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Grassmann manifolds of Jordan algebras

By

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Abstract. We show that, in a JB-algebra, the projections form a Banach manifold and also, the rank-n projections in a JBW-factor form a Riemannian symmetric space of compact type, for $n \in \mathbb{N} \cup \{0\}$.

1. Introduction. The close connection between Jordan algebras and geometry is well-known (cf. [10]). Recently, various differentiable manifolds associated with a JB*-triple have been studied in [1], [5], [6], [7], [8]. These manifolds can be regarded as infinite dimensional analogues of the Grassmann manifolds. In particular, the manifolds of finite rank projections in the algebra $B(H)$ of bounded operators on a Hilbert space H have been studied in [1], [5], via the complex JB*-structures of $B(H)$. Since these manifolds are contained in the self-adjoint part $B(H)_{sa}$ of $B(H)$, which is a real JB-algebra, it is desirable to study them via the real structures of $B(H)_{sa}$ without complexification, and moreover, to tackle the wider question of such manifolds in arbitrary JB-algebras. The object of this paper is to address these issues, and indeed, we study manifolds of projections in JB-algebras using only real Jordan algebraic structures. The merit of this alternative approach may lie in its simplicity and generality. It also unifies and clarifies some results in [1], [5]. For convenience, we regard a point as a "0-dimensional manifold".

We first show that, in any JB-algebra, the projections form a real Banach manifold $\mathcal{P}$, and the finite rank projections, as well as the infinite rank projections, in a JBW-algebra form submanifolds of $\mathcal{P}$. In a JBW-factor $\mathcal{A}$, the manifold of finite rank projections consists of a sequence of connected components:

$$\{\mathcal{P}_n\}_{n=0}^{k} \quad (k \in \mathbb{N} \cup \{\infty\})$$

where $\mathcal{P}_n$ is the subspace of rank-n projections in $\mathcal{A}$. We show that each of these components carries the structure of a Riemannian symmetric space, which can be infinite dimensional. This result generalizes Hirzebruch's result [4] on the manifold of minimal projections in a finite dimensional formally real simple Jordan algebra, and is analogous to Nomura's result

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[12] on manifolds of rank-n projections in a topologically simple Jordon-Hilbert algebra. In fact, we develop our method by unifying the ideas in [4], [12] and extending them to the setting of infinite dimensional JB-algebras.

The manifolds considered in this paper also provide some natural examples of non-associative vector bundles discussed in [2]. We use [9], [11], [16] for references for infinite dimensional Banach manifolds.

We recall that a *Jordan algebra* is a commutative, but not necessarily associative, algebra $(\mathcal{A}, \circ)$ satisfying the *Jordan identity* : $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$. We restrict our attention to only real algebras and always use $\circ$ for the product in a Jordan algebra. Every associative algebra $\mathcal{B}$ is a Jordan algebra in the canonical Jordan product

$$a \circ b = \frac{1}{2}(ab + ba) \quad (a, b \in \mathcal{B}) \tag{1.1}$$

where the product on the right is the original product in $\mathcal{B}$. A Jordan algebra $\mathcal{A}$ is called *special* if it is isomorphic to, and hence identified with, a Jordan subalgebra of an associative algebra $\mathcal{B}$ with respect to the Jordan product in (1.1). In this case, we will use the canonical Jordan product in (1.1) for $\mathcal{A}$, omitting mentioning $\mathcal{B}$ explicitly. A real Banach space $\mathcal{A}$ is called a *JB-algebra* if it is a Jordan algebra and the norm satisfies

$$\|a \circ b\| \leqq \|a\|\|b\|, \quad \|a^2\| = \|a\|^2, \quad \|a^2\| \leqq \|a^2 + b^2\|$$

for all $a, b \in \mathcal{A}$. The self-adjoint part of a C*-algebra is a JB-algebra.

A JB-algebra $\mathcal{A}$ is called a *JBW-algebra* if it is the dual of a Banach space in which case the predual of $\mathcal{A}$ is unique, the weak* topology on $\mathcal{A}$ is unambiguous and $\mathcal{A}$ must have an identity, denoted by $\mathbf{1}$. A JBW-algebra is called a *JBW-factor* if its *centre* $Z = \{z \in \mathcal{A} : z \circ (a \circ b) = (z \circ a) \circ b \; \forall \, a, b \in \mathcal{A}\}$ is trivial, that is, $Z = \{\gamma \mathbf{1} : \gamma \in \mathbb{R}\}$.

The finite dimensional formally real Jordan algebras are exactly the finite dimensional JB-algebras [3]. Hence Hirzebruch's result [4] states that the manifold of minimal projections in a finite dimensional *simple* JB-algebra form a compact Riemannian symmetric space. The infinite dimensional generalization of finite dimensional simple JB-algebras are the JBW-factors. Our goal is a complete generalization of Hirzebruch's result, using only real Jordan algebraic methods to show that the rank-n projections in a JBW-factor form a Riemannian symmetric space of compact type.

