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A parametric type of KKM theorem in *FC*-spaces with applications *

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Abstract The paper first proves a characteristic property in FC-spaces. By the use of the connectedness of sets, a parametric type of KKM theorem is then established in noncompact FC-spaces by introducing a linear ordered space. As a consequence, some recent results, such as noncompact minimax inequalities, saddle point theorem, and section theorem, are improved. The results generalize the corresponding results in the literatures.

Key words FC-subspace, FC-parametric-quasiconvex mapping, minimax inequality

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1 Introduction and preliminaries

In 1929, Knaster et al.[1] had proved the well-known KKM theorem on n-simplex. In 1961, Fan[2] had generalized the KKM theorem in the infinite dimensional topological vector space. Later, the KKM theorem and related topics, such as the matching theorem, fixed point theorem, coincidence theorem, variational inequalities, minimax inequalities and so on, were presented.

Chang et al.[3] obtained a parametric type of KKM theorem in generalized interval spaces. Ding[4] introduced the concept of finitely continuous spaces (FC-spaces) without any convexity assumptions, which include many topological spaces with abstrcat convexity structures, such as C-spaces (or H-spaces), G-convex spaces, and L-convex spaces. From then on, many authors have studied lots of KKM theorems and corresponding applications in FC-spaces[5-7].

Basing on the above research, in this paper, we expand the parametric type of KKM theorem from generalized interval spaces to FC-spaces and prove a parametric type of KKM theorem in FC-spaces. We obtain the noncompact minimax inequalities, the saddle point theorem, and the section problems. Our results improve and extend other corresponding results (see Refs. [3, 8-10] and the references therein).

Definition 1.1[4] *$(Y, \{\varphi_N\})$ is said to be a finitely continuous space (FC-space) if Y is a topological space and for each $N = \{y_0, y_1, \cdots, y_n\} \in \langle Y\rangle$ where some elements may be the same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset B of Y is said to be an FC-subspace of Y if for each $N = \{y_0, y_1, \cdots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\} \subset B \cap N, \varphi_N(\Delta_k) \subset B$, where $\Delta_k = co(\{e_{i_0}, \cdots, e_{i_k}\})$.*

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Definition 1.2[10] *A linear ordered space Z is said to be order-complete if, for any subset C of Z, there exists a minimal upper bound of C. A linear ordered space Z is said to be order-dense if, for any $z_1, z_2 \in Z$ with $z_1 \prec z_2$, there exists $z_3 \in Z$ such that $z_1 \prec z_3 \prec z_2$.*

Definition 1.3 *Let $(Y, \{\varphi_N\})$ be an FC-space and Z be a linear ordered space. A mapping $f : Y \to Z$ is said to be FC-parametric-quasiconvex (resp., FC-parametric-quasiconcave) if, for each $z \in Z$, the set $\{y \in Y : f(y) \preceq z\}$ (resp., $\{y \in Y : f(y) \succeq z\}$) is an FC-subspace of Y.*

Remark 1.1 Definition 1.3 generalizes Definition 4.3 of Ding[5] from real spaces to linear ordered spaces.

Definition 1.4[11] *Let Y be a topological space and Z be a linear ordered space. A mapping $f : Y \to Z$ is said to be upper semi-continuous (resp., lower semi-continuous) if, for each $z \in Z$, the set $\{y \in Y : f(y) \succeq z\}$ (resp., $\{y \in Y : f(y) \preceq z\}$) is a closed set in Y.*

Definition 1.5[12] *Let X and Y be two topological spaces. A mapping $G : Y \to 2^X$ is called transfer closed valued in Y if, for any $y \in Y, x \notin G(y)$, there exists $y' \in Y$ such that $x \notin \overline{G(y')}$.*

Remark 1.2 It is obvious that if G is closed valued in Y, then G is transfer closed valued in Y.

