{vrailim}_{t\to+\infty}\sum_{i=1}^{n}p_i(t)=0,$$

then there exist unbounded solutions of equation (2.10).

For equation

$$x''(t)+p(t)x(t-\tau(t))=0, \quad t\in[0,+\infty), \tag{5.3}$$

$$x(\xi)=0 \text{ for } \xi<0,$$

the following assertion was obtained.

THEOREM 5.3. *[43] Let the function $t-\tau(t)$ do not decrease and*

$$\lim_{t\to\infty}[t/(\int_0^t W(s)ds)^2]\int_0^t p(s)ds=0, \tag{5.4}$$

then there exists unbounded solutions of equation (5.3).

6. Estimates of Wronskian. In order to use the results of Part 5 we have to obtain estimates of Wronskian.

THEOREM 6.1. *[14] Let h-condition be fulfilled for equation (2.10). Then Wronskian $W(t)$ of the fundamental system satisfies the following differential inequality*

$$W'(t)\geq\sum_{i=1}^{n}p_i(t)C(t,h_i(t))W(h_i(t)), \quad t\in[0,+\infty), \tag{6.1}$$

where $W(s)=0$ for $s<0$, $C(\cdot,s)=0$ for $s<0$.

¿From Theorem 6.1 the estimate of Wronskian

$$W(t)\geq W(0)(1+\int_0^t\sum_{i=1}^{n}p_i(s)C(s,h_i(s))\,ds), \tag{6.2}$$

where $C(t,h_i(t))=0$ if $h_i(t)<0$, follows.

THEOREM 6.2. *[14] Let h-condition be fulfilled for equation (2.10). If there exists function $v(\cdot,h_i(\cdot))\colon[\nu,+\infty)\mapsto[0,+\infty)$ so that*

1) $v(\cdot,s)$ for each fixed s has an absolutely continuous derivative on each segment $[s,b]$;

2) $v(\cdot,h_i(\cdot))\colon[\nu+\infty)\mapsto[0,+\infty)$ is measurable for $i=1,\dots,n$;

3)

$$v(t,h_i(s))\begin{cases}>0 & t\in(h_i(s),s], \quad h_i(s)\in[\nu,s),\\ =0 & t=h_i(s),\\ =0 & t\in[\nu,+\infty), \quad h_i(s)\notin[\nu,s),\end{cases}$$

$$v'(h_i(s),h_i(s))=\begin{cases}1, & h_i(s)\in[\nu,s),\\ 0, & h_i(s)\notin[\nu,s),\end{cases}$$

and

$$\psi(t)\equiv v''(t,h_i(s))+\sum_{j=1}^{n}p_j(t)v(h_j(t),h_i(s))\leq 0$$

for $i=1,\dots,n$ and almost all $t\in[h_i(s),s]$.

Then

$$W(t)\geq W(\nu)(1+\int_\nu^t\sum_{i=1}^{n}p_i(s)v(s,h_i(s))\,ds), \quad t\in[\nu,+\infty).$$

Choosing the function v we obtain in Part 7 assertions about unboundedness of solutions (see, the recent paper [14]).

7. Asymptotic Behavior of Delay Equations. Delay equation is generally known to inherit oscillation properties of a corresponding ordinary equation

$$x''(t)+p(t)x(t)=0. \tag{7.1}$$

For example, it was proved by J. J. A. M. Brands [7] that for each bounded delay $\tau(t)$ equation

$$x''(t)+p(t)x(t-\tau(t))=0, \tag{7.2}$$

is oscillatory if and only if corresponding ordinary differential equation (7.1) is oscillatory. The asymptotic behavior of ordinary equation (7.1) is not inherited by (7.2). Thus, A.D.Myshkis [41] proved that there exists unbounded solution of equation

$$x''(t)+px(t-\varepsilon)=0, \quad t\in[0,+\infty),$$

for each couple of positive constants p and ε. The problem of solutions unboundedness in case of nonconstant coefficients was formulated in [41] as one to be solved. The first results in this subject were obtained in [13]: if there exists a positive constant ε such that $\tau_i(t) > \varepsilon$, then there exist unbounded solution to equation

$$(7.3) \qquad (\mathcal{L}x)(t) \equiv x''(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) = 0,$$

$$p_i(t) \geq 0, \quad t \in [0,+\infty), \quad x(\xi) = 0 \text{ for } \xi < 0,$$

