Cette matrice est conjuguée dans H à la matrice:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \otimes \mathrm{id} \otimes \mathrm{id}.$$

Comme le produit de ces deux matrices agit comme l'homothétie de rapport λ, on en déduit que $\pi_1(w) = -2g_{\mathrm{HT}}(\imath)$, ce qui prouve le corollaire aussi dans ce cas.

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Matrix-valued harmonic functions on groups

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Abstract. We study the basic structures of matrix-valued harmonic functions on locally compact groups. We show that the bounded matrix-valued harmonic functions on a group form a Jordan triple system and we determine its structure. We also show that Liouville property implies amenability of the group. We characterize the unbounded matrix-valued harmonic functions on abelian groups.

1. Introduction

In this paper, we embark on a systematic study of matrix-valued harmonic functions on groups. Our first task is to understand the basic structures of matrix-valued harmonic functions. We extend some well-known results for real (complex) harmonic functions on groups to the matrix-valued case, but in contrast, we show that the bounded matrix-valued harmonic functions form a ternary Jordan algebra, that is, a Jordan triple system. The latter introduces a new aspect of non-associative structure into the theory of harmonic functions and provides an interesting link between non-associative functional analysis and harmonic analysis.

We begin with some background. Let G be a Lie group and let Δ be the Laplace operator on G. A function $f \in C^\infty(G)$ is *harmonic* if $\Delta f = 0$. It is well-known (cf. [22], [26]) in this case that there exists a family $\{\sigma_t\}_{t>0}$ of absolutely continuous probability measures on G such that f satisfies the following convolution equations

$$f = f * \sigma_t \quad (t > 0)$$

which motivates the following definition. Given a probability measure σ on a locally compact group G, a Borel function $f\colon G \to \mathbb{R}$ is called *σ-harmonic* if $f = f * \sigma$.

Harmonic functions on groups play an important role in many areas of mathematics. Recently, matrix-valued harmonic functions on groups have been studied in [8], [10], [33], with some applications to problems concerning the L^p-dimension of vector-valued self-similar measures. As in the scalar case, the matrix-valued harmonic functions on groups arise naturally in the following way.

Let G be a Lie group and let M_n be the C*-algebra of $n \times n$ complex matrices. Let λ be the left-invariant Haar measure on G. Let $L^2(G, M_n)$ be the usual Banach space of (equivalence classes of) M_n-valued L^2-(Bochner) integrable functions on G (w.r.t. λ) (cf. [15], p. 97). Naturally $L^2(G, M_n)$ is a left Hilbert M_n-module with the M_n-valued inner product

$$\langle f, g \rangle = \int_G f(x) g(x)^* \, d\lambda(x).$$

Let $C_c^\infty(G, M_n)$ be the space of M_n-valued infinitely differentiable functions on G with compact support, where a function $f = (f_{ij})\colon G \to M_n$ is in $C_c^\infty(G, M_n)$ if and only if each f_{ij} is in $C_c^\infty(G)$. Then $C_c^\infty(G, M_n)$ is dense in $L^2(G, M_n)$.

Let $w \in C^\infty(G, M_n)$ be pointwise positive and invertible, and let

$$L_w^2(G, M_n) = \{hw^{-1} \colon h \in L^2(G, M_n)\}$$

which is a left Hilbert M_n-module with inner product $\langle\langle f, g \rangle\rangle = \langle fw, gw \rangle$.

Let $\mathscr{L}\colon C_c^\infty(G, M_n) \to L_w^2(G, M_n)$ be an (unbounded) operator which generates a one-parameter semigroup of bounded operators

$$H_t\colon L_w^2(G, M_n) \to L_w^2(G, M_n) \quad (t \geqq 0).$$

Let $\mathscr{L}$ be left-invariant, that is, $\mathscr{L}$ commutes with the left translation

$$L_u\colon L_w^2(G, M_n) \to L_w^2(G, M_n)$$

defined by

$$(L_u f)(y) = f(u^{-1}y) \quad (u, y \in G).$$

Suppose that $\mathscr{L}$ is a left M_n-module map. Then so is H_t. If, for any $t > 0$ and $x \in G$, the left M_n-module map

$$f \in L_w^2(G, M_n) \mapsto H_t f(x) \in M_n$$

is bounded and $H_t f$ is continuous, then there exists $\Psi_{t,x} \in L_w^2(G, M_n)$ (cf. [20], Proposition 4.4) such that

$$H_t f(x) = \int_G f(y) w^2(y) \Psi_{t,x}(y) \, d\lambda(y).$$

Since $L_u \mathscr{L} = \mathscr{L} L_u$, we have $(L_u H_t) f(x) = H_t f(u^{-1}x)$ which gives, for λ-almost all $y \in G$,

$$w^2(y) \Psi_{t,u^{-1}x}(y) = w^2(uy) \Psi_{t,x}(uy) \quad (u \in G).$$

Define $h_t = w^2 \Psi_{t,e}$. Then $h_t(x^{-1}y) = w^2(x^{-1}y) \Psi_{t,e}(x^{-1}y) = w^2(y) \Psi_{t,x}(y)$.

We call a function $f \in C_c^\infty(G, M_n)$ *$\mathscr{L}$-harmonic* if $\mathscr{L} f = 0$. Given such a function f, we have

$$f(x) = H_t f(x) = \int_G f(y) h_t(x^{-1}y) \, d\lambda(y) = f * (\tilde{h}_t . \lambda)(x) \quad (t > 0)$$

where $\tilde{h}_t(z) = h_t(z^{-1})$ and $h_t . \lambda$ is an M_n-valued measure on G. Thus f satisfies a family of matrix-valued convolution equations.

Let G be a locally compact group and let σ be an M_n-valued measure on G. A function $f\colon G \to M_n$ is called *σ-harmonic* if it satisfies the following convolution equation

$$f = f * \sigma$$

where the convolution can be computed in the following way. Given $f = (f_{ij})$ and $\sigma = (\sigma_{ij})$ where $f_{ij}\colon G \to \mathbb{C}$ and σ_{ij} is a complex-valued measure on G, the function $f * \sigma\colon G \to M_n$ has the ij-th entry

$$(f * \sigma)_{ij} = \sum_k f_{ik} * \sigma_{kj}.$$

Examples of $\mathscr{L}$ can be constructed from different *'weights'* w. For simple illustration, if we take $w = 1$, then $\mathscr{L}\colon C_c^\infty(G, M_n) \to L^2(G, M_n)$ and we can have $\mathscr{L} = \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix}$. So $H_t = \begin{pmatrix} e^{t\Delta} & & \\ & \ddots & \\ & & e^{t\Delta} \end{pmatrix}$ (cf. [13], p. 149) and (f_{ij}) is $\mathscr{L}$-harmonic if and only if each f_{ij} is Δ-harmonic. We note that however, in general, if $f = (f_{ij})$ is σ-harmonic for some measure $\sigma = (\sigma_{ij})$, each f_{ij} need not be σ_{ij}-harmonic.

We now give a brief review of the paper. We first develop in Section 2, for completeness, some basic tools for matrix-valued measures and integration, such as polar representation of a measure, Riesz Representation Theorem, Fubini Theorem and convolution, which will be used for later computation. We give a brief introduction in Section 3 to Jordan structures in Banach spaces and prove some structure results for the ranges of contractive projections on type I finite Jordan triple systems. These results are motivated by a later application. In Section 4, we study bounded matrix-valued harmonic functions. Given an M_n-valued measure σ on a locally compact group G with $\|\sigma\| = 1$, we construct a contractive projection P on the space $L^\infty(G, M_n)$ of (essentially) bounded M_n-valued functions on G such that the range of P is the space $H_\sigma(G, M_n)$ of bounded M_n-valued σ-harmonic functions on G. This extends a result in [9]. Using the non-associative analysis in Section 3, we determine the structure of $H_\sigma(G, M_n)$ completely as follows: $H_\sigma(G, M_n)$ is linearly isometric to a finite ℓ^∞-sum $\bigoplus_k L^\infty(\Omega_k) \otimes C_k$ where C_k is a finite-dimensional Cartan factor of the following type:

(i) M_{pq}, the space of complex $p \times q$-matrices;

(ii) S_p, the space of complex $p \times p$ symmetric matrices;

(iii) A_p, the space of complex $p \times p$ skew symmetric matrices;

(iv) V_p, the spin factor of dimension at least 3, consisting of complex $p \times p$ matrices such that $a \in V_p$ implies $a^* \in V_p$ and a^2 is a scalar multiple of the identity matrix.

Further, $H_\sigma(G, M_n)$ is a Jordan algebra if there is a unitary σ-harmonic function on G. The above result generalizes the familiar one in the scalar case [1], [9], namely, $H_\sigma(G, \mathbb{C})$ is isometric to $L^\infty(\Omega)$ which gives a Poisson representation of $H_\sigma(G, \mathbb{C})$, with the spectrum of $L^\infty(\Omega)$ as the Poisson boundary.

We next study the Liouville property. Using a matrix-valued Fourier transform and the Peter-Weyl Theorem, we show that, as in the scalar case, every continuous M_n-valued σ-harmonic function on a compact group is constant if σ is positive, adapted and $\|\sigma\| = 1$. A similar result for abelian groups has been proved in [8] and one expects that it should generalize to some other groups including nilpotent groups. Using the contractive projection P, we also show that a group G is necessarily amenable if there is a positive, norm-one M_n-valued measure σ on G such that all bounded M_n-valued σ-harmonic functions on G are constant.

Section 5 concerns unbounded harmonic functions. Our objective is to extend Schwartz's result [41] for mean periodic functions on $\mathbb{R}$ to the matrix-valued case. For this, we introduce a useful device, namely, the *determinant* of a matrix-valued measure which enables us to reduce some arguments to the scalar case. Given an M_n-valued measure σ on an abelian group G with compact support, we make use of [23], [19] to extend Schwartz's result by showing that the continuous M_n-valued σ-harmonic functions on G are synthesized from the M_n-valued *exponetial polynomials*. Finally in Section 6, we extend Choquet and Deny's method in [4], [14] to show that the (unbounded) positive matrix-valued σ-harmonic functions on abelian groups, with range commuting with that of σ, are integrals of matrix-valued exponential functions. Naturally several directions can be followed, an obvious next step is to examine other classes of groups and to extend, for instance, the results in [7], [9], [11] to the matrix-valued case. This will be considered elsewhere. One can also consider harmonic functions taking values in subspaces of M_n, for example, in matrix groups.

2. Matrix-valued measures and integration

For future reference and to clarify terminology as well as avoiding possible measure theoretic pitfalls, we first develop a self-contained theory of matrix-valued measures and integration which may also be of some independent interest. Let G be a locally compact space and $\mathscr{B}$ be the σ-algebra of Borel subsets of G. Let M_n be the C*-algebra of complex $n \times n$ matrices and let M_n^+ be the positive cone of M_n, consisting of all self-adjoint matrices with non-negative eigenvalues. The trace Tr: $M_n \to \mathbb{C}$ is a positive linear functional of norm n. Every continuous linear functional φ: $M_n \to \mathbb{C}$ corresponds to a unique matrix $A_\varphi \in M_n$ such that $\varphi(B) = Tr(A_\varphi B)$ and $\|\varphi\| = Tr(|A_\varphi|)$ where $|A_\varphi| = \sqrt{A_\varphi A_\varphi^*}$. We will identify the dual M_n^*, via the map $\varphi \in M_n^* \mapsto A_\varphi \in M_n$, with the complex vector space M_n equipped with the *trace norm* $\|A\|_1 = Tr(|A|) \geqq \|A\|$. Given $A = (a_{ij}) \in M_n$, we have

$$\|A\| \leqq \Big(\sum_{ij} |a_{ij}|^2\Big)^{\frac{1}{2}} \leqq \sqrt{n}\|A\|.$$

Hence norm convergence in M_n is equivalent to entry-wise convergence. We also note that $Tr(|A|^2) = Tr(AA^*) = \sum_{ij} |a_{ij}|^2$ and $\|A\|_1 \leqq \sqrt{n}\, Tr(|A|^2)^{\frac{1}{2}} \leqq n\|A\|$.

By an *M_n-valued measure* μ on G, we mean a (norm) countably additive function μ: $\mathscr{B} \to M_n$. Since the trace norm $\|\cdot\|_1$ is equivalent to the C*-algebra norm on M_n and $M_n^* = (M_n, \|\cdot\|_1)$, we can also regard an M_n-valued measure on G as an M_n^*-valued measure, and *vice versa*. A measure μ is said to be *positive* if it is M_n^+-valued. A complex measure ν: $\mathscr{B} \to \mathbb{C}$ is also regarded as the M_n-valued measure $\nu(\cdot)I_n$ where I_n (or simply I if n is understood) always denotes the identity matrix in M_n. Likewise, a complex function f: $G \to \mathbb{C}$ is regarded as the M_n-valued function $f(\cdot)I_n$.

If we use the matrix notation $\mu = (\mu_{ij})$, then each μ_{ij} is a complex-valued measure on G. We note that complex-valued measures are not only bounded, but also of bounded variation [38], Theorem 6.4. The *variation* $|\nu|$ of a Banach space-valued measure ν on G is defined, as in [15], p. 2, by

$$|\nu|(E) = \sup_{\mathscr{P}} \Big\{ \sum_{E_i \in \mathscr{P}} \|\nu(E_i)\| \Big\}$$

where the supremum is taken over all partitions $\mathscr{P}$ of E into a finite number of pairwise disjoint Borel sets. One can verify directly that $|\nu|$ is a monotone non-negative extended real-valued, finitely additive set function. We say that ν is of *bounded variation* if $|\nu|(G) < \infty$ in which case we define the *norm* of ν to be $\|\nu\| = |\nu|(G)$.

Remark 1. The positive matrix $|\mu(E)| = \sqrt{\mu(E)\mu(E)^*}$ should not be confused with the positive number $|\mu|(E)$.

Lemma 2. *Let μ be an M_n-valued measure on G. Then μ is of bounded variation and $|\mu|$ is countably additive.*

Proof. Let $\mu = (\mu_{ij})$. If $\{E_k\}$ is a finite partition of G, then

$$\|\mu(E_k)\| \leqq \Big(\sum_{ij} |\mu_{ij}(E_k)|^2\Big)^{\frac{1}{2}} \leqq \sum_{ij} |\mu_{ij}(E_k)|$$

which gives $\sum_k \|\mu(E_k)\| \leqq \sum_k \sum_{ij} |\mu_{ij}(E_k)| \leqq \sum_{ij} |\mu_{ij}|(G)$. Hence $|\mu|(G) < \infty$. The countable additivity of $|\mu|$ follows from [15], p. 3. □

Throughout the paper, all M_n-valued measures μ are assumed to be *regular* which means that $\rho \circ \mu$ is a regular complex Borel measure for every $\rho \in M_n^*$ (cf. [38], p. 131). It follows that the variation $|\mu|$ is also regular. We will write $\rho\mu$ for $\rho \circ \mu$.

For an M_n-valued measure μ, its total variation norm is given by $\|\mu\| = |\mu|(G)$. If we regard μ as M_n^*-valued, then its variation $|\mu|_1$ with respect to the trace norm $\|\cdot\|_1$ is also bounded and we denote by $\|\mu\|_1 = |\mu|_1(G)$ the corresponding total variation norm of μ.