Lemma 1.1[12] *Let X and Y be two topological spaces and $G : Y \to 2^X$ be a set-valued mapping. Then G is transfer closed valued on Y if and only if $\cap_{y\in Y}\overline{G(y)} = \cap_{y\in Y} G(y)$.*

Lemma 1.2 *Let $(Y, \{\varphi_N\})$ be an FC-space, X be a topological space, and $L : Y \to 2^X$. $Y\backslash L^{-1}(x)$ is an FC-subspace for each $x \in X$ if and only if, for each $N = \{y_0, y_1, \cdots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\} \subset N$, we have*

$$L(y) \subset \cup_{m=0}^{k} L(y_{i_m}) \text{ for all } y \in \varphi_N(\Delta_k).$$

Proof Necessity: Suppose the contrary: there exist $N = \{y_0, y_1, \cdots, y_n\} \in \langle Y\rangle$, $\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\} \subset N$, and $y' \in \varphi_N(\Delta_k)$ such that $L(y') \not\subset \cup_{m=0}^{k} L(y_{i_m})$. Hence, there exists $x' \in L(y')$, but $x' \notin L(y_{i_m}), m = 0, 1, \cdots, k$, i.e., $\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\} \subset Y \setminus L^{-1}(x')$. Thus, $\varphi_N(\Delta_k) \subset Y \setminus L^{-1}(x')$. Since $y' \in \varphi_N(\Delta_k)$, $y' \in Y \setminus L^{-1}(x')$, i.e., $x' \notin L(y')$. This contradicts the choice of x'. Therefore, for each $N = \{y_0, y_1, \cdots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\} \subset N$, we have

$$L(y) \subset \cup_{m=0}^{k} L(y_{i_m}) \text{ for all } y \in \varphi_N(\Delta_k).$$

Sufficiency: For $x\in X$, $N=\{y_0, y_1, \cdots, y_n\} \in \langle Y\rangle$, any $\{y_{i_0}, y_{i_1}, \cdots, y_{i_k}\} \subset (Y\backslash L^{-1}(x))\cap N$, and $y \in \varphi_N(\Delta_k)$, we have $L(y) \subset \cup_{m=0}^{k} L(y_{i_m})$. $y_{i_j} (j = 0, 1, \cdots, k) \notin L^{-1}(x)$, i.e., $x \notin \cup_{m=0}^{k} L(y_{i_m})$, implies that $x \notin L(y)$ for all $y \in \varphi_N(\Delta_k)$, i.e., $y \in Y\backslash L^{-1}(x)$ for all $y \in \varphi_N(\Delta_k)$. Therefore, $\varphi_N(\Delta_k) \subset Y\backslash L^{-1}(x)$. This shows that $Y\backslash L^{-1}(x)$ is an FC-subspace for each $x \in X$. This completes the proof.

Remark 1.3 Lemma 1.1 extends Lemma 2.2 of Chang et al.[3] from generalized interval spaces to FC-spaces.

2 A parametric type of KKM theorem in FC-spaces

The following theorem is a parametric type of KKM theorem in FC-spaces.

Theorem 2.1 *Let $(Y, \{\varphi_N\})$ be an FC-space, X be a topological space, Z be a linear ordered space, and $F, G : Y \times Z \to 2^X$ be two set-valued mappings such that F has nonempty values. If the following conditions are satisfied:*

(i) *for each $N \in \langle Y\rangle$ and each $z \in Z$, $\cap_{y\in N} F(y, z)$ is a connected or empty set; and for any $y_0, y_1 \in N$, there exist $y_0', y_1' \in \varphi_N(\Delta_1)$ such that $F(y_0', z) \subset F(y_0, z)$, $F(y_1', z) \subset F(y_1, z)$;*

(ii) *for each $x \in X$ and each $z \in Z$, $\{y \in Y : x \notin F(y, z)\}$ is a closed FC-subspace;*

(iii) *for each* $(y,z)\in Y\times Z$ *and* $F(y,z)\subset\overline{G(y,z)}$, $z_1\preceq z_2$ *implies that* $F(y,z_2)\subset F(y,z_1)$ *for all* $y\in Y$;

(iv) *for each* $z\in Z$, *there exists* $\hat{z}\in Z$ *such that* $\overline{G(y,\hat{z})}\subset F(y,z)$ *for all* $y\in Y$, *then*

(1) $\{\overline{G(y,z)}:y\in Y,z\in Z\}$ *has the finite intersection property*;

(2) *if* G *is transfer closed valued and there exists* $(u,v)\in Y\times Z$ *such that* $\overline{G(u,v)}$ *is compact*, *then* $\cap_{y\in Y,z\in Z}G(y,z)\neq\emptyset$.

