MATH. SCAND. 59 (1986), 131–142

COMPLETE HOLOMORPHIC VECTOR FIELDS ON THE SECOND DUAL OF A BANACH SPACE

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Summary.

We show that a uniformly bounded collection of complete holomorphic vector fields on B_{E_i} (the unit ball of the Banach space E_i), $i \in I$, can be combined to define a complete holomorphic vector field on the unit ball of an ultraproduct of $(E_i)_{i \in I}$. Using this result the principle of local reflexivity and a contractive projection property due to Kaup and Stacho we prove

(a) the second dual of a JB* triple system is also a JB* triple system

(b) each biholomorphic automorphism of the unit ball of a Banach space E extends to a biholomorphic automorphism of the unit ball of E''.

The result (a) was proved for J* algebras in [3].

1.

In recent years the research of R. Braun, L. Harris, W. Kaup, L. Stacho, J. P. Vigué, H. Upmeier and others has shown that, for bounded symmetric domains in Banach spaces, the study of biholomorphic automorphisms, complete holomorphic vector fields and JB* triple system are equivalent. To a lesser extent this is still true of convex balanced bounded domains which are not necessarily symmetric. Thus the problems we discuss in this article may be approached from three different points of view. We adopt the vector field approach although in some cases (e.g. Corollary 11) the JB* approach might have been easier.

We refer to [4] for basic concepts and results in infinite dimensional holomorphy and to [10], [11], [16] for properties of symmetric domains and JB* triple systems.

In this section we sketch the relationship between the various approaches and recall specific results that we shall use in later sections. We also discuss Banach space ultraproducts.

Received April 17, 1985.

Notation.

E	a complex Banach space.
B_E	the open unit ball of E.
$\mathscr{L}(^nE)$	The Banach space of all continuous symmetric n linear mappings from E into E endowed with the topology of uniform convergence on B_E (for $n=1$ we use the notation $\mathscr{L}(E)$).
$P(^nE)$	the Banach space of all continuous n-homogeneous E-valued polynomials on E endowed with the topology of uniform convergence on B_E.
$h(E)$	the (real) vector subspace of $\mathscr{L}(E)$ consisting of all Hermitian linear operators (φ is hermitian if and only if $\|e^{it\varphi}(x)\| = \|x\|$ for all $t \in \mathsf{R}$ and all $x \in E$).
$H(B_E)$	the set of all E-valued holomorphic functions on B_E (we identify by means of the restriction mapping $P(^nE)$ with a subspace of $H(B_E)$).
$G(B_E)$	the group of all biholomorphic automorphisms of B_E.
$G(B_E)(0)$	the orbit of the origin under the action of $G(B_E)$.
$g(B_E)$	the (real) vector space of all complete holomorphic vector fields on B_E (if $X \in g(B_E)$, then $X = h\,\delta/\delta z$ where $h \in H(B_E)$ and z is the variable in E).
$g^+(B_E)$	$\{h\,\delta/\delta z \in g(B_E)\colon h(0)=0\}$.
$g^-(B_E)$	$\{h\,\delta/\delta z \in g(B_E)\colon h'(0)=0\}$.

PROPOSITION 1.

(a) (See [14], [16].) *If $h\,\delta/\delta z \in g(B_E)$, then $h = \xi + l + p$, where $\xi \in E$, $l \in \mathscr{L}(E)$, $p \in P(^2E)$ and, moreover, p is determined by ξ and hence written as p_ξ. We also have $\xi + p_\xi \in g^-(B_E)$, $l \in g^+(B_E)$ and $g(B_E) = g^+(B_E) \oplus g^-(B_E)$ is a direct sum decomposition.*

(a) $i\,h(E) = g^+(B_E)$.

(c) (See [14].) *$V = \{h(0) \mid h\,\delta/\delta z \in g(B_E)\}$ is a closed complex Banach subspace of E and $g(B_E)$ is tangent to V (that is, $h(V) \subset V$ for all $h\,\delta/\delta z \in g(B_E)$).*

(d) (See [14].) *$V \cap B_E = B_V = G(B_E)(0)$ and B_V is a symmetric domain* (i.e. *for all $a \in B_V$ there exists $s_a \in H(B_V)$ such that $s_a^2 = I$ (identity mapping) and a is the unique fixed point of s_a).*

(e) (See [14].) $p_{\lambda\xi} = \bar{\lambda} p_\xi$.