2. Jordan algebras. We begin by recalling some basic properties of projections in a JB-algebra $(\mathcal{A}, \circ)$. On $\mathcal{A}$, one defines the *Jordan triple product* by

$$\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$$

and the multiplication operator $L(a) : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$L(a)(x) = a \circ x.$$

A *projection* $p \in \mathcal{A}$, that is, an element satisfying $p^2 = p$, gives rise to the *Peirce decomposition* of $\mathcal{A}$ when it is unital:

$$\mathcal{A} = \mathcal{A}_0(p) \oplus \mathcal{A}_1(p) \oplus \mathcal{A}_2(p)$$

where

$$\mathcal{A}_k(p) = \{x \in \mathcal{A} : 2p \circ x = kx\}$$

is the k-eigenspace of the operator $2L(p)$ for $k = 0, 1, 2$, with the corresponding *Peirce projection* $P_k(p) : \mathcal{A} \longrightarrow \mathcal{A}_k(p)$ given by

$$P_0(p)(\cdot) = \{\mathbf{1} - p, \cdot, \mathbf{1} - p\}, \quad P_1(p) = 4L(p) - 4L(p)^2,$$
$$P_2(p)(\cdot) = \{p, \cdot, p\}.$$

We note that

$$\mathcal{A}_0(p) \circ \mathcal{A}_2(p) = \{0\} \quad \text{and} \quad \mathcal{A}_1(p) \circ \mathcal{A}_1(p) \subset \mathcal{A}_0(p) \oplus \mathcal{A}_2(p).$$

A JB-algebra may contain only the trivial projection 0 and possibly the identity **1**. However, a JBW-algebra contains an abundance of projections which form an orthomodular lattice.

A non-zero projection p in a JB-algebra $\mathcal{A}$ is called *minimal* if $\{p, \mathcal{A}, p\} = \mathbb{R}p$. Given a *non-zero* projection p in a JBW-algebra $\mathcal{A}$, we say that p has *infinite rank* if there are infinitely many mutually orthogonal non-zero projections in $\{p, \mathcal{A}, p\}$; otherwise, p is said to have *finite rank* and the unique maximal cardinality of mutually orthogonal non-zero projections in $\{p, \mathcal{A}, p\}$ is defined to be the rank of p, denoted by $\operatorname{rank}(p)$, in which case, p is a sum of mutually orthogonal minimal projections $p_1, \ldots, p_n$ with $n = \operatorname{rank}(p)$. The minimal projections are exactly the rank-1 projections. We regard 0 as a finite rank projection with $\operatorname{rank}(0) = 0$. It follows that, if $\mathcal{A}$ is a JBW-algebra, then the non-zero finite rank projections are all contained in the type *I* summand $\mathcal{A}_I$ of $\mathcal{A}$ since minimal projections are abelian. The rank of a JBW-algebra $\mathcal{A}$, $\operatorname{rank}(\mathcal{A})$, is defined to be the rank of the identity. We refer to [3, 5.3.9] for more details of the type *I*, type *II* and type *III* summands of a JBW-algebra.

Lemma 2.1. *Let $(\mathcal{A}, \circ)$ be a unital JB-algebra and let $p \in \mathcal{A}$ be a minimal projection with Peirce decomposition*

$$\mathcal{A} = \mathcal{A}_0(p) \oplus \mathcal{A}_1(p) \oplus \mathcal{A}_2(p).$$

Then for every $x \in \mathcal{A}_1(p)\backslash\{0\}$, we have $x^2 \in \mathcal{A}_0(p) \oplus \mathcal{A}_2(p)$ and the Jordan subalgebra $\mathcal{A}(p, x)$ in $\mathcal{A}$ generated by p and x is 3-dimensional.

Proof. Note that $\{p, x, p\} = 2p \circ (p \circ x) - p \circ x = 0$. By the *Shirshov-Cohn theorem* [3, 7.2.5], $\mathcal{A}(p, x)$ is special and we have $x = 2(p \circ x) = xp + px$ which gives $x^2 = x^2p + xpx$ and, by minimality, $px^2 = px^2p + pxpx = px^2p = \gamma p$ for some $\gamma \in \mathbb{R}$. Likewise $x^2p = \gamma p$ and hence $p \circ x^2 = \gamma p = \{p, x^2, p\}$. Moreover $x^2 \circ (p \circ x) = (x^2 \circ p) \circ x$ gives $x^3 = \gamma x$. Hence $\mathcal{A}(p, x)$ is the linear span of $\{p, x, x^2\}$ which can be seen readily to be linearly independent, using the identities derived above. □

An element s in a unital JB-algebra $\mathcal{A}$ is called a *symmetry* if $s^2 = \mathbf{1}$. Two projections p and q in $\mathcal{A}$ are called *Jordan equivalent* it they are exchanged by a symmetry s, that is,

$p = \{s, q, s\}$ which implies $q = \{s, p, s\}$. We note that any two minimal projections in a JBW-factor are Jordan equivalent, by the comparison theorem for projections [3, 5.2.13].

Lemma 2.2. *Let p and q be two Jordan equivalent orthogonal projections in a unital JB-algebra $\mathcal{A}$. Then there is an element $x \in \mathcal{A}_1(p) \cap \mathcal{A}_1(q)$ such that $x^2 = p + q$.*

Proof. Let $q = \{t, p, t\}$ for some symmetry $t \in \mathcal{A}$. Let $s = 2q - \mathbf{1}$. Then s is a symmetry and we have $\{s, \{t, p, t\}, s\} = q$. We define $x = 2\{p, t, s\}$. Following the computation in [3, p. 125], one finds $x^2 = p + q$. Further, we have

$$\begin{aligned} x &= 2\{p, t, 2q - \mathbf{1}\} = 4\{p, t, \{t, p, t\}\} - 2\{p, t, \mathbf{1}\} \\ &= 4p \circ t - 2p \circ t = 2p \circ t \end{aligned}$$

which gives $p \circ x = 2p \circ (p \circ t) = p \circ t + \{p, t, p\}$ where, by orthogonality of p and q, we have

$$\begin{aligned} \{p, t, p\} &= \{p, t, \{t, q, t\}\} \\ &= \{\{p, t, t\}, q, t\} - \{t, \{t, p, q\}, t\} + \{t, q, \{p, t, t\}\} \\ &= 2\{\{p, t, t\}, q, t\} = 2\{p, q, t\} = 0. \end{aligned}$$