Proof Since F has nonempty values, by condition (iii), we know that $\overline{G(y,z)}\neq\emptyset$ for each $(y,z)\in Y\times Z$. If the intersection of any n elements of $\{\overline{G(y,z)}:y\in Y,z\in Z\}$ is nonempty, then next we prove that, for any $n+1$ elements of $\{\overline{G(y,z)}:y\in Y,z\in Z\}$, their intersection is also nonempty, where $n\geq 2$.

If there exist $(y_i,z_i)\in Y\times Z$ $(i=0,1,\cdots,n)$ such that $\cap_{i=0}^n\overline{G(y_i,z_i)}=\emptyset$ with $z_0\succeq z_1\succeq\cdots\succeq z_n$, then, by letting $H=\cap_{i=2}^nF(y_i,z_0)$, we have

$$\overline{H\cap F(y_0,z_0)}\cap\overline{H\cap F(y_1,z_0)}\subset\cap_{i=2}^n\overline{F(y_i,z_i)}\cap\overline{F(y_0,z_0)}\cap\overline{F(y_1,z_1)}\subset\cap_{i=0}^n\overline{G(y_i,z_i)}=\emptyset.$$

If y_0 and y_1 are the same, then $H\cap F(y_0,z_0)\subset\overline{H\cap F(y_0,z_0)}=\emptyset$. By condition (iv), there exists $\hat{z}\in Z$ such that, for any $y\in Y$,

$$H\cap F(y,z_0)=\cap_{i=2}^nF(y_i,z_0)\cap F(y,z_0)\supset\cap_{i=2}^n\overline{G(y_i,\hat{z})}\cap\overline{G(y,\hat{z})}\neq\emptyset.$$

This is a contradiction. Then $\{\overline{G(y,z)}:y\in Y,z\in Z\}$ has the finite intersection property.

If y_0 and y_1 are different, then $H\cap F(y_0,z_0)$ and $H\cap F(y_1,z_0)$ are separated. Define a mapping $L:Y\to 2^X$ by $L(y):=F(y,z_0)$. Then, by condition (ii), $Y\backslash L^{-1}(x)=\{y\in Y:x\notin F(y,z_0)\}$ is a closed FC-subspace for each $x\in X$. Letting $N=\{y_0,y_1,\cdots,y_n\}$ and the fixed $\{y_0,y_1\}\subset N$, in view of Lemma 1.2, we have $L(y)\subset L(y_0)\cup L(y_1)$ for any $y\in\varphi_N(\Delta_1)$, i.e., $F(y,z_0)\subset F(y_0,z_0)\cup F(y_1,z_0)$ for all $y\in\varphi_N(\Delta_1)$. Hence,

$$H\cap F(y,z_0)\subset(H\cap F(y_0,z_0))\cup(H\cap F(y_1,z_0))\text{ for all }y\in\varphi_N(\Delta_1).$$

Since $H\cap F(y,z_0)$ is connected by condition (i), for any $y\in\varphi_N(\Delta_1)$, $H\cap F(y,z_0)\subset H\cap F(y_0,z_0)$ or $H\cap F(y,z_0)\subset H\cap F(y_1,z_0)$.

Let $E_i=\{y\in\varphi_N(\Delta_1):H\cap F(y,z_0)\subset H\cap F(y_i,z_0)\}$, where $i=0,1$. By condition (i), $E_i\neq\emptyset$ $(i=0,1)$, and then $E_0\cup E_1=\varphi_N(\Delta_1)$. Since $\varphi_N(\Delta_1)$ is connected, at least one of $E_0\cap\overline{E}_1$ and $\overline{E}_0\cap E_1$ is nonempty. Supposing $E_0\cap\overline{E}_1\neq\emptyset$ and taking $\hat{y}\in E_0\cap\overline{E}_1$, we have $H\cap F(\hat{y},z_0)\subset H\cap F(y_0,z_0)$. And there exists a net $\{y_\alpha\}_{\alpha\in I}\subset E_1$ such that $y_\alpha\to\hat{y}$, and

$$H\cap F(y_\alpha,z_0)\subset H\cap F(y_1,z_0)\quad\text{for all }\alpha\in I.$$

Taking $\hat{x}\in H\cap F(\hat{y},z_0)$, we know that $\hat{x}\notin H\cap F(y_1,z_0)$. Hence, $\hat{x}\notin H\cap F(y_\alpha,z_0)$ for all $\alpha\in I$. Therefore, $\{y_\alpha\}\subset Y\backslash L^{-1}(\hat{x})$; and then $\hat{y}\in Y\backslash L^{-1}(\hat{x})$, i.e., $\hat{x}\notin F(\hat{y},z_0)$. This contradicts the choice of $\hat{x}$. So $\{\overline{G(y,z)}:y\in Y,z\in Z\}$ has the finite intersection property.