Let $M(G, M_n^*)$ be the space of all M_n^*-valued measures on G, equipped with the total variation norm $\|\cdot\|_1$. It is clearly isomorphic to the space $M(G, M_n)$ of M_n-valued measures on G, equipped with the total variation norm $\|\cdot\|$. Let $C_0(G, M_n)$ be the space of continuous M_n-valued functions vanishing at infinity, equipped with the supremum norm. We will show that $M(G, M_n^*)$ can be identified as the dual of $C_0(G, M_n)$. We need to define matrix-valued integration first.

Given $\mu \in M(G, M_n)$, using the natural bilinear map

$$(A, B) \in M_n \times M_n \mapsto AB \in M_n,$$

as in [2], [10], one can define the *μ-integrable* functions $f: G \to M_n$ and the *bilinear vector integrals* $\int_E f\, d\mu$ for $E \in \mathscr{B}$. For our purpose, we extend the notion of a complex-valued μ-integrable *Borel* function, which is stronger than the definition of a μ-integrable complex function, to the matrix-valued case. Given $\mu = (\mu_{ij})$, a function $f = (f_{ij}): G \to M_n$ is said to be *μ-integrable* if each f_{ij} is a Borel function and the integrals $\int_G f_{ij}\, d\mu_{k\ell}$ exist for all i, j, k, ℓ in which case, we define

$$\int_E f\, d\mu = \left(\sum_{k=1}^{n} \int_E f_{ik}\, d\mu_{kj} \right) \in M_n \quad (E \in \mathscr{B}).$$

By a *simple function* $f: G \to M_n$, we mean $f = \sum_k A_k \chi_{E_k}$ where $A_k \in M_n$, χ_{E_k} is the characteristic function of $E_k \in \mathscr{B}$ and $\{E_k\}$ is a partition of G. For such a function f, we have

$$\int_E f\, d\mu = \sum_k A_k \mu(E \cap E_k).$$

Given any μ-integrable function $f: G \to M_n$, entry-wise consideration yields a sequence (f_m) of simple functions on G such that $\lim_{m\to\infty} \|f_m(x) - f(x)\| = 0$ for each $x \in G$ and $\int_E f\, d\mu = \lim_{m\to\infty} \int_E f_m\, d\mu$ for every $E \in \mathscr{B}$. From this and using [10], p. 161 and Lemma 2.3, if $f: G \to M_n^+$ is μ-integrable, then the sequence (f_m) can be chosen to be M_n^+-valued and $f_m \leqq f_{m+1} \leqq f$. An M_n^+-valued function will be called *positive*.

Since $\|\mu(E)\| \leqq |\mu|(E)$ for every $E \in \mathscr{B}$, μ is *absolutely continuous* with respect to $|\mu|$ in the sense of [15], p. 10. By the Radon-Nikodym property of M_n (cf. [15], p. 82), there is a Bochner $|\mu|$-integrable function $\omega: G \to M_n$ such that

$$\mu(E) = \int_E \omega\, d|\mu| \quad (E \in \mathscr{B}).$$

We denote this by $\mu = \omega \cdot |\mu|$ and call it the *polar representation* (or *decomposition*) of μ. We refer to [15], p. 44 for the definition of a Bochner integral. The set $F = \{x \in G : \|\omega(x)\| > 1\}$ is $|\mu|$-measurable and by [15], p. 46, we have

$$|\mu|(E) = \int_E \|\omega(x)\|\, d|\mu|(x) \quad (E \in \mathscr{B})$$

which implies that $|\mu|(F) = 0$. On the other hand, $\int_E (1 - \|\omega(x)\|)\, d|\mu|(x) = 0$ for every $E \in \mathscr{B}$ implies that $\|\omega(x)\| = 1$ $|\mu|$-almost everywhere. By redefining ω, we can therefore assume that $\|\omega(x)\| = 1$ for every $x \in G$. Likewise, there is a Bochner $|\mu|_1$-integrable function $\omega': G \to M_n$ such that $\|\omega'(x)\|_1 = 1$ for every $x \in G$ and

$$\mu(E) = \int_E \omega'\, d|\mu|_1 \quad (E \in \mathscr{B}).$$

We note that if μ is M_n^+-valued, then $\omega(x) \in M_n^+$ for $|\mu|$-almost all $x \in G$. This follows from the fact that there is a sequence $\{\rho_k\}$ of positive linear functionals on M_n such that $A \in M_n^+$ if, and only if, $\rho_k(A) \geqq 0$ for all k, and that $0 \leqq \rho_k \mu(E) = \int_E \rho_k \omega\, d|\mu|$ for all $E \in \mathscr{B}$ implies $\rho_k \omega(x) \geqq 0$ for $|\mu|$-almost all $x \in G$.

Let $f: G \to M_n$ be μ-integrable. Then we have

$$\int_E f\, d\mu = \int_E f\omega\, d|\mu| \quad (E \in \mathscr{B}).$$

Indeed we have $\left(\int_E f\, d\mu\right)_{ij} = \sum_k \int_E f_{ik}\, d\mu_{kj} = \sum_k \int_E f_{ik}\omega_{kj}\, d|\mu| = \left(\int_E f\omega\, d|\mu|\right)_{ij}$ since

$$\mu_{kj} = \omega_{kj} \cdot |\mu|$$

as complex-valued measures. We note that f is μ-integrable if, and only if, it is Bochner $|\mu|$-integrable. This fact and the above formula enable us to use the theory of Bochner integrals.

We have the following matrix-valued version of Fatou's Lemma.

Lemma 3. *Let μ be a positive M_n-valued measure on G and let (f_k) be a sequence of M_n^+-valued μ-integrable functions on G, converging pointwise to a μ-integrable function $f: G \to M_n^+$. Then*

$$Tr\left(\int_G f\, d\mu\right) \leqq \liminf_{k\to\infty} Tr\left(\int_G f_k\, d\mu\right).$$

Proof. Let $\mu = \omega \cdot |\mu|$ be the polar decomposition. Then $\omega(x) \geqq 0$ for $|\mu|$-almost all $x \in G$. The sequence $\{Tr(f_k(x)\omega(x))\}_{k=1}^{\infty}$ consists of $|\mu|$-almost everywhere non-negative functions on G, converging pointwise to $Tr(f(x)\omega(x))$. By Fatou's Lemma, we have

$$Tr\left(\int_G f\,d\mu\right) = Tr\left(\int_G f\omega\,d|\mu|\right)$$

$$= \int_G Tr(f\omega)\,d|\mu| \leqq \liminf_{k\to\infty} \int_G Tr(f_k\omega)\,d|\mu|$$

$$= \liminf_{k\to\infty} Tr\left(\int_G f_k\,d\mu\right). \quad \Box$$

Lemma 4. *Let μ be an M_n-valued measure and let $f: G \to M_n$ be μ-integrable and bounded. Then for $E \in \mathscr{B}$, we have*

$$\left\|\int_E f\,d\mu\right\| \leqq \|f\|\,|\mu|(E) \quad and \quad \left\|\int_E f\,d\mu\right\|_1 \leqq \|f\|_1|\mu|_1(E).$$

Proof. By [15], p. 46, we have

$$\left\|\int_E f\omega\,d|\mu|\right\| \leqq \int_E \|f\omega(x)\|\,d|\mu|(x) \leqq \|f\omega\|_E|\mu|(E). \quad \Box$$

We are now ready to show that $C_0(G, M_n)^*$ identifies with $M(G, M_n^*)$ which is a matrix-valued version of the Riesz Representation Theorem.

Lemma 5. *The map $\mu \in M(G, M_n^*) \mapsto \mu(\cdot) \in C_0(G, M_n)^*$ defined by*

$$\mu(f) = Tr\left(\int_G f\,d\mu\right) \quad (f \in C_0(G, M_n))$$

is a linear isometric order-isomorphism.

Proof. The map is clearly linear. We first show that it is an isometry. To see that $\|\mu(\cdot)\| \leqq |\mu|_1(G)$, let $f \in C_0(G, M_n)$. Then

$$\left|Tr\left(\int_G f\,d\mu\right)\right| = \left|Tr\left(\int_G f\omega'\,d|\mu|_1\right)\right| = \left|\int_G Tr(f(x)\omega'(x))\,d|\mu|_1(x)\right|$$

$$\leqq \int_G \|f(x)\|\,\|\omega'(x)\|_1\,d|\mu|_1(x) \leqq \|f\|\,|\mu|_1(G)$$

which gives $\|\mu(\cdot)\| \leqq |\mu|_1(G)$. To reverse the inequality, let $\varepsilon > 0$. Let $\{E_i\}_{i=1}^k$ be a partition of G and choose compact sets $K_i \subset E_i$ such that $|\mu|(E_i\backslash K_i) < \frac{\varepsilon}{kn}$. Choose disjoint open sets $V_i \supset K_i$, with compact closure, such that $|\mu|(V_i\backslash K_i) < \frac{\varepsilon}{kn}$. There are continuous functions $f_i: G \to [0,1]$ such that $f_i(K_i) = \{1\}$ and $f_i(G\backslash V_i) = \{0\}$. Let $|\mu(E_i)| = u_i\mu(E_i)$ be the usual polar decomposition in M_n, where u_i is a partial isometry in M_n. Define a function $f \in C_0(G, M_n)$ by $f = \sum_i u_i f_i$. Then $\|f\| \leqq 1$ and

$$|\mu(E_i)| = u_i\mu(E_i) = u_i\mu(K_i) + u_i\mu(E_i\backslash K_i)$$

$$= u_i\int_{V_i} f_i\,d\mu - u_i\int_{V_i\backslash K_i} f_i\,d\mu + u_i\mu(E_i\backslash K_i).$$

We have

$$\left|\sum_i \|\mu(E_i)\|_1 - Tr\left(\int_G f\,d\mu\right)\right| \leqq \sum_i \left|\,\|\mu(E_i)\|_1 - Tr\left(\int_G u_i f_i\,d\mu\right)\right|$$

$$= \sum_i \left|Tr\left(|\mu(E_i)| - \int_G u_i f_i\,d\mu\right)\right|$$

$$\leqq \sum_i n\left\|\,|\mu(E_i)| - \int_G u_i f_i\,d\mu\right\|$$

$$= n\sum_i \left\|-u_i\int_{V_i\backslash K_i} f_i\,d\mu + u_i\mu(E_i\backslash K_i)\right\|$$

$$\leqq n\sum_i (\|u_i\|\,|\mu|(V_i\backslash K_i) + \|u_i\|\,|\mu|(E_i\backslash K_i)) < 2\varepsilon$$

which gives $\|\mu(\cdot)\| \geqq |\mu|_1(G)$.

To show surjectivity, let $\varphi \in C_0(G, M_n)^*$. We note that $C_0(G, M_n)$ identifies with the injective tensor product $C_0(G) \check{\otimes} M_n$. Let $\{e_{ij} : i, j = 1, \ldots, n\}$ be the canonical basis in M_n. Then each $f \in C_0(G, M_n)$ can be expressed as $f = (f_{ij}) = \sum_{ij} f_{ij} \otimes e_{ij}$ with $f_{ij} \in C_0(G)$. By [24], Proposition 32, there is a Borel measure $m \in C_0(G)^*$ and a Bochner m-integrable function $g: G \to M_n^*$ such that

$$\varphi(f) = \varphi\left(\sum_{ij} f_{ij} \otimes e_{ij}\right)$$

$$= \sum_{ij}\int_G f_{ij}(x)\,Tr(e_{ij}g(x))\,dm(x)$$

$$= \sum_{ij}\int_G f_{ij}(x)g(x)_{ji}\,dm(x).$$

Define an M_n^*-valued measure μ on G by

$$\mu(E) = \int_E g\,dm \quad (E \in \mathscr{B}).$$

Then $Tr\left(\int_G f\,d\mu\right) = \sum_i \left(\int_G fg\,dm\right)_{ii} = \sum_i\sum_j \int_G f_{ij}g_{ji}\,dm = \varphi(f)$ which shows $\varphi = \mu(\cdot)$.

Finally, we show that μ is positive if, and only if, $\mu(\cdot)$ is positive. Let μ be positive. Then for a positive simple function $f = \sum_i A_i \chi_{E_i}$ with $A_i \geqq 0$, we have

$$\mu(f) = Tr\left(\int_G f\,d\mu\right) = \sum_i Tr(A_i\mu(E_i)) = \sum_i Tr(\mu(E_i)^{\frac{1}{2}}A_i\mu(E_i)^{\frac{1}{2}}) \geqq 0.$$

Suppose, conversely, $\mu(\cdot)$ is positive. Let $E \in \mathscr{B}$. Given any $A \in M_n^+$, the function $f = A\chi_E$ is positive and hence $Tr(A\mu(E)) = Tr\left(\int_G f\,d\mu\right) \geqq 0$ using standard approximation of $A\chi_E$ by continuous functions. As A was arbitrary, we have $\mu(E) \geqq 0$. □

For the space $M(G, M_n)$ of M_n-valued measures, we have the following identification.

Lemma 6. *The map $\mu \in M(G, M_n) \mapsto \mu(\cdot) \in C_0(G, M_n^*)^*$ defined by*

$$\mu(f) = Tr\left(\int_G f\,d\mu\right) \quad (f \in C_0(G, M_n^*))$$

is a linear isometric order-isomorphism.

Proof. The arguments are similar to those in the proof of Lemma 5, the only difference is that, in proving $\|\mu(\cdot)\| \geqq |\mu|(G)$, we choose, for a partition $\{E_i\}_{i=1}^k$ of G, $\varphi_i \in M_n^*$ with $\varphi_i(\cdot) = Tr(u_i\cdot)$ and $\|u_i\|_1 \leqq 1$ such that $\|\mu(E_i)\| \leqq |\varphi_i(\mu(E_i))| + \frac{\varepsilon}{k}$, and define, instead, $f \in C_0(G, M_n^*)$ by $f = \sum_i u_i f_i$ which gives

$$\left|\sum_i |\varphi(\mu(E_i))| - \left|Tr\left(\int_G f\,d\mu\right)\right|\right| \leqq \sum_i \left|Tr(u_i\mu(E_i)) - Tr\left(\int_G u_i f_i\,d\mu\right)\right|$$
$$\leqq n\sum_i (\|u_i\|\,|\mu|(V_i\backslash K_i) + \|u_i\|\,|\mu|(E_i\backslash K_i)) < 2\varepsilon$$

and therefore $\sum_i \|\mu(E_i)\| \leqq \|\mu(\cdot)\| + 3\varepsilon$. □

Given $\mu, \sigma \in M(G, M_n)$, to define their convolution if G is a group, we first define the *product measure* $\mu \times \sigma$ on $G \times G$. Let $\mathscr{B}^2$ be the σ-algebra in $G \times G$ generated by the Borel rectangles $E \times F$ with $E, F \in \mathscr{B}$. Given M_n-valued measures $\mu = (\mu_{ij})$ and $\sigma = (\sigma_{ij})$ on G, we define the product measure $\mu \times \sigma: \mathscr{B}^2 \to M_n$ by

$$(\mu \times \sigma)_{ij} = \sum_k \mu_{ik} \times \sigma_{kj}$$

where $\mu_{ik} \times \sigma_{kj}$ is the product measure of the complex-valued measures μ_{ik} and σ_{kj}. For $Q \in \mathscr{B}^2$ and $y \in G$, we define the *y-section* $Q^y = \{x \in G : (x, y) \in Q\}$. Then the function $y \in G \mapsto \mu(Q^y) \in M_n$ is σ-integrable and $(\mu \times \sigma)(Q) = \int_G \mu(Q^y)\,d\sigma(y)$ since

$$(\mu \times \sigma)(Q)_{ij} = (\mu \times \sigma)_{ij}(Q) = \sum_k (\mu_{ik} \times \sigma_{kj})(Q)$$
$$= \sum_k \int_G \mu_{ik}(Q^y)\,d\sigma_{kj}(y)$$
$$= \left(\int_G \mu(Q^y)\,d\sigma(y)\right)_{ij}.$$

For $E, F \in \mathscr{B}$, we have $(\mu \times \sigma)(E \times F) = \mu(E)\sigma(F)$. We note that $\mu \times \sigma \neq \sigma \times \mu$ in general.