Therefore we obtain $p \circ x = \frac{1}{2}x$, that is, $x \in \mathcal{A}_1(p)$. Since $q \circ t = \{t, p, t\} \circ t = p \circ t$, we also have $x \in \mathcal{A}_1(q)$. $\square$

The JBW-factors, generalizing finite dimensional simple JB-algebras, are classified as follows:

type I_2: spin factors $H \oplus \mathbb{R}$,
type I_3: $H_3(\mathcal{O})$,
type I_n: $B(H)_{sa}$ $(\dim H = n \in \mathbb{N} \cup \{\infty\} \backslash \{2, 3\})$,
type *II*: semifinite and continuous,
type *III*: purely infinite,

where a spin factor $H \oplus \mathbb{R}$ is a direct sum of a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $\mathbb{R}$, with Jordan product

$$(x \oplus \zeta)(y \oplus \eta) = (\eta x + \zeta y) \oplus (\langle x, y \rangle + \zeta \eta)$$

and norm

$$\|(x \oplus \zeta\| = \|x\|_H + |\zeta|,$$

$H_3(\mathcal{O})$ is the Jordan algebra of 3×3 Hermitian matrices over the octonions $\mathcal{O}$ and $B(H)_{sa}$ is the Jordan algebra of self-adjoint bounded linear operators on a real, complex or quaternionic Hilbert space H. The Jordan product in $H_3(\mathcal{O})$ and $B(H)_{sa}$ is given by

$$a \circ b = \frac{1}{2}(ab + ba)$$

where the product on the right is the usual product of matrices or operators. The exceptional Jordan algebra $H_3(\mathcal{O})$ is equipped with an order-unit norm and $B(H)_{sa}$ is equipped with the operator norm. We need not discuss the details of type *II* and type *III* factors, it suffices to remark that they cannot contain minimal, and hence non-zero finite rank, projections [3, 5.3.1].

Given a finite-dimensional (type *I*) JBW-factor $\mathcal{A}$ with dimension n, we define $\lambda_1 : \mathcal{A} \longrightarrow \mathbb{R}$ to be the trace

$$\lambda_1(x) = \frac{\operatorname{rank}(\mathcal{A})}{n} \operatorname{trace}(L(x))$$

(see also [4]) so that $\lambda_1(p) = 1$ for every minimal projection p in $\mathcal{A}$.

If $\mathcal{A}$ is an infinite-dimensional type *I* JBW-factor, then $\mathcal{A}$ is of type I_2 or type I_∞. In the former case, say $\mathcal{A} = H \oplus \mathbb{R}$, we define $\lambda_2 : \mathcal{A} \longrightarrow \mathbb{R}$ by

$$\lambda_2(x \oplus \zeta) = 2\zeta.$$

In the type I_∞ case, we have $\mathcal{A} = B(H)_{sa}$ and define $\lambda_\infty : \mathcal{A} \longrightarrow \mathbb{R} \cup \{\infty\}$ by

$$\lambda_\infty(x) = \begin{cases} \operatorname{trace}(x) & \text{if } x \text{ is of trace class} \\ \infty & \text{otherwise.} \end{cases}$$

We have $\lambda_\infty(p) = 1$ for every minimal projection p and $\lambda_\infty(x) < \infty$ for each x in the Peirce 1-space $\mathcal{A}_1(p)$ since the trace-class operators in $B(H)$ form an ideal.

In a type I_2 JBW-factor, an element $p = x \oplus \zeta \in H \oplus \mathbb{R}$ is a minimal projection if, and only if, $\|x\|_H = \frac{1}{2} = \zeta$. Hence we also have $\lambda_2(p) = 1$ for a minimal projection p in $H \oplus \mathbb{R}$.

Given a type *I* JBW-factor $\mathcal{A}$, we now define a function $\lambda : \mathcal{A} \longrightarrow \mathbb{R} \cup \{\infty\}$, called the *canonical trace*, by

$$\lambda = \begin{cases} \lambda_1 & \text{if } \dim \mathcal{A} < \infty, \\ \lambda_2 & \text{if } \mathcal{A} \text{ is an infinite-dimensional spin factor,} \\ \lambda_\infty & \text{if } \mathcal{A} \text{ is of type } I_\infty. \end{cases} \tag{2.1}$$

It is readily verified that λ is associative, that is

$$\lambda((x \circ y) \circ z) = \lambda(x \circ (y \circ z))$$

if $\lambda(x) < \infty$. We also note that $\lambda(\{x, y, x\}) = \lambda(x^2 \circ y)$ if $\lambda(x) < \infty$.