In addition, if there exists $(u,v)\in Y\times Z$ such that $\overline{G(u,v)}$ is compact, then it is easy to prove that $\cap_{y\in Y,z\in Z}\overline{G(y,z)}\neq\emptyset$. Since G is transfer closed valued, by Lemma 1.1, $\cap_{y\in Y,z\in Z}G(y,z)\neq\emptyset$. This completes the proof.

Remark 2.1 Theorem 2.1 extends Theorem 2.3 of Chang et al.[3]

Corollary 2.1 *Let* $(Y,\{\varphi_N\})$ *be an* FC*-space*, X *be a topological space*, Z *be a linear ordered space*, *and* $F,G:Y\times Z\to 2^X$ *be two set-valued mappings such that* F *has nonempty values and* G *has closed values. If the following conditions are satisfied*:

(i) *for each* $N\in\langle Y\rangle$ *and each* $z\in Z$, $\cap_{y\in N}F(y,z)$ *is a connected or empty set*; *and for any* $y_0,y_1\in N$, *there exist* y_0' *and* $y_1'\in\varphi_N(\Delta_1)$ *such that* $F(y_0',z)\subset F(y_0,z)$ *and* $F(y_1',z)\subset F(y_1,z)$;

(ii) *for each* $x \in X$ *and each* $z \in Z$, $\{y \in Y : x \notin F(y,z)\}$ *is a closed* FC*-subspace*;

(iii) *for each* $(y,z) \in Y \times Z$ *and* $F(y,z) \subset G(y,z)$, $z_1 \preceq z_2$ *implies that* $F(y,z_2) \subset F(y,z_1)$ *for all* $y \in Y$;

(iv) *for each* $z \in Z$, *there exists* $\hat{z} \in Z$ *such that* $G(y,\hat{z}) \subset F(y,z)$ *for all* $y \in Y$,

then

(1) $\{G(y,z) : y \in Y, z \in Z\}$ *has the finite intersection property*;

(2) *if there exists* $(u,v) \in Y \times Z$ *such that* $G(u,v)$ *is compact*, *then* $\cap_{y\in Y,z\in Z} G(y,z) \neq \emptyset$.

3 Applications to minimax inequalities

In this section, we shall use the results presented in Section 2 to study the minimax inequalities. We have the following results.

Theorem 3.1 *Let* $(Y,\{\varphi_N\})$ *be an* FC*-space*, X *be a topological space*, $\tilde{Z}$ *be an order-complete and order-dense linear ordered space*, *and* $f,g : X\times Y \to \tilde{Z}$ *be two mappings satisfying the following conditions*:

(i) *For each* $N \in \langle Y\rangle$ *and each* $z \in \tilde{Z}$, $\cap_{y\in N}\{x \in X : f(x,y) \succ z\}$ *is a connected or empty set*; *and for any* $y_0, y_1 \in N$, *there exist* $y_0', y_1' \in \varphi_N(\Delta_1)$ *such that* $f(x,y_i') \preceq f(x,y_i)$ *for all* $x \in X$, $i = 0,1$.

(ii) (a) *For each* $x \in X$, $f(x,\cdot)$ *is* FC*-parametric-quasiconvex and lower semi-continuous.*

(b) *For each* $y \in Y$, $f(\cdot,y)$ *and* $g(\cdot,y)$ *are upper semi-continuous.*

(iii) *There exist* $v \prec \inf_{y\in Y}\sup_{x\in X} f(x,y)$, $u \in Y$ *and a compact subset* $H \subset X$ *such that* $g(x,u) \prec v$ *for all* $x \in X\backslash H$.

(iv) $f(x,y) \preceq g(x,y)$ *for all* $(x,y) \in X\times Y$.

Then $z_* := \sup_{x\in X}\inf_{y\in Y} g(x,y) \succeq \inf_{y\in Y}\sup_{x\in X} f(x,y) := z^*$.