(f) (See [3], [13], [15, Theorem 2.3].) *If* $h\,\delta/\delta z \in g^-(B_E)$ *and* φ *is a contractive projection on E then*

$$\varphi \circ h|_{\varphi(E)}\delta/\delta v \in g^-(B_{\varphi(E)})$$

(φ *is a contractive projection if* $\varphi^2 = \varphi$ *and* $\|\varphi\| \leqq 1$). *In particular, using Proposition* 4 *below, we see that if* B_E *is symmetric then* $B_{\varphi(E)}$ *is also symmetric.*

Definition 2. (See [10], [12].) A Banach space E is called a JB* triple system if there exists a (necessarily unique) continuous mapping $\varphi: E \times E \to \mathscr{L}(E)$ such that

(i) $\{x\,y\,z\} := \varphi(x,y)(z)$ is linear in x, antilinear in y and symmetric in x and z.
(ii) $[\varphi(x,y), \varphi(u,v)] = \varphi(\{x\,y\,u\},v) - \varphi(u,\{v\,x\,y\})$ for all x,y,u,v in E (Jordan triple identity).
(iii) $\|\varphi(z,z)\| = \|z\|^2$ for all $z \in E$.
(iv) $\varphi(z,z) \in h(E)$ and the spectrum of $\varphi(z,z) \subset \mathrm{R}^+$ for all z in E.

Proposition 3. (See [12].) *A Banach space E is a* JB* *triple system if and only if* B_E *is a symmetric domain.*

In this case we have

$$\begin{aligned} p_\xi(x,y) = \varphi(x,\xi)(y) &= (x \,\square\, \xi^*)(y) \quad \text{in the notation of [12]} \\ &= -Z(\xi,x,y) \quad \text{in the notation of [16].} \end{aligned}$$

When B_E is not symmetric we may use $p_\xi(x,y)$ to define a partial JB* triple system (see [1], [9]). Kaup [12] shows that in a JB* triple system E we have

$$\|\varphi(x,y)\| \leqq 2\,\|x\| \cdot \|y\| \quad \text{for all } x,y \in E, \tag{1.1}$$

and conjectures that we can replace the constant 2 by 1 in (1.1).

By [16] the unit ball of a Banach space E is symmetric if and only if it is homogeneous (i.e. $G(B_E)(0) = B_E$). Hence Proposition 1 (d) and Proposition 3 imply the following:

Proposition 4. *A Banach space E is a* JB* *triple system if and only if*

$$\begin{aligned} E &= \{h(0) \mid h\,\delta/\delta z \in g(B_E)\} \\ &= \{h(0) \mid h\,\delta/\delta z \in g^-(B_E)\}. \end{aligned}$$

If $(E_i)_{i\in\Gamma}$ is a collection of Banach spaces, we let $l^\infty\{(E_i)_{i\in\Gamma}\}$ denote the set of all $(x_i)_{i\in\Gamma}$, $x_i \in E_i$, such that $\sup_i \|x_i\| < \infty$ and we endow this space with the norm

$$\|(x_i)_{i\in\Gamma}\| = \sup_i \|x_i\|.$$

If each E_i is a JB* triple system, then $l^\infty\{(E_i)_{i\in\Gamma}\}$ is also a JB* triple system (see [12]).

Let $\mathcal{U}$ be an ultrafilter on the set Γ and let

$$N_{\mathcal{U}} = \{(x_i)_{i\in\Gamma}: x_i \in E_i \text{ and } \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The (Banach space) ultraproduct $(E_i)_{\mathcal{U}}$ of the family $(E_i)_{i\in\Gamma}$ with respect to $\mathcal{U}$ is the quotient space $l^\infty\{(E_i)_{i\in\Gamma}\}/N_{\mathcal{U}}$. $(E_i)_{\mathcal{U}}$ is a Banach space and we let $(x_i)_{\mathcal{U}}$ denote the element (i.e. equivalence class) of $(E_i)_{\mathcal{U}}$ which contains $(x_i)_{i\in\Gamma}$. For any $(x_i)_{i\in\Gamma}$ in $l^\infty\{(E_i)_{i\in\Gamma}\}$ we have

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

If $E_i = E$ for all $i \in \Gamma$ we write $E^{\mathcal{U}}$ in place of $(E_i)_{\mathcal{U}}$ and call $E^{\mathcal{U}}$ an ultrapower of E. The mapping

$$x \in E \to (x_i = x)_{\mathcal{U}} \in E^{\mathcal{U}}$$

is an isometric embedding which we denote by $I_{\mathcal{U}}$.

