Normal contractive projections preserve type

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Abstract

Given a JBW*-triple Z and a normal contractive projection $P : Z \longrightarrow Z$, we show that the (Murray-von Neumann) type of each summand of $P(Z)$ is dominated by the type of Z.

Introduction

Contractive projections play a useful role in the theory of operator algebras and Banach spaces. The ranges of contractive projections on C*-algebras form an important subclass of those complex Banach spaces whose open unit balls are bounded symmetric domains. An important feature of these spaces is that they are equipped with a Jordan *triple* product, induced by the Lie algebra of the automorphism group of the open unit ball. Known as JB^*-triples, they have been shown to be the appropriate category in which to study contractive projections; indeed the fact that the category of JB^*-triples is stable under contractive projections played a key role in their structure theory.

Recently, contractive projections on von Neumann algebras have arisen in the study of operator spaces as well as the theory of harmonic functions on locally compact groups. In [22], a family of Hilbertian operator spaces were studied and used to classify, in an appropriate sense, the ranges of contractive projections on $B(H)$ which are atomic as Banach spaces. In [5], it was shown that the Banach space of bounded matrix-valued harmonic functions on a locally compact group is the range of a contractive projection

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on a type I finite von Neumann algebra. It has also been shown in [6] that the Banach space of harmonic functionals on the Fourier algebra of a locally compact group G is the range of a contractive projection on the group von Neumann algebra $VN(G)$.

There is a Murray-von Neumann type classification for JBW^*-triples, that is, JB^*-triples which are the dual of a Banach space. In view of the fact, noted above, that the range of a contractive projection on a JB^*-triple is again a JB^*-triple [12, 21, 27], the above investigations point to a natural and important question, namely, how is the Murray-von Neumann classification of the domain affected by a contractive projection? More precisely, given a JBW^*-triple Z of type X, where $X = I, II$, or III, is the range of a normal contractive projection on Z of type Y with $Y \leq X$, meaning each summand of the range is of type $\leq X$? In this paper, we answer this question affirmatively. We shall see that it suffices to prove this for JW^*-triples, that is, for JBW^*-triples which are linearly isometric to a weak operator closed subspace of $B(H)$, stable for the triple product $xy^*z + zy^*x$, where $B(H)$ is the von Neumann algebra of bounded operators on a Hilbert space H.

Tomiyama [31] has analysed the type structure of the range of a contractive projection which is a von Neumann subalgebra of the domain. His arguments depend on the crucial fact that the range is a *subalgebra*. In our investigation, the range, which automatically has an algebraic structure, need not be a subalgebra nor even a *subtriple*. This adds both generality and complexity to our question.

This paper is organized as follows. Section 1 is devoted to background and motivation for the problem. In section 2 we consider, as a preliminary tool, contractive projections on JW^*-algebras. Propositions 2.5 and 2.6 show that if the image of a normal contractive projection on a JW^*-algebra is a JW^*-subalgebra (not necessarily with the same identity), then the properties of being semifinite or of type I are passed on from the domain to the image. In section 3 we study normal contractive projections on a von Neumann algebra of type I and show in Proposition 3.5 that the image is isometric to a JW^*-triple of type I. It is necessary first to prove this (in Proposition 3.1) in the special case when the projection is the Peirce 2-projection with respect to a partial isometry. Our main results, that normal contractive projections on JW^*-triples preserve both type I and semifiniteness, appear in section 4 as Theorems 4.2 and 4.4. Again, Propositions 4.1 and 4.3 deal with the special case of a Peirce 2-projection. Although Propositions 2.5 through 4.3 are each a special case of Theorem 4.2 or 4.4, they are essential steps in the proofs of these theorems and they are new and of interest. In section 5, we extend Theorems 4.2 and 4.4 to arbitrary JBW^*-triples, and consider the case of atomic JBW^*-triples.

1 Motivation and Background

Let M be a von Neumann algebra and let N be a von Neumann subalgebra of M containing the identity element of M. A positive linear map $E : M \to N$ satisfying $Ex = x$ for $x \in N$ and $E(axb) = aE(x)b$ for $x \in M$ and $a, b \in N$ is called a *conditional expectation*. Conditional expectations have played some fundamental roles in the theory of von Neumann algebras, for instance in V. Jones' theory of subfactors. Work in the 1950s of Tomiyama and Nakamura-Takesaki-Umegaki established that conditional expectations are idempotent, contractive, and completely positive mappings, and they preserve type when normal; see the survey paper of Stormer [29]. Conversely ([19, 10.5.85]), a unital contractive projection from one C^*-algebra onto a unital C^*-subalgebra extends to a normal conditional expectation on the universal enveloping von Neumann algebra, and is in particular a conditional expectation on the C^*-algebra.

A type theory for weakly closed Jordan operator algebras, based on modularity of the lattice of projections, and parallel to the type classification theory for von Neumann algebras, was introduced and developed in the 1960s by Topping [32] and Stormer [28]. In particular, Stormer showed that a JW-algebra is of type I if and only if its enveloping von Neumann algebra is of type I. This was extended to types II and III by Ayupov in 1982 [1]. In some cases the JW-algebra in these results is required to be reversible.

A special case of a result of Choi-Effros in 1977 [3], of fundamental importance in the rapidly advancing theory of operator spaces, states that the range of a unital completely positive projection on a C^*-algebra, while not in general a subalgebra, nevertheless carries the structure of a C^*-algebra. The proof hinges on a conditional expectation formula (needed to prove that the abstract product is associative) which is established using the Kadison-Schwarz inequality for positive linear maps. We note such a projection is completely contractive.

A special case of a result of Effros-Stormer in 1979 [8] states that the range of a unital positive projection on a C^*-algebra, while not in general a Jordan subalgebra, carries a natural Jordan algebra structure. As before, the proof depends on a conditional expectation formula (needed to prove that the abstract product satisfies the Jordan identity), and such a projection is contractive.

The above results raised the question of what algebraic structure existed in the range of an arbitrary contractive projection on a C^*-algebra. A special case of a result of Friedman and Russo in 1983 states that the range of such a projection is linearly isometric to a subspace, closed under the triple product $xy^*z + zy^*x$, of the second dual of the C^*-algebra. Because of the lack of an order structure and hence the unavailability of the Kadison-Schwarz inequality, new techniques were needed and developed by Friedman-Russo in their theory of "operator algebras without order" ([10]), including some conditional expectation formulas for the triple product ([11, Corollary 1]).

During the 1980s, the theory of JB^*-triples was developed extensively; for a summary, see the survey [24]. In particular, a type I theory was developed for JBW^*-triples by Horn in his thesis in 1984. In this theory, idempotents (projections) were replaced by tripotents (which are abstraction of partial isometries), and the reduced algebra pAp was replaced by the Peirce 2-space of a tripotent. Of special importance here is the algebraic fact that such a Peirce 2-space has an abstract structure of a Jordan algebra, and moreover Horn has proved that a JBW^*-triple is of type I if, and only if, it contains a complete tripotent whose Peirce 2-space is a Jordan algebra of type I. The remarkable structure theorem of Horn states that type I JBW^*-triples are isometric to direct sums of tensor products of a commutative von Neumann algebra by a Cartan factor.