Proof By the completeness of $\tilde{Z}$, z_* and z^* both exist. Let $Z = \{z \in \tilde{Z} : z \prec z^*\}$. Then Z is an order-dense linear ordered space. For any $(y,z) \in Y\times Z$, let $F(y,z) = \{x \in X : f(x,y) \succ z\}$, $G(y,z) = \{x \in X : f(x,y) \succeq z\}$, and $P(y,z) = \{x \in X : g(x,y) \succeq z\}$. It follows from (b) of condition (ii) that $G(y,z)$ and $P(y,z)$ are both closed for all $(y,z) \in Y\times Z$.

Now we prove that mappings F and G satisfy all conditions of Corollary 2.1. In fact, by the definition of z^*, $F : Y\times Z \to 2^X$ is nonempty valued; and F and G satisfy condition (iii) in Corollary 2.1. For each $z \in Z$ with $z \prec z^*$, by the denseness of $\tilde{Z}$, there exists $\hat{z} \in \tilde{Z}$ such that $z \prec \hat{z} \prec z^*$. And then $\hat{z} \in Z$ and $F(y,z) \supset G(y,\hat{z})$ for all $y \in Y$. This implies that condition (iv) in Corollary 2.1 is satisfied. For each $x \in X$ and $z \in Z$, $\{y \in Y : x \notin F(y,z)\} = \{y \in Y : f(x,y) \preceq z\}$ is a closed FC-subspace by (a) of condition (ii). Then condition (ii) in Corollary 2.1 is satisfied. By condition (i), we know that condition (i) in Corollary 2.1 is also satisfied. By conclusion (1) of Corollary 2.1, $\{G(y,z) : y \in Y, z \in Z\}$ has the finite intersection property.

By condition (iv), we know $G(y,z) \subset P(y,z)$ for all $(y,z) \in Y\times Z$. Then $\{P(y,z) : y \in Y, z \in Z\}$ is a family of closed sets having the finite intersection property. By condition (iii), there exists $(u,v) \in Y\times Z$ such that $P(u,v) \subset H$. Since H is compact and $P(u,v)$ is closed, $P(u,v)$ is compact. Hence

$$\cap_{y\in Y,z\in Z} P(y,z) = \cap_{y\in Y,z\in Z} P(y,z) \cap P(u,v) \neq \emptyset.$$

Take $\hat{x} \in \cap_{y\in Y,z\in Z} P(y,z)$. Then $g(\hat{x},y) \succeq z$ for all $(y,z) \in Y\times Z$. Thus, $z_* = \sup_{x\in X}\inf_{y\in Y} g(x,y) \succeq z$ for all $z \in Z$. By the denseness of $\tilde{Z}$, we have

$$\sup_{x\in X}\inf_{y\in Y} g(x,y) \succeq z^* = \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

This completes the proof.

Corollary 3.1 *Let $(Y, \{\varphi_N\})$ be an FC-space, X be a topological space, and Z be an order-complete and order-dense linear ordered space. If $f : X \times Y \to Z$ satisfies the following conditions*:

(i) *for each $N \in \langle Y \rangle$ and each $z \in Z$, $\cap_{y \in N}\{x \in X : f(x,y) \succ z\}$ is a connected or empty set*; *and for any $y_0, y_1 \in N$, there exist $y_0', y_1' \in \varphi_N(\Delta_1)$ such that $f(x, y_i') \preceq f(x, y_i)$ for all $x \in X$, $i = 0, 1$*;

(ii) (a) *for each $x \in X$, $f(x, \cdot)$ is FC-parametric-quasiconvex and lower semi-continuous*;

(b) *for each $y \in Y$, $f(\cdot, y)$ is upper semi-continuous*;

(iii) *there exist $v \prec \inf_{y \in Y} \sup_{x \in X} f(x,y)$, $u \in Y$, and a compact subset $H \subset X$ such that $f(x,u) \prec v$ for all $x \in X \backslash H$*,

then $\sup_{x \in X} \inf_{y \in Y} f(x,y) = \inf_{y \in Y} \sup_{x \in X} f(x,y)$.