The approach of the work [13] was based on estimates of Wronskian. Then this approach was developed by D.V.Paatashvili [43] for one- term equation (7.2). In the recent paper by Yu.Dolgii and S.G.Nikolaev [11] the following system of delay equations on the whole axis t was considered: $y''(t) + P(t)y(t-\omega) = 0$, $t \in (-\infty,+\infty)$, where $y : R \to R^n$, $\omega > 0$ and $P(t)$ is an ω−periodic symmetric matrix function.Using the monodromy operator (see monograph by J.Hale and S.Lunel [24]), to be a fundamental in the theory of periodic systems, the authors obtained instability of this system in case: det $P_\omega \neq 0$, where $P_\omega = \frac{1}{\omega}\int_{-\omega}^{0} P(t)dt$. In the monograph by S.B.Norkin [42] the following boundary value problem on semiaxis is considered: $x''(t)+\lambda x(t)+p(t)x(t-\tau(t)) = 0$, $t \in [0,+\infty)$, $x(0)\cos\alpha + x'(0)\sin\alpha = 0$, $x(t-\tau(t)) = x(0)\varphi(t-\tau(t))$ for $t-\tau(t) < 0, t \in [0,+\infty)$, $\sup|x(t)| < \infty$, where $\varphi(t)$ is continuous bounded function on the initial set $(-\infty, 0)$ such that $\varphi(0) = 1$, λ and α are real numbers. If $|p(t)|$ is a summable function on semiaxis, then every positive parameter λ is eigenvalue of this problem [42]. We can interpret this result as the one concerning solutions boundedness of delay equations. Results on boundedness of delay equations solution in which the "smallness" of coefficient $p(t)$ is combined with the "smallness" of delay $\tau(t)$ were obtained by D.V. Izjumova [29]. The asymptotic formula of solutions of second order equation with a summable delay $\tau(t)$ was obtained by M.Pinto [44]. Note also that investigation of equation $x''(t)+p(t)x(t-\tau(t)) = 0$, with nonpositive coefficient $p(t)$, was started by G.A.Kamenskii [30,31]. Assertions on existence of bounded solutions, their uniqueness and oscillation were obtained in the monograph by G.S.Ladde, V.Lakshmikantham and B.Zhang [38, pp. 130-139]. Several possible types of solutions' behavior of this equation in case $p(t)$ and $\tau(t)$ are bounded functions on semiaxis and $\int_0^\infty |p(t)|\,dt = \infty$, can be only as following : a) $|x(t)| \to \infty$ for $t \to \infty$; b) $x(t)$ oscillates; c) $x(t) \to 0$, $x'(t) \to 0$ for $t \to \infty$. Existence and uniqueness of solutions of each of these types were obtained by R.G.Koplatadze [34],

A.L.Skubachevskii [50] and M.G.Shmul'yan [49] . S.M.Labovskii [37] proved that nonvanishing of Wronskian $W(t)$ on semiaxis was necessary and sufficient for existence of positive decreasing solution to equation (7.3) with nonpositive coefficients p_i $(i = 1, ..., m)$ and obtained several coefficient tests of $W(t) \neq 0$ for $t \in [0,+\infty)$. Solutions tending to zero were considered in the paper by T.A.Burton and J.R.Haddock [8]. Note that an approach for studying of asymptotic properties of equations with linear transformations of arguments $x''(t) = \sum_{j=-l,j\neq 0}^{l} a_j x(q^j t) + \lambda x(t)$, $t \in (-\infty,+\infty)$, where a_j, q and λ are constants, was proposed by E.Yu.Romanenko and A.N.Sharkovskii [46] and developed by G.A.Derfel and S.A.Molchanov [10]. In [10] equations with combination of delayed and advanced arguments are considered. Systematic study of advanced equations $(\tau(t) \leq 0)$ can be found in the recent paper by Z.Dosla and I.Kiguradze [18] in which results on boundedness, stability and asymptotic representations of solutions are obtained. The criteria of boundedness of all solutions of the equation (7.2) were obtained in the recent paper [14] on base of Wronskian's estimates.