We now recall some definitions. A *Jordan triple system* is a complex vector space V with a *Jordan triple product* $\{\cdot,\cdot,\cdot\} : V \times V \times V \longrightarrow V$ which is symmetric and linear in the outer variables, conjugate linear in the middle variable and satisfies the Jordan triple identity

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

A complex Banach space Z is called a *JB^*-triple* if it is a Jordan triple system such that for each $z \in Z$, the linear map

$$z\square z : v \in Z \mapsto \{z,z,v\} \in Z$$

is Hermitian, that is, $\|e^{it(z\square z)}\| = 1$ for all $t \in \mathbb{R}$, with non-negative spectrum and $\|z\square z\| = \|z\|^2$. A JB^*-triple Z is called a *JBW^*-triple* if it is a dual Banach space, in which case its predual is unique, denoted by Z_*, and the triple product is separately weak* continuous. The second dual Z^{**} of a JB^*-triple is a JBW^*-triple. A norm-closed subspace of a JB*-triple is called a *subtriple* if it is closed with respect to the triple product. A JBW*-triple is called a *JW*-triple* if it can be embedded as a subtriple of some $B(H)$.

The JB^*-triples form a large class of Banach spaces which include C^*-algebras, Hilbert spaces and spaces of rectangular matrices. The triple product in a C*-algebra $\mathcal{A}$ is given by

$$\{x, y, z\} = \frac{1}{2}\,(xy^*z + zy^*x).$$

In fact, $\mathcal{A}$ is a Jordan algebra in the product

$$x \circ y = \frac{1}{2}(xy + yx)$$

and we have $\{x, y, z\} = (x \circ y^*) \circ z + (y^* \circ z) \circ x - (z \circ x) \circ y^*$. A norm-closed subspace of a C^*-algebra is called a JC*-*algebra* if it is also closed with respect to the involution $*$ and the Jordan product $\circ$ given above. A JC^*-algebra is called a *JW^*-algebra* if it is a dual Banach space.

An element e in a JB*-triple Z is called a *tripotent* if $\{e, e, e\} = e$ in which case the map $e\square e : Z \longrightarrow Z$ has eigenvalues 0, $\frac{1}{2}$ and 1, and we have the following decomposition in terms of eigenspaces

$$Z = Z_2(e) \oplus Z_1(e) \oplus Z_0(e)$$

which is called the *Peirce decomposition* of Z. The $\frac{k}{2}$-eigenspace $Z_k(e)$ is called the *Peirce k-space*. The *Peirce projections* from Z onto the Peirce k-spaces are given by

$$P_2(e) = Q^2(e), \qquad P_1(e) = 2(e\square e - Q^2(e)), \qquad P_0(e) = I - 2e\square e + Q^2(e)$$

where $Q(e)z = \{e, z, e\}$ for $z \in Z$. The Peirce projections are contractive.

In later computation, we will use frequently the *Peirce rules*

$$\{Z_i(e)\, Z_j(e)\, Z_k(e)\} \subset Z_{i-j+k}(e)$$

where $Z_l(e) = \{0\}$ for $l \neq 0, 1, 2$. We note that the Peirce 2-space $Z_2(e) = P_2(e)(Z)$ is a Jordan Banach algebra with identity e, the Jordan product $a \circ b = \{a, e, b\}$ and involution $a^\# = \{e, a, e\}$ which satisfy

$$\|a^\#\| = \|a\|; \qquad \|\{a, a^\#, a\}\| = \|a\|^3$$

where $\{x, y, z\} = (x \circ y^*) \circ z + (y^* \circ z) \circ x - (z \circ x) \circ y^*$, in other words, $Z_2(e)$ is a unital *JB*-algebra*. A JB*-algebra having a predual is called a *JBW*-algebra*. As shown in [33], the self-adjoint parts of JB*-algebras (resp JBW*-algebras) are exactly the *JB-algebras* (resp *JBW-algebras*). For definitions and basic results about JB-algebras, we refer the reader to [15]. If $Z = Z_2(e)$, then e is called *unitary*. If $Z_0(e) = \{0\}$, then the tripotent e is called *complete*. Two tripotents u and v are said to be *orthogonal* if $u \square v = 0$. The elements of the predual Z_* of a JBW*-triple Z are exactly the *normal* functionals on Z, that is, the continuous linear functionals on Z which are additive on orthogonal tripotents.

Given an orthogonal family of tripotents $\{e_i\}_{i\in\Lambda}$ in a JB*-triple Z, we can form a *joint Peirce decomposition*

$$Z = \bigoplus_{i,j\in\Lambda} Z_{ij}$$

where Peirce spaces Z_{ij} are defined by

$$Z_{ii} = Z_2(e_i)\,, \quad Z_{ij} = Z_1(e_i) \cap Z_1(e_j) \quad (i \neq j)$$

$$Z_{i0} = Z_1(e_i) \cap \bigcap_{j\neq i} Z_0(e_j)\,, \quad Z_{00} = \bigcap_i Z_0(e_i).$$

We have, for $z_{ij} \in Z_{ij}$ and $e = \sum e_i$,

$$(e_k \square e)(z_{ij}) = (e_k \square e_k)(z_{ij}) = \begin{cases} 0 & \text{if } k \notin \{i, j\} \\ \frac{1}{2} z_{ij} & \text{if } k \in \{i, j\}. \end{cases}$$

JBW*-triples have an abundance of tripotents. In fact, given a JBW*-triple Z and f in the predual Z_*, there is a unique tripotent $v_f \in Z$, called the *support tripotent* of f, such that $f \circ P_2(v_f) = f$ and the restriction $f|_{Z_2(v_f)}$ is a *faithful positive* normal functional.

The Murray-von Neumann classification of the von Neumann algebras can be extended to that of JBW*-triples and, a JBW*-triple can be decomposed into a direct sum of type j ($j = I, II, III$) summands (see [16, 18]). A JBW*-triple is called *continuous* if it does not contain a type I summand in which case, it is a direct sum of a JW*-algebra $H(A, \alpha)$ and a weak* closed right ideal of a continuous von Neumann algebra, as shown in [18], where

$$H(A, \alpha) = \{a \in A : \alpha(a) = a\}$$

is the fixed-point set of a period 2 weak* continuous antiautomorphism α of a von Neumann algebra A. It follows that continuous JBW*-triples are JW*-triples.

A JBW*-triple Z is called *type* I if it contains an abelian tripotent e such that $Z = U(e)$ where $U(e)$ denotes the weak* closed triple ideal generated by e. We recall that a tripotent e is said to be *abelian* if the Peirce 2-space $P_2(e)(Z)$ is an abelian triple which is equivalent to saying that $P_2(e)(Z)$ is an associative JBW*-algebra in the usual Jordan product $x \circ y = \{x, e, y\}$. Horn [17, 4.14] has shown that a JBW*-triple is type I if, and only if, every weak*-closed triple ideal contains an abelian tripotent.

An important class of type I JBW*-triples are the following six types of *Cartan factors*:

type 1 $B(H, K)$ *with triple product* $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$,
type 2 $\{z \in B(H, H) : z^t = -z\}$,
type 3 $\{z \in B(H, H) : z^t = z\}$,
type 4 spin factor,
type 5 $M_{1,2}(\mathcal{O})$ *with triple product* $\{x, y, z\} = \frac{1}{2}(x(y^*z) + z(y^*x))$,
type 6 $M_3(\mathcal{O})$

where $B(H, K)$ is the Banach space of bounded linear operators between complex Hilbert spaces H and K, and z^t is the transpose of z induced by a conjugation on H. Cartan factors of type 2 and 3 are subtriples of $B(H, H)$, the latter notation is shortened to $B(H)$. The type 3 and 4 are Jordan algebras with the usual Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. A *spin factor* is a Banach space that is equipped with a complete inner product $\langle \cdot, \cdot \rangle$ and a conjugation j on the resulting Hilbert space, with triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x - \langle x, jz\rangle jy)$$

such that the given norm and the Hilbert space norm are equivalent.