Lemma 2.3. *Let $\mathcal{A}$ be a JB-algebra and let $\mu : \mathcal{A} \longrightarrow \mathbb{R}$ be an associative positive linear functional. For any element $a \geqq 0$, we have*

$$|\mu(x \circ a)| \leqq \|x\| \mu(a) \qquad (x \in \mathcal{A}).$$

Proof. We may assume $\mathcal{A}$ has an identity $\mathbf{1}$. By associativity, we have $\mu(\{x, y, x\}) = \mu(x^2 \circ y)$ for all $x, y \in \mathcal{A}$. The linear functional $\psi(x) = \mu(x \circ a)$ is positive since $x \geqq 0$ implies

$$\mu(x \circ a) = \mu((x^{1/2})^2 \circ a) = \mu(\{x^{1/2}, a, x^{1/2}\}) \geqq 0$$

as $a \geqq 0$. Hence we have

$$|\mu(x \circ a)| = |\psi(x)| \leqq \|x\| \|\psi\| = \|x\| \psi(\mathbf{1}) = \|x\| \mu(a). \qquad \square$$

In what follows, we denote by M the subspace of minimal projections in a JBW-factor $\mathcal{A}$. We note that M may be empty; but if it is non-empty, then $\mathcal{A}$ must be of type I and hence admits the canonical trace λ. The following result generalizes [4, Satz 2.1].

Proposition 2.4. *Let $\mathcal{A}$ be a JBW-factor and let p be a minimal projection in M. For any x in the Peirce-1 space $\mathcal{A}_1(p)$ satisfying $\lambda(x^2) = 2$, we have*

$$M \cap \mathcal{A}(p, x) = \left\{ (\cos 2\theta) p + \left(\frac{1}{2} \sin 2\theta \right) x + \frac{1}{2} (1 - \cos 2\theta)\, x^2 : \theta \in \mathbb{R} \right\}.$$

Proof. Since $\mathcal{A}$ contains a minimal projection, it is of type I. Let $\lambda : \mathcal{A} \longrightarrow \mathbb{R} \cup \{\infty\}$ be the canonical trace defined in (2.1). We first note that $\lambda(x) = 0$ since $\lambda(x) = 2\lambda(p \circ x) = 2\lambda(p \circ (p \circ x)) = \lambda(p \circ x) = \frac{1}{2}\lambda(x)$. As in the proof of Lemma 2.1, we have $p \circ x^2 = \gamma p$ for some $\gamma \in \mathbb{R}$. Since $\lambda(p \circ x^2) = \lambda((p \circ x) \circ x) = \frac{1}{2}\lambda(x^2) = 1$, we have $\gamma = 1$.

Now let $q = \zeta p + \eta x + \kappa x^2 \in M \cap \mathcal{A}(p, x)$. Then $\{p, q, p\} = \zeta p + \kappa\{p, x^2, p\} = (\zeta + \kappa) p$ implies $0 \leqq \zeta + \kappa \leqq 1$. Also $1 = \lambda(q) = \zeta + 2\kappa$ implies $-1 \leqq -\kappa \leqq \zeta \leqq 1 - \kappa$. On the other hand, we have

$$\begin{aligned} \zeta p + \eta x + \kappa x^2 &= (\zeta p + \eta x + \kappa x^2)^2 \\ &= (\zeta^2 + 2\zeta\kappa) p + (\zeta\eta + 2\eta\kappa) x + (\eta^2 + \kappa^2) x^2 \end{aligned}$$

which implies $\kappa = \eta^2 + \kappa^2 \geqq 0$. Therefore $|\zeta| \leqq 1$ and $\zeta = \cos 2\theta$ for some $\theta \in \mathbb{R}$ which gives $\kappa = \frac{1}{2}(1 - \cos 2\theta)$ and $\eta = \frac{1}{2} \sin 2\theta$.

Conversely, given any

$$z = (\cos 2\theta) p + \left(\frac{1}{2} \sin 2\theta \right) x + \frac{1}{2} (1 - \cos 2\theta)\, x^2$$

for some $\theta \in \mathbb{R}$, it is evident that $z^2 = z$ by the above arguments. Since $\lambda(z) = 1$, it follows that z is a minimal projection and hence $z \in M \cap \mathcal{A}(p, x)$. $\square$

Corollary 2.5. *Let M be the subspace of minimal projections in a JBW-factor $\mathcal{A}$. Then M is path connected.*

Proof. By definition, the empty set is path connected. Fix $p \in M$. We show that any other $q \in M$ is of the form

$$q = (\cos 2\theta) p + \left(\frac{1}{2} \sin 2\theta \right) x + \frac{1}{2} (1 - \cos 2\theta) x^2$$

for some $\theta \in \mathbb{R}$ and $x \in \mathcal{A}_1(p)$, and hence q is joined to p by a continuous path of projections in M. Note that p and q are Jordan equivalent as remarked before. If q and

p are orthogonal, then by Lemma 2.2, we have $q = -p + x^2 \in M \cap \mathcal{A}(p, x)$ for some $x \in \mathcal{A}_1(p)$ and we are done by Proposition 2.4.

Suppose q and p are not orthogonal. Then the Peirce-1 component $q_1 = P_1(p)(q) = 2(p \circ q - P_2(p)(q))$ is in the Jordan algebra $\mathcal{A}(p, q)$ generated by p and q. Therefore we have $\mathcal{A}(p, q_1) \subset \mathcal{A}(p, q)$ where $\dim \mathcal{A}(p, q) = 3$ since $p \circ q \neq 0$. We have $q_1 \neq 0$ for otherwise, $p \circ q = P_2(p)(q) = \gamma p$ for some $\gamma \in \mathbb{R}$ which is impossible since p and q are two distinct minimal projections. It follows from Lemma 2.1 that $\dim \mathcal{A}(p, q_1) = 3$. Hence $\mathcal{A}(p, q_1) = \mathcal{A}(p, q)$ and $q \in \mathcal{A}(p, q_1)$. By Proposition 2.4, q is joined to p by a continuous path of minimal projections. □

Remark 2.6. The above result is false for JBW-algebras. In fact, it is even false for the abelian algebra $\mathbb{R}^2$ in which the space of minimal projections consists of two points $\{(1, 0), (0, 1)\}$ which is not connected.