Let $\mu = \omega^\mu \cdot |\mu|$ and $\sigma = \omega^\sigma \cdot |\sigma|$ be the polar representations. Then entrywise calculation gives $d(\mu \times \sigma)(x, y) = \omega^\mu(y)\omega^\sigma(x)\,d(|\mu| \times |\sigma|)(x, y)$. It follows that $|\mu \times \sigma| \leqq |\mu| \times |\sigma|$. In contrast to the scalar case, we need not have equality. This is due to the fact that the product of two nonzero matrix-valued measures could be zero. For instance, if μ and σ are nonzero M_n-valued measures with orthogonal ranges, then $\mu \times \sigma = 0$ but $\mu \times |\sigma| \neq 0$. Hence $\mu \times \sigma$-integrability need not imply $\mu \times |\sigma|$-integrability while the latter implies the former by entry-wise inspection. We have the following version of the Fubini Theorem for matrix-valued integrals.

Proposition 7. *Let $f: G \times G \to M_n$ be $\mu \times |\sigma|$-integrable. Then*

(i) *the function $x \in G \mapsto f(x, y)$ is μ-integrable for $|\sigma|$-almost every $y \in G$;*

(ii) *the function $y \in G \mapsto \int_G f(x, y)\,d\mu(x)$, defined $|\sigma|$-almost everywhere, is σ-integrable;*

(iii) $\int_{G\times G} f\,d(\mu \times \sigma) = \int_G \left(\int_G f(x, y)\,d\mu(x)\right) d\sigma(y).$

Proof. Since f is $\mu_{ij} \times |\sigma|$-integrable, where $\mu_{ij} \times |\sigma| = \omega_{ij}^\mu \cdot (|\mu| \times |\sigma|)$, the function $f(x, y)\omega_{ij}^\mu(x)$ is Bochner $|\mu| \times |\sigma|$-integrable and by [17], p. 190,

$$\int_G f(x, y)\,d\mu_{ij}(x) = \int_G f(x, y)\omega_{ij}^\mu(x)\,d|\mu|(x)$$

exists for $|\sigma|$-almost all $y \in G$. So $\int_G f(x, y)\,d\mu(x)$ exists for $|\sigma|$-almost all $y \in G$ which proves (i). By [17], p. 193, the integral

$$\int_G\left(\int_G f(x, y)\,d\mu_{ij}(x)\right) d|\sigma|(y)$$

exists and equals $\int_{G\times G} f\,d(\mu_{ij} \times |\sigma|)$. By boundedness of $\omega_{k\ell}^\sigma$ for all k, ℓ, the integral

$$\int_G\left(\int_G f(x, y)\,d\mu_{ij}(x)\right) d\sigma_{k\ell}(y) = \int_G\left(\int_G f(x, y)\,d\mu_{ij}(x)\right)\omega_{k\ell}^\sigma\,d|\sigma|(y)$$

and the integral $\int_{G\times G} f\, d(\mu_{ij} \times \sigma_{k\ell})$ exist and are equal. Therefore the integral $\int_G \left(\int_G f(x,y)\, d\mu(x) \right) d\sigma(y)$ exists and

$$\left(\int_G \left(\int_G f(x,y)\, d\mu(x) \right) d\sigma(y) \right)_{ij} = \int_G \sum_k \left(\int_G f(x,y)\, d\mu(x) \right)_{ik} d\sigma_{kj}(y)$$

$$= \sum_k \int_G \left(\int_G \sum_\ell f_{i\ell}(x,y)\, d\mu_{\ell k}(x) \right) d\sigma_{kj}(y)$$

$$= \sum_k \sum_\ell \int_G \left(\int_G f_{i\ell}(x,y)\, d\mu_{\ell k}(x) \right) d\sigma_{kj}(y)$$

$$= \sum_{k,\ell} \int_{G\times G} f_{i\ell}\, d(\mu_{\ell k} \times \sigma_{kj})$$

$$= \sum_\ell \int_{G\times G} f_{i\ell}\, d(\mu \times \sigma)_{\ell j}$$

$$= \left(\int_{G\times G} f\, d(\mu \times \sigma) \right)_{ij}. \quad \square$$

Let G be a locally compact group. We now define the *convolution* $\mu * \sigma$ of two M_n-vlaued measures σ and μ on G by

$$(\mu * \sigma)(E) = (\mu \times \sigma)\{(x,y) \in G \times G : xy \in E\}.$$

Clearly positivity is not preserved by product nor convolution since the product of two positive matrices need not be positive unless they commute.

Lemma 8. *Let μ and σ be positive M_n-valued measures on G. Then $Tr(\mu \times \sigma)(Q) \geqq 0$ and $Tr(\mu * \sigma)(E) \geqq 0$ for all $Q \in \mathscr{B}^2$ and $E \in \mathscr{B}$. Further, if $\mu(\mathscr{B})$ and $\sigma(\mathscr{B})$ commute, then both $\mu \times \sigma$ and $\mu * \sigma$ are positive M_n-valued.*

Proof. The first assertion follows from Lemma 5 since $(\mu \times \sigma)(Q) = \int_G \mu(Q^y)\, d\sigma(y)$. If μ and σ have commuting ranges, then $(\mu \times \sigma)(f) \geqq 0$ for every positive simple function, and hence for every positive $\mu \times \sigma$-integrable function f. This gives the second assertion. $\square$

Let $f \in C_0(G, M_n)$ and let $\mu, \sigma \in M(G, M_n)$. Since $\mu * \sigma$ is the image measure of $\mu \times \sigma$ under the continuous transformation $\Psi\colon G \times G \to G$ given by $\Psi(x,y) = xy$ and $f \circ \Psi$ is $\mu \times |\sigma|$-integrable, entry-wise change of variable implies that

$$\int_G f\, d(\mu * \sigma) = \int_G f\, d\big((\mu \times \sigma) \circ \Psi^{-1}\big) = \int_{G\times G} f \circ \Psi\, d(\mu \times \sigma)$$

$$= \int_G \left(\int_G f(xy)\, d\mu(x) \right) d\sigma(y).$$

We also have $(\mu * \sigma)_{ij} = \sum_k \mu_{jk} * \sigma_{kj}$. Given that G is an abelian group and $\mu \in M(G, M_n)$, we define its *Foureir transform* on the dual group $\hat{G}$ by

$$\hat{\mu}(\gamma) = \int_G \gamma(-x)\, d\mu(x) \quad (\gamma \in \hat{G})$$

which denotes the μ-integral of the function $x \in G \mapsto \gamma(-x)I \in M_n$. If $G = (\mathbb{R}, +)$, then $\hat{G} = \mathbb{R}$ and $\hat{\mu}(\gamma) = \int_{\mathbb{R}} e^{-i\gamma x}\, d\mu(x)$.

We will refer to the following result in Section 6.

Proposition 9. *Let σ be a positive M_n-valued measure on $\mathbb{R}$ such that $\sigma(\mathbb{R}) = I$. If there exists $\mu \in M(G, M_n)$ with $\mu(\mathbb{R}) = I$ and $\mu * \sigma = \mu$, then $\sigma = \delta_0 I$ where δ_0 is the unit mass at* 0.

Proof. We have $\hat{\mu}\hat{\sigma} = \hat{\mu}$ and $\hat{\mu}(0) = \mu(\mathbb{R}) = I$. By continuity, there is an interval $(-a,a)$ in $\mathbb{R}$ such that $\|\hat{\mu}(\gamma) - I\| = \|\hat{\mu}(\gamma) - \hat{\mu}(0)\| < 1$ for all $\gamma \in (-a,a)$. This implies that $\hat{\mu}(\gamma)$ is invertible and $\hat{\sigma}(\gamma) = I$ for all $\gamma \in (-a,a)$. Therefore we have

$$\int_{\mathbb{R}} e^{-i\gamma x}\, d\sigma_{ii}(x) = 1 = \int_{\mathbb{R}} \cos \gamma x\, d\sigma_{ii}(x) \quad (i = 1, \ldots, n).$$

Since σ_{ii} is a probability measure on $\mathbb{R}$, we have $\frac{\gamma x}{2\pi} \in \mathbb{Z}$ for σ_{ii}-almost all x in $\mathbb{R}$. So $\sigma_{ii}\left(\frac{2\pi}{\gamma}\mathbb{Z}\right) = 1$ for all $\gamma \in (-a,a)$. Choose $\gamma, \gamma' \in (-a,a)$ with irrational quotient. Then $\frac{2\pi}{\gamma}\mathbb{Z} \cap \frac{2\pi}{\gamma'}\mathbb{Z} = \{0\}$ and $\sigma_{ii}\left(\frac{2\pi}{\gamma}\mathbb{Z}\right) = \sigma_{ii}\left(\frac{2\pi}{\gamma'}\mathbb{Z}\right) = 1$ give $\sigma_{ii} = \delta_0$. Hence, for every Borel set A not containing 0, we have $\sigma(A) \geqq 0$ and $Tr(\sigma(A)) = 0$ which implies that $\sigma_{ij}(A) = 0$ for all i, j. It follows that $\sigma_{ij} = t_{ij}\delta_0$ for some $t_{ij} \in \mathbb{C}$ and

$$\sigma = \begin{pmatrix} 1 & t_{12} & \cdots & t_{1n} \\ t_{21} & 1 & \cdots & t_{2n} \\ \vdots & & \ddots & \vdots \\ t_{n1} & \cdot & \cdots & 1 \end{pmatrix} \delta_0.$$

Finally $I = \sigma(\mathbb{R})$ gives $\sigma = \delta_0 I$. $\square$

Remark. Clearly the condition $\mu(\mathbb{R}) = I$ in the above proposition can be replaced by the condition that $\mu(\mathbb{R})$ is invertible.

3. Jordan structures in Banach spaces

In this section, we give a brief introduction to Jordan algebras and Jordan triple systems, and prove some structure results for the range of a contractive projection on a type I finite Jordan triple system, for later application. References for Jordan theory and Banach manifolds can be found in [6], [39], [43].

We will only consider algebras over the complex field. A *Jordan algebra* is a commutative but not necessarily associative algebra whose elements satisfy the Jordan identity

$$a(ba^2) = (ab)a^2.$$

A *Jordan triple system* is a complex vector space V with a *Jordan triple product* $\{\cdot,\cdot,\cdot\}: V \times V \times V \to V$ which is symmetric and linear in the outer variables, conjugate linear in the middle variable and satisfies the Jordan triple identity

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

A Jordan algebra with involution $*$ is a Jordan triple system in the Jordan triple product

$$\{a,b,c\} = (ab^*)c + (b^*c)a - (ca)b^*.$$

A complex Banach space Z is called a JB*-*triple* if it is a Jordan triple system such that for each $z \in Z$, the linear map

$$D(z,z): v \in Z \mapsto \{z,z,v\} \in Z$$

is Hermitian, that is, $\|e^{itD(z,z)}\| = 1$ for all $t \in \mathbb{R}$, with non-negative spectrum and $\|D(z,z)\| = \|z\|^2$. A JB*-triple Z is called a JBW*-*triple* if it is a dual Banach space, in which case its predual is unique, denoted by Z_*, and the triple product is separately w*-continuous. The second dual Z^{**} of a JB*-triple is a JBW*-triple. A subspace of a JB*-triple is called a *subtriple* if it is closed with respect to the triple product.

The JB*-triples form a large class of Banach spaces. They include for instance, C*-algebras, Hilbert spaces and spaces of rectangular matrices. The triple product in a C*-algebra $\mathscr{A}$ is given by

$$\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x).$$

In fact, $\mathscr{A}$ is a Jordan algebra in the product

$$x \circ y = \frac{1}{2}(xy + yx)$$

and we have $\{x,y,z\} = (x \circ y^*) \circ z + (y^* \circ z) \circ x - (z \circ x) \circ y^*$. A norm-closed subspace of a C*-algebra is called a JC*-*algebra* if it is closed with respect to the involution $*$ and the Jordan product $\circ$ given above. A JC*-algebra is called a JW*-*algebra* if it is a dual Banach space.

Jordan structures occur in symmetric Banach manifolds and operator algebras. In geometry, JB*-triples arise as tangent spaces to complex symmetric Banach manifolds, the latter are infinite-dimensional generalization of the Hermitian symmetric spaces classified by E. Cartan [3] using Lie groups. The non-compact Hermitian symmetric spaces are the bounded symmetric domains in $\mathbb{C}^n$ and the irreducible ones are, up to biholomorphic equivalence, the open unit balls of one of the following six types of finite-dimensional complex normed vector spaces of matrices:

type 1: $p \times q$ complex matrices,

type 2: $p \times p$ skew symmetric complex matrices,

type 3: $p \times p$ symmetric complex matrices,

type 4: spin factor,

type 5: $M_{1,2}(\mathcal{O}) = 1 \times 2$ matrices over the Cayley algebra $\mathcal{O}$,

type 6: $M_3(\mathcal{O}) = 3 \times 3$ hermitian matrices over $\mathcal{O}$.

Spin factor is defined below. The infinite-dimensional generalization of the above spaces are the following six types of JBW*-triples, called the *Cartan factors*:

type 1: $B(H,K)$ with triple product $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$,

type 2: $\{z \in B(H,H): z^t = -z\}$,

type 3: $\{z \in B(H,H): z^t = z\}$,

type 4: spin factor,

type 5: $M_{1,2}(\mathcal{O})$ with triple product $\{x,y,z\} = \frac{1}{2}(x(y^*z) + z(y^*x))$,

type 6: $M_3(\mathcal{O})$,

where $B(H,K)$ is the Banach space of bounded linear operators between complex Hilbert spaces H and K, and z^t is the transpose of z induced by a conjugation on H. Cartan factors of type 2 and 3 are subtriples of $B(H,H)$, the latter notation is shortened to $B(H)$. The type 3 and 4 are Jordan algebras with the usual Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. A *spin factor* is a Banach space that is equipped with a complete inner product $\langle\cdot,\cdot\rangle$ and a conjugation j on the resulting Hilbert space, with triple product

$$\{x,y,z\} = \frac{1}{2}(\langle x,y\rangle z + \langle z,y\rangle x - \langle x,jz\rangle jy)$$

such that the given norm and the Hilbert space norm are equivalent. The Cartan factors $M_{1,2}(\mathcal{O})$ and $M_3(\mathcal{O})$ are *exceptional* which means they can not be embedded as a subtriple of $B(H)$. A JBW*-triple is called a JW*-*triple* if it can be embedded as a subtriple of some $B(H)$. If a JW*-triple Z admits a *unitary element* u, that is, an element u such that $\{u,u,x\} = x$ for all $x \in Z$, then Z is a JW*-algebra is the following Jordan product and involution:

$$x \circ y = \{x,u,x\}, \quad x^* = \{u,x^*,u\}.$$

Cartan's classification can be extended to the infinite-dimensional case in that the irreducible bounded symmetric domains in complex Banach spaces are (biholomorphically equivalent to) the open unit balls of the Cartan factors [29].