If $T_i \in P(^nE_i)$ (respectively $\mathscr{L}(^nE_i)$) and $\sup_i \|T_i\| < \infty$, then we define $(T_i)_{\mathcal{U}}$ by

$$(T_i)_{\mathcal{U}}(x_i)_{\mathcal{U}} = \big(T_i(x_i)\big)_{\mathcal{U}}$$

(respectively $(T_i)_{\mathcal{U}}\big((x_i^1)_{\mathcal{U}}, \ldots, (x_i^n)_{\mathcal{U}}\big) = \big(T_i(x_i^1, \ldots, x_i^n)\big)_{\mathcal{U}}$). It is easily seen that $(T_i)_{\mathcal{U}}$ is well defined and belongs to $P\big(^n(E_i)_{\mathcal{U}}\big)$ (respectively $\mathscr{L}\big(^n(E_i)_{\mathcal{U}}\big)$ and also that

$$\|(T_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_i\|.$$

We let E'' denote the second dual of a Banach space E and let J denote the canonical isometric embedding of a Banach space into its second dual.

PROPOSITION 5. (See [2], [8].) (Principle of local reflexivity). *If E is a Banach space there exist an index set I, an ultrafilter $\mathcal{U}$ on I, an isometric* *embedding* $J_{\mathscr{U}}$ *of* E'' *into* $E^{\mathscr{U}}$, *and a contractive projection* $P_{\mathscr{U}}$ *on* $E^{\mathscr{U}}$ *(given by* $P_{\mathscr{U}}((x_i)_{\mathscr{U}}) = J(\sigma(E'', E)) - \lim_{\mathscr{U}} x_i)$ *such that*

(1.2) $P_{\mathscr{U}}(E^{\mathscr{U}}) = J_{\mathscr{U}}(E'')$.

(1.3) $J_{\mathscr{U}} \circ J = I_{\mathscr{U}}$ *(that is* $J_{\mathscr{U}}$ *"extends"* J*).*

(1.4) $P_{\mathscr{U}} \circ J_{\mathscr{U}} = J_{\mathscr{U}}$.

2.

In this section we prove some estimates which will be used in the final section to prove our main results. The following proposition is essentially contained in Lemma 2 of [14].

PROPOSITION 6. *If* $X = (\xi + p_\xi)\delta/\delta z \in g^-(B_E)$, *then*

$$\|p_\xi\| = \sup_{\|x\| \leqq 1} \|p_\xi(x)\| \leqq 450 \|\xi\|.$$

PROOF. We show that whenever

$$\sup_{\|x\| \leqq \frac{2}{3}} \|\xi + p_\xi(x)\| \leqq \tfrac{1}{54} \tag{2.1}$$

then

$$\sup_{\|x\| \leqq \frac{2}{3}} \|\xi + p_\xi(x)\| \leqq 198 \|\xi\|.$$

Then, since $p_{\lambda\xi} = \bar{\lambda} p_\xi$ (Proposition 1 (e)), we have

$$\begin{aligned}\|p_\xi\| = 9/4 \sup_{\|x\| \leqq \frac{2}{3}} \|p_\xi(x)\| &\leqq 9/4 \sup_{\|x\| \leqq \frac{2}{3}} (\|\xi + p_\xi(x)\| + \|\xi\|) \\ &\leqq 9/4(198+1)\|\xi\| \leqq 450 \|\xi\|.\end{aligned}$$

