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MINIMAX THEOREMS IN PROBABILISTIC METRIC SPACES

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In this paper, new minimax theorems for mixed lower-upper semicontinuous functions in probabilistic metric spaces are given. As applications, we utilise these results to show the existence of solutions of abstract variational inequalities, implicit variational inequalities and saddle point problems, and the existence of coincidence points in probabilistic metric spaces.

I. INTRODUCTION AND PRELIMINARIES

The minimax problem is of fundamental importance in nonlinear analysis and, especially, plays an important role in mathematical economics and game theory.

The purpose of this paper is to obtain some minimax theorems for mixed lower-upper semi-continuous functions in probabilistic metric spaces which extend the minimax theorems of von Neumann types [**1, 3, 4, 5, 6, 8, 10, 11, 12**]. As applications, we utilise these results to study the existence problems of solutions for variational inequalities and implicit variational inequalities in probabilistic metric spaces and to show the existence of coincidence points and saddle points in probabilistic metric spaces.

Throughout this paper, let $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$.

DEFINITION 1.1: A mapping $F : R \to R^+$ is called a *distribution function* if it is nondecreasing and left-continuous with $\inf F(t) = 0$ and $\sup F(t) = 1$.

In what follows we always denote by $\mathcal{D}$ the set of all distribution functions and by H the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ 1, & \text{if } t > 0. \end{cases}$$

DEFINITION 1.2: A *probabilistic metric space* (briefly, a PM-space) is an ordered pair $(X, \mathcal{F})$, where X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$. We denote the distribution function $\mathcal{F}(x,y)$ by $F_{x,y}$ and $F_{x,y}(t)$ represents the value of $F_{x,y}$ at $t \in R$. The function $F_{x,y}$ is assumed to satisfy the following conditions :

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

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(PM-2) $F_{x,y}(0) = 0$,
(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$,
(PM-4) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$.

DEFINITION 1.3: A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if it satisfies the following conditions:

(T-1) $T(a,1) = a$,
(T-2) $T(a,b) = T(b,a)$,
(T-3) $T(c,d) \geqslant T(a,b)$ for $c \geqslant a$ and $d \geqslant b$,
(T-4) $T(T(a,b),c) = T(a,T(b,c))$.

DEFINITION 1.4: A *Menger PM-space* is a triplet $(X,\mathcal{F},T)$, where $(X,\mathcal{F})$ is a PM-space and T is a t-norm satisfying the following triangle inequality:

$$F_{x,z}(t_1 + t_2) \geqslant T(F_{x,y}(t_1), F_{y,z}(t_2))$$

for all $x, y, z \in X$ and $t_1, t_2 \geqslant 0$.

Schweizer and Sklar [9] have proved that if $(X,\mathcal{F},T)$ is a Menger PM-space with a continuous t-norm T, then $(X,\mathcal{U})$ is a Hausdorff topological space in the topology $\mathcal{T}$ induced by the family of neighbourhoods

$$\{U_p(\varepsilon,\lambda) : p \in X,\ \varepsilon > 0,\ \lambda > 0\},$$

where

$$U_p(\varepsilon,\lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1 - \lambda\}.$$

DEFINITION 1.5: Let $(X,\mathcal{F})$ be a PM-space. A subset D of X is said to be *chainable* if for any $a, b \in D$, $\lambda \in (0,1]$ and $\varepsilon > 0$, there exists a finite set $\{a = p_0, p_1, \cdots, p_n = b\} \subset D$ such that

$$F_{p_i,p_{i-1}}(\varepsilon) > 1 - \lambda, \quad i = 1, 2, \cdots, n.$$

The set $\{p_0, p_1, \cdots, p_n\}$ is called a (ε,λ)-*chain joining* a and b.

In the sequel, we consider the empty set ϕ to be chainable.

DEFINITION 1.6: Let $(X,\mathcal{F},T)$ be a Menger PM-space with a continuous t-norm T. $(X,\mathcal{F},T)$ is called a *problabilistic interval space* if there exists a mapping $[\cdot,\cdot] : X \times X \to CI(X)$, the family of all chainable subsets of X, such that for any $x_1, x_2 \in X$, $[x_1,x_2]$ is a compact chainable subset of X and $[x_1,x_2] = [x_2,x_1] \supset \{x_1,x_2\}$. In this case, the set $[x_1,x_2]$ is called a *probabilistic interval* in X.

REMARK. It should be pointed out that if $(X,\mathcal{F},T)$ is a Menger PN-space with a continuous t-norm T, then $(X,\mathcal{F},T)$ must be a probabilistic interval space. In fact, for any $x_1, x_2 \in X$, letting $[x_1, x_2] = co\{x_1, x_2\}$, then it is obvious that $[x_1, x_2]$ is a chainable comapct subset of X containing x_1, x_2 and $[x_1, x_2] = [x_2, x_1]$.

DEFINITION 1.7: Let $(X, \mathcal{F}, T)$ be a probabilistic interval space with a continuous t-norm T. A subset D of X is called *W-chainable* if for any $x_1, x_2 \in D$, the probabilistic interval $[x_1, x_2] \subset D$. A function $f : X \to R$ is said to be *probabilistic quasi-convex* (respectively, *probabilistic quasi-concave*) if $\{x \in X : f(x) \leqslant r\}$ (respectively, $\{x \in X : f(x) \geqslant r\}$) is W-chainable for all $r \in R$. $f : X \to R$ is said to be *upper-compact* (respectively, *lower-compact*) if for any $r \in R$, the set $\{x \in X : f(x) \geqslant r\}$ (respectively, $\{x \in X : f(x) \leqslant r\}$) is compact in X.