Given two projections p and q in a JBW-algebra $\mathcal{A}$, their supremum $p \vee q$ is the range projection $r(p + q)$ of $p + q$ [3, 4.2.8]. For a positive element $a \in \mathcal{A}$, its range projection $r(a)$ is the weak* limit of the sequence $((a + \frac{1}{m})^{-1} \circ a)$ where $(a + \frac{1}{m})^{-1}$ is the inverse of $a + \frac{1}{m}$ in the JBW-algebra $W(a)$ generated by a (cf. [13, p. 23]). By continuity of the inverse and Jordan product, we see that if (a_k) is a sequence of positive elements norm converging to some $a \in \mathcal{A}$, then $(r(a_k))$ weak* converges to $r(a)$. In particular, if $\mathcal{A}$ is finite dimensional, then this convergence is equivalent to norm convergence.

Corollary 2.7. *The subspace $\mathcal{P}_n$ of rank-n projections in a JBW-factor $\mathcal{A}$ is path connected.*

Proof. Let $n \neq 0$ and let $p, q \in \mathcal{P}_n$ with $p \neq q$. Then p and q are rank-n projections in the finite dimensional JBW-factor $\{(p \vee q),\ \mathcal{A},\ (p \vee q)\}$, each is an orthogonal sum of n minimal projections:

$$p = p_1 + \cdots + p_n, \quad q = q_1 + \cdots + q_n.$$

By Corollary 2.5, each p_k is joined to q_k by a continuous path $\{p_k(\theta)\}$ of minimal projections, with parametrization $\theta \in [0, 1]$. By the above remark, the path

$$p(\theta) = p_1(\theta) \vee \ldots \vee p_n(\theta) = r(p_1(\theta) + \cdots + p_n(\theta))$$

is a continuous path of rank-n projections with $p(0) = p$ and $p(1) = q$. □

3. Manifolds of projections. The aim of this section is to show that various manifolds of projections in JBW-algebras, possibly infinite dimensional, admit structures of a Riemannian symmetric space which are closely related to the underlying Jordan algebraic structures. Recall that a *Riemannian symmetric space* X is a connected Riemannian manifold in which every point is an isolated fixed-point of an involutive isometry of X.

We first consider the manifold $\mathcal{P}$ of projections in a JB-algebra $\mathcal{A}$. Given a projection p in $\mathcal{A}$ with Peirce decomposition

$$\mathcal{A} = \mathcal{A}_0(p) \oplus \mathcal{A}_1(p) \oplus \mathcal{A}_2(p)$$

and given $v \in \mathcal{A}_0(p)$, we define a linear map $p_v : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$p_v = 4[L(v), L(p)]$$

where $[\cdot, \cdot]$ denotes the usual Lie algebra product. The exponential $\exp p_v : \mathcal{A} \longrightarrow \mathcal{A}$ is a Jordan algebra automorphism, in particular, $(\exp p_v)(z)$ is a projection if, and only if, z is such.

Lemma 3.1. *Let q be a non-zero projection in $\mathcal{A}_2(p) \oplus \mathcal{A}_0(p)$. Then $\|q - p\| \geqq 1$ if $q \neq p$.*

Proof. Write $q - p = z_2 \oplus z_0 \in \mathcal{A}_2(p) \oplus \mathcal{A}_0(p)$. Then z_0 and z_2 cannot be both 0 and we have

$$\begin{aligned} p + z_2 + z_0 &= q = q^2 = (p + z_2 + z_0)^2 \\ &= p + z_2^2 + z_0^2 + 2p \circ z_2 + 2p \circ z_0 \\ &= p + z_2^2 + z_0^2 + 2z_2 \end{aligned}$$

which gives $z_0 = z_0^2 + (z_2^2 + z_2)$. Therefore $z_0 = z_0^2$ and $z_2^2 + z_2 = 0$. It follows that, if $z_0 \neq 0$, then

$$\|q - p\|^2 = \|(q - p)^2\| = \|z_2^2 + z_0^2\| \geqq \|z_0^2\| = \|z_0\| = 1.$$

If $z_0 = 0$, we also have $\|q - p\| \geqq 1$. □

We show below that the projections in a JB-algebra form a Banach manifold. The proof makes use of an argument in [14, p. 25].

Proposition 3.2. *Let $\mathcal{A}$ be a JB-algebra. The subspace $\mathcal{P}$ of projections in $\mathcal{A}$ is a submanifold of $\mathcal{A}$.*

Proof. Let $p \in \mathcal{P}$ and write

$$V = \mathcal{A}_1(p) \quad \text{and} \quad W = \mathcal{A}_2(p) \oplus \mathcal{A}_0(p).$$