We now prove (2.1). Let g_t (that is $\mathrm{Exp}\,(tX)$) denote the one parameter subgroup of $G(B_E)$ associated with X, let $g_t'(z) = \delta/\delta z(g_t(z))$ and let $u(t) = (g_t(0), g_t'(0))$. We endow $E \times \mathscr{L}(E)$ with the sup norm topology. By [14] we have, if

$$\sup_{\|x\| \leqq \frac{2}{3}} \|\xi + p_\xi(x)\| \leqq 1/54$$

that

$$\|u(t) - u(0)\| \leqq |t| \, \|\xi\| \quad \text{for } |t| \leqq 1.$$

Let

$$h_t(z) = \frac{g_t(z)-g_t(0)}{1+\|g_t(0)\|}.$$

Since $g_t(z) \in B_E$ we have $\|h_t(z)\| < 1$ for all $z \in B_E$ and hence h_t is a holomorphic mapping from B_E into itself. Since $h_t(0) = 0$ we have by [5, equation (5)]

$$\|h_t(z) - h_t'(0)(z)\| \leqq \frac{8\|z\|^2}{(1-\|z\|)^2}\|I_E - h'(0)\| \quad \text{for } |z| < 1.$$

For $\|z\| \leqq \frac{2}{3}$ we have

$$\|g_t(z) - g_t(0) - g_t'(0)z\| \leqq \frac{8(4/9)}{(1/3)^2}\|I - \|g_t(0)\|I - g_t'(0)\|.$$

Since $g_0(0) = 0$ and $g_0' = I$, we thus have

$$\|g_t(z) - g_t(0) - g_t'(0)z\| \leqq 64\|u(t) - u(0)\| \quad \text{for } \|z\| \leqq \tfrac{2}{3}.$$

Hence for $|t| \leqq 1$ and $\|z\| \leqq \frac{2}{3}$

$$\begin{aligned}\|g_t(z) - z\| &\leqq \|g_t(z) - g_t(0) - g_t'(0)z\| + \|g_t'(0)(z) - g_0'(0)z\| + \\ &\quad + \|g_t(0) - g_0(0)\| \\ &\leqq (64+2)\|u(t) - u(0)\| \leqq 198t\|\xi\|.\end{aligned}$$

Thus

$$\|\lim_{t\to 0} t^{-1}(g_t(z) - z)\| = \|\xi + p_\xi(z)\| \leqq 198\|\xi\| \quad \text{for } \|z\| \leqq \tfrac{2}{3}.$$

Hence (2.1) holds and this completes the proof.

PROPOSITION 7. *For each* $\rho \in (0,1)$, $M \in \mathbf{R}^+$, $\alpha \in \mathbf{R}^+$, *there exists* $A = A(\rho, M, \alpha) \in (0,1)$ *such that for any Banach space* E *and any* $X_\xi = (\xi + p_\xi)\,\delta/\delta z \in g^-(B_E)$ *we have*

(2.2) $\sup\{\|\operatorname{Exp}(tX_\xi)(x)\|;\ \|\xi\| \leqq M,\ \|x\| \leqq \rho,\ |t| \leqq \alpha\} \leqq A.$

PROOF. Since $tX_\xi = X_{t\xi}$ (Proposition 1 (e)), it suffices to show

$$\sup\{\|\operatorname{Exp}(X_\xi)(x)\|;\ \|\xi\| \leqq M,\ \|x\| \leqq \rho\} < 1.$$

We first consider the case, where E is a JB* triple system. Suppose for each n there exists $\theta_n \in (0,1)$, E_n a JB* triple system, $\xi_n \in E_n$, $\|\xi_n\| \leqq M$, $x_n \in E_n$, $\|x_n\| \leqq \rho$ such that

$$\|\operatorname{Exp}(X_{\xi_n})(x_n)\| = \theta_n \to 1 \quad \text{as } n \to \infty.$$

Let $\xi = (\xi_n)_{n=1}^\infty$, now $E = l^\infty\{(E_n)_{n=1}^\infty\}$ is a JB* triple system and

$$X_\xi = \{(\xi_n + p_{\xi_n})\,\delta/\delta z_n\}_n$$

is a complete holomorphic vector field on B_E. Hence

$$\|(\operatorname{Exp}(X_{\xi_n})(x_n))_{n=1}^\infty\| = \|\operatorname{Exp}(X_\xi)(x_n)_{n=1}^\infty\| < 1.$$