By Horn's result in [16], a JBW*-triple Z is of type I if, and only if, it is linearly isometric to an ℓ^∞-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha) \otimes C_\alpha$ where C_α is a Cartan factor. Such a type I JBW*-triple will be called *type* I_{fin} if each Cartan factor C_α is finite-dimensional. It has been shown in [7] that a JBW*-triple Z is type I_{fin} if, and only if, its predual Z_* has the Dunford-Pettis property. We recall that a Banach space W has the *Dunford-Pettis property* if every weakly compact operator on W is completely continuous. Such property is inherited by complemented subspaces.

Horn's type I structural result above also shows that a JBW*-algebra is type I as a JBW*-triple if and only if its self-adjoint part is a type I JBW-algebra in the sense of [15].

Lemma 1.1 *Let Z be a JBW*-subtriple of a type I_{fin} JBW*-triple. Then Z is type I_{fin}.*

Proof. By [7, Corollary 6], Z_* has the Dunford-Pettis property. □

We will begin our investigation of contractive projections in the next section. A contractive projection $P : Z \longrightarrow Z$ on a JB*-triple Z is a bounded linear map such that $P^2 = P$ and $\|P\| \leq 1$. We will exclude the trivial case of $P = 0$ which then implies $\|P\| = 1$. Given such a contractive projection P on Z with triple product $\{\cdot, \cdot, \cdot,\}$, one can show, using the holomorphic characterization of JB*-triples [21, 27], that the range $P(Z)$ is also a JB*-triple in the triple product

$$[x, y, z] = P\{x, y, z\} \qquad (x, y, z \in P(Z)).$$

Moreover, one has the following conditional expectation formula:

$$P\{Px, Py, Pz\} = P\{Px, y, Pz\} \qquad (x, y, z \in Z).$$

The above result has also been proved in [12] for subtriples of C*-algebras, via an operator algebra approach which also yields the formula

$$P\{Px, Py, Pz\} = P\{x, Py, Pz\}.$$

A weak* continuous projection on a JBW*-triple is called *normal*.

2 Contractive projections on JW^*-algebras

In this section, we consider a JW*-algebra $A \subset B(H)$ with positive part A^+, inheriting various topologies of $B(H)$. A positive linear functional φ

of A is called a *trace* if $\varphi(sxs) = \varphi(x)$ for all symmetries $s \in A$ and all $x \in A^+$, where a *symmetry* in A is a self-adjoint element s such that s^2 is the identity in A. By [1], every normal trace on A can be extended to a normal trace on its enveloping von Neumann algebra. Further, if φ is faithful, so is its extension. In the sequel, our JW^*-subalgebras need not have the same identity element as the JW^*-algebras which contain them.

The following lemma is a special case of Lemma 1.1, but the proof below is *intrinsic* without using the Dunford-Pettis property.

Lemma 2.1 *Every JW*-subalgebra of a type I_{fin} JW*-algebra is of type I_{fin}.*

Proof. Let A be a JW*-subalgebra of a type I_{fin} JW*-algebra B. Then A is finite since it is a subalgebra of a finite algebra. Let $p \in A$ be a projection. Then pAp is a subalgebra of the type I_{fin} algebra pBp. Suppose, for contradiction, that pAp contains no abelian projection. By cutting down to a homogeneous summand, we may assume that pBp is homogeneous. Then by [32, Theorem 17], p can be decomposed into any number of mutually orthogonal and strongly equivalent projections in pAp. Since equivalent projections in pAp are also equivalent in pBp, and since in a homogeneous type I_{fin} algebra, there are at most a fixed number of mutually orthogonal and strongly equivalent projcetions, we have a contradiction. So pAp contains an abelian projection and A is type I_{fin}. $\square$

Lemma 2.2 *Let $(A, \circ)$ be a JW*-algebra with identity 1 and let $P : A \longrightarrow A$ be a contractive projection. If $P(A)$ contains a unitary tripotent u of $P(A)$, then $P1 = P(uu^*u^*u) = P(u \circ u^*)$. In addition, if u is a projection in A, then $P1 = u$.*

Proof. Recall that the triple product in $P(A)$ is given by

$$[x, y, z] = P\{x, y, z\}.$$

Since u is a unitary tripotent in $P(A)$, we have by the main identity

$$\begin{aligned} P1 &= [P1, u, [u, u, u]] \\ &= [[P1, u, u], u, u] - [u, [u, P1, u], u] + [u, u, [P1, u, u]] \\ &= P1 - [u, [u, P1, u], u] + P1 \end{aligned}$$

and by the conditional expectation formula,

$$\begin{aligned} P1 &= [u,[u,P1,u],u] \\ &= P\{u,P\{u,P1,u\},u\} \\ &= P\{u,P(u^2),u\} = P\{u,u^2,u\} = P(uu^*u^*u). \end{aligned}$$

Also, $P1 = [u,u,P1] = P\{u,u,P1\} = P\{u,u,1\} = P(u \circ u^*)$. $\square$

Remark. The above result shows that there is at most one unitary tripotent in $P(A)$ which is a projection in A. If $P(A)$ is a JW*-subalgebra of A, then the identity in $P(A)$ is a projection in A and $P(1_A) = 1_{P(A)}$.

As in [32], A is said to be *modular* if its projections form a modular lattice in which case A admits a centre-valued trace, and therefore a separating family of normal traces. It has been shown in [1] that a JW*-algebra is modular if, and only if, its enveloping von Neumann algebra is finite. For this reason, we propose from now on to replace the term "modular" by the more common term "finite" throughout. A projection p in a JW*-algebra A is called *finite* if the JW*-algebra pAp is finite.

We recall that, for a net (x_α) in a von Neumann algebra, we have

$$x_\alpha \longrightarrow 0 \textit{ strongly} \Leftrightarrow x_\alpha^* x_\alpha \longrightarrow 0 \textit{ weakly}$$

$$x_\alpha \longrightarrow 0 \textit{ strongly}^* \Leftrightarrow x_\alpha^* x_\alpha + x_\alpha x_\alpha^* \longrightarrow 0 \textit{ weakly}.$$

Plainly, strong* convergence implies strong convergence.

Lemma 2.3 *Let A be a JW*-algebra. The following conditions are equivalent:*

(i) A is finite.

(ii) The map $x \in A \mapsto x^* \in A$ is strongly continuous on bounded spheres in the enveloping von Neumann algebra of A.

Proof. $(i) \Rightarrow (ii)$. Let $\mathcal{F}$ be a separating family of normal traces of A. Then the family $\tilde{\mathcal{F}} = \{\tilde{\varphi} : \varphi \in \mathcal{F}\}$ of normal tracial extensions of traces in $\mathcal{F}$ is separating on the enveloping von Neumann algebra $\mathcal{A}$ of A (cf. [1, proof of Theorem 3]). We have $x_\alpha \to 0$ strongly $\Rightarrow x_\alpha^* x_\alpha \to 0$ weakly $\Rightarrow \tilde{\varphi}(x_\alpha x_\alpha^*) = \tilde{\varphi}(x_\alpha^* x_\alpha) \to 0$ for all $\tilde{\varphi} \in \tilde{\mathcal{F}}$. Hence $x_\alpha x_\alpha^* \to 0$ weakly, that is, $x_\alpha^* \to 0$ strongly.