We define a differentiable map $\varphi : V \times W \longrightarrow \mathcal{A}$ by

$$\varphi(v, w) = (\exp p_v)(w).$$

We have $\varphi(0, p) = p$ and at $(0, p)$, the derivative $d\varphi(0, p) : V \times W \longrightarrow \mathcal{A}$ is given by

$$d\varphi(0, p)(v, w) = v + w$$

(cf. [14, p. 25]) and is therefore an isomorphism. Hence, by the *inverse mapping theorem* [11, p. 13], φ is a diffeomorphism on an open set $O_1 \times O_2$ in $V \times W$, containing $(0, p)$. Let

$$N = \{w \in W : \|w - p\| < 1\}$$

and let $\Omega = \varphi(O_1 \times N)$. Then Ω is an open neighbourhood of p in $\mathcal{A}$ and we have

$$\Omega \cap \mathcal{P} = \varphi(O_1 \times \{p\}).$$

Indeed, given $(v, p) \in O_1 \times \{p\}$, we have $\varphi(v, p) = (\exp p_v)(p)$ which is a projection in Ω. Conversely, for $q \in \Omega \cap \mathcal{P}$ with $q = \varphi(v, w)$ and $(v, w) \in O_1 \times N$, we have $q = (\exp p_v)(w)$ which implies that w is a projection in W. Since $\|w - p\| < 1$, we must have $w = p$ by Lemma 3.1. Therefore we have proved that $\mathcal{P}$ is a submanifold of $\mathcal{A}$. □

We now consider projections in JBW-algebras. We first show that the space of finite rank projections and the space of infinite rank projections both admit Banach manifold structures.

Proposition 3.3. *Let $\mathcal{A}$ be a JBW-algebra. Then the subspace $\mathcal{P}_f$ of finite rank projections in $\mathcal{A}$ is an open subset of the manifold $\mathcal{P}$ of projections in $\mathcal{A}$. Also, the subspace $\mathcal{P}_\infty$ of infinite rank projections in $\mathcal{A}$ is open in $\mathcal{P}$.*

Proof. The openness of $\mathcal{P}_f$ follows from the fact that, for each $p \in \mathcal{P}_f$, the set

$$\{q \in \mathcal{P} : \|q - p\| < 1\}$$

is an open subset of $\mathcal{P}_f$ because $\|q - p\| < 1$ implies that q and p are Jordan equivalent, by [15, Proposition 7] and by considering the special JBW-algebra generated by p and q, if necessary.

Likewise $\mathcal{P}_\infty$ is open in $\mathcal{P}$. □

The Banach manifolds $\mathcal{P}_f$ and $\mathcal{P}_\infty$ need not be connected, and $\mathcal{P}_\infty$ need not have a Riemannian structure. However, these structures occur in JBW-factors.

Theorem 3.4. *Let $\mathcal{A}$ a JBW-algebra. Then the subspace $\mathcal{P}_n$ of projections of rank n in $\mathcal{A}$ is a submanifold of $\mathcal{P}$, for $n \in N \cup \{0\}$. Further, if $\mathcal{A}$ is a JBW-factor, then $\mathcal{P}_n$ is a Riemannian symmetric space and the tangent space $T_p\mathcal{P}_n$ of $\mathcal{P}_n$ at each $p \in \mathcal{P}_n$ identifies with the Peirce 1-space $\mathcal{A}_1(p)$.*

Proof. As in the proof of Proposition 3.3, $\mathcal{P}_n$ is an open subset of $\mathcal{P}$ and hence the first assertion follows.

Now let $\mathcal{A}$ be a JBW-factor. Ignore the trivial case of $n = 0$ and suppose $\mathcal{P}_n \neq \emptyset$ for some n. Then $\mathcal{A}$ must be of type *I*. Let $p \in \mathcal{P}_n$ and let

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{P}_n \subset \mathcal{A}$$

be a differentiable curve with $\alpha(0) = p$. The derivative $\alpha'(0) : \mathbb{R} \longrightarrow \mathcal{A}$ satisfies

$$\alpha'(0) = 2\alpha(0) \circ \alpha'(0)$$

since $\alpha(t)^2 = \alpha(t)$. In particular, $\alpha'(0)(1) \in \mathcal{A}_1(p)$. On the other hand, given $v \in \mathcal{A}_1(p)$, we can define a differentiable curve $\beta : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{P}_n$ by

$$\beta(t) = \exp(4t[L(v), L(p)])\,(p).$$

Then $\beta(0) = p$ and the derivative $\beta'(0) : \mathbb{R} \longrightarrow \mathcal{A}$ is given by

$$\beta'(0)(t) = 4t[L(v), L(p)](p)$$

and we have $\beta'(0)(1) = v$ since $4L(v)L(p)p - 4L(p)L(v)p = 4v \circ p^2 - 4p \circ (v \circ p) = 4v \circ p - 4p \circ \left(\frac{1}{2}v\right) = 2p \circ v = v$.

Hence the tangent space $T_p\mathcal{P}_n$ identifies with $\{\alpha'(0)(1) : p = \alpha(0) \text{ for some curve } \alpha\} = \mathcal{A}_1(p)$.

To see that $\mathcal{P}_n$ has a Riemannian structure, we let, by a minor abuse of notation,

$$\lambda : \mathcal{A}_1(p) \longrightarrow \mathbb{R}$$

be the restriction of the canonical trace $\lambda : \mathcal{A} \longrightarrow R \cup \{\infty\}$ defined in (2.1), where

$$\lambda(v) = 2\lambda(p \circ v) \leqq 2\lambda(p)\|v\| = 2n\|v\|$$

by Lemma 2.3. On the tangent space $\mathcal{A}_1(p)$, we can define an inner product

$$\langle \cdot, \cdot \rangle_p : \mathcal{A}_1(p) \longrightarrow \mathbb{R}$$

by

$$\langle u, v \rangle_p = \lambda(u \circ v).$$

The inner product norm $|v|_p = \lambda(v^2)^{1/2}$ is equivalent to the JBW-algebra norm on $\mathcal{A}_1(p)$. Indeed, we have, by Lemma 2.3 again,

$$\|v\|^2 = \|v^2\| \leqq |v|_p^2 = \lambda(v^2) = 2\lambda((p \circ v) \circ v) = 2\lambda(p \circ v^2) \leqq 2n\|v^2\|.$$

It is clear that the inner product $\langle \cdot, \cdot \rangle_p$ depends smoothly on $p \in \mathcal{P}_n$ and defines a Riemannian metric.