This is impossible and hence

$$\sup\{\|\operatorname{Exp}(X_\xi)(x)\| : \xi, x \in E,\ \|\xi\| \leqq M,\ \|x\| \leqq \rho,\ E \text{ a JB* triple system}\} = A_1 < 1.$$

We now suppose E is arbitrary.
Let

$$V = \{h(0) \mid h\,\delta/\delta z \in g(B_E)\}.$$

If

$$X_\xi = (\xi + p_\xi)\,\delta/\delta z \in g^-(B_E),$$

then

$$\xi \in V \quad \text{and} \quad (\xi + p_{\xi|V})\,\delta/\delta v \in g^-(B_V)$$

(Proposition 1 (c)) and V is a JB* triple system (Proposition 1 (d) and Proposition 3). Hence $\|\operatorname{Exp}(X_\xi)(0)\| \leqq A_1$.

Let k denote the Kobayashi distance on B_E (see [3], [6]), then since $\operatorname{Exp}(X_\xi) \in G(B_E)$ we have

$$\begin{aligned} k(0, \operatorname{Exp}(X_\xi)(x)) &\leqq k(0, \operatorname{Exp}(X_\xi)(0)) + k(\operatorname{Exp}(X_\xi)(0), \operatorname{Exp}(X_\xi)(x)) \\ &= k(0, \operatorname{Exp}(X_\xi)(0)) + k(0, x) \\ &\leqq \frac{1}{2}\log\frac{1+A_1}{1-A_1} + \frac{1}{2}\log\frac{1+\rho}{1-\rho} \end{aligned}$$

(see [4, p. 87]). Hence, by [7, Proposition 23], there exists $A < 1$ (which depends only on A_1 and ρ) such that $\|\operatorname{Exp}(X_\xi)(x)\| \leqq A$. This completes the proof.

PROPOSITION 8. *For each* $\rho \in (0,1)$, $M \in \mathrm{R}^+$, $\alpha \in \mathrm{R}^+$, $\varepsilon \in \mathrm{R}^+$, *there exists* $\delta > 0$ *such that for any Banach space E and*

$$X_\xi = (\xi + p_\xi)\,\delta/\delta z = h(z)\,\delta/\delta z \in g^-(B_E)$$

we have

$$\sup\{\|s^{-1}(\mathrm{Exp}((t+s)X_\xi)(x) - \mathrm{Exp}(tX_\xi)(x)) - h(\mathrm{Exp}(tX_\xi)(x))\| : \\ : \|\xi\| \leqq M,\ \|x\| \leqq \rho,\ |t| \leqq \alpha,\ |s| \leqq \delta\} \leqq \varepsilon.$$

PROOF. Let φ_x denote the integral curve to X_ξ with initial point x, that is $\varphi_x(t) = \mathrm{Exp}(tX_\xi)(x)$ for all $t \in \mathrm{R}$, $\varphi_x'(t) = h(\varphi_x(t))$ and $\varphi_x(0) = x$. Hence

$$\varphi_x(t) = x + \int_0^t h(\varphi_x(w))dw$$

and

$$s^{-1}(\varphi_x(t+s) - \varphi_x(t)) - h(\varphi_x(t)) = \frac{1}{s}\int_0^s \{h(\varphi_x(t+u)) - h(\varphi_x(t))\}du$$
$$= \frac{1}{s}\int_0^s \{P_\xi(\varphi_x(t+u)) - P_\xi(\varphi_x(t))\}du.$$

Hence

$$\|s^{-1}(\varphi_x(t+s) - \varphi_x(t)) - h(\varphi_x(t))\|$$
$$\leqq \sup_{|u| \leqq s} \|p_\xi(\varphi_x(t+u)) - p_\xi(\varphi_x(t))\|$$
$$\leqq 2\|p_\xi\| \cdot \sup_{|u| \leqq s} \|\varphi_x(t+u)\| \cdot \sup_{|u| \leqq s} \|\varphi_x(t+u) - \varphi_x(t)\|$$
$$\leqq 2 \cdot 450\,\|\xi\| \cdot \sup_{|u| \leqq s} \left\|\int_0^u h(\varphi_x(t+w))dw\right\| \quad \text{(Proposition 6)}$$
$$\leqq 900 \cdot \|\xi\| \cdot (\|\xi\| + \|p_\xi\|) \cdot |s|$$
$$\leqq 900 \cdot 451 \cdot \|\xi\|^2 \cdot |s| \quad \text{(Proposition 6)}$$

This completes the proof.