$(ii) \Rightarrow (i)$. If A is not finite, then by [32, Lemma 23], there is an infinite orthogonal sequence $\{p_n\}$ of projections in A such that, for every n,

$$p_1 = s_n p_n s_n$$

where s_n is a symmetry. Given any normal state ψ of the enveloping von Neumann algebra of A, we have

$$\sum \psi(p_n) = \psi(\sum p_n) \leq \psi(1) < \infty.$$

So $\psi((s_n p_n)^*(s_n p_n)) = \psi(p_n) \to 0$. But $\psi((s_n p_n)(s_n p_n)^*) = \psi(p_1) \not\to 0$. So the map $x \mapsto x^*$ is not strongly continuous on the unit ball. $\square$

Lemma 2.4 *Let $P : A \longrightarrow A$ be a contractive projection on a JW*-algebra A such that $P(A)$ is a JW*-subalgebra of A. Then*

(i) $P(x \circ x^*) \geq 0$ for all $x \in A$;

(ii) $P(a \circ x) = P(a) \circ x$ for $a \in A, x \in P(A)$;

(iii) If P is normal, then P is strongly continuous on bounded spheres.*

Proof. (i) By Lemma 2.2, $P1$ is the identity in $P(A)$. Let φ be a state of $P(A)$. Then $\varphi \circ P(1) = \varphi(P1) = 1$ implies that $\varphi \circ P$ is a state of A. Hence $\varphi(P(x \circ x^*)) \geq 0$. As φ was arbitrary, we have $P(x \circ x^*) \geq 0$. This implies that P is self-adjoint.

(ii) This is proved in [8]. We give a short alternative proof here. We have $P(a \circ x) = P(a \circ (x \circ P1)) = \frac{1}{2}P\{a, x, P1\} + \frac{1}{2}P\{a, P1, x\} = \frac{1}{2}(P\{Pa, x, P1\} + P\{Pa, P1, x\}) = \frac{1}{2}(\{Pa, x, P1\} + \{Pa, P1, x\}) = Pa \circ x$.

(iii) Let $x_\alpha \to 0$ strongly*. Then $x_\alpha \circ x_\alpha^* \to 0$ weakly and hence $P(x_\alpha \circ x_\alpha^*) \to 0$ weakly. Using (i), (ii), and the self-adjointness of P, we have

$$0 \leq P((Px_\alpha - x_\alpha) \circ (Px_\alpha - x_\alpha)^*) = P(x_\alpha \circ x_\alpha^*) - P(x_\alpha) \circ P(x_\alpha)^*$$

which implies that $P(x_\alpha) \circ P(x_\alpha)^* \to 0$ weakly. $\square$

A JW*-algbra A is *semifinite* if every nonzero projection in A contains a nonzero finite projection. This is equivalent to saying that A does not contain any type III summand.

Proposition 2.5 *Let A be a semifinite (resp. finite) JW*-algebra and P a normal contractive projection on A such that $P(A)$ is a JW*-subalgebra of A. Then $P(A)$ is semifinite (resp. finite).*

Proof. Suppose $P(A)$ is of type III. We show that $P(A) = 0$. Let $e \in A$ be a finite projection. Suppose $P(e) \neq 0$. We have $P(e) \geq 0$ and by spectral theory there exists a nonzero projection $p \in P(A)$ such that $\lambda p \leq P(e)$ for some $\lambda > 0$. Let (x_α) be a bounded net in $pP(A)p$ converging to 0 strongly, in the enveloping von Neumann algebra $\mathcal{A}$ of A. Since e is also a finite projection in $\mathcal{A}$ (cf.[1, Corollary 3.2]), by [25, p.97-98], the nets $(x_\alpha^* e)$ and $(e x_\alpha^*)$ converge to 0 strongly in $\mathcal{A}$. Since the nets $(e x_\alpha)$ and $(x_\alpha e)$ both converge to 0 strongly, we have $(x_\alpha^* e)$ and $(e x_\alpha^*)$ both converging to 0 strongly*. Therefore $\{e, x_\alpha, P(e)\} \longrightarrow 0$ strongly* in A. Since P is strongly* continuous on bounded spheres and $P(A)$ is in particular a subtriple of A, we have

$$P(e)x_\alpha^* P(e) = P\{Pe, x_\alpha, P(e)\} = P\{e, x_\alpha, P(e)\} \longrightarrow 0$$

strongly*, and hence strongly. It follows that

$$[pP(e)p + (1-p)]x_\alpha^*[pP(e)p + (1-p)] = pP(e)x_\alpha^* P(e)p \longrightarrow 0$$

strongly which gives

$$x_\alpha^* = [pP(e)p + (1-p)]^{-1} pP(e)x_\alpha^* P(e)p[pP(e)p + (1-p)]^{-1} \longrightarrow 0$$

strongly, implying that $pP(A)p$ is finite and contradicting that $P(A)$ is type III. Hence P vanishes on every finite projection in A and $P(A) = 0$.

If we apply the above argument to the identity element of a finite A, we obtain that $P(A)$ is finite. $\square$

It will follow from Theorems 4.2 and 4.4 in section 4 that Proposition 2.5, and Proposition 2.6 which follows, remain true without the assumption that $P(A)$ is a subalgebra. The proof of Proposition 2.6 is an adaptation to the Jordan algebra setting of the proof for von Neumann algebras in [31].

Proposition 2.6 *Let P be a normal contractive projection on a type I JW^*-algebra A and suppose $P(A)$ is a JW^*-subalgebra of A. Then $P(A)$ is of type I.*

Proof. By Proposition 2.5, $P(A)$ is a semifinite JW^*-algebra. Suppose that $P(A)$ contains a type II summand. By following P by the projection onto the type II part, we can assume that $P(A)$ is of type II. We show $P(A) = 0$. It suffices to show that for any finite projection q in $P(A)$, we have $qP(A)q = 0$. By following P with the projection $q \cdot q$, we may further assume that $P(A)$ is of type II_1. Suppose $B = P(A) \neq 0$, we deduce a contradiction.

By [1, Theorems 2 and 5] there are faithful normal semifinite traces $\tau, \tau_0, \tilde{\tau}_0$ on B, A, $\tilde{A}$ respectively, where $\tilde{A}$ is the von Neumann algebra generated by A, such that τ is finite and $\tilde{\tau}_0$ is an extension of τ_0. Since $\tau \circ P$ is a normal positive functional on A, by the Radon-Nikodym theorem [2, Theorem 2.4], there is an operator $h \in L^1(A, \tau_0)^+$ such that

$$\tau \circ P(x) = \tau_0(\{h^{1/2} x h^{1/2}\}) \text{ for } x \in A. \tag{1}$$

Note that for self-adjoint $x \in A$ and $y \in B$, $\tau \circ P(y^2 \circ x) = \tau(P(y^2 \circ x)) = \tau(y^2 \circ Px)$ by Lemma 2.4(ii). On the other hand, $\tau \circ P(\{yxy\}) = \tau(P\{yxy\}) = \tau(\{y, Px, y\}) = \tau(y^2 \circ Px)$, the latter by [23]. Hence $\tau \circ P(\{yxy\}) = \tau \circ P(y^2 \circ Px)$, for self-adjoint $x \in A, y \in B$. Applying this to a projection $p \in B$ and using the extension property and (1), we have

$$\tilde{\tau}_0\left(h\left[\frac{xp + px}{2} - pxp\right]\right) = 0 \text{ for every } x \in A.$$

Expanding $\tilde{\tau}_0(h \circ (p \circ x)) = \tilde{\tau}_0(h \circ (pxp))$ and using the associative trace properties of $\tilde{\tau}_0$ yields

$$\tilde{\tau}_0\left(\frac{xhp + xph + xhp + xph}{4}\right) = \tilde{\tau}_0\left(\frac{xphp + xphp}{2}\right).$$

Hence,

$$\tilde{\tau}_0(x(ph + hp - 2php)) = 0,$$

which is the same as $\tau_0(x \circ (ph + hp - 2php)) = 0$. Since this is true for all $x \in A$, we have $ph = php = hp$, so that h is affiliated with B', the commutant of B (see [2]). Note that, since p is a finite projection, all of the strong products above are in $L^1(\tilde{A}, \tilde{\tau}_0)$.