Finally we show that $\mathcal{P}_n$ is a symmetric space. By Corollary 2.7, $\mathcal{P}_n$ is connected.

Given $p \in \mathcal{P}_n$, the element $\mathbf{1} - 2p$ is a symmetry in $\mathcal{A}$ and the map $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\sigma(a) = \{\mathbf{1} - 2p, a, \mathbf{1} - 2p\}$$

is a Jordan automorphism of $\mathcal{A}$. Its restriction $\sigma_p : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ is an isometry with p as an isolated fixed point. This proves that $\mathcal{P}_n$ is a symmetric space. $\square$

Corollary 3.5. *In a JBW-factor, the connected components of the manifold $\mathcal{P}_f$ of finite rank projections are exactly the manifolds*

$$\{\mathcal{P}_n\}_{n=0}^{k} \qquad (k \in N \cup \{\infty\})$$

where $\mathcal{P}_0 = \{0\}$ and $k = \infty$ if, and only if, the factor is of type I_∞.

Proof. In a type *II* or *III* factor, we have $P_f = \{0\}$. For a type I factor, we only need to observe that two projections in a connected component, which is now path connected, must be of the same rank since they can be joined by a continuous path of projections $\{p(\theta)\}$ which can be subdivided into smaller paths such that $\|p(\theta) - p(\theta')\| < 1$ on each of them and it follows that these projections are all Jordan equivalent. □

We now consider the curvature of $\mathcal{P}_n$. Denote by $\mathfrak{X}\mathcal{P}_n$ the space of vector fields on $\mathcal{P}_n$. First, we define an affine connection $\nabla : \mathfrak{X}\mathcal{P}_n \times \mathfrak{X}\mathcal{P}_n \longrightarrow \mathfrak{X}\mathcal{P}_n$ by, as in [1], [12],

$$(\nabla_X Y)_p = P_1(p)(dY_p(X(p))) \quad (p \in \mathcal{P}_n)$$

where we regard the vector field Y as a differentiable mapping $Y : \mathcal{P}_n \longrightarrow \mathcal{A}$ and $dY_p : T_p\mathcal{P}_n \longrightarrow T_{Y(p)}\mathcal{A} = \mathcal{A}$ is the differential

$$dY_p(X(p)) = \frac{d}{dt}Y(\alpha(t))|_{t=0}$$

for a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{P}_n$ with $\alpha(0) = p$ and $\alpha'(0) = X(p)$. We always identify the tangent space $T_p\mathcal{P}_n$ with the Peirce 1-space $\mathcal{A}_1(p)$.

It can be verified that ∇ is torsionfree and is compatible with the Riemannian metric on $\mathcal{P}_n$ defined above. Hence it is the Levi-Civita connection on $\mathcal{P}_n$.

We compute the Ricci curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \qquad (X, Y, Z \in \mathfrak{X}\mathcal{P}_n).$$

Although *some* of the following computations are similar to [12], we include the crucial main steps for completeness and clarity. We first compute the differential

$$d(P_1)_p : \mathcal{A}_1(p) \longrightarrow B(\mathcal{A})$$

of the Peirce 1-projection

$$P_1 : \mathcal{P}_n \longrightarrow B(\mathcal{A})$$

where $B(\mathcal{A})$ is the space of bounded linear self-maps on $\mathcal{A}$. To simplify notation, we write $P'(p)$ for $d(P_1)_p$ and consider it as a bilinear map $P'(p) : \mathcal{A}_1(p) \times \mathcal{A} \longrightarrow \mathcal{A}$.

Lemma 3.6. *For $(x, a) \in \mathcal{A}_1(p) \times \mathcal{A}$, we have*

(i) $P'(p)(x, a) = 4x \circ a - 4p \circ (x \circ a) - 4x \circ (p \circ a)$,
(ii) $P'(p)(x, a) = P_1(p)P'(p)(x, a) + P'(p)(x, P_1(p)a)$.

Proof. (i) Recall that $P_1(p) = 4L(p) - 4L(p)^2$. Let $p = \alpha(0)$ and $x = \alpha'(0)$ for some differentiable curve α in $\mathcal{P}_n$. Then we have

$$\begin{aligned} P'(p)(x, a) &= \lim_{t\to 0} \frac{P_1(\alpha(t))a - P_1(\alpha(0))a}{t} \\ &= \lim_{t\to 0} \frac{4}{t}(\alpha(t) \circ a - \alpha(t) \circ (\alpha(t) \circ a) - \alpha(0) \circ a - \alpha(0) \circ (\alpha(0) \circ a)) \\ &= \lim_{t\to 0} \frac{4}{t}\{\alpha(t) \circ a - \alpha(0) \circ a - \alpha(t) \circ (\alpha(t) \circ a - \alpha(0) \circ a) \\ &\qquad -\alpha(t) \circ (\alpha(0) \circ a) - \alpha(0) \circ (\alpha(0) \circ a)\} \\ &= 4x \circ a - 4p \circ (x \circ a) - 4x \circ (p \circ a). \end{aligned}$$