THEOREM 7.1. *[14] All solutions of equation (7.2) with positive nondecreasing and bounded coefficient $p(t)$ and nondecreasing $h(t) \equiv t - \tau(t)$ are bounded if and only if*

$$(7.4) \qquad \int_0^\infty \tau(t)\,dt < \infty.$$

Note that sufficiency was proved by D.V.Izjumova [29]. The following result shows that solutions of delay equation (7.2) only in case of summable delay τ are getting closer and closer to solutions of corresponding ordinary equation (7.1).

THEOREM 7.2. *[14] Assume that $p(t) = c^2 > 0$ and $h(t) \equiv t - \tau(t)$ does not decrease. Then any solution $x(t)$ of equation(7.2) satisfies the formulas*

$$x(t) = (\alpha + o(1))\sin ct + (\beta + o(1))\cos ct,$$

$$x'(t) = c(\alpha + o(1))\cos ct - c(\beta + o(1))\sin ct,$$

for $t \to \infty$, where α and β are constants, if and only if the condition (7.4) is fulfilled.

Note that sufficiency was proved by M.Pinto [44]. In the paper [14] several criteria of existence of unbounded solutions to equation (7.3) are obtained. The following examples demonstrate some of them. If $\varepsilon = 0$, then all solutions of the equations

$$x''(t) + e^t x(t-\varepsilon) = 0, \tag{7.5}$$

$$x''(t) + t^2 x(t) + t^{3/2} x\left(t - \frac{\varepsilon}{t}\right) = 0, \tag{7.6}$$

$$x''(t) + t^\alpha x\left(t - \frac{\varepsilon}{t^\beta}\right) = 0, \quad \alpha + 2 > 2\beta, \tag{7.7}$$

$$x''(t) + x(t) + \frac{1}{\sqrt{t}} x\left(t - \frac{\varepsilon}{\sqrt{t}}\right) = 0, \tag{7.8}$$

$$x(\xi) = 0 \text{ for } \xi < 0,$$

are bounded on $(1, +\infty)$, and for equations (7.5)- (7.7) they even tend to zero when $t \to +\infty$ (See the monograph by V.N.Shevelo [48, p. 24]). If $\varepsilon > 0$, then there exist unbounded solutions to equations (7.5) and (7.7). If in addition ε is small enough, then there exist unbounded solution to equations (7.6) and (7.8). Note that the delays in equations (7.6)-(7.8) tend to zero when $t \to +\infty$, but even these "very small" delays totally change the asymptotic behavior of solutions.

Let us formulate results on unboundedness of solutions of equation (7.3). All of them are based on estimates of Wronskian.

THEOREM 7.3. *[14] Let h-condition be fulfilled for equation (7.3),*

$$M \equiv \underset{t\in[0,+\infty)}{vraisup} \sum_{j=1}^{n} p_j(t) < +\infty$$

and there exist $i \in \{1, \ldots, n\}$ so that

$$\int_0^\infty p_i(t)\tau_i(t)\left(2\sqrt{2}/\sqrt{M} - \tau_i(t)\right) dt = +\infty.$$

then there exist unbounded solutions of equation (7.3).

For equation

$$\begin{aligned} &x''(t) + p_1(t)x(t) + p_2(t)x(t - \tau_2(t)) = 0, \quad t \in [0, +\infty), \\ &x(\xi) = 0 \text{ for } \xi < 0, \end{aligned} \tag{7.9}$$

the following result is obtained.