Corollary 3.2 *Let $(Y, \{\varphi_N\})$ be a compact FC-space, X be a compact topological space, and Z be an order-complete and order-dense linear ordered space. If $f : X \times Y \to Z$ satisfies the following conditions*:

(i) *for each $N \in \langle Y \rangle$ and each $z \in Z$, $\cap_{y \in N}\{x \in X : f(x,y) \succ z\}$ is a connected or empty set*; *and for any $y_0, y_1 \in N$ (y_0 and y_1 may be the same), there exist $y_0', y_1' \in \varphi_N(\Delta_1)$ such that $f(x, y_i') \preceq f(x, y_i)$ for all $x \in X$, $i = 0, 1$*;

(ii) (a) *for each $x \in X$, $f(x, \cdot)$ is FC-parametric-quasiconvex and lower semi-continuous*;

(b) *for each $y \in Y$, $f(\cdot, y)$ is upper semi-continuous*,

then f has a saddle $(\hat{x}, \hat{y}) \in X \times Y$.

Remark 3.1 Corollary 3.2 contains the famous von Neumann theorem in mathematical economy and game theory.

Theorem 3.2 *Let $(Y, \{\varphi_N\})$ be an FC-space, X be a topological space, Z be an order-complete and order-dense linear ordered space, and $f, g : X \times Y \to Z$ be two mappings satisfying $f(x,y) \preceq g(x,y)$ for all $(x,y) \in X \times Y$ and the following conditions*:

(i) *For each $x \in X$, $f(x, \cdot)$ is FC-parametric-quasiconvex and lower semi-continuous.*

(ii) *For each $y \in Y$, $f(\cdot, y)$ and $g(\cdot, y)$ are upper semi-continuous.*

(iii) *There exist a nonempty subset $K \subset X$ and a compact subset $H \subset Y$ such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) \preceq \inf_{y \in Y \backslash H} \sup_{x \in K} f(x,y);$$

and for any finite subset $F \subset X$, there exists a compact set $K(F) \supset K \cup F$ such that, for each $N \in \langle Y \rangle$ and $z \in Z$, $\cap_{y \in N}\{x \in K(F) : f(x,y) \succ z\}$ is a connected or empty set; *and for any $y_0, y_1 \in N$, there exist $y_0', y_1' \in \varphi_N(\Delta_1)$ such that $f(x, y_i') \preceq f(x, y_i)$ for all $x \in X$, $i = 0, 1$.*
Then $\sup_{x \in X} \inf_{y \in Y} g(x,y) \succeq \inf_{y \in Y} \sup_{x \in X} f(x,y)$.

Proof Let $z_* = \sup_{x \in X} \inf_{y \in Y} g(x,y)$ and $z^* = \inf_{y \in Y} \sup_{x \in X} f(x,y)$. By the completeness of Z, z_* and z^* both exist. If $z_* \prec z^*$, then, by the density of Z, there exists $\bar{z} \in Z$ such that $z_* \prec \bar{z} \prec z^*$.

For any $x \in X$, let $L(x) = \{y \in Y : f(x,y) \preceq \bar{z}\}$. By condition (i), $L(x)$ is closed. For any $x \in X$, let $M(x) = L(x) \cap (\cap_{z \in K} L(z))$. Then $M(x)$ is closed. For any $y \in Y \backslash H$, by condition (iii), there exists $x_0 \in K$ such that $f(x_0, y) \succ \bar{z}$, i.e., $y \notin \cap_{z \in K} L(z)$. This implies that $\cap_{z \in K} L(z) \subset H$, and then $M(x) \subset H$ for all $x \in X$.

Now, we prove that $\{M(x) : x \in X\}$ has the finite intersection property. By condition (iii), for any finite $F \subset X$, there exists a compact set $K(F) \supset K \cup F$ such that, for each $N \in \langle Y \rangle$ and $z \in Z$, $\cap_{y \in N}\{x \in K(F) : f(x,y) \succ z\}$ is a connected or empty set. From Theorem 3.1, we have

$$\sup_{x \in K(F)} \inf_{y \in Y} g(x,y) \succeq \inf_{y \in Y} \sup_{x \in K(F)} f(x,y),$$

and

$$\inf_{y \in Y} \sup_{x \in K(F)} f(x,y) \preceq \sup_{x \in X} \inf_{y \in Y} g(x,y) = z_* \prec \bar{z};$$