Let Z be a JB*-triple and let $P: Z \to Z$ be a contractive projection, that is, P is linear, $P^2 = P$ and $\|P\| \leqq 1$. (For later application, we do not exclude the trivial case $P = 0$.) Kaup [30] and Stacho [42] have shown that the range $P(Z)$ is linearly isometric to a JB*-triple, although $P(Z)$ need not be a subtriple of Z. For closed subtriples Z of C*-algebras, this result has also been proved by Friedman and Russo [21].

Let $Z \subset B(H)$ and $W \subset B(K)$ be JW*-triples. Then their algebraic tensor product $Z \odot W$ identifies naturally as a subtriple of $B(H \otimes K)$, where $H \otimes K$ is the usual Hilbert space tensor product. The ultraweak closure $Z \otimes W$ of $Z \odot W$ in $B(H \otimes K)$ is a JW*-triple.

A JBW*-triple Z is of type I if, and only if, it is linearly isometric to an ℓ^∞-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha) \otimes C_\alpha$ where C_α is a Cartan factor, and if C_α is exceptional, $L(\Omega_\alpha^\infty) \otimes C_\alpha$ denotes the injective tensor product $C(S_\alpha) \check{\otimes} C_\alpha$ where $C(S_\alpha)$ is the space of complex continuous functions on the spectrum of $L^\infty(\Omega_\alpha)$ [25]. Such a type I JBW*-triple is called *type* I *finite* if each Cartan factor C_α is finite-dimensional. It has been shown in [12] that a JBW*-triple Z is type I finite if, and only if, its predual Z_* has the Dunford-Pettis property. We recall that a Banach space W has the *Dunford-Pettis property* if every weakly compact operator on W is completely continuous. Such property is inherited by complemented subspaces.

Proposition 10. *Let $P: Z \to Z$ be a weak*-continuous contractive projection on a type* I *finite* JBW**-triple Z. Then its range $P(Z)$ is (linearly isometric to) a type* I *finite* JBW*-*triple.*

Proof. We note that $P(Z)$ is norm-closed. By weak*-continuity and the Krein-Smulyan Theorem, $P(Z)$ is also weak*-closed. Also, P induces a contractive projection $P_*: f \in Z_* \mapsto f \circ P \in Z_*$ on the predual Z_*. As remarked above, the predual Z_* has the Dunford-Pettis property. The predual of $P(Z)$ identifies with $Z_*/P_*^{-1}(0)$ which is linearly isometric to the complemented subspace $P_*(Z_*)$ of Z_*, and therefore has the Dunford-Pettis property. Hence $P(Z)$ is linearly isometric to a type I finite JBW*-triple by the above result from [12]. □

Without weak*-continuity, we have the following result which will be used later to determine the structure of the space of bounded matrix-valued harmonic functions.

Proposition 11. *Let Z be a finite-dimensional* JW*-*triple and let*

$$P: L^\infty(\Omega) \otimes Z \to L^\infty(\Omega) \otimes Z$$

be a contractive projection such that its range is weak-closed. Then the range is either $\{0\}$ or is (linearly isometric to) an ℓ^∞-sum $\sum_{k=1}^{n} L^\infty(\Omega_k) \otimes C_k$ where C_k is a finite-dimesional Cartan factor.*

Proof. By [21], the range $P(L^\infty(\Omega) \otimes Z)$ can be regarded, via a linear isometry, as a subtriple of the second dual $(L^\infty(\Omega) \otimes Z)^{**}$ which is in turn a subtriple of $(L^\infty(\Omega) \otimes M_n)^{**}$ for some n, by the finite-dimensionality of Z. But $(L^\infty(\Omega) \otimes M_n)^{**}$ is of the form $L^\infty(\Omega') \otimes M_n$, by [27], which has the Dunford-Pettis property. By [12], Corollary 6, the subtriple $P(L^\infty(\Omega) \otimes Z)$ also has the Dunford-Pettis property and is therefore either $\{0\}$ or of the form $\sum_\alpha L^\infty(\Omega_\alpha) \otimes C_\alpha$ where C_α is a Cartan factor and $\sup_\alpha \dim C_\alpha < \infty$, by [12], Theorem 14. The latter implies that there are only a finite number of distinct Cartan factors. Rearranging terms, we can write $P(L^\infty(\Omega) \otimes Z) = \sum_{k=1}^{n} L^\infty(\Omega_k) \otimes C_k$ where C_k is a finite-dimesional Cartan factor. □

4. Bounded matrix-valued harmonic functions

Throughout, we will denote by λ the left invariant Haar measure on G. Let $f = (f_{ij}): G \to M_n$ be a Borel function, that is, each f_{ij} is a Borel function. Let $\sigma \in M(G, M_n)$. We define the convolution $f * \sigma: G \to M_n$, if it exists, by

$$(f * \sigma)(x) = \int_G f(xy^{-1})\, d\sigma(y) \quad (x \in G).$$

Remark 12. As in Section 1, the usual definition of the convolution $f * \sigma$ includes the modular function $\triangle_G(y^{-1})$ in the integral, here we omit it for convenience, but this would not affect the theory of harmonic functions for, considering σ-harmonic functions in terms of the usual convolution amongs to considering $(\triangle_G^{-1}.\sigma)$-harmonic functions in our setting, and the adjustments to the results are minor. Of course, the distinction disappears if G is unimodular.

Let $f: G \to M_n$ be Bochner λ-integrable and let $\mu = f.\lambda$. Simple entry-wise computation shows, as in the scalar case, that

$$\mu * \sigma = (f * (\triangle_G^{-1}.\sigma)).\lambda.$$

We also define the convolution $\sigma * f: G \to M_n$ by

$$(\sigma * f)(x) = \int_G f(y^{-1}x)\, d\sigma(y) \quad (x \in G).$$

Unlike the scalar case, $\sigma * (f.\lambda)$ need not equal $(\sigma * f).\lambda$. In fact, they are equal if, and only if, $\sum_k \sigma_{kj} * f_{ik} = \sum_k \sigma_{ik} * f_{kj}$ for all i, j. For this reason, we will mostly work with the convolution $f * \sigma$.

Example 13. Let $\sigma = \begin{pmatrix} \delta_0 & \delta_y \\ 0 & \delta_z \end{pmatrix} \in M(\mathbb{R}, M_2)$ and $f = \begin{pmatrix} h & 0 \\ h & 0 \end{pmatrix}$ be an M_2-valued Bochner λ-integrable function on $\mathbb{R}$. Then we have

$$(\sigma * f).\lambda = \begin{pmatrix} h & h(y^{-1}\cdot) \\ h & h(z^{-1}\cdot) \end{pmatrix}.\lambda$$

and

$$\sigma * (f.\lambda) = \begin{pmatrix} h + h(y^{-1}\cdot) & 0 \\ h(z^{-1}\cdot) & 0 \end{pmatrix}.\lambda.$$

A complex-valued function h on G is called *locally λ-measurable* if for every Borel set $B \subset \mathbb{C}$, the set $h^{-1}(B) \cap E$ is Borel for every Borel set $E \subset G$ with $\lambda(E) < \infty$. A function $f: G \to M_n$ is called *weakly locally λ-measurable* if $\phi \circ f: G \to \mathbb{C}$ is locally λ-measurable for every $\phi \in M_n^*$. We note that if (G, λ) is σ-finite, then local λ-measurability is the same as Borel measurability.

Let $L^1(G, M_n^*)$ be the Banach space of all (equivalence classes of) M_n^*-valued Bochner λ-integrable functions on G. Then the dual of $L^1(G, M_n^*)$ is the Banach space $L^\infty(G, M_n)$ of all M_n-valued essentially bounded weakly locally λ-measurable functions on G (modulo the locally null functions), and $L^\infty(G, M_n)$ is a von Neumann algebra under the pointwise multiplication and involution (cf. [40], Theorem 1.22.13). The identity in $L^\infty(G, M_n)$ is the function $\mathbb{1}: G \to M_n$ such that $\mathbb{1}(x)$ is the identity matrix for all $x \in G$. We note that $L^1(G, M_n^*)$ is the projective tensor product $L^1(G) \otimes_\gamma M_n^*$ and that $L^\infty(G, M_n)$ is the tensor product $L^\infty(G) \otimes M_n$ defined before [40], p. 68. The w*-topology on $L^\infty(G, M_n)$ is the weak topology $w(L^\infty(G, M_n), L^1(G, M_n^*))$ with respect to the duality

$$\langle g, f\rangle = \int_G Tr(g(x)f(x))\, d\lambda(x) \quad (g \in L^1(G, M_n^*), f \in L^\infty(G, M_n)).$$

For $f = (f_{ij}) \in L^\infty(G, M_n)$ and $g = (g_{ij}) \in L^1(G, M_n^*)$, we have $f_{ij} \in L^\infty(G)$ and $g_{ij} \in L^1(G)$. Let $\{f_\alpha\}$ be a net in $L^\infty(G, M_n)$ w*-convergent to $f \in L^\infty(G, M_n)$. Then $(f_\alpha)_{ij}$ is a net in $L^\infty(G)$ and is $w(L^\infty(G), L^1(G))$-convergent to $f_{ij} \in L^\infty(G)$.

Given $\mu \in M(G, M_n)$, we define

$$d\tilde{\mu}(y) = \triangle_G(y^{-1})\, d\mu(y^{-1}).$$

Then for $g \in L^1(G, M_n^*)$, we have

$$\begin{aligned}\langle g, f_\alpha * \mu\rangle &= \int_G Tr(g(x)(f_\alpha * \mu)(x))\, d\lambda(x) \\ &= \sum_{i,j,k} \int_G g_{ik}(x)(f_\alpha)_{kj} * \mu_{ji}(x)\, d\lambda(x) \\ &= \sum_{i,j,k} \int_G (g_{ik} * \tilde{\mu}_{ji})(x)(f_\alpha)_{kj}(x)\, d\lambda(x) \\ &\to \sum_{i,j,k} \int_G (g_{ik} * \tilde{\mu}_{ji})(x) f_{kj}(x)\, d\lambda(x) = \langle g, f * \mu\rangle \quad \text{as } \alpha \to \infty.\end{aligned}$$

That is, $(f_\alpha * \mu)$ is w*-convergent to $f * \mu$. It may be useful to note that, if G is abelian, the above computation also yields

$$\langle g, f * \mu\rangle = \langle \tilde{\mu} * g, f\rangle$$

for $g \in L^1(G, M_n^*)$ and $f \in L^\infty(G, M_n)$.

Let $\sigma \in M(G, M_n)$. A Borel function $f: G \to M_n$ is called *σ-harmonic*, or *harmonic* for short, if it satisfies the convolution equation

$$f * \sigma = f.$$

Let (G, λ) be σ-finite. By a slight abuse of language, we call the functions in

$$H_\sigma(G, M_n) = \{f \in L^\infty(G, M_n) : f * \sigma = f\}$$

the *bounded* σ-harmonic functions on G. Evidently $H_\sigma(G, M_n)$ is w*-closed. Our first task in this section is to show that $H_\sigma(G, M_n)$ is a JW*-triple, for $\|\sigma\| = 1$. In fact, we show that $H_\sigma(G, M_n)$ is the range of a contractive projection on $L^\infty(G, M_n)$, and therefore admits a Jordan triple structure.

Proposition 14. *Let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then there is a contractive projection P: $L^\infty(G, M_n) \to L^\infty(G, M_n)$ with range $H_\sigma(G, M_n)$.*

Proof. For $m = 1, 2, \ldots$, we define a map Λ_m: $L^\infty(G, M_n) \to L^\infty(G, M_n)$ by

$$\Lambda_m(f) = f * \overbrace{\sigma * \cdots * \sigma}^{m\text{-times}}.$$

By Lemma 4, $\|\Lambda_m\| \leqq 1$ since $\|\sigma\| = 1$, and by the above remarks, Λ_m is w*-continuous. Let $\mathscr{K} = \overline{co}\{\Lambda_m : m = 1, 2, \ldots\}$ be the closed convex hull of $\{\Lambda_m : m = 1, 2, \ldots\}$ with respect to the product topology $\mathscr{T}$ of $L^\infty(G, M_n)^{L^\infty(G, M_n)}$ where $L^\infty(G, M_n)$ is equipped with the w*-topology. Then $\mathscr{K}$ is compact. Define Φ: $\mathscr{K} \to \mathscr{K}$ by

$$\Phi(\Lambda)(f) = \Lambda(f) * \sigma \quad (\Lambda \in \mathscr{K}, f \in L^\infty(G, M_n)).$$

It is straightforward to verify that Φ is well-defined, affine and $\mathscr{T}$-continuous. Therefore, by the Markov-Kakutani fixed-point theorem, there exists $P \in \mathscr{K}$ such that $\Phi(P) = P$. We have clearly $P(f) = f$ for $f \in H_\sigma(G, M_n)$ since $\Lambda_m(f) = f$ for all m. Given $f \in L^\infty(G, M_n)$, we have $P(f) = \Phi(P)(f) = P(f) * \sigma$, that is, $P(f) \in H_\sigma(G, M_n)$. This proves that $P(L^\infty(G, M_n)) = H_\sigma(G, M_n)$ and $P^2 = P$. Since each Λ_m is contractive, so is P. □

Corollary 15. *Let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then $H_\sigma(G, M_n)$ is a* JW*-*triple. Further, if $H_\sigma(G, M_n)$ contains a unitary element in $L^\infty(G, M_n)$, then it is a* JW*-*algebra.*

Proof. Let P: $L^\infty(G, M_n) \to L^\infty(G, M_n)$ be the contractive projection in Proposition 14. By [21], [30], $H_\sigma(G, M_n)$ is a JB*-triple with the triple product

$$\{f, g, h\} = \frac{1}{2} P(fg^*h + hg^*f)$$

where $g^*(x) =: g(x)^* \in M_n$ for each $x \in G$. Let $u \in H_\sigma(G, M_n)$ be unitary in $L^\infty(G, M_n)$. Then we have

$$\{u, u, h\} = P(h) = h$$

for $h \in H_\sigma(G, M_n)$. Hence, by the remarks in Section 3, $H_\sigma(G, M_n)$ is a JW*-algebra with respect to the following Jordan product and involution:

$$f \circ g = \{f, u, g\} = \frac{1}{2}P(fu^*g + gu^*f), \quad f^* = \{u, f^*, u\} = P(uf^*u). \quad \square$$

Example 16. Let $G = \{e\}$ and $\sigma(G)$ be any proper projection p in M_n. Then $H_\sigma(G, M_n)$ identifies with $\{A \in M_n : A = Ap\} = M_n p$ which is a left ideal of M_n, but not a JW*-algebra. The projection $P: L^\infty(G, M_n) \to H_\sigma(G, M_n)$ is given by $P(A) = Ap$. If $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2$ say, then $H_\sigma(G, M_2)$ identifies with the 2×1 complex matrices. This is a special case of the following result.