THEOREM 7.4. *[14] Let p_1 and p_2 be bounded on $[0, +\infty)$, $\tau_2(t) \underset{t\to+\infty}{\longrightarrow} 0$ and*

$$\int_0^\infty p_2(t)\tau_2(t)\, dt = +\infty. \tag{7.10}$$

Then there exist unbounded solutions of equation (7.3).

Example 7.1. Equation

$$x''(t) + p(t)x(t) + \frac{1}{t^\alpha} x\left(t - \frac{\varepsilon}{t^\beta}\right) = 0, \quad t \in [1, +\infty), \tag{7.11}$$

has unbounded solution if $\alpha + \beta \leq 1$, $\alpha \geq 0$, $\beta \geq 0$. Unboundedness of solution of equation (7.8) follows from the above assertion in case $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $p = 1$.

Denote

$$\tau(t) = \min_{1\leq i\leq n} \tau_i(t).$$

THEOREM 7.5. *[14] Let h-condition be fulfilled for equation (7.3) and there be index i so that*

$$\int_0^\infty p_i(t)\tau(t)\, dt = \infty. \tag{7.12}$$

Assume that at least one of the following two conditions a) or b) are fulfilled:

a) there exists $\varepsilon > 0$ so that $\tau(t) \geq \varepsilon$ for $t \geq \nu \geq 0$;

b) $\underset{t\in[\nu,+\infty)}{vrai\ sup} \sum_{i=1}^{n} p_i(t) < \infty$.

Then there exist unbounded solutions of equation (7.3).

Existence of unbounded solutions of equation (7.5) results from Theorem 7.5 (condition a).

THEOREM 7.6. *[13] Let h-condition be fulfilled. If there exists index i such that at least one of the following conditions*

$$\lim_{t\to\infty} \tau(t) \int_0^t p_i(t)\tau(t)\, dt = +\infty. \tag{7.13}$$

or

$$\lim_{t\to\infty} \int_0^t p_i(s)\tau(s)ds) / \sqrt{\sum_{j=1}^{n} p_j(t)} = +\infty, \tag{7.14}$$

satisfies, then there exist unbounded solutions of equation (7.3).

Existence of unbounded solutions of equation (7.7) follows from Theorem 7.6. The following assertion was proved in [43] on the base of estimate of Wronskian (6.2) firstly obtained in [13] and Theorem 5.3.

THEOREM 7.7. *[43] Let the function $t-\tau(t)$ do not decrease and at least one of the following conditions be fulfilled*

(7.15) $$\lim_{t\to\infty}[t/(\int_0^t (t-s)\tau(s)ds)^2]\int_0^t p(s)ds = 0,$$

or

(7.16) $$\lim_{t\to\infty}\{p(t)/(1+\int_0^t p(s)\tau(s)ds)^2\} = 0,$$

then there exist unbounded solutions of one-term equation (7.2).

8. Unboundedness of all the Solutions. Asymptotic properties of solutions of the delay equation (7.3) can be very distinct. The problem of similar asymptotic behavior of all solutions to the same equation has not been solved yet even with ordinary second order equation For example, H.Milloux [40] discovered that if $p(t)\to\infty$ for $t\to\infty$, then there exists solution of ODE (7.1) tending to zero when $t\to\infty$. There are also several examples of other solutions without tending to zero. The problem to find conditions under which all solutions tend to zero remains one of highlighted in the qualitative theory of differential equations (see, the recent papers by A.Elbert [19], L.Hatvani and L.Stacho [27,28]). If coefficient $p(t)\to 0$ for $t\to+\infty$, then there exist unbounded solutions of ordinary equation (7.1) (see monograph by I.T.Kiguradze and T.A.Chanturia [32]). The equation $x''(t)+\frac{2}{t^2(t-1)}x(t)=0, t\in[2,+\infty)$, is an example, when the second solution $x(t)=\frac{t-1}{t}$ is bounded. In almost all statements of Part 7 it is said about existence of certain unbounded solution of equation

(8.1) $$x''(t)+p(t)x(t-\tau(t))=0, p(t)\geq 0, t\in[0,+\infty).$$

It is not true to conclude about unboundedness of **all solutions** according to the following example.