3.

We prove our main results in this section.

THEOREM 9. *If* $(E_i)_{i\in I}$ *is a collection of Banach spaces indexed by the set* I, $\mathcal{U}$ *is an ultrafilter on* I,.

$$X_i = (\xi_i + p_{\xi_i})\frac{\delta}{\delta z_i} = h_i \frac{\delta}{\delta z_i} \in g^-(B_{E_i})$$

for all i *and* $\sup_i \|\xi_i\| < \infty$, *then*

$$X_{(\xi_i)_{\mathcal{U}}} := \big((\xi_i)_{\mathcal{U}} + (p_{\xi_i})_{\mathcal{U}}\big)\frac{\delta}{\delta z} \in g^-(B_{(E_i)_{\mathcal{U}}}) .$$

PROOF. By Proposition 6,

$$(p_{\xi_i})_{\mathcal{U}} \in P(^2(E_i)_{\mathcal{U}})$$

and hence $X_{(\xi_i)_{\mathcal{U}}}$ is a holomorphic vector field on $B_{(E_i)_{\mathcal{U}}}$. Let $(x_i)_{\mathcal{U}} \in B_{(E_i)_{\mathcal{U}}}$. We may suppose without loss of generality that there exists $\rho < 1$ such that $\|x_i\| \leqq \rho < 1$ for all i. For each i let $\varphi_{x_i}(t)$ be the integral curve to X_{ξ_i} with initial point x_i. By Proposition 7,

$$\sup_i \|\varphi_{x_i}(t)\| < 1$$

and hence $(\varphi_{x_i}(t))_{\mathcal{U}} \in B_{(E_i)_{\mathcal{U}}}$. By Proposition 8 for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for fixed t we have

$$\begin{aligned}&\|s^{-1}\big((\varphi_{x_i}(t+s))_{\mathcal{U}} - (\varphi_{x_i}(t))_{\mathcal{U}}\big) - (h_i)_{\mathcal{U}}\big((\varphi_{x_i}(t))_{\mathcal{U}}\big)\| \\ &\qquad \leqq \sup_i \|s^{-1}\big((\varphi_{x_i}(t+s)) - (\varphi_{x_i}(t))\big) - h_i(\varphi_{x_i}(t))\| \leqq \varepsilon\end{aligned}$$

whenever $|s| \leqq \delta$. Hence the mapping $(\varphi_{x_i})_{\mathcal{U}}$ is an integral curve to $X_{(\xi_i)_{\mathcal{U}}}$ with initial point $(x_i)_{\mathcal{U}}$. Thus $X_{(\xi_i)_{\mathcal{U}}}$ is a complete holomorphic vector field and this completes the proof.

REMARK. If $T \in h(E)$, then $''T \in h(E'')$, where $''T$ is the second transpose of T. Hence using Theorem 9, it is easily seen that if

$$X_i = (\xi_i + l_i + p_{\xi_i})\frac{\delta}{\delta z_i}, \quad \sup_i \|\xi_i\| + \sup_i \|l_i\| < \infty,$$

then

$$(X_i)_{\mathcal{U}} \in g(B_{(E_i)_{\mathcal{U}}}).$$

COROLLARY 10. *The ultraproduct of* JB* *triple systems is a* JB* *triple system.*

PROOF. If $E = (E_i)_{\mathcal{U}}$, then it suffices to apply Proposition 4 to each E_i, then to apply Theorem 9 to E and finally to apply, once more, Proposition 4 to E.

Theorem 9 also shows that the JB* structure on $(E_i)_{\mathcal{U}}$ is given by $(\varphi_i(x_i, y_i)(z_i))_{\mathcal{U}}$, where φ_i gives the JB* structure on E_i for each i.

COROLLARY 11. *If E is a* JB* *triple system, then E'' is also a* JB* *triple system (or equivalently if B_E is a symmetric domain, then $B_{E''}$ is also a symmetric domain).*

PROOF. Choose I and $\mathcal{U}$ as in Proposition 5. By Corollary 10, $E^{\mathcal{U}}$ is a JB* triple system. By Proposition 5, Proposition 1 (f) and Proposition 4, E'' is a JB* triple system.