We can now describe the structure of $H_\sigma(G, M_n)$.

Corollary 17. *Let G be a locally compact group and let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then $H_\sigma(G, M_n)$ is either $\{0\}$ or linearly isometric to a finite ℓ^∞-sum $\bigoplus_k L^\infty(\Omega_k) \otimes C_k$ where C_k is a finite-dimensional Cartan factor of the following type:*

(i) *M_{pq}, the space of complex $p \times q$-matrices;*

(ii) *S_p, the space of complex $p \times p$ symmetric matrices;*

(iii) *A_p, the space of complex $p \times p$ skew symmetric matrices;*

(iv) *V_p, the spin factor of dimension at least 3, consisting of complex $p \times p$ matrices such that $a \in V_p$ implies $a^* \in V_p$ and a^2 is a scalar multiple of the identity matrix.*

Proof. By Proposition 14, $H_\sigma(G, M_n) = P(L^\infty(\Omega) \otimes M_n)$ which is either $\{0\}$ or an ℓ^∞-sum as in Proposition 11, in which the Cartan factors can not be exceptional. $\square$

A positive measure $\sigma \in M(G, M_n)$ is called *adapted* if the support of $\rho \circ \sigma$ generates a dense subgroup of G for every pure state $\rho \in M_n^*$. Given the polar representation $\sigma = \omega \cdot |\sigma|$, we have $\rho \circ \sigma = (\rho \circ \omega)|\sigma|$ and so $\operatorname{supp} \rho \circ \sigma \subset \operatorname{supp}|\sigma|$. It follows that if σ is adapted, then $|\sigma|$ is adapted in the usual sense.

Example 18. Let G be an abelian group and let $\sigma \in M(G, M_n)$ be a positive measure such that $\sigma(G)$ is the identity matrix I. Then by a recent generalization in [8] of the Choquet-Deny Theorem [4], every bounded σ-harmonic M_n-valued function on G is constant if, and only if, σ is adapted. For any group G, we show below that the absence of non-constant bounded M_n-valued harmonic functions on G implies that G is amenable. This result is known for $n = 1$ (cf. [28], [37]) with a different proof.

Corollary 19. *If there is a positive $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$ such that all bounded σ-harmonic M_n-valued functions on G are constant, not all 0, then G is amenable.*

Proof. We have $H_\sigma(G, M_n) = L\mathbb{1}$ where $L = \{A \in M_n : A\sigma(G) = A\}$ is a nonzero closed left ideal of M_n. So there is a nonzero projection $q \in M_n$ such that $L = M_n q$ (cf. [40], 1.10.1). In particular, $q\sigma(G) = q$. Let $P: L^\infty(G, M_n) \to H_\sigma(G, M_n)$ be the contractive projection in Proposition 14 where $\Lambda_m(\mathbb{1}) = \sigma(G)^m \mathbb{1}$ implies $q\Lambda_m(\mathbb{1}) = q\mathbb{1}$ for all m and therefore $qP(\mathbb{1}) = q\mathbb{1}$. On the other hand, $P(\mathbb{1}) = A\mathbb{1}$ for some positive $A \in M_n q$. It follows that $A = Aq = qA = q$.

Given an M_n-valued function f on G, we denote by $f_z(\cdot) = f(z^{-1}\cdot)$ the left translate of f by $z \in G$. Since $f_z * \sigma = (f * \sigma)_z$ for $f \in L^\infty(G, M_n)$, we have $P(f_z) = P(f)_z = P(f)$, the latter equality holds because $P(f)$ is a constant function.

Since $q\mathbb{1}$ is a projection in $L^\infty(G, M_n)$, we can find a state φ of $L^\infty(G, M_n)$ such that $\varphi(q\mathbb{1}) = 1$. It follows that $\varphi \circ P$ is a state of $L^\infty(G, M_n)$ since

$$(\varphi \circ P)(\mathbb{1}) = \varphi(P(\mathbb{1})) = \varphi(q\mathbb{1}) = 1 = \|\varphi \circ P\|.$$

Given a positive function $h \in L^\infty(G)$, the function $h \otimes I \in L^\infty(G, M_n)$ is M_n^+-valued and it is now readily seen that the map $m: L^\infty(G) \to \mathbb{C}$ defined by

$$m(h) = \varphi(P(h \otimes I))$$

is a left-invariant mean on $L^\infty(G)$. $\square$

We next show that the continuous σ-harmonic M_n-valued functions are constant on compact groups for adapted positive M_n-valued measures σ with $\|\sigma\| = 1$, as in the scalar case [32]. For this, we will apply the Peter-Weyl Theorem and we need to extend the notion of Fourier transform to the matrix-valued case, for compact groups. Given $\sigma \in M_n(G, M_n)$, we define its *amplification* $\sigma \otimes 1_m$ to be the measure in $M(G, M_{nm})$ given by

$$(\sigma \otimes 1_m)(E) = \sigma(E) \otimes I_m = \begin{pmatrix} \sigma(E) & & & \\ & \sigma(E) & & \\ & & \ddots & \\ & & & \sigma(E) \end{pmatrix} \quad (E \in \mathscr{B})$$

where the tensor product $M_n \otimes M_m$ is naturally identified with M_{nm}. We note that if σ is adapted, then so is $\sigma \otimes 1_m$. Indeed, given a pure state $\rho(\cdot) = \langle \cdot \zeta, \zeta \rangle$ of M_{nm} where $\zeta = \sum_{i=1}^m \xi_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m$ is a unit vector and $\{e_1, \ldots, e_m\}$ is the standard basis in $\mathbb{C}^m$, we have $\rho \circ (\sigma \otimes 1_m) = \sum_i \langle \sigma(\cdot)\xi_i, \xi_i \rangle$ which implies that $\operatorname{supp}\langle \sigma(\cdot)\xi_i, \xi_i \rangle \subset \operatorname{supp} \rho \circ (\sigma \otimes 1_m)$.

Let $\hat{G}$ be the dual space of G, consisting of the equivalence classes of continuous unitary irreducible representations of G. Let G be compact. Then G is unimodular and every $\pi \in \hat{G}$ is finite-dimensional and so $\pi = (\pi_{ij}): G \to M_m$ is a continuous function, where $m = \dim \pi$. We define its *amplification* $1_n \otimes \pi: G \to M_{nm}$ by

$$(1_n \otimes \pi)(x) = I_n \otimes \pi(x) = (\pi_{ij}(x) I_n) = \begin{pmatrix} \pi_{11}(x) & & & \cdots & \pi_{1n}(x) & & \\ & \ddots & & & & \ddots & \\ & & \pi_{11}(x) & \cdots & & & \pi_{1n}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{n1}(x) & & & \cdots & \pi_{nn}(x) & & \\ & \ddots & & & & \ddots & \\ & & \pi_{n1}(x) & \cdots & & & \pi_{nn}(x) \end{pmatrix}.$$

For $\sigma \in M(G, M_n)$, we define its *Fourier transform* by

$$\hat{\sigma}(\pi) = \int_G (1_n \otimes \pi)(x)\, d(\sigma \otimes 1_{\dim \pi})(x)$$

for $\pi \in \hat{G}$. Given $f = (f_{ij}) \in L^1(G, M_n)$, we define its *Fourier transform* by

$$\hat{f}(\pi) = \int_G f(x) \otimes \pi(x)\, d\lambda(x) \in M_{nm}$$

where $m = \dim \pi$. We have

$$\begin{aligned}
\widehat{f * \sigma}(\pi) &= \int_G (f * \sigma)(x) \otimes \pi(x)\, d\lambda(x) \\
&= \int_G \int_G f(xy^{-1})\, d\sigma(y) \otimes \pi(x)\, d\lambda(x) \\
&= \int_G \int_G f(xy^{-1}) \otimes \pi(x)\, d(\sigma(y) \otimes 1_m)\, d\lambda(x) \\
&= \int_G \int_G f(z) \otimes \pi(zy)\, d(\sigma(y) \otimes 1_m)\, d\lambda(zy) \\
&= \int_G \int_G f(z) \otimes \pi(z)\pi(y)\, d(\sigma(y) \otimes 1_m)\, d\lambda(z) \\
&= \int_G \int_G (f(z) \otimes \pi(z))(1_n \otimes \pi(y))\, d(\sigma(y) \otimes 1_m)\, d\lambda(z) \\
&= \int_G (f(z) \otimes \pi(z)) \int_G (1_n \otimes \pi(y))\, d(\sigma(y) \otimes 1_m)\, d\lambda(z) \\
&= \int_G (f(z) \otimes \pi(z))\hat{\sigma}(\pi)\, d\lambda(z) = \hat{f}(\pi)\hat{\sigma}(\pi).
\end{aligned}$$

Lemma 20. *Let σ be a positive M_n-valued measure on a compact group G such that $\|\sigma\| = 1$. If σ is adapted, then $I_{nm} - \hat{\sigma}(\pi)$ is invertible for every $\pi \in \hat{G}$ with $\pi \neq \iota$ where ι is the trivial one-dimensional representation and $m = \dim \pi$.*

Proof. We show that 1 is not an eigenvalue of $\hat{\sigma}(\pi)$. Suppose otherwise, there exists a unit vector $\zeta = \sum_{i=1}^{n} e_i \otimes \xi_i \in \mathbb{C}^n \otimes \mathbb{C}^m$ such that $\hat{\sigma}(\pi)\zeta = \zeta$, where $\{e_1, \dots, e_n\}$ is the standard basis in $\mathbb{C}^n$ and $\sum_{i=1}^{n} \|\xi_i\|^2 = 1$. Let $\sigma = \omega \cdot |\sigma|$ be the polar decomposition. Then we have $|\sigma \otimes 1_m| = |\sigma|$ and $\sigma \otimes 1_m = (\omega \otimes 1_m) \cdot |\sigma|$. Also,

$$\begin{aligned}
1 = \langle \hat{\sigma}(\pi)\zeta, \zeta \rangle &= \left\langle \int_G (1_n \otimes \pi)\, d(\sigma \otimes 1_m)\zeta, \zeta \right\rangle \\
&= \left\langle \left(\int_G (1_n \otimes \pi)(\omega \otimes 1_m)\, d|\sigma| \right)\zeta, \zeta \right\rangle \\
&= \int_G \langle (1_n \otimes \pi)(\omega \otimes 1_m)\zeta, \zeta \rangle\, d|\sigma|
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Re}\langle (1_n \otimes \pi(x))(\omega(x) \otimes 1_m)\zeta, \zeta \rangle &\leqq |\langle (\omega(x) \otimes 1_m)^{1/2}(1_n \otimes \pi(x))(\omega(x) \otimes 1_m)^{1/2}\zeta, \zeta \rangle| \\
&\leqq \langle (\omega(x) \otimes 1_m)\zeta, \zeta \rangle \leqq 1.
\end{aligned}$$

Therefore $\langle (\omega(x) \otimes 1_m)\zeta, \zeta \rangle = 1$ and hence $(\omega(x) \otimes 1_m)\zeta = \zeta$ for $|\sigma|$-almost all $x \in G$. This gives

$$\int_G \langle (1_n \otimes \pi)\zeta, \zeta \rangle\, d|\sigma| = \int_G \langle (1_n \otimes \pi)(\omega \otimes 1_m)\zeta, \zeta \rangle\, d|\sigma| = 1$$

where $|\sigma|$ is an adapted probability measure on G.

Let $x \in \mathrm{supp}|\sigma|$. If $\mathrm{Re}\langle (1_n \otimes \pi)(x)\zeta, \zeta \rangle < 1$, then there is an open set $V \subset G$ containing x such that $\mathrm{Re}\langle (1_n \otimes \pi)(y)\zeta, \zeta \rangle < 1$ for all $y \in V$. Therefore we have

$$\begin{aligned}
1 &< \int_V d|\sigma| + \int_{G \setminus V} \mathrm{Re}\langle (1_n \otimes \pi)\zeta, \zeta \rangle\, d|\sigma| \\
&\leqq |\sigma|(V) + |\sigma|(G \setminus V) = 1
\end{aligned}$$

which is impossible. Hence $\langle (1_n \otimes \pi)(x)\zeta, \zeta \rangle = 1$ and so $(1_n \otimes \pi)(x)\zeta = \zeta$ for all $x \in \mathrm{supp}|\sigma|$. Since $\mathrm{supp}|\sigma|$ generates a dense subgroup of G, we have $(1_n \otimes \pi)(x)\zeta = \zeta$ for all $x \in G$, that is,

$$\sum_i (1_n \otimes \pi)(x)(e_i \otimes \xi_i) = \sum_i e_i \otimes \pi(x)\xi_i = \sum_i e_i \otimes \xi_i.$$

Hence we have $\pi(x)\xi_i = \xi_i$, and in particular, for some $\xi_i \neq 0$, which gives $\pi = \iota$ by irreducibility of π, a contradiction. □

Proposition 21. *Let σ be an adapted positive M_n-valued measure on a compact group G with $\|\sigma\| = 1$. Then every continuous σ-harmonic M_n-valued function on G is constant.*

Proof. Let $f = (f_{ij})$ be continuous and σ-harmonic on G. Then we have $\hat{f}(\pi) = \widehat{f * \sigma}(\pi) = \hat{f}(\pi)\hat{\sigma}(\pi)$ for all $\pi \in \hat{G}$. By Lemma 20, we have $\hat{f}(\pi) = 0$ for all $\pi \in \hat{G}$ with $\pi \neq \iota$ which implies

$$\int_G f_{ij}(x)\pi_{k\ell}(x)\, d\lambda(x) = 0 \quad (i, j = 1, \dots, n; k, \ell = 1, \dots, \dim \pi).$$

By the Peter-Weyl Theorem, we have in $L^2(G)$, for $f'_{ij}(x) = f_{ij}(x^{-1})$,

$$f'_{ij} = \sum_{\pi \in \hat{G}} \sum_{1 \leqq k, \ell \leqq \dim \pi} (\dim \pi)\widehat{f'_{ij}}(\pi)_{\ell k}\pi_{k\ell}$$

where $\widehat{f'_{ij}}(\pi)_{\ell k} = \int_G f'_{ij}(x)\pi_{\ell k}(x^{-1})\, d\lambda(x)$ is the ordinary Fourier transform and is zero for $\pi \neq \iota$. It follows that f_{ij} is a constant function by continuity. Hence f is constant. □

5. Continuous matrix-valued harmonic functions

In this section, we study continuous, but not necessary bounded, matrix-valued harmonic functions. We describe the M_n-valued continuous σ-harmonic functions on abelian groups, for an M_n-valued measure σ with compact support. The complex valued harmonic functions on $\mathbb{R}$ were first characterized by Schwartz [41], and on $\mathbb{R}^m$ by Malgrange [35] and Ehrenpreis [18]. Lefranc [34] has proved similar results for $\mathbb{Z}^m$. Their results have been extended to discrete abelian groups by Elliott [19], and to locally compact abelian groups by Gilbert [24]. We extend Gilbert's result to the matrix-valued case.