and then we assert $\cap_{x\in K(F)}M(x)\neq\emptyset$. Suppose the contrary, then $Y=\cup_{x\in K(F)}(Y\backslash M(x))$. Thus, for each $y\in Y$, there exists $x(y)\in K(F)$ such that $y\in Y\backslash M(x(y))$, i.e., $y\notin M(x(y))=L(x(y))\cap(\cap_{z\in K}L(z))$. Hence, $y\notin L(x(y))$ or $y\notin\cap_{z\in K}L(z)$. If $y\notin L(x(y))$, then $f(x(y),y)\succ\bar{z}$; if $y\notin\cap_{z\in K}L(z)$, then there exists $\hat{x}(y)\in K\subset K(F)$ such that $y\notin L(\hat{x}(y))$, i.e., $f(\hat{x}(y),y)\succ\bar{z}$. This implies that there exists a mapping $\hat{u}:Y\to K(F)$ such that $f(\hat{u}(y),y)\succ\bar{z}$ for all $y\in Y$. Then $\sup_{x\in K(F)}f(x,y)\succ\bar{z}$ for all $y\in Y$. Hence, $\inf_{y\in Y}\sup_{x\in K(F)}f(x,y)\succeq\bar{z}$, this contradicts $\inf_{y\in Y}\sup_{x\in K(F)}f(x,y)\prec\bar{z}$. Therefore, $\cap_{x\in K(F)}M(x)\neq\emptyset$. Since $\cap_{x\in F}M(x)\supset\cap_{x\in K(F)}M(x)$, $\{M(x):x\in X\}$ has the finite intersection property. Then $\cap_{x\in X}M(x)\neq\emptyset$. Hence, there exists $\bar{y}\in\cap_{x\in X}M(x)$, and $\bar{y}\in L(x)$ for all $x\in X$, i.e., $f(x,\bar{y})\preceq\bar{z}$. So $z^*\preceq\bar{z}$. This contradicts the choice of $\bar{z}$. Therefore, $z_*\succeq z^*$. This completes the proof.

Corollary 3.3 *Let $(Y,\{\varphi_N\})$ be an FC-space, X be a topological space, Z be an order-complete and order-dense linear ordered space, and $f:X\times Y\to Z$ be a mapping satisfying the following conditions*:

(i) *For each $x\in X, f(x,\cdot)$ is FC-parametric-quasiconvex and lower semi-continuous.*

(ii) *For each $y\in Y, f(\cdot,y)$ is upper semi-continuous.*

(iii) *There exist a nonempty subset $K\subset X$ and a compact subset $H\subset Y$ such that*

$$\inf_{y\in Y}\sup_{x\in X}f(x,y)\preceq\inf_{y\in Y\backslash H}\sup_{x\in K}f(x,y);$$

and for any finite subset $F\subset X$, there exists a compact set $K(F)\supset K\cup F$ such that, for each $N\in\langle Y\rangle$ and $z\in Z$, $\cap_{y\in N}\{x\in K(F):f(x,y)\succ z\}$ is a connected or empty set; and for any $y_0,y_1\in N$, there exist $y_0',y_1'\in\varphi_N(\Delta_1)$ such that $f(x,y_i')\preceq f(x,y_i)$ for all $x\in X$, $i=0,1$.
Then $\sup_{x\in X}\inf_{y\in Y}f(x,y)=\inf_{y\in Y}\sup_{x\in X}f(x,y)$.

Corollary 3.4 *Let $(Y,\{\varphi_N\})$ be a compact FC-space, X be a compact topological space, Z be an order-complete and order-dense linear ordered space, and $f:X\times Y\to Z$ be a mapping satisfying the following conditions*:

(i) *For each $x\in X, f(x,\cdot)$ is FC-parametric-quasiconvex and lower semi-continuous.*

(ii) *For each $y\in Y, f(\cdot,y)$ is upper semi-continuous.*

(iii) *There exist a nonempty subset $K\subset X$ and a compact subset $H\subset Y$ such that*

$$\inf_{y\in Y}\sup_{x\in X}f(x,y)\preceq\inf_{y\in Y\backslash H}\sup_{x\in K}f(x,y);$$

and for any finite subset $F\subset X$, there exists a compact set $K(F)\supset K\cup F$ such that, for each $N\in\langle Y\rangle$ and $z\in Z$, $\cap_{y\in N}\{x\in K(F):f(x,y)\succ z\}$ is a connected or empty set; and for any $y_0,y_1\in N$, there exist $y_0',y_1'\in\varphi_N(\Delta_1)$ such that $f(x,y_i')\preceq f(x,y_i)$ for all $x\in X$, $i=0,1$.
Then f has a saddle $(\hat{x},\hat{y})\in X\times Y$.