COROLLARY 12. *If $\varphi \in G(B_E)$, then there exists $\tilde{\varphi} \in G(B_{E''})$ such that $\tilde{\varphi} \circ J = J \circ \varphi$.*

PROOF. Let

$$\tilde{G} = \{\psi \in G(B_{E''}) : \psi \circ J(B_E) = J(B_E)\}$$

and let

$$\tilde{g} = \{X \in g(B_{E''}) : \operatorname{Exp}(X) \subset \tilde{G}\}.$$

$\tilde{G}$ is a closed subgroup of $G(B_{E''})$, when $G(B_{E''})$ is endowed with the topology of local uniform convergence.

If $\varphi(0) = \xi$, then $X_\xi = (\xi + p_\xi)\,\delta/\delta z \in g^-(B_E)$. An examination of the construction given in Theorem 9 and an application of the properties listed in Proposition 5 shows that

$$X_{J(\xi)} = \big(J(\xi) + p_{J(\xi)}\big)\frac{\delta}{\delta z''} \in g^-(B_{E''})$$

and that moreover,

$$X_{J(\xi)} \circ J = J \circ X_\xi.$$

Hence

$$(J \circ \varphi)'(t) = J(\varphi'(t)) = J\big(X_\xi(\varphi(t))\big) = X_{J(\xi)}(J \circ \varphi(t))$$

for any integral curve φ to X_ξ. This implies

$$\operatorname{Exp}(t X_{J(\xi)})(J(B_E)) \subset J(B_E) \quad \text{for all } t \in \mathbb{R}.$$

Hence $\operatorname{Exp}(X_{J(\xi)}) \subset \tilde{G}$ and $X_{J(\xi)} \in \tilde{g}$.

By [1, Theorem 1.2], $J(\xi) \in \tilde{G}(0)$. Choose $w \in \tilde{G}$ such that $w(J(\xi)) = 0$. Let $w_1 = J^{-1} \circ w \circ J$. Then $w_1 \in G(B_E)$ and $J \circ w_1^{-1} = w^{-1} \circ J$. We have $w_1 \circ \varphi \in G(B_E)$ and $w_1 \circ \varphi(0) = 0$. Hence, by Schwarz's lemma [see 5], $T := w_1 \circ \varphi$ is a linear isometry of E. Let $''T$ denote the second transpose of T and let $\tilde{\varphi} = w^{-1} \circ {}''T$. Since $''T$ is an isometry of E'' we have $\tilde{\varphi} \in G(B_{E''})$. Using the fact that $J \circ T = {}''T \circ J$, we see that

$$\tilde{\varphi} \circ J = w^{-1} \circ {}''T \circ J = w^{-1} \circ J \circ T = J \circ w_1^{-1} \circ T = J \circ \varphi$$

and this completes the proof.

Corollary 12 answers problem V (i) posed by Kaup in [11].

Remarks. Corollary 11 can also be proved using J* ideals or by showing directly from the axioms of Definition 2 that the ultraproduct of JB* triple system is a JB* triple system – this just uses the fact that φ is continuous rather than Proposition 6 (this was our original approach). We can also prove Corollary 11 by first proving Theorem 9 for $E = E_i$ for all i. This still requires a weak version of Propositions 8 and 9. In addition to the corollaries already given this also shows

(i) if F is a subspace of E and $G(B_E)(0) \supset B_F$, then $G(B_{E''})(0) \supset B_{F''}$.

(ii) the ultrapower of Partial JB* triple systems (see [1], [9]) is a Partial JB* triple system.

If we require E_i to vary in Theorem 9 we also need Proposition 6, which in addition to the above also shows

(iii) the ultraproduct of Partial JB* triple systems is a Partial JB* triple system and in addition

$$\|\varphi(x, y)\| \leqq 900 \|x\| \cdot \|y\|$$

for any Partial JB* triple system.

(iv) in [9] the authors show that biholomorphic mappings of the unit ball can be extended to holomorphic mappings on a strictly larger ball. Using Theorem 9 we see that the functions which can be extended to a ball of prescribed radius can be chosen independently of the underlying Banach space.

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