Given an M_n-valued measure $\sigma = (\sigma_{ij})$ on a group G, with polar representation $\sigma = \omega \cdot |\sigma|$, we define the *support* of σ, $\operatorname{supp}\sigma$, to be the support of the positive measure $|\sigma|$. Since $\sigma_{ij} = \omega_{ij} \cdot |\sigma|$, we have $|\sigma_{ij}| = |\omega_{ij}| \cdot |\sigma|$ (cf. [38], p. 126) and so $\operatorname{supp}\sigma_{ij} \subset \operatorname{supp}\sigma$. In particular, if $\operatorname{supp}\sigma$ is compact, then $\operatorname{supp}\sigma_{ij}$ is also compact.

We will show that the continuous matrix-valued harmonic functions on abelian groups are 'synthesized' from the exponential polynomials which we now define. First, a *real character* on any group G is a continuous homomorphism from G to the additive group $\mathbb{R}$. For an abelian group G, an *exponential polynomial* on G is a complex-valued function of the form

$$p(\chi_1(x), \ldots, \chi_j(x))\pi(x) \quad (x \in G)$$

where $p(\cdot)$ is a polynomial with a finite number of variables and complex coefficients, $\chi_1, \ldots, \chi_j$ are real characters on G and π is a generalized character on G, that is, a continuous homomorphism from G to the multiplicative group $\mathbb{C}\backslash\{0\}$. Abusing the notation slightly, we write $p(x) = p(\chi_1(x), \ldots, \chi_j(x))\pi(x)$.

Let $C(G)$ be the linear space of complex-valued continuous functions on G, equipped with the topology of uniform convergence on compact sets in G. Then $C(G)$ is a complete locally convex space. If G is a countable union of compact sets, then $C(G)$ is metrizable, that is, $C(G)$ is a Fréchet space (cf. [31], p. 81). A separable and metrizable locally compact group is a countable union of compact sets.

Given a complex-valued measure μ on G, we let

$$P_\mu(G) = \{p : p * \mu = 0\}$$

where p is an exponential polynomial on G. Let $H_\mu(G) = \{f \in C(G) : f * \mu = 0\}$. If G is abelian and μ has compact support, then, by [23], Theorem 3.2 and [19], spectral synthesis holds for the left-invariant space $H_\mu(G)$ in that the linear span of $P_\mu(G)$ is dense in $H_\mu(G)$. Now we consider the matrix-valued harmonic functions on G.

Given $\nu \in M(G, M_n)$, we define its *determinant* $\det \nu$, which is a complex-valued measure, by convolution

$$\det \nu = \sum_\tau \operatorname{sgn}(\tau)\nu_{1\tau(1)} * \cdots * \nu_{n\tau(n)}$$

where τ is a permutation of $\{1, \ldots, n\}$. Let $\sigma \in M(G, M_n)$ and let $\delta_e I = \delta_e(\cdot)I \in M(G, M_n)$ where δ_e is the unit mass at the identity e of G. We define the complex-valued measure

$$\tilde{\sigma} = \det(\sigma - \delta_e I) = \det(\sigma_{ij} - \delta_{ij}\delta_e)$$

where δ_{ij} is the Kronecker delta. Let $\{e_{ij} : i, j = 1, \ldots, n\}$ denote the canonical matrix unit in M_n. Given a complex-valued function h on G, we denote by $h \otimes e_{ij}$ the M_n-valued function whose ij-th entry is h, and 0 elsewhere. Thus we can write $f = \sum_{ij} f_{ij} \otimes e_{ij}$ for an M_n-valued function $f = (f_{ij})$.

Let D_{ij} be a directed partially ordered set, for $i, j = 1, \ldots, n$. We define $M_n(D)$ to be the following set of $n \times n$ matrices:

$$M_n(D) = \{(\alpha_{ij}) : \alpha_{ij} \in D_{ij}\}$$

which is a directed partially ordered set under entry-wise ordering. Given nets $\{f_{\alpha_{ij}}\}_{\alpha_{ij} \in D_{ij}}$, $i, j = 1, \ldots, n$, in a vector space V, the net

$$f_\alpha = \sum_{i,j} f_{\alpha_{ij}}, \quad \alpha = (\alpha_{ij}) \in M_n(D)$$

is well-defined in V.

Proposition 22. *Let σ be an M_n-valued measure on an abelian group G, with compact support, and let $f\colon G \to M_n$ be a continuous σ-harmonic function. Then there is a net $\{p_\alpha\}$ in the linear span of $\{p \otimes e_{ij} : p \in P_{\tilde{\sigma}}(G), i, j = 1, \ldots, n\}$ such that f is the uniform limit of $\{p_\alpha\}$ on compact sets in G.*

Proof. Let $f = \sum_{ij} f_{ij} \otimes e_{ij}$ be continuous and σ-harmonic. Then we have $f * (\sigma - \delta_e I) = 0$. Let $\nu = (\nu_{ij})$ be the M_n-valued measure on G defined as the adjoint matrix of $\sigma - \delta_e I = (\sigma_{ij} - \delta_{ij}\delta_e)$, using convolution, so that

$$(\sigma - \delta_e I) * \nu = \begin{pmatrix} \det(\sigma - \delta_e I) & & 0 \\ & \ddots & \\ 0 & & \det(\sigma - \delta_e I) \end{pmatrix}.$$

Then we have

$$f * \begin{pmatrix} \tilde{\sigma} & & 0 \\ & \ddots & \\ 0 & & \tilde{\sigma} \end{pmatrix} = f * (\sigma - \delta_e I) * \nu = 0$$

which gives

$$f_{ij} * \tilde{\sigma} = 0$$

for all i, j.

Given two complex-valued measures γ_1 and γ_2 on a group G, we have

$$\operatorname{supp}(\gamma_1 + \gamma_2) \subset \operatorname{supp}\gamma_1 \cup \operatorname{supp}\gamma_2,$$

$$\operatorname{supp}(\gamma_1 * \gamma_2) \subset \overline{(\operatorname{supp}\gamma_1)(\operatorname{supp}\gamma_2)}.$$

Since each $\sigma_{ij} - \delta_{ij}\delta_e$ has compact support, it follows that $\tilde{\sigma} = \det(\sigma - \delta_e I)$ also has compact support. By the spectral synthesis for $H_{\tilde{\sigma}}(G)$ stated before, there is a net $(h_{\alpha_{ij}})$ in the linear span of $P_{\tilde{\sigma}}(G)$ such that $f_{ij} = \lim_{\alpha_{ij}} h_{\alpha_{ij}}$ in the topology of uniform convergence on compact sets in G. Define

$$p_\alpha = \sum_{i,j} h_{\alpha_{ij}} \otimes e_{ij}$$

for $\alpha = (\alpha_{ij})$. Then p_α is in the linear span of $\{p \otimes e_{ij} : p \in P_{\tilde{\sigma}}(G), i, j = 1, \ldots, n\}$ and $f = \lim_\alpha p_\alpha$ uniformly on compact sets in G. □

6. Positive matrix-valued harmonic functions

Throughout this section, we assume that G is separable and metrizable so that G is a union of compact sets G_k $(k = 1, 2, \ldots)$ and G_k is contained in the interior of G_{k+1}.

We will apply Choquet's integral representation theory to characterize the positive (unbounded) matrix-valued harmonic functions on abelian groups. For this we need to introduce the concept of an *extended* matrix-valued measure. Let $K(G, M_n^*)$ be the linear space of continuous M_n^*-valued functions on G, with compact support. The notation $K(G, \mathbb{C})$ is used for $n = 1$. We recall that M_n^* is identified with M_n with the trace norm $\|\cdot\|_1$. Let $K(G_k, M_n^*)$ be the subspace of $K(G, M_n^*)$, consisting of functions with support in G_k. With the supremum norm, $K(G_k, M_n^*)$ is a Banach space, and its dual $K(G_k, M_n^*)^*$ identifies with $M(G_k, M_n)$ by Lemma 6. With the inductive topology, $K(G, M_n^*)$ is the strict inductive limit $\varinjlim_k K(G_k, M_n^*)$ of the increasing sequence $\{K(G_k, M_n^*)\}_{k=1}^\infty$ (cf. [5], 20.11) and the dual $K(G, M_n^*)^*$ is the projective limit $\varprojlim_k K(G_k, M_n^*)^* = \varprojlim_k M(G_k, M_n)$, in the weak*-topology [31], p. 151. Elements in $K(G, M_n^*)^*$ are regarded as the *extended* matrix-valued measures on G.

Let $\mu \in K(G, M_n^*)^*$. Then $\mu = (\mu_k) \in \varprojlim_k M(G_k, M_n)$ where, in the notation of Lemma 6, $\mu_k(\cdot) \in K(G_k, M_n^*)^*$ is the restriction of the functional μ to the space $K(G_k, M_n^*)$. Given $f \in K(G, M_n^*)$ with $\operatorname{supp} f \subset G_k$, we define

$$\int_G f\, d\mu = \int_{G_k} f\, d\mu_k \in M_n$$

which is well-defined since $\mu_j = \mu_k$ on G_k for $j \geqq k$. Likewise, given any compact set $F \subset G$, we have $F \subset G_k$ for some k, and we can define $\mu(F) = \mu_k(F)$. Also, for any positive functional $\varphi \in M_n^*$, $\varphi\mu$ denotes the measure $\varphi\mu = (\varphi \circ \mu_k) \in \varprojlim_k M(G_k, \mathbb{C}) = K(G, \mathbb{C})^*$.

Now given $\sigma \in M(G, M_n)$ and $\mu \in K(G, M_n^*)^*$, we can define their *convolution* $\mu * \sigma$ as an element in $K(G, M_n^*)^*$ by

$$(\mu * \sigma)(f) = Tr\left(\int_G \left(\int_G f(xy)\, d\mu(x)\right) d\sigma(y)\right)$$

for $f \in K(G, M_n^*)$.

Given a positive $\sigma \in M(G, M_n)$ with $\sigma(G) = I$, we have seen in Proposition 9 that the equation $\mu * \sigma = \mu$ may have few '*bounded*' solutions μ in $M(G, M_n)$. This is the reason why we introduce $K(G, M_n^*)^*$ and seek '*unbounded*' solutions $\mu \in K(G, M_n^*)^*$ for the equation.

The self-adjoint part $K(G, M_n^*)_{\text{sa}}$ of $K(G, M_n^*)$ is the real linear subspace consisting of $f \in K(G, M_n^*)$ such that $f(x)$ is a self-adjoint matrix for all $x \in G$. The space $K(G, M_n^*)_{\text{sa}}$ is partially ordered by the cone

$$K(G, M_n^*)^+ = \{f \in K(G, M_n^*) : f(x) \in M_n^+ \ \forall x \in G\}.$$

A linear functional $\mu \in K(G, M_n^*)^*$ is called *positive* if $\mu(f) = Tr\left(\int_G f\, d\mu\right) \geqq 0$ for all $f \in K(G, M_n^*)^+$. As shown in [10], the positive linear functionals on $K(G, M_n^*)$ can be regarded as positive *extended* M_n-valued measures on G. If $\mu = (\mu_k) \in K(G, M_n^*)^*$ is positive and if σ is a positive M_n-valued measure on G such that μ and σ *have commuting ranges*, that is, $\mu_k(E)\sigma(F) = \sigma(F)\mu_k(E)$ for each k and $E, F \in \mathscr{B}$, then $\mu * \sigma$ is a positive functional on $K(G, M_n^*)$ (cf. Lemma 8).

Example 23. Let $h \in K(G, M_n^*)^+$ and λ be the Haar measure on G. We define $h.\lambda \in K(G, M_n^*)^*$ by $h.\lambda = (\mu_k) \in \varprojlim_k M(G_k, M_n)$ where

$$\mu_k(f) = Tr\left(\int_{G_k} fh\, d\lambda\right)$$

for $f \in K(G_k, M_n^*)$. Then $h.\lambda$ is a positive linear functional on $K(G, M_n^*)$ and for $\sigma \in M(G, M_n)$, we have $(h.\lambda) * \sigma = (h * \Delta_G^{-1}\sigma).\lambda$ since, for $f \in K(G, M_n^*)$,

$$\begin{aligned}
(h.\lambda) * \sigma(f) &= Tr\left(\int_G \left(\int_G f(xy)\, d(h.\lambda)(x)\right) d\sigma(y)\right) \\
&= Tr\left(\int_G \left(\int_{(\operatorname{supp} f)y^{-1}} f(xy)h(x)\, d\lambda(x)\right) d\sigma(y)\right) \\
&= Tr\left(\int_G \left(\int_{\operatorname{supp} f} f(z)h(zy^{-1})\, d\lambda(zy^{-1})\right) d\sigma(y)\right) \\
&= Tr\left(\int_{\operatorname{supp} f} f(z)\left(\int_G h(zy^{-1})\Delta_G(y^{-1})\, d\sigma(y)\right) d\lambda(z)\right).
\end{aligned}$$

Given an adapted positive M_n-valued measure on G, we will use Choquet's integral representation theory to describe the positive M_n-valued σ-harmonic functions on abelian groups. Extending Choquet and Deny's method [4], [14] for positive real harmonic functions on abelian groups, we will show that, given any adapted positive M_n-valued measure σ on an abelian group G, the positive continuous σ-harmonic M_n-valued functions on G, with range commuting with that of σ, are integrals of M_n-valued exponential functions. The results in this section improve considerably some results in [10] where σ is assumed to take values in the centre of a C*-algebra which is too restrictive in the setting of M_n.

Henceforth we fix an adapted positive M_n-valued measure σ on an abelian group G which is separable and metrizable. By [10], Lemma 5.3, the cone $K(G, M_n^*)_+^*$ of positive linear functionals on $K(G, M_n^*)$ is weak* complete.

Lemma 24. *The cone $K(G, M_n^*)_+^*$ is weak* metrizable.*

Proof. Since $K(G_k, M_n^*)$ is separable, the cone $K(G_k, M_n^*)^+$ is also separable. Let $\{f_{k,m}\}_{m=1}^{\infty}$ be dense in $K(G_k, M_n^*)^+$. By similar arguments as in [5], 12.10, one can show that $K(G, M_n^*)_+^*$ is homeomorphic to a subspace of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ via the homeomorphism $\mu \in K(G, M_n^*)_+^* \mapsto \tilde{\mu} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ where $\tilde{\mu}(k,m) = \mu(f_{k,m})$. □

Let $K_\sigma(G, M_n^*)_+^* = \{\mu \in K(G, M_n^*)_+^* : \mu$ and σ have commuting ranges$\}$. Then $K_\sigma(G, M_n^*)_+^*$ is also a weak* complete metrizable cone. Indeed, it is weak* closed in $K(G, M_n^*)_+^*$ which follows from the lemma below.

Lemma 25. *Let (μ_α) be a net of positive measures in $M(G_k, M_n) = K(G_k, M_n^*)^*$, weak* converging to $\mu \in M(G_k, M_n)$. If each μ_α and σ have commuting ranges, then μ and σ have commuting ranges.*

Proof. For $\varphi \in M_n^*$ and any real continuous function f on G_k, we have $\int_{G_k} f\, d(\varphi \circ \mu_\alpha) = \varphi\left(\int_{G_k} f\, d\mu_\alpha\right) = Tr\left(\int_{G_k} A_\varphi f\, d\mu_\alpha\right) \to Tr\left(\int_{G_k} A_\varphi f\, d\mu\right) = \int_{G_k} f\, d(\varphi \circ \mu)$, that is, the complex measures $(\varphi \circ \mu_\alpha)$ converge weakly to $\varphi \circ \mu$. Hence, given any $\xi \in \mathbb{C}^n$ and Borel set $E \subset G_k$, the net $(\psi \circ \mu_\alpha)$ converges weakly to $\psi \circ \mu$ where $\psi(\cdot) = \langle \cdot\, \sigma(E)\xi, \xi\rangle \in M_n^*$.