Since $h \in L^1(A, \tau_0)$, we may pick a nonzero finite projection $e \in A \cap B'$ (a spectral projection of h). It is easy to see that eB is a JW-subalgebra of A and that $eB'' = (eB)''$. By [1, Theorem 8], B of type $II_1 \Rightarrow B''$ of type $II_1 \Rightarrow eB''$ of type $II_1 \Rightarrow (eB)''$ of type $II_1 \Rightarrow eB$ of type II_1, the latter since eB is reversible. But $eB = eBe \subset eAe$ is of type I_{fin} by Lemma 2.1, giving a contradiction. Hence $B = 0$. □

3 Contractive projections on von Neumann algebras

Proposition 3.1 *Let M be a von Neumann algebra of type I and let e be a partial isometry of M. Then the Peirce 2-space $P_2(e)M$ is a JW^*-algebra of type I.*

Proof. We note that $P_2(e)M$ is a von Neumann algebra with identity e under the product $x \cdot y = xe^*y$ and involution $x^\sharp = ex^*e$ as well as a JW^*-algebra under $x \circ y = \{xey\} = (xe^*y + ye^*x)/2$ and $x^\sharp$. Also, $(P_2(e)M, \cdot)$ is a von Neumann algebra of type I if and only if $(P_2(e)M, \circ)$ is a JW^*-algebra of type I.

Now suppose that v is a nonzero central projection in $(P_2(e)M, \cdot)$. Below we shall verify the following:

(i) v is a tripotent in M.

(ii) $v \cdot P_2(e)M = P_2(v)M$ as sets.

(iii) The identity map : $(v \cdot P_2(e)M, \cdot) \to (P_2(v)M, \times)$, where $x \times y = xv^*y$ and $x \mapsto vx^*v$ is the involution in $(P_2(v)M, \times)$, is a *-isomorphism of von Neumann algebras.

(iv) $(P_2(v)M, \times)$ has a non-zero abelian projection.

Assuming that (i)-(iv) have been proved, if there is a nonzero central projection v such that $v \cdot P_2(e)M$ is a continuous von Neumann algebra, we obtain a contradiction that it contains a nonzero abelian projection. So $(P_2(e)M, \cdot)$ is a von Neumann algebra of type I.

It remains to verify (i)-(iv) above.

Since $v = v \cdot v = v^\sharp$, we have $v = v \cdot v \cdot v = ve^*ve^*v = v(ev^*e)^*v = vv^*v$. This proves (i).

Since $v = ee^*ve^*e$, we have $v = ee^*v = ve^*e$ so that $vv^*e = v(ev^*e)^*e = ve^*ve^*e = ve^*v = v$ and similarly $ev^*v = v$. Hence, for $y \in M$, $v \cdot P_2(e)y = v \cdot P_2(e)y \cdot v = ve^*(P_2(e)y)e^*v = vv^*ee^*(P_2(e)y)e^*ev^*v = P_2(v)y$ (since $v^*v \le e^*e$ and $vv^* \le ee^*$). This proves (ii).

For $x, y \in P_2(v)M$, we have, by (ii), $x = xe^*v$ and $y = ve^*y$. Therefore $x \cdot y = xe^*y = xe^*ve^*ve^*y = xv^*ve^*y = xv^*y = x \times y$. As for the involution, $ex^*e = e(ve^*xe^*v)^*e = ev^*ex^*ev^*e = vx^*v$. This proves (iii).

Let $p = vv^*$. We shall show that

(a) There is a non-zero abelian projection $h \in M$ with $h \le p$ and $c(h) = c(p)$, where $c(\cdot)$ denotes central support.

(b) With $z = hv$, z is a non-zero abelian projection in the von Neumann algebra $(P_2(v)M, \times)$. This will prove (iv).

Since pMp is of type I, there is a non-zero abelian projection $h \in pMp$ such that $c_{pMp}(h) = p$. Since $h \le p$, we have $c(h) \le c(p)$. To show equality here, take a central projection $r \in M$ with $h \le r$. Then pr is a central projection in pMp, so that $pr = prp \ge c_{pMp}(h) = p$ and thus $p \le r$ gives $c(p) \le r$. Taking $r = c(h)$, we get $c(p) \le c(h)$. This proves (a).

We next show that z is a non-zero projection in $(P_2(v)M, \times)$. We have $z = hv = phvv^*v = vv^*hvv^*v \in P_2(v)M$, $vz^*v = v(vv^*hvv^*v)^*v = vv^*hv = hv = z$, $z \times z = zv^*z = hvv^*hv = hphv = hv = z$, and $zz^* = hvv^*h = h \ne 0$.

It remains to show that for $x, y \in M$, we have

$$\begin{aligned} &[z \times (vv^*xv^*v) \times z]v^*[z \times (vv^*yv^*v) \times z] = \\ &= \;\; [z \times (vv^*yv^*v) \times z]v^*[z \times (vv^*xv^*v) \times z]. \end{aligned} \tag{2}$$

The left and right sides of (2) collapse to hxv^*hyv^*hv and hyv^*hxv^*hv respectively, which are equal since hMh is an abelian subalgebra of M. For example, the left side is equal to

$$zv^*vv^*xv^*vv^*zv^*zv^*vv^*yv^*vv^*z = hvv^*xv^*hvv^*hvv^*yv^*hv = hxv^*hyv^*hv.$$

This proves (b), hence (iv) and the Proposition. □

Let P be a normal contractive projection on a JBW^*-triple Z and let $f \in P_*(Z_*)$ have the support tripotent (partial isometry in this case) v_f.

Let $P_k = P_k(v_f)$ denote the Peirce projections induced by v_f. The following commutativity formulas were proved in [14]. These will be used freely in the remainder of the paper.

- $P_2 P = P_2 P P_2 \,, \quad P P_2 = P P_2 P \,;$
- $P P_0 = P_0 P P_0 = P P_0 P \,;$
- $P P_1 = P P_1 P \,, \quad P_1 P P_0 = 0 \,.$

In the next lemma, we shall use these formulas to extend the first two of them to the case where the tripotent is not assumed to be the support of a normal functional. We shall use the fact that, by Zorn's lemma, every tripotent in a JBW^*-triple Z is the sum of an orthogonal family of tripotents which are support tripotents of normal functionals on Z.

The following lemma is needed in the next section. In this section, it will be used only in the case that Z is a von Neumann algebra, considered as a JW^*-triple under $\frac{1}{2}(xy^*z + zy^*x)$.