Example 8.1. A function $x = sin\ t$ is one of solutions of the equation

$$x''(t)+x(t-\tau(t))=0, \quad t\in[0,+\infty),$$

where

$$\tau(t)=\begin{cases}0, & 0\leq t\leq\frac{\pi}{2},\\ 2t-\pi, & \frac{\pi}{2}<t<\pi,\end{cases}$$

$$\tau(t+\pi)=\tau(t).$$

Other solutions are unbounded by Theorem 7.1. Note that in this example the distance between adjoint zeros (π) is equal to the period of coefficients (π). It has some logical ground. The idea to connect oscillation and asymptotic properties of solutions of a second order ODE (7.1) appeared in Lyapunov's investigation on stability. The classical Lyapunov's results says that all solutions of second order ordinary differential equation

$$x''(t)+p(t)x(t)=0, \quad t\in[0,+\infty),$$

$$p(t)=p(t+\omega)\geq c>0,$$

with $\omega-$ periodic coefficient are bounded on semiaxis if ω is less than distance between two adjoint zeros (see, the book by N.E.Zhukovskii [51]). The classical estimate of distance between two adjoint zeros

(8.2) $$\int_0^\omega p(t)dt\leq\frac{4}{\omega},$$

implies that all the solutions are bounded. It will be obtained that in contrast with the ordinary differential equation all the solutions of the delay equation with $\omega-$periodic coefficients $p(t)$ and $\tau(t)$ are unbounded if distance between zeros of solutions is different from 2ω. Consider the following equation with periodic coefficients

(8.3) $$x''(t)+\sum_{i=1}^n p_i(t)x(t-\tau_i(t))=0,\ p_i(t)\geq 0, t\in[0,+\infty).$$

where $p_i(t)=p_i(t+\omega)$, $\tau_i(t)=\tau_i(t+\omega)$, $t-\tau_i(t)\geq 0$, $i=1,...,n$. For this equation there exist solutions satisfying the condition

$$x(t+\omega)=\lambda x(t).$$

Using the Floquet theory [47] we can construct the following equation

(8.4) $$\lambda^2-(x_1(\omega)+x_2'(\omega))\lambda+W(\omega)=0,$$

where $W(0)=1$, for this λ.

If λ_1 is a real root of equation (8.4), then a corresponding solution has the following representation:

$$y(t) = g(t)\exp\{\frac{\ln|\lambda_1|}{\omega}t\}, \tag{8.5}$$

where g is a $\omega-$periodic function if $\lambda_1 > 0$ and a $2\omega-$periodic function if $\lambda_1 < 0$.

If equation (8.4) has two complex roots $\lambda_1 = |\lambda_1|\exp(i\theta)$ and $\lambda_2 = |\lambda_1|\exp(-i\theta)$, then corresponding solutions are of the following form

$$y_1(t) = [g_1(t)\cos\frac{\theta t}{\omega} - g_2(t)\sin\frac{\theta t}{\omega}]\exp\{\frac{\ln|\lambda_1|}{\omega}t\}, \tag{8.6}$$

$$y_2(t) = [g_2(t)\cos\frac{\theta t}{\omega} + g_1(t)\sin\frac{\theta t}{\omega}]\exp\{\frac{\ln|\lambda_1|}{\omega}t\}, \tag{8.7}$$

where g_1 and g_2 are $\omega-$periodic functions.

Remark 8.1. Theorems 3.1 and 3.2 imply that the Wronskian $W(t)$ does not decrease if the $h-$condition is fulfilled. Moreover it can be proved that $W(\omega) > W(0) = 1$ if there exists an index i such that

$$\int_0^{\omega} p_i(t)\tau_i(t)dt > 0. \tag{8.8}$$

Now it is clear that a bounded solution of equation (8.3) can exist only in the following case: λ_1 and λ_2 are real and one of them is equal or less than one. In Example 8.1 we have: $\lambda_2 = 1$ and $\lambda_1 = W(\omega) > 1$. Note that the unbounded Floquet solution has representation (8.5). The conditions of the following assertion exclude existence of real roots of equation (8.4).