4 Applications to section theorem

In this section, we shall use the results presented in Section 2 to study the section theorem. We have the following results.

Theorem 4.1 *Let $(Y,\{\varphi_N\})$ be an FC-space, X be a topological space, Z be a linear ordered space, and B and C be two subsets of $X\times Y\times Z$ satisfying the following conditions*:

(i) *For any $(y,z)\in Y\times Z$, the sections $B_{(y,z)}:=\{x\in X:(x,y,z)\in B\}\neq\emptyset$ and $C_{(y,z)}:=\{x\in X:(x,y,z)\in C\}$ are transfer closed valued, and there exists $(u,v)\in Y\times Z$ such that $\overline{C}_{(u,v)}$ is compact.*

(ii) *For each $N\in\langle Y\rangle$ and each $z\in Z$, $\cap_{y\in N}B_{(y,z)}$ is a connected or empty set; and for any $y_0,y_1\in N$, there exist $y_0',y_1'\in\varphi_N(\Delta_1)$ such that $B_{(y_0',z)}\subset B_{(y_0,z)}$ and $B_{(y_1',z)}\subset B_{(y_1,z)}$.*

(iii) *For each $(x,z)\in X\times Z$, the set $\{y\in Y:(x,y,z)\notin B\}$ is a closed FC-subspace.*

(iv) *$B\subset C$ and $z_1\preceq z_2$ implies that $B_{(y,z_2)}\subset B_{(y,z_1)}$ for all $y\in Y$.*

(v) *For each* $z \in Z$, *there exists* $\hat{z} \in Z$ *such that* $\overline{C}_{(y,\hat{z})} \subset B_{(y,z)}$ *for all* $y \in Y$.
Then there exists $\hat{x} \in X$ *such that* $\{\hat{x}\} \times (Y \times Z) \subset C$.

Proof Let $F(y,z) = B_{(y,z)}$ and $G(y,z) = C_{(y,z)}$. F and G satisfy all conditions in Theorem 2.1. By conclusion (2) in Theorem 2.1, $\cap_{y \in Y, z \in Z} G(y,z) \neq \emptyset$. Hence, there exists $\hat{x} \in X$ such that $\hat{x} \in G(y,z)$ for all $(y,z) \in Y \times Z$, i.e., $(\hat{x}, y, z) \in C$ for all $(y,z) \in Y \times Z$. Therefore, $\{\hat{x}\} \times (Y \times Z) \subset C$.

Corollary 4.1 *Let* $(Y, \{\varphi_N\})$ *be an* FC*-space,* X *be a topological space,* Z *be a linear ordered space, and* B *and* C *be two subsets of* $X \times Y \times Z$ *satisfying the following conditions*:

(i) *For any* $(y,z) \in Y \times Z$, *the sections* $B_{(y,z)} := \{x \in X : (x,y,z) \in B\} \neq \emptyset$ *and* $C_{(y,z)} := \{x \in X : (x,y,z) \in C\}$ *are closed valued, and there exists* $(u,v) \in Y \times Z$ *such that* $C_{(u,v)}$ *is compact.*

(ii) *For each* $N \in \langle Y \rangle$ *and each* $z \in Z$, $\cap_{y \in N} B_{(y,z)}$ *is a connected or empty set*; *and for any* $y_0, y_1 \in N$, *there exist* $y_0', y_1' \in \varphi_N(\Delta_1)$ *such that* $B_{(y_0',z)} \subset B_{(y_0,z)}$ *and* $B_{(y_1',z)} \subset B_{(y_1,z)}$.

(iii) *For each* $(x,z) \in X \times Z$, *the set* $\{y \in Y : (x,y,z) \notin B\}$ *is a closed* FC*-subspace.*

(iv) $B \subset C$ *and* $z_1 \preceq z_2$ *implies that* $B_{(y,z_2)} \subset B_{(y,z_1)}$ *for all* $y \in Y$.

(v) *For each* $z \in Z$, *there exists* $\hat{z} \in Z$ *such that* $C_{(y,\hat{z})} \subset B_{(y,z)}$ *for all* $y \in Y$.
Then there exists $\hat{x} \in X$ *such that* $\{\hat{x}\} \times (Y \times Z) \subset C$.

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