Lemma 3.2 *Let P be a normal contractive projection on a JBW^*-triple Z and suppose that v is a tripotent of the JBW^*-triple $P(Z)$. Choose a set $S = \{f_i : i \in I\}$ of pairwise orthogonal normal functionals on $P(Z)$ such that $v = \sum_{i \in I} v_{f_i}$, where v_{f_i} is the support tripotent of f_i in $P(Z)$. Let w_i be the support tripotent of f_i in Z, necessarily pairwise orthogonal, and let w be the partial isometry $\sum_{i \in I} w_i$. Then*

$$P_2(w)P = P_2(w)PP_2(w)\,, \quad PP_2(w) = PP_2(w)P.$$

Proof. Since $w = \sum w_i$, we have $P_2(w) = \sum_i P_2(w_i) + \sum_{j \neq k} P_1(w_j)P_1(w_k)$ and therefore

$$\begin{aligned} &P_2(w)PP_2(w) \\ &\quad = \sum_{i,i'} P_2(w_i)PP_2(w_{i'}) + \sum_{j \neq k, j' \neq k'} P_1(w_j)P_1(w_k)PP_1(w_{j'})P_1(w_{k'}) \\ &\quad + \sum_{j' \neq k', \text{ all } i} P_2(w_i)PP_1(w_{j'})P_1(w_{k'}) + \sum_{j \neq k, \text{ all } i'} P_1(w_j)P_1(w_k)PP_2(w_{i'}). \end{aligned}$$

Because $P_2(w_i)P = P_2(w_i)PP_2(w_i)$, by properties of the joint Peirce decomposition, the first sum reduces to $\sum_i P_2(w_i)P$ and each term in the third sum is zero.

Each term in the fourth sum is zero as well. Indeed, since in the following we may assume $k \neq i'$,

$$\begin{aligned}
&P_1(w_j)P_1(w_k)PP_2(w_{i'}) \\
&\quad = P_1(w_j)[I - P_2(w_k) - P_0(w_k)]PP_2(w_{i'}) \\
&\quad = P_1(w_j)[PP_2(w_{i'}) - P_2(w_k)PP_2(w_{i'}) - P_0(w_k)PP_2(w_{i'})] \\
&\quad = P_1(w_j)[PP_2(w_{i'}) - 0 - P_0(w_k)PP_0(w_k)P_2(w_{i'})] \\
&\quad = P_1(w_j)[PP_2(w_{i'}) - PP_0(w_k)P_2(w_{i'})] \\
&\quad = P_1(w_j)[PP_2(w_{i'}) - PP_2(w_{i'})] = 0.
\end{aligned}$$

The second sum reduces to $\sum_{j\neq k} P_1(w_j)P_1(w_k)P$. Indeed, if $k \notin \{j', k'\}$, then $[P_1(w_k)PP_1(w_{j'})]P_1(w_{k'}) = 0$ since $P_1(w_{j'})P_1(w_{k'})Z \subset P_0(w_k)Z$ and $P_1(w_k)PP_0(w_k) = 0$. Thus the second sum is reduced to $\sum_{j\neq k} P_1(w_j)P_1(w_k)PP_1(w_j)P_1(w_k)$. However,

$$\begin{aligned}
&P_1(w_j)P_1(w_k)PP_1(w_j)P_1(w_k) \\
&\quad = P_1(w_j)P_1(w_k)P[I - P_2(w_j) - P_0(w_j)]P_1(w_k) \\
&\quad = P_1(w_j)P_1(w_k)PP_1(w_k) - P_1(w_j)P_1(w_k)PP_2(w_j)P_1(w_k) \\
&\qquad -P_1(w_j)P_1(w_k)PP_0(w_j)P_1(w_k) \\
&\quad = P_1(w_j)P_1(w_k)P[I - P_2(w_k) - P_0(w_k)] + 0 + 0 \\
&\quad = P_1(w_j)P_1(w_k)P - P_1(w_j)P_1(w_k)PP_2(w_k) - P_1(w_j)P_1(w_k)PP_0(w_k) \\
&\quad = P_1(w_j)P_1(w_k)P - P_1(w_k)P_1(w_j)PP_2(w_k) \\
&\quad = P_1(w_j)P_1(w_k)P.
\end{aligned}$$

This proves the first formula.

For the second formula, we have

$$\begin{aligned}
PP_2(w)P &= P\left(\sum_i P_2(w_i) + \sum_{j\neq k} P_1(w_j)P_1(w_k)\right)P \\
&= \sum_i PP_2(w_i)P + \sum_{j\neq k} PP_1(w_j)P_1(w_k)P \\
&= \sum_i PP_2(w_i) + \sum_{j\neq k} PP_1(w_j)P_1(w_k) = PP_2(w)
\end{aligned}$$

since

$$\begin{aligned} PP_1(w_j)P_1(w_k)P &= (PP_1(w_j)P)P_1(w_k)P \\ &= PP_1(w_j)(PP_1(w_k) \\ &= PP_1(w_j)P_1(w_k).\Box \end{aligned}$$

The following lemma is probably known. We include a proof for completeness.

Lemma 3.3 *Let p be a projection in a JB^*-algebra A and let $A_1(p)$ be the Peirce 1-space. Then $A_1(p) \cap A^+ = 0$*

Proof. If $x \in A_1(p) \cap A^+$, then let $y = x^{\frac{1}{2}} \in A_{sa}$ and let $y = y_2 + y_1 + y_0$ be its Peirce decomposition with respect to p. Then $x = y_2^2 + y_1^2 + y_0^2 + 2(y_2 + y_0)y_1$. Since $x \in A_1(p)$, we have $y_2^2 + y_1^2 + y_0^2 = 0$ and because the JB-algebra A_{sa} is formally real, $y = 0$. $\Box$

Lemma 3.4 *Let P, Z, v, S, w be as in Lemma 3.2. Then*

(a) The map $Q = P_2(w)P : Z_2(w) \to Z_2(w)$ is a normal faithful unital contractive projection with range $P_2(w)P(Z)$.

(b) The map $P_2(w)$ is a linear surjective isometry of $P(Z)_2(v)$ onto $P_2(w)P(Z)$

Proof. (a) By Lemma 3.2, $Q^2 = P_2(w)PP_2(w)P = P_2P = Q$ and $Q(Z_2(w)) = P_2(w)PP_2(w)(Z) = P_2(w)P(Z)$. To show that Q is unital, note first that by [9, Lemma 2.7], $v_i = w_i + P_0(w_i)v_i$ so that $w_i \perp (v_i - w_i)$. By taking sums and limits, one obtains $(v - w) \perp w$ and $\|v - w\| \leq 1$. Indeed, it is easy to see that for any finite set F of indices, $\sum_F w_i$ is the support tripotent of the normal functional $\sum_F f_i$. Hence, $\sum_F w_i \perp \sum_F (v_i - w_i)$ so that $\sum_F (v_i - w_i) \in Z_0(\sum_F w_i)$ and $\| \sum_F w_i \pm \sum_F (v_i - w_i)\| = 1$. By passing to the limit and noting that each f_i has the value 1 on $w \pm (v - w)$, we have $\|w \pm (v - w)\| = 1$, and since $P_2(w)$ is contractive, $\|w \pm P_2(w)(v - w)\| \leq 1$, and since w is an extreme point, $P_2(w)(v - w) = 0$, that is, $P_2(w)v = w$. Now $v = w + P_1(w)v + P_0(w)v$, so by [13, Lemma 1.6], $P_1(w)v = 0$ and thus $v = w + P_0(w)v$ and $v = Pv = Pw + PP_0(w)x$ so that $Qw = P_2(w)Pw = P_2(w)(v - PP_0(w)v) = P_2(w)v = w$ and Q is unital.