THEOREM 8.1. *[13] Let h-condition be fulfilled for equation (8.3) and p_i and τ_i satisfy condition (8.8) for at least one index i, then all solutions of the delay equation (8.3) are unbounded if distance between zeros of solutions is different from 2ω.*

Using the integral estimate of nonoscillation interval from below we obtain the following assertion.

THEOREM 8.2. *[13] Let h-condition be fulfilled for equation (8.3), p_i and τ_i satisfy condition (8.8) for at least one index i, and*

$$\int_0^{\omega}\sum_{j=1}^{n} p_j(t)dt \le \frac{1}{\omega}.$$

Then all solutions of delay equation (8.3) are unbounded.

On the base of the lower and upper estimates of distance between zeros (see, Theorems 4.1 and 4.2) we obtain the following assertion in which notations (4.4) are used.

THEOREM 8.3. *[13] Let h− condition be fulfilled for equation (8.3), p_i and τ_i satisfy condition (8.8) for at least one index i, and for a natural k the following inequalities be fulfilled*

$$(k-1)(\frac{\pi}{\sqrt{P}} + \tau + 2g) < \frac{2\sqrt{2}}{\sqrt{R}}k, \tag{8.9}$$

and

$$\omega \in (0, \frac{\sqrt{2}}{\sqrt{R}}] \cup [\frac{1}{2}(\frac{\pi}{\sqrt{P}} + \tau + 2g), \frac{2\sqrt{2}}{\sqrt{R}}] \cup \ldots \tag{8.10}$$

$$\ldots \cup [\frac{k-1}{2}(\frac{\pi}{\sqrt{P}} + \tau + 2g), \frac{\sqrt{2}}{\sqrt{R}}k].$$

Then all solutions of equation (8.3) are unbounded.

COROLLARY 8.1. *Let h-condition be fulfilled for equation (8.3), $p_1 = p_2 = 1, \tau_1 = 0$ and $\tau_2 \le 0.5$. If a period ω satisfies the condition $\omega \in (0,1]U[\frac{\pi}{2} + \frac{1}{4}, 2]$, then all the solutions of equation (8.3) are unbounded.*

Example 8.2. If a period ω of the delay τ in the following equation

$$x''(t) + 100x(t) + x(t-\tau(t)) = 0, \tau' \le 1,\ \tau \le 0.002,\ t-\tau(t) \ge 0,\ t \in [0,+\infty),$$

satisfies the condition $0.15808(i-1) \le \omega \le 0.1406i$, where $i = 1, 2, \ldots 9$, then all its solutions are unbounded.

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ON SOLVABILITY OF A MINIMIZATION PROBLEM FOR A QUADRATIC FUNCTIONAL WITH LINEAR RESTRICTIONS IN HILBERT SPACE *

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Abstract. A method for investigation of solvability of a quadratic functional with linear restrictions

$$\frac{1}{2}\langle Uz, z\rangle - \langle z, f\rangle \longrightarrow \min, \qquad \ell z = \alpha,$$

is developed. The result is obtained with the help of an orthogonal projection on the kernel of the vector functional ℓ.

Key Words. Variational problem, boundary value problem, functional differential equation, sufficient conditions

AMS(MOS) subject classification. 49K25, 34K10

Consider the problem of minimization of a quadratic functional with linear restrictions in a real Hilbert space $\mathbf{H}$

$$J(z) = \frac{1}{2}\langle Uz, z\rangle - \langle z, f\rangle \longrightarrow \min, \qquad \ell z = \alpha, \tag{1}$$

where $U : \mathbf{H} \to \mathbf{H}$ is a linear bounded self-adjoint operator, $f \in \mathbf{H}$, $\ell : \mathbf{H} \to \mathbb{R}^{\mathbf{k}}$ is a linear bounded vector-functional with linearly independent components, $\alpha \in \mathbb{R}^k$.

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