By commuting ranges, each $\psi \circ \mu_\alpha$ is positive which implies that $\psi \circ \mu$ is also positive. In particular, for any Borel set $F \subset G_k$, we have

$$\langle \mu(F)\sigma(E)\xi, \xi\rangle = (\psi \circ \mu)(F) \geqq 0.$$

Therefore $\mu(F)\sigma(E) \in M_n^+$ and so $\mu(F)\sigma(E) = \sigma(E)\mu(F)$. □

We let

$$C_\sigma = \{\mu \in K_\sigma(G, M_n^*)_+^* : \mu * \sigma \leqq \mu\},$$

$$H_\sigma = \{\mu \in K_\sigma(G, M_n^*)_+^* : \mu * \sigma = \mu\}$$

which are subcones of $K_\sigma(G, M_n^*)_+^*$.

Lemma 26. *The cone C_σ is weak* complete.*

Proof. Let $\sigma = \omega \cdot |\sigma|$ be the polar decomposition. It suffices to show that C_σ is weak* closed in $K(G, M_n^*)_+^*$. Let (μ_k) be a sequence in C_σ weak* converging to $\mu \in K(G, M_n^*)_+^*$. Let $f \in K(G, M_n^*)^+$. Then

$$(\mu * \sigma)(f) = Tr\left(\int_G \left(\int_G f(xy)\, d\mu(x)\right) d\sigma(y)\right)$$

$$= Tr\left(\int_G \left(\int_G f(xy)\, d\mu(x)\right) \omega(y)\, d|\sigma|(y)\right)$$

$$= \int_G Tr\left(\int_G f(xy)\, d\mu(x)\omega(y)\right) d|\sigma|(y)$$

$$= \int_G Tr\left(\omega(y) \int_G f(xy)\, d\mu(x)\right) d|\sigma|(y)$$

where, for fixed $y \in G$, we have

$$Tr\left(\omega(y) \int_G f(xy)\, d\mu(x)\right) = Tr\left(\int_G \omega(y) f(xy)\, d\mu(x)\right) = \lim_{k\to\infty} Tr\left(\int_G \omega(y) f(xy)\, d\mu_k(x)\right)$$

since $\omega(y) f(\cdot y) \in K(G, M_n^*)$. By Fatou's Lemma, we have

$$(\mu * \sigma)(f) \leqq \liminf_{k\to\infty} \int_G Tr\left(\int_G f(xy)\, d\mu_k(x)\omega(y)\right) d|\sigma|(y)$$

$$= \liminf_{k\to\infty} Tr\left(\int_G \left(\int_G f(xy)\, d\mu_k(x)\right) d\sigma(y)\right)$$

$$= \liminf_{k\to\infty} (\mu_k * \sigma)(f) \leqq \liminf_{k\to\infty} \mu_k(f) = \mu(f).$$

Therefore $\mu \in C_\sigma$ and C_σ is weak* closed. □

By [10], Proposition 5.5, the cone C_σ is well-capped which means that C_σ is a union of caps, where a cap of C_σ is a weak* compact convex subset C containing 0 such that $C_\sigma \backslash C$ is convex. We recall that $\nu \in C_\sigma$ is called *extremal* if whenever $\nu' \in C_\sigma$ satisfies $\nu - \nu' \in C_\sigma$, then $\nu' = \alpha\nu$ for some $\alpha \geqq 0$. We denote by ∂C_σ the set of extremal elements in C_σ. The extremal elements of any other cone are defined and denoted likewise.

Since C_σ is weak* complete and metrizable, by Choquet's representation theory (cf. [5], 36), every $\mu \in C_\sigma$ has an integral representation

$$\mu = \int_{\partial C_\sigma} \nu\, dP(\nu)$$

which means that $\mu(f) = \int_{\partial C_\sigma} v(f)\, dP(v)$ for all $f \in K(G, M_n^*)$, where P is a probability measure on ∂C_σ which is a Borel set.

We have $\partial H_\sigma = \partial C_\sigma \cap H_\sigma$ as in [10], Lemma 5.2. Since

$$H_\sigma = \bigcap_{m=1}^{\infty} \{\mu \in C_\sigma : (\mu * \sigma)(f_m) = \mu(f_m)\},$$

it is a Borel set, where $\{f_m\}$ is a countable dense subset of $K(G, M_n^*)$. For $\mu \in H_\sigma$ with representation $\mu = \int_{\partial C_\sigma} v\, dP(v)$, we have $P(\partial C_\sigma \backslash H_\sigma) = 0$ since

$$(\mu * \sigma)(f) = \int_{\partial C_\sigma} (v * \sigma)(f)\, dP(v) \leqq \int_{\partial C_\sigma} v(f)\, dP(v) = \mu(f)$$

for all $f \in K(G, M_n^*)$. Hence we have

$$\mu = \int_{\partial H_\sigma} v\, dP(v).$$

Therefore, to describe the cone H_σ, it suffices to describe ∂H_σ. We prove some lemmas first.

Lemma 27. *Let G be abelian, $\mu \in K(G, M_n^*)_+^*$ and $\sigma \in M(G, M_n)$ be positive. Then we have $(\mu * \sigma)(f) = \overline{(\sigma * \mu)(f)}$ for all $f \in K(G, M_n^*)^+$. In particular, if $\mu * \sigma$ is positive, then $\mu * \sigma = \sigma * \mu$.*

Proof. Let $\sigma = \omega \cdot |\sigma|$ be the polar decomposition. Then

$$\begin{aligned}(\mu * \sigma)(f) &= Tr\Big(\int_G\int_G f(xy)\, d\mu(x)\, d\sigma(y)\Big) = \int_G Tr\Big(\int_G f(xy)\, d\mu(x)\omega(y)\Big)\, d|\sigma|(y) \\ &= \int_G Tr\Big(\omega(y)\int_G f(xy)\, d\mu(x)\Big)\, d|\sigma|(y) = \int_G Tr\Big(\int_G \omega(y) f(xy)\, d\mu(x)\Big)\, d|\sigma|(y) \\ &= \overline{\int_G Tr\Big(\int_G f(xy)\omega(y)\, d\mu(x)\Big)\, d|\sigma|(y)} = \overline{Tr\Big(\int_G\int_G f(xy)\, d\sigma(y)\, d\mu(x)\Big)} \\ &= \overline{(\sigma * \mu)(f)}. \quad \square\end{aligned}$$

Given any measure $\sigma \in M(G, M_n)$ and $A \in M_n$, the measure $v \in M(G, M_n)$ defined by $v(\cdot) = A\sigma(\cdot)A$ satisfies

$$v(f) = Tr\Big(\int_G Af(x)A\, d\sigma(x)\Big)$$

for $f \in K(G, M_n^*)$. For $\mu \in K(G, M_n^*)^*$, the measure $A\mu(\cdot)A$ can be defined by the above

identity. For each $x \in G$, we will write δ_x, if no confusion is likely, for the measure $\delta_x(\cdot)I \in M(G, M_n)$:

$$\delta_x(E) = \begin{cases} I & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Given $\mu \in K(G, M_n^*)_+^*$, we have $(\mu * \delta_x)(f) = \mu(f_x)$ where f_x denotes the right-translation of f by x: $f_x(\cdot) = f(\cdot x)$. Therefore we have $(\mu * \delta_x) * \delta_y = \mu * \delta_{xy}$. We denote by $V_k(x)$ the open sphere in G, centred at x with radius $\frac{1}{k}$ for $k = 1, 2, \ldots$.

Since $\omega(x)$ may not be invertible in the next lemma, we consider instead the inverse of $\omega(x) + \varepsilon I$ for $\varepsilon > 0$.

Lemma 28. *Let G be any separable and metrizable group. Let $\sigma \in M(G, M_n)$ be positive with polar representation $\sigma = \omega \cdot |\sigma|$. Let $\varepsilon > 0$. Then, for $|\sigma|$-almost all $x \in \operatorname{supp}\sigma$, the sequence of measures $\{v_k^x\}_{k=1}^{\infty}$ in $M(G, M_n)$ defined by*

$$dv_k^x(y) = |\sigma|(V_k(x))^{-1}(\omega(x) + \varepsilon)^{-1/2}\, d\sigma_k^x(y)(\omega(x) + \varepsilon)^{-1/2}$$

weak-converges to the measure $(\omega(x) + \varepsilon)^{-1}\omega(x)\delta_x$, where σ_k^x is the restriction of σ to $V_k(x)$ and εI is shortened to ε.*

Proof. Let $\{f_m\}$ be a countable dense set in $C_0(G, M_n^*)$. Then for each f_m, we have, for $|\sigma|$-almost all $x \in \operatorname{supp}\sigma$,

$$\begin{aligned}v_k^x(f_m) &= Tr\Big(\int_G f_m\, dv_k^x\Big) \\ &= Tr\Big(\int_{V_k(x)} |\sigma|(V_k(x))^{-1}(\omega(x) + \varepsilon)^{-1/2} f_m(y)(\omega(x) + \varepsilon)^{-1/2}\, d\sigma(y)\Big) \\ &= |\sigma|(V_k(x))^{-1} \int_{V_k(x)} Tr\big((\omega(x) + \varepsilon)^{-1/2} f_m(y)(\omega(x) + \varepsilon)^{-1/2}\omega(y)\big)\, d|\sigma|(y) \\ &\to Tr\big((\omega(x) + \varepsilon)^{-1/2} f_m(x)(\omega(x) + \varepsilon)^{-1/2}\omega(x)\big) = (\omega(x) + \varepsilon)^{-1}\omega(x)\delta_x(f_m)\end{aligned}$$

as $k \to \infty$. Hence there is a Borel $|\sigma|$-null set $E \subset G$ such that for all m, $v_k^x(f_m) \to (\omega(x) + \varepsilon)^{-1}\omega(x)\delta_x(f_m)$ for all $x \in G \backslash E$. The density of $\{f_m\}$ concludes the proof. $\square$

Given $A \in M_n$ and $\mu \in K(G, M_n^*)^*$ with $\mu = (\mu_k)$ and $\mu_k \in M(G_k, M_n)$, we say that A *commutes with the range of* μ if A commutes with the range of each μ_k. We note that, for $A, B \in M_n$, we have $Tr(B^*AB) \leqq \|A\|\, Tr(B^*B)$ which is used in the following proof.

Lemma 29. *Let $\sigma \in M(G, M_n)$ be positive and let $\mu \in \partial H_\sigma$. Let $A \in M_n$ be positive and commute with the range of μ. Then $A\mu = \alpha\mu$ for some number $\alpha \geqq 0$.*

Proof. We have $0 \leqq A \leqq \|A\| I$ and we may assume $A \neq 0$. Let

$$\nu(\cdot) = A\mu = A^{1/2}\mu(\cdot)A^{1/2}.$$

Then, for $f \in K(G, M_n^*)^+$, we have

$$0 \leqq \nu(f) = Tr\left(\int_G A^{1/2} f A^{1/2}\, d\mu\right) = Tr\left(\left(\int_G f\, d\mu\right) A\right) \leqq \|A\|\, Tr\left(\int_G f\, d\mu\right) = \|A\|\mu(f),$$

where the last inequality follows by considering simple functions and the fact that A commutes with the range of μ. So $0 \leqq \nu \leqq \|A\|\mu$. We also have $\nu * \sigma = \nu$. It follows that ν is a scalar multiple of μ since μ is extremal in H_σ. □

We recall that the self-adjoint part M_n^{sa} of M_n is isometrically order-isomorphic to the partially ordered Banach space $A(S)$ of real continuous affine functions on the state space $S = \{\varphi \in M_n^* : \varphi(I) = \|\varphi\| = 1\}$ of M_n, via the evaluation map $A \in M_n^{sa} \mapsto \tilde{A} \in A(S)$ where $\tilde{A}(\varphi) = \varphi(A)$ for $\varphi \in S$. We also recall that a projection $p \in M_n$ is *minimal* if $pM_np = \mathbb{C}p$. We are now ready to characterize ∂H_σ. A function $g: G \to (0, \infty)$ is called *exponential* if $g(xy) = g(x)g(y)$.

Proposition 30. *Let G be a metrizable and separable abelian group and let σ be an adapted positive M_n-valued measure on G. Let $\mu \in K_\sigma(G, M_n^*)_+^*$. Then the following conditions are equivalent:*

(i) $\mu \in \partial H_\sigma$;

(ii) $\mu = cgp.\lambda$ *where* $c \geqq 0$, $p \in M_n$ *is a minimal projection and* $g: G \to (0, \infty)$ *is a continuous exponential function such that* $\left(\int_G g(x^{-1})\, d\sigma(x)\right) p = p$.

Proof. (i) ⇒ (ii). Let $\sigma = \omega \cdot |\sigma|$ be the polar representation and let $\mu \in \partial H_\sigma \backslash \{0\}$. Since σ and μ have commuting ranges, by Lemma 29, we have $\omega(x)\mu = \tilde{\omega}(x)\mu$ for some number $\tilde{\omega}(x) \geqq 0$. Let $\tilde{\sigma} = \tilde{\omega} \cdot |\sigma|$. Then $\operatorname{supp}\tilde{\sigma} \subset \operatorname{supp}|\sigma|$. Since $\mu = \mu * \sigma$, we can find a compact set $F \subset G$ such that $\mu(F)\sigma \neq 0$ and $\langle \mu(F)\xi, \xi\rangle > 0$ for some $\xi \in \mathbb{C}^n$. We note that $\mu(F)\sigma = \mu(F)\tilde{\sigma}$. Let $\rho(\cdot) = \langle \cdot\, \mu(F)^{1/2}\xi, \mu(F)^{1/2}\xi\rangle$. Then $\operatorname{supp} \rho \circ \sigma = \operatorname{supp}\tilde{\sigma}$ since for any open set $V \subset G$, we have $(\rho \circ \sigma)(V) = \langle \mu(F)\sigma(V)\xi, \xi\rangle = \langle \mu(F)\tilde{\sigma}(V)\xi, \xi\rangle = \tilde{\sigma}(V)\langle \mu(F)\xi, \xi\rangle$. Therefore $\tilde{\sigma}$ is adapted. Let

$$U = \{x \in \operatorname{supp}\tilde{\sigma} : \omega(x)\mu \neq 0\} = \{x \in \operatorname{supp}\tilde{\sigma} : \tilde{\omega}(x) \neq 0\}.$$

Then $\tilde{\sigma}(G\backslash\overline{U}) \leqq \tilde{\sigma}(G\backslash U) = 0$ implies $\operatorname{supp}\tilde{\sigma} \subset \overline{U}$ and it follows that U and hence $\{x \in \operatorname{supp}\sigma : \omega(x)\mu \neq 0\}$ generates a dense subgroup of G. By considering the latter set, we may therefore, without loss of generality, assume that $\omega(x)\mu \neq 0$ for $x \in \operatorname{supp}\sigma$.