Finally we show that Q is faithful. Suppose that $b \in Z$, $P_2(w)b \geq 0$, and $P_2(w)Pb = 0$ We shall show that $P_2(w)b = 0$. In the first place, since $P_2(w_i)$ is a positive operator on the JB^*-algebra $P_2(w)Z$ ([15, 3.3.6]), $P_2(w_i)b = P_2(w_i)P_2(w)b \geq 0$ for every $i \in I$. Since $P_1(w_k)P_1(w_l)b \perp w_i$, we have

$$\begin{aligned} 0 &= \langle P_2(w)Pb, f_i\rangle = \langle PP_2(w)Pb, f_i\rangle = \langle PP_2(w)b, f_i\rangle = \langle P_2(w)b, f_i\rangle \\ &= \sum_j \langle P_2(w_j)b, f_i\rangle + \sum_{k\neq l}\langle P_1(w_k)P_1(w_l)b, f_i\rangle = \langle P_2(w_i)b, f_i\rangle. \end{aligned}$$

Hence $P_2(w_i)b = 0$ for all i. Therefore $P_2(w)b = \sum_i P_2(w_i)b + \sum_{j\neq k} P_1(w_j)P_1(w_k)b = \sum_{j\neq k} P_1(w_j)P_1(w_k)b = 0$ by Lemma 3.3, since each $P_1(w_k)P_1(w_l)b$ must be positive. This proves that Q is faithful, and hence (a) holds.

(b) Let B denote the JBW^*-algebra $P(Z)_2(v)$. Then by definition, $B = \{\{v, \{vxv\}_{P(Z)}, v\}_{P(Z)} : x \in P(Z)\}$. But

$$\{v, \{vxv\}_{P(Z)}, v\}_{P(Z)} = P\{v, P\{vxv\}, v\} = P\{v, \{vxv\}, v\} = PQ(v)^2x$$

so that $B = PQ(v)^2P(Z)$ and

$$P_2(w)B = P_2(w)PQ(v)^2P(Z) = P_2(w)PP_2(w)Q(v)^2P(Z) = P_2(w)P(Z).$$

Now let F_v be the normal state space of B, that is

$$F_v = \{\ell \in B_* : \|\ell\| = 1 = \ell(v)\}.$$

Recall from the first part of the proof that $v = Pw + PP_0(w)v$. Also, $P(w) = P\{v, Pw, v\} = P(w)^\sharp$ implies that for $\ell \in F_v$, $\ell(P(w))$ is real and $1 = \ell(v) = \ell(P(w)) + \ell(PP_0(w)v)$. Therefore $\ell(P(w)) \geq 0$ so that in fact $0 \leq P(w) \leq v$, that is, $v - P(w) \in B^+$. Now for each i, we have $f_i(v - Pw) = f_i(PP_0(w)v) = f_i(P_0(w)v) = f_i(P_2(w_i)P_0(w)v) = 0$. It follows, using Lemma 3.3 as above, that $v - Pw = 0$.

Now for arbitrary $\ell \in F_v$, as $Pw = v$, we have $\ell(w) = \ell(P(w)) = \ell(v) = 1$ and by [9, Lemma 3.1],

$$\ell = P_2(w)_*\ell. \tag{3}$$

By linearity and the Jordan decomposition for self-adjoint functionals, (3) extends to all $\ell \in B_*$. Hence for $b \in B$, we have $\|b\| = \sup\{|\ell(b)| : \|\ell\| = 1, \ell \in B_*\} = \sup\{|\ell(P_2(w)b)| : \|\ell\| = 1, \ell \in B_*\} \leq \|P_2(w)b\|$. This proves (b). $\square$

Proposition 3.5 *Let P be a normal contractive projection on a von Neumann algebra M of type I. Then $P(M)$ is a JW^*-triple of type I.*

Proof. Let v be any nonzero tripotent of $P(M)$ and choose $w \in M$ as in Lemma 3.2. By Proposition 3.1, $M_2(w)$ is a JW^*-algebra of Type I. By Lemma 3.4 (a) and [8, Corollary 1.5], $P_2(w)P(M) = Q(M_2(w))$ is a JW^*-subalgebra of $M_2(w)$, where $Q = P_2(w)P$. By Proposition 2.6, $Q(M_2(w))$ is a JBW^*-algebra of type I, and by Lemma 3.4(b), $P(M)_2(v)$ (the Peirce 2-space of the tripotent v of the JW^*-triple $P(M)$) is also of type I since a unital surjective linear isometry is a Jordan *-isomorphism. One can now choose v to be a complete tripotent of $P(Z)$ to obtain from [17, 4.14] that $P(M)$ is a JW^*-triple of type I. $\square$

4 Contractive projections on JW^*-triples

Proposition 4.1 *Let Z be a JBW^*-triple of type I and let v be a tripotent in Z. Then $P_2(v)Z$ is a JBW^*-algebra of type I.*

Proof. By Horn's structure theorem, we may assume that $Z = L^\infty(\Omega, C)$ where C is a Cartan factor. If C is of types 1,2, or 3, then there is a normal contractive projection Q on $L^\infty(\Omega, \tilde{C})$, where $\tilde{C}$ is the von Neumann envelope of C, with range Z. Since $P_2(v)Q$ is a normal contractive projection from the type I von Neumann algebra $L^\infty(\Omega, \tilde{C})$ onto $P_2(v)Z$, the latter is of type I by Proposition 3.5. If C is of type 4, then $P_2(v)Z = L^\infty(\Omega_2, C) \oplus L^\infty(\Omega_1)$, where $\Omega_k = \{\omega \in \Omega : \text{ rank of } v(\omega) \text{ is } k\}$, $k = 0, 1, 2$. Indeed, if $f \in Z$ and $g = P_2(v)f$, then $g = 0$ on Ω_0, $g(\omega) = \langle f(\omega), \widehat{v(\omega)}\rangle v(\omega)$ for $\omega \in \Omega_1$ and $g = f$ on Ω_2. Here we use the notation $\hat{v}$ for the normal functional with support tripotent v. It follows that the map $g = P_2(v)f \in P_2(v)Z \mapsto (g_2, g_1) \in L^\infty(\Omega_2, C) \oplus L^\infty(\Omega_1)$, where $g_1(\omega) = \langle f(\omega), \widehat{v(\omega)}\rangle$ for $\omega \in \Omega_1$ and $g_2 = g|\Omega_2$, is a surjective linear isometry.

If C is of types 5 or 6, then it is finite-dimensional and $L^\infty(\Omega, C)$ is of type I_{fin}. By Lemma 1.1, the subtriple $P_2(v)(Z)$ is of the same type. $\square$

Theorem 4.2 *Let P be a normal contractive projection on a JW^*-triple Z of type I. Then $P(Z)$ is of type I.*

Proof. By [17, 4.14], we need only show that $P(Z)_2(v)$ is of type I for a complete tripotent $v \in P(Z)$. Choose $w \in Z$ as in Lemma 3.2. By Proposition 4.1, $P_2(w)Z$ is a JW^*-algebra of type I. One can now argue exactly as in the proof of Proposition 3.5, using Lemma 3.4, to show that $P_2(w)P$ is a faithful, normal, unital contractive projection of $P_2(w)Z$ onto $P_2(w)P(Z)$ (which is a again subalgebra by [8, Corollary 1.5]) and that $P_2(w)$ is a unital isometry of $P(Z)_2(v)$ onto $P_2(w)P(Z)$. As in the proof of Proposition 3.5 and using Proposition 2.6, $P_2(w)P(Z)$ is of type I and so is $P(Z)_2(v)$. $\square$

Proposition 4.3 *Let Z be a semifinite JW^*-triple and let v be a partial isometry in Z. Then $Z_2(v)$ is a semifinite JW^*-algebra.*

Proof. We prove this first in the case that Z is a von Neumann algebra M. If $M_2(v)$ had a type III part, we could follow $P_2(v)$ by the projection of $M_2(v)$ onto that type III part and obtain a Peirce 2-space of M of type III. So we may assume that $M_2(v)$ is of type III. Let p be a finite nonzero projection in M dominated by v^*v. Then vp is a nonzero projection in $M_2(v)$ dominated by v (cf. Proposition 3.1). We shall show that $M_2(vp)$ is finite by showing that its involution is strongly continuous on bounded spheres.