For (ii), we differentiate $P_1(\alpha(t)) = P_1(\alpha(t))^2$ at $t = 0$ to obtain the formula. □

Returning to the curvature tensor, we have, for $p \in \mathcal{P}_n$,

$$\begin{aligned} \nabla_X(\nabla_Y Z)(p) = P_1(p)\left(d(\nabla_Y Z)_p(X(p))\right) &= P_1(p)\left(\frac{d}{dt}\nabla_Y Z(\alpha(t))|_{t=0}\right) \\ &= P_1(p)P'(p)(X(p), dZ_p(Y(p))) \\ &\quad + P_1(p)(d^2 Z_p((X(p), Y(p)) - dZ_p(dY_p(X(p))). \end{aligned}$$

It follows that

$$\begin{aligned} R(X,\ Y)\,Z\,(p) = &P_1(p)P'(p)(X(p), dZ_p(Y(p))) \\ &-P_1(p)P'(p)(Y(p), dZ_p(X(p))) \end{aligned}$$

where, by Lemma 3.6 (ii), we have

$$P_1(p)P'(p)(X(p), dZ_p(Y(p))) = P'(p)(X(p), (I - P_1(p))dZ_p(Y(p))).$$

Differentiating $P_1(\alpha(t))Z(\alpha(t)) = Z(\alpha(t))$ at $t = 0$, we obtain

$$P'(p)(X(p), (I - P_1(p))dZ_p(Y(p))) = P'(p)(Y(p), Z(p))$$

and hence

$$\begin{aligned} R(X,\ Y)\,Z\,(p) = &P'(p)(X(p),\ P'(p)(Y(p), Z(p))) \\ &-P'(p)(Y(p),\ P'(p)(X(p), Z(p))). \end{aligned}$$

We can now define the curvature operator $R_p(x, y) : T_p\mathcal{P}_n \longrightarrow T_p\mathcal{P}_n$ by

$$R_p(x, y)z = P'(p)(x, P'(p)(y, z)) - P'(p)(y, P'(p)(x, z))$$

for $z \in T_p\mathcal{P}_n = \mathcal{A}_1(p)$. The sectional curvature $K_p(x, y)$ of the subspace spanned by two independent vectors $x, y \in T_p\mathcal{P}_n$ is given by

$$K_p(x, y) = \frac{\langle R_p(x, y)x, y\rangle_p}{\langle x, x\rangle_p\langle y, y\rangle_p - \langle x, y\rangle_p^2}.$$

We conclude that the symmetric space $\mathcal{P}_n$ is of compact type although it is not compact if $\mathcal{A}$ is infinite dimensional.

Theorem 3.7. *The manifold $\mathcal{P}_n$ of rank-n projections in a JBW-factor $\mathcal{A}$ is a Riemannian symmetric space of compact type.*

Proof. We show that $\mathcal{P}_n$ has non-negative sectional curvature. Let $x, y \in T_p\mathcal{P}_n = \mathcal{A}_1(p)$ be two orthogonal vectors with $|x|_p = |y|_p = 1$. Given $x, y, z \in \mathcal{A}_1(p)$, we have $x \circ (y \circ z) \in \mathcal{A}_1(p)$. Using this fact and Lemma 3.6, one obtains

$$P'(p)(x, P'(p)(y, z)) = 4x \circ (y \circ z)$$

and therefore

$$\begin{aligned}\langle R_p(x, y)y, x\rangle_p &= \langle 4x \circ y^2 - 4y \circ (x \circ y), x\rangle_p \\ &= 4\lambda((x \circ y^2) \circ x) - 4\lambda((y \circ (x \circ y)) \circ x) \\ &= 4\lambda(x^2 \circ y^2) - 4\lambda((x \circ y)^2)\end{aligned}$$

where, by the Cauchy-Schwarz inequality and Lemma 2.3, we have

$$\begin{aligned}\lambda((x \circ y)^2) \;&=\; \frac{1}{2}\lambda(x \circ \{y, x, y\}) + \frac{1}{4}\lambda(\{x, y^2, x\}) + \frac{1}{4}\lambda(\{y, x^2, y\}) \\ &=\; \frac{1}{2}\lambda(x \circ \{y, x, y\}) + \frac{1}{2}\lambda(x^2 \circ y^2) \\ &\leqq\; \frac{1}{2}\lambda(x^2)\lambda(\{y, x, y\}^2) + \frac{1}{2}\lambda(x^2 \circ y^2) \\ &=\; \frac{1}{2}\lambda(\{y, \{x, y^2, x\}, y\}) + \frac{1}{2}\lambda(x^2 \circ y^2) \\ &=\; \frac{1}{2}\lambda(y^2 \circ \{x, y^2, x\}) + \frac{1}{2}\lambda(x^2 \circ y^2) \\ &\leqq\; \frac{1}{2}\|y^2\|\lambda(\{x, y^2, x\}) + \frac{1}{2}\lambda(x^2 \circ y^2) \\ &\leqq\; \frac{1}{2}|y|_p^2\lambda(\{x, y^2, x\}) + \frac{1}{2}\lambda(x^2 \circ y^2) = \lambda(x^2 \circ y^2).\end{aligned}$$

Hence $K_p(x, y) \geqq 0$. □

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