For $x \in \operatorname{supp}\sigma$ and $m, k = 1, 2, \ldots$, we let

$$d\nu_k^{x,m}(\cdot) = |\sigma|(V_k(x))^{-1}\left(\omega(x) + \frac{1}{m}\right)^{-1/2} d\sigma_k^x(\cdot)\left(\omega(x) + \frac{1}{m}\right)^{-1/2}$$

be the measure defined in Lemma 28.

We have $0 \leqq \sigma_k^x \leqq \sigma$. Since the range of σ commutes with that of μ, the range of σ_k^x also does so and it follows that $0 \leqq \mu * \sigma_k^x \leqq \mu * \sigma = \mu$. By Lemma 27, we have $(\mu * \sigma_k^x) * \sigma = \sigma * (\mu * \sigma_k^x) = (\sigma * \mu) * \sigma_k^x = \mu * \sigma_k^x$. So $\mu * \sigma_k^x \in H_\sigma$. By extremality of μ, we have $\mu * \sigma_k^x = \alpha_k \mu$ for some $\alpha_k \geqq 0$. By commuting ranges again, we have

$$\mu * \nu_k^{x,m} = |\sigma|(V_k(x))^{-1}\left(\omega(x) + \frac{1}{m}\right)^{-1/2}(\mu * \sigma_k^x)\left(\omega(x) + \frac{1}{m}\right)^{-1/2}$$

$$= \alpha_k|\sigma|(V_k(x))^{-1}\left(\omega(x) + \frac{1}{m}\right)^{-1}\mu.$$

Hence $\mu * \nu_k^{x,m} = \beta_k^{x,m}\mu$ for some $\beta_k^{x,m} \geqq 0$, by Lemma 29. By Lemma 28, $\mu * \nu_k^{x,m}$ weak*-converges to $\mu * \left(\omega(x) + \frac{1}{m}\right)^{-1}\omega(x)\delta_x$ for $|\sigma|$-almost all $x \in \operatorname{supp}\sigma$. So $g_m(x) = \lim_{k\to\infty}\beta_k^{x,m}$ exists and we have $\mu * \left(\omega(x) + \frac{1}{m}\right)^{-1}\omega(x)\delta_x = g_m(x)\mu$ for $|\sigma|$-almost all $x \in \operatorname{supp}\sigma$. Hence there is a Borel set $E \subset \operatorname{supp}\sigma$ with $|\sigma|(G\backslash E) = 0$ such that for all m and $x \in E$,

$$\mu * \left(\omega(x) + \frac{1}{m}\right)^{-1}\omega(x)\delta_x = g_m(x)\mu.$$

It is well-known that $p_x = \lim_{m\to\infty}\left(\omega(x) + \frac{1}{m}\right)^{-1}\omega(x)$ is the range projection of $\omega(x)$. We therefore have

$$\mu * p_x\delta_x = g(x)\mu$$

for all $x \in E$, where $g(x) = \lim_{m\to\infty} g_m(x)$ and $\mu * p_x\delta_x = p_x\mu * \delta_x$. We note that $g(x) \neq 0$ for otherwise the above would imply that $\omega(x)\mu = \omega(x)p_x\mu = 0$. We have

$$g(x)p_x\mu = p_x(g(x)\mu) = p_x(\mu * p_x\delta_x) = \mu * p_x\delta_x = g(x)\mu$$

which gives $p_x\mu = \mu$ and hence $\mu * \delta_x = g(x)\mu$. It follows that, for any $y \in E$ with $xy \in E$, we have $g(xy)\mu = \mu * \delta_{xy} = (\mu * \delta_x) * \delta_y = g(x)g(y)\mu$ which gives $g(xy) = g(x)g(y)$. Since $|\sigma|(G\backslash\overline{E}) = 0$, we have $\operatorname{supp}\sigma \subset \overline{E}$ and therefore E generates a dense subgroup of G. So we can extend g to a continuous function, still denoted by g, on G such that $g(xy) = g(x)g(y)$ and $\mu * \delta_x = g(x)\mu$ for all $x, y \in G$. Define $\nu \in K(G, M_n^*)_+^*$ by

$$d\nu(y) = g(y^{-1})\, d\mu(y).$$

Then $d\nu_x(y) =: d\nu(yx) = g(x^{-1}y^{-1})\, d\mu(yx) = d\nu(y)$.

For every state φ of M_n, we have a translation invariant scalar measure

$$\varphi\nu_x = \varphi\nu$$

which implies that $\varphi\nu = a(\varphi)\lambda$ for some $a(\varphi) \geqq 0$. It is evident that $a(\varphi)$ is a positive continuous affine function of the states φ of M_n, and therefore it identifies uniquely with a positive self-adjoint element $A \in M_n$.

We have thus established that

$$g(y^{-1})\,d\mu(y) = d\nu(y) = A\,d\lambda(y)$$

which gives $d\mu(y) = g(y)\,d\nu(y) = g(y)A\,d\lambda(y)$. Since A commutes with the range of μ, Lemma 29 implies that $A\mu = \alpha\mu$ for some $\alpha > 0$ which gives

$$\alpha\,d\mu(y) = g(y)A^2\,d\lambda(y) = \alpha g(y)A\,d\lambda(y).$$

In particular, $A^2 = \alpha A$, that is, $\frac{1}{\alpha}A$ is a projection. We show further that A is in fact a scalar multiple of a minimal projection. By [10], Proposition 4.2, this is equivalent to showing that A is an extremal element in the cone M_n^+. Let $b \in M_n^+$ and $b \leqq A$. By [10], Lemma 4.1, we have $\alpha b = bA = Ab$ and so $b\,d\mu(y) = d\mu(y)b$. By Lemma 29, $b\mu = \beta\mu$ for some $\beta \geqq 0$ which gives

$$bg(y)A\,d\lambda(y) = b\,d\mu(y) = \beta g(y)A\,d\lambda(y)$$

and hence $\alpha b = bA = \beta A$, showing that A is extremal and so $A = cp$ for some minimal projection $p \in M_n$ and $c > 0$. Therefore we have

$$d\mu(y) = cg(y)p\,d\lambda(y).$$

Finally, we show that $\left(\int_G g(y^{-1})\,d\sigma(y)\right)p = p$. We first note that p commutes with the range of σ because μ does. Let $f \in K(G, M_n^*)$. Then

$$\begin{aligned}\mu(f) = (\mu * \sigma)(f) &= Tr\big(\textstyle\iint cf(xy)g(x)p\,d\lambda(x)\,d\sigma(y)\big)\\ &= Tr\big(\textstyle\iint cf(z)g(z)g(y^{-1})p\,d\lambda(z)\,d\sigma(y)\big)\\ &= Tr\Big(\big(\textstyle\int f(z)\,d\mu(z)\big)\big(\textstyle\int g(y^{-1})p\,d\sigma(y)\big)\Big)\end{aligned}$$

which gives $d\mu = \left(\int_G g(y^{-1})\,d\sigma(y)\right)p\,d\mu$, that is,

$$cg(y)p\,d\lambda(y) = \big(\textstyle\int g(y^{-1})p\,d\sigma(y)\big)cg(y)p\,d\lambda(y),$$

giving $p = \left(\int_G g(y^{-1})\,d\sigma(y)\right)p$.

(ii) $\Rightarrow$ (i) Let $d\mu = cgp.d\lambda$ be as given. We show that $\mu \in \partial H_\sigma$. Let $\nu \in H_\sigma$ be such that $0 \leqq \nu(\cdot) \leqq \mu(\cdot)$. We show that ν is a scalar multiple of μ.

Define $d\tilde{\nu}(x) = g(x^{-1})\,d\nu(x)$ and $d\tilde{\sigma}(x) = g(x^{-1})\,d\sigma(x)$. Then it is straightforward to verify that $\tilde{\nu} * \tilde{\sigma} = \tilde{\nu}$ as g is exponential.

Consider $\nu = (\nu_k) \in \varprojlim_k K(G_k, M_n^*)^*$. We have, as functionals,

$$\nu_k(\cdot) \leqq \mu(\cdot)|_{K(G_k, M_n^*)} = cg(x)p\,d\lambda(x)|_{K(G_k, M_n^*)}.$$

So, by Lemma 6, for every Borel set $E \subset G_k$, we have

$$0 \leqq \nu_k(E) \leqq \left(\int_E cg(x)\,d\lambda(x)\right)p$$

and therefore, by [10], Lemma 4.1, $\nu_k(E) = \nu_k(E)p = p\nu_k(E)$ which is a nonnegative scalar multiple of p since p is a minimal projection. It follows that $\nu_k(\cdot) = \nu_k'(\cdot)p$ where ν_k' is a positive extended real-valued measure on G. Hence $\nu(\cdot) = \nu'(\cdot)p$ and ν' is a positive extended real-valued measure on G.

Let φ be a pure state of M_n supported by p, that is, $\varphi(p) = 1$. Then $\varphi(A) = \varphi(Ap)$ for all $A \in M_n$.

We have $\varphi\nu = \nu'$ and $d\varphi\tilde{\nu}(x) = g(x^{-1})\,d\nu'(x)$. We next show that $\varphi\tilde{\nu}$ is translation invariant. Let $x \in \tilde{G}$ and let $\varphi\tilde{\nu}_x$ be the translation of $\varphi\tilde{\nu}$ by x. Fix an arbitrary $f \in K(G, \mathbb{C})$ and define $F\colon G \to \mathbb{C}$ by

$$F(x) = \int_G f\,d(\varphi\tilde{\nu}_x) = \int_G f(z)g(x^{-1}z^{-1})\,d\nu'(zx).$$

Then we have

$$\begin{aligned}F * \varphi\tilde{\sigma}(x) &= \int_G F(xy^{-1})\,d\varphi\tilde{\sigma}(y)\\ &= \varphi\left(\int_G F(xy^{-1})\,d\tilde{\sigma}(y)\right)\\ &= \varphi\left(\int_G F(xy^{-1})\,d\tilde{\sigma}(y)p\right)\\ &= \varphi\left(\int_G \left(\int_G f(zyx^{-1})g(z^{-1})\,d\nu'(z)\right)p\,d\tilde{\sigma}(y)\right)\\ &= \varphi\left(\int_G \left(\int_G f(zyx^{-1})\,d\tilde{\nu}(z)\right)d\tilde{\sigma}(y)\right)\\ &= \varphi\left(\int_G f_{x^{-1}}\,d(\tilde{\nu} * \tilde{\sigma})\right)\\ &= \varphi\left(\int_G f_{x^{-1}}\,d\tilde{\nu}\right) = \int_G f\,d\varphi\tilde{\nu}_x = F(x).\end{aligned}$$

Therefore the function F is $\varphi\tilde{\sigma}$-harmonic where

$$\varphi\tilde{\sigma}(G) = \varphi(\tilde{\sigma}(G)p) = \varphi\left(\int_G g(y^{-1})\,d\sigma(y)p\right) = \varphi(p) = 1$$

and $\varphi\tilde{\sigma}$ is adapted. Moreover F is bounded since, given $v \leqq cgp.\lambda$ and $\operatorname{supp} f \subset G_k$, we have

$$|F(x)| = \left|\int_{G_k} f(z)g(x^{-1}z^{-1})\,d\varphi v(zx)\right| \leqq c\int_{G_k} |f(z)|\,d\lambda(zx) \leqq c\|f\|\lambda(G_k).$$

Hence, by the Choquet-Deny Theorem [4], F is constant and in particular,

$$\int_G f\,d\varphi\tilde{v}_x = F(x) = F(e) = \int_G f\,d\varphi\tilde{v}.$$

As $f \in K(G, \mathbb{C})$ was arbitrary, we have $\varphi\tilde{v}_x = \varphi\tilde{v}$. Hence invariance gives $\varphi\tilde{v} = \beta\lambda$ for some $\beta \geqq 0$, and

$$dv'(x) = d(\varphi v)(x) = g(x)\,d\varphi\tilde{v}(x) = \beta g(x)\,d\lambda(x).$$

Therefore we have $dv(x) = dv'(x)p = \beta g(x)p\,d\lambda(x)$ which is a scalar multiple of $d\mu(x)$. This proves that $\mu \in \partial H_\sigma$. □

Now let $f\colon G \to M_n$ be a positive σ-harmonic function with range commuting with that of σ. The measure $\mu = f.\lambda \in K_\sigma(G, M_n^*)_+^*$ satisfies $\mu * \sigma = \mu$ (cf. Example 23) and therefore $\mu = \int_{\partial H_\sigma} v\,dP(v)$ for some probability measure P on ∂H_σ. Using Proposition 30, we can now describe f as follows.

Theorem 31. *Let G be a metrizable and separable abelian group and let σ be an adapted positive M_n-valued measure on G. Let $f\colon G \to M_n$ be a positive σ-harmonic function with range commuting with that of σ. Then there exists a probability measure P on*

$$\mathscr{E} = \left\{cgp : c \geqq 0, \left(\int_G g(y^{-1})\,d\sigma(y)\right)p = p\right\}$$

where p is a minimal projection in M_n and $g\colon G \to (0, \infty)$ is continuous and exponential, such that

$$f(x) = \int_{\mathscr{E}} h(x)\,dP(h) \quad (\lambda\text{-a.e.}).$$

Note added in proof. A Liouville theorem for matrix-valued harmonic functions on nilpotent groups has been proved recently by the author. It has also been shown that a normal contractive projection on a JBW*-triple preserves types, in a recent paper entitled "*Normal contractive projections preserve types*" by C.-H. Chu, M. Neal and B. Russo.

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Surgery and the spectrum of the Dirac operator

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Abstract. We show that for generic Riemannian metrics on a simply-connected closed spin manifold of dimension $\geqq 5$ the dimension of the space of harmonic spinors is not larger than it must be by the index theorem. The same result holds for periodic fundamental groups of odd order.

The proof is based on a surgery theorem for the Dirac spectrum which says that if one performs surgery of codimension $\geqq 3$ on a closed Riemannian spin manifold, then the Dirac spectrum changes arbitrarily little provided the metric on the manifold after surgery is chosen properly.

0. Introduction

Classical Hodge-deRham theory establishes a tight link between the analysis of the Laplace operator acting on differential forms of a compact Riemannian manifold and its topology. Specifically, the dimension of the space of harmonic k-forms is a topological invariant, the k^{th} Betti number.

The question arises whether a similar relation holds for other elliptic geometric differential operators such as the Dirac operator on a compact Riemannian spin manifold. It is not hard to see that the dimension h_g of the space of harmonic spinors is a conformal invariant, it does not change when one replaces the Riemannian metric g by a conformally equivalent one ([10], Prop. 1.3). Moreover, the Atiyah-Singer index theorem implies a topological lower bound on h_g.

Berger metrics on spheres of dimension $4k+3$ provide examples showing that in general h_g depends on the metric and is not topological, see [10], Prop. 3.2 and [3], Thm. 3.1. Also for surfaces of genus at least 3 the number h_g varies with the choice of metric ([10], Thm. 2.6). All known examples indicate that the following two conjectures should be true. On the one hand, we should have

Conjecture A. *Harmonic spinors are not topologically obstructed, i.e. on any compact spin manifold of dimension at least three there is a metric g such that $h_g > 0$.*