Let x_α be a bounded net in $M_2(vp)$. Then

$$x_\alpha \xrightarrow{s} 0 \text{ in } M_2(vp) \Rightarrow vpx_\alpha^* vp(vp)^* x_\alpha \xrightarrow{w} 0 \Rightarrow vpx_\alpha^* x_\alpha \xrightarrow{w} 0 \Rightarrow$$

$$px_\alpha^* x_\alpha \xrightarrow{w} 0 \Rightarrow px_\alpha^* x_\alpha p \xrightarrow{w} 0 \Rightarrow x_\alpha p \xrightarrow{s} 0 \Rightarrow (\text{ by [25, p. 97-98]})$$

$$x_\alpha^* = px_\alpha^* \xrightarrow{s} 0 \Rightarrow x_\alpha x_\alpha^* \xrightarrow{w} 0 \Rightarrow x_\alpha x_\alpha^* vp \xrightarrow{w} 0 \Rightarrow x_\alpha (vp)^* vpx_\alpha^* vp \xrightarrow{w} 0$$

$$\Rightarrow x_\alpha \circ x_\alpha^\sharp \xrightarrow{w} 0 \Rightarrow x_\alpha^\sharp \xrightarrow{s} 0 \text{ in } M_2(vp).$$

Thus vp is a finite projection which is a contradiction.

To prove the general case, write $Z = Z_I \oplus Z_{II}$ where Z_I is of type I and Z_{II} is of type II. Since $P_2(v)Z = P_2(v_1)Z_I \oplus P_2(v_2)Z_{II}$ for suitable partial isometries $v_1 \in Z_I$ and $v_2 \in Z_{II}$, and we already know that $P_2(v_1)Z_I$ is of type I, we may assume by [18] that Z is triple isomorphic to $pM \oplus H(N, \alpha)$, where M and N are von Neumann algebras of type II. Accordingly $v_2 = v_2' + v_2''$ so that $P_2(v_2)Z_{II} = P_2(v_2')(pM) \oplus P_2(v_2'')(H(N, \alpha)) = M_2(v_2'') \oplus H(N_2(v_2''), \alpha)$ and $Q(N_2(v_2'')) = H(N_2(v_2''), \alpha)$ where Q is the projection $Q(x) = (x+\alpha(x))/2$ for $x \in N$. By the first part of the proof, both $M_2(v_2')$ and $N_2(v_2'')$ are semifinite. Then by Proposition 2.5, $P_2(v_2'')H(N, \alpha)$ is semifinite and the result follows. $\square$

Theorem 4.4 *Let P be a normal contractive projection on a semifinite JW^*-triple Z. Then $P(Z)$ is a semifinite JW^*-triple.*

Proof. By passing to the type III part of $P(Z)$, assuming it is nonzero for contradiction, and using [18], we may assume that $P(Z) = pM \oplus H(N, \alpha)$ where M and N are von Neumann algebras of type III. As in the proof of Proposition 3.5, using Lemma 3.4, Proposition 4.3, and Proposition 2.5, one shows that $P(Z)_2(v)$ is semifinite for any tripotent v of $P(Z)$. Choosing $v = 0 \oplus 1_N$ leads to $P(Z)_2(v) = H(N, \alpha)$, a contradiction unless $H(N, \alpha) = 0$. Choosing $v = p \oplus 0$ leads to $P(Z)_2(v) = pMp$, again a contradiction unless $pMp = 0$, which implies that $p = 0$, another contradiction. □

5 Contractive projections on JBW^*-triples

In this section we extend Theorems 4.2 and 4.4 to arbitrary JBW^*-triples and make some remarks on the atomic case.

A close examination of the proof of Theorem 4.2 reveals that it carries over to the case of JBW^*-triples if we can show that the range of a faithful normal positive unital projection on a Type I JBW*-algebra is Type I. For this, one only needs to decompose a JBW-algebra A into a direct sum of a JC-algebra A_{sp} and an "exceptional algebra" A_{ex} of the form $L^\infty(\Omega, M_3^8)$ where M_3^8 denotes the JBW-algebra whose complexification is the Cartan factor of type 6 ([26]). Since both of these summands are unital subalgebras, it follows easily that any positive unital projection on A restricts to a positive unital projection on each summand. The image of the restriction of a faithful projection to A_{sp} is a subalgebra of A_{sp} by [8, Corollary 1.5], and we can apply Proposition 2.6. The image of the restriction of the projection to A_{ex} is of type I by the following lemma which applies *verbatim* to type I_{fin} JBW-algebras of which A_{ex} is one.

Lemma 5.1 *Let Z be a type I_{fin} JBW^*-triple and let $P : Z \longrightarrow Z$ be a normal contractive projection. Then $P(Z)$ is a type I_{fin} JBW^*-triple.*

Proof. We note that $P(Z)$ is norm-closed. By weak* continuity of P and the Krein-Smulyan Theorem, $P(Z)$ is also weak* closed. Also, P induces a contractive projection $P_* : f \in Z_* \mapsto f \circ P \in Z_*$ on the predual Z_*. By [7], Z_* has the Dunford-Pettis property. The predual of $P(Z)$ identifies with

$Z_*/P_*^{-1}(0)$ which is linearly isometric to the complemented subspace $P_*(Z_*)$ of Z_*, and therefore has the Dunford-Pettis property. Hence by [7] again, $P(Z)$ is of type I_{fin}. □

Now, proceeding exactly as in the proof of Theorem 4.2 we have the result for JBW*-triples.

Theorem 5.2 *Let P be a normal contractive projection on a JBW*-triple Z of type I. Then $P(Z)$ is of type I.*

As noted before, a type II JBW*-triple is a JW*-triple. It follows from Proposition 4.1 and Theorem 4.4 that if Z is a semifinite JBW*-triple, then $P_2(e)Z$ is also a semifinite JBW*-algebra. Using this fact, now there is no difficulty of extending the proof of Theorem 4.4 to the case of JBW*-triples.

Theorem 5.3 *Let P be a normal contractive projection on a semifinite JBW*-triple Z. Then $P(Z)$ is a semifinite JBW*-triple.*

Tomiyama [31] has proved that the a von Neumann algebra which is the range of a normal contractive projection on an atomic von Neumann algebra is itself atomic. It is also known (see [30, Exercise 8, p.334]) that a von Neumann algebra $M \subset B(H)$ is atomic if and only if there is a faithful family of normal conditional expectations of $B(H)$ onto M. We end with a very simple proof of the following result which extends Tomiyama's theorem to *JBW**-triples. The proof follows from a result in [4] which states that a JBW*-triple is atomic if, and only if, its predual has the Radon-Nikodym property. The following result is clearly false without the normality assumption on P.

Proposition 5.4 *Let Z be an atomic JBW*-triple and let $P : Z \longrightarrow Z$ be a normal contractive projection. Then the range $P(Z)$ is (linearly isometric to) an atomic JBW*-triple.*

Proof. As in the proof of Lemma 5.1, the predual of $P(Z)$ is linearly isometric to a complemented subspace of the predual Z_* which has the Radon-Nikodym property. So $P(Z)$ is atomic by [4]. □

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