DEFINITION 1.8: Let X and Y be two topological spaces. A function $f : X \times Y \to R$ is said to be *lower-upper semi-continuous* if $x \mapsto f(\cdot, y)$ and $y \mapsto f(x, \cdot)$ are lower and upper semi-continuous, respectively.

DEFINITION 1.9: Let $(X, \mathcal{F}, T)$ and $\left(Y, \widetilde{\mathcal{F}}, \widetilde{T}\right)$ be two probabilistic interval spaces. A function $f : X \times Y \to R$ is said to be *probabilistic quasi-convex-concave* if $x \mapsto f(\cdot, y)$ and $y \mapsto f(x, \cdot)$ are probabilistic quasi-convex and probabilistic quasi-concave, respectively.

PROPOSITION 1.1. *In probabilistic interval spaces,*

(1) *the intersection of all W-chainable subsets is still a W-chainable subset (the empty set ϕ is assumed to be W-chainable).*
(2) *Each W-chainable subset is a chainable subset.*

PROPOSITION 1.2. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with a continuous t-norm T. If $\{p_n\}$ and $\{q_n\}$ are two sequences of X satisfying $p_n \to p$, $q_n \to q$ and for all n,*

$$F_{p_n, q_n}(1/n) > 1 - (1/n),$$

then $p = q$.

PROOF: For any $\varepsilon > 0$ and $\lambda > 0$, by the continuity of T, there exists $\lambda' \in (0, 1]$ such that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$. Hence there exists a positive integer N such that for $n \geqslant N$

$$1/n < \min\{\varepsilon/2, \lambda'\}, \quad F_{p_n, p}(\varepsilon/2) > 1 - \lambda'.$$

Thus, for any $n \geqslant N$, we have

$$F_{p, q_n}(\varepsilon) \geqslant T(F_{p, p_n}(\varepsilon/2), F_{p_n, q_n}(\varepsilon/2)) \geqslant T(1 - \lambda', 1 - \lambda') > 1 - \lambda,$$

which implies that $q_n \to p$. Since $q_n \to q$, we have $p = q$. This completes the proof. $\square$

2. Minimax Theorems in PM-spaces

Now, we are ready to give our main theorems.

Theorem 2.1. *Let $(X, \mathcal{F}, T)$ be a probabilistic interval space with a continuous t-norm T and $\left(Y, \widetilde{\mathcal{F}}, \widetilde{T}\right)$ be a compact Menger PM-space with a continuous t-norm $\widetilde{T}$. Let $f : X \times Y \to R$ be a function satisfying the following conditions:*

(i) *$y \mapsto f(x, \cdot)$ is upper semi-continuous,*
(ii) *for any finite set $A \subset X$ and for all $r \in R$, $\bigcap_{x \in A} \{y \in Y : f(x,y) > r\}$ is chainable,*
(iii) *$x \mapsto f(\cdot, y)$ is probabilistic quasi-convex and lower semi-continuous on any probabilistic interval of X.*

Then

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) = \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

Proof: Let

$$r_* r \sup_{y \in Y} \inf_{x \in X} f(x,y)$$

and

$$r^* = \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

It is obvious that $r^* \geqslant r_*$. Next we prove that $r_* \geqslant r^*$. In fact, for any $x \in X$ and for any $r < r^*$, letting

$$\widehat{M}(x,r) = \{y \in Y : f(x,y) \geqslant r\}, \quad M(x,r) = \{y \in Y : f(x,y) > r\},$$

by the definition of r^*, we have $M(x,r) \neq \phi$. By the condition (i), $\{\widehat{M}(x,r) : x \in X,\ r < r^*\}$ is a family of nonempty closed sets in Y. Now we prove that the family $\{\widehat{M}(x,r) : x \in X,\ r < r^*\}$ has the finite intersection property.

In fact, for any $x \in X$ and for any $r < r^*$, by the above discussion, $\widehat{M}(x,r) \neq \phi$. Now, by induction, we assume that for any n elements in $\{\widehat{M}(x,r) : x \in X,\ r < r^*\}$, $n \geqslant 2$, their intersection is nonempty. Then we prove that for any $n+1$ elements in $\{\widehat{M}(x,r) : x \in X,\ r < r^*\}$, their intersection is also nonempty. Suppose the contrary. Then there exist $\{x_1, \cdots, x_{n+1}\} \subset X$ and $\{r_1, \cdots, r_{n+1}\} \subset R$ with $r^* > r_1 \geqslant r_2 \geqslant \cdots \geqslant r_{n+1}$ such that

$$\bigcap_{i=1}^{n+1} \widehat{M}(x_i, r_i) = \phi. \tag{2.1}$$

By the density of R, there exists $\widehat{r} \in R$ such that $r_1 < \widehat{r} < r^*$. Letting

$$T(x) = M(x, r_1), \quad \widehat{T}(x) = \widehat{M}(x, r_1) \quad \text{and} \quad H = \bigcap_{i=3}^{n+1} T(x_i)$$

for any $x \in X$, in view of the inductive assumption, we have

$$H \cap T(x) = \left(\bigcap_{i=3}^{n+1} M(x_i, r_1)\right) \cap M(x, r_1) \supset \left(\bigcap_{i=3}^{n+1} \widehat{M}(x_i, \widehat{r})\right) \cap \widehat{M}(x, \widehat{r}) \neq \phi.$$

On the other hand, by the condition (i), we have

$$\overline{(H \cap T(x_1))} \cap \overline{(H \cap T(x_2))} \subset \bigcap_{i=1}^{n+1} \widehat{M}(x_i, r_i) = \phi. \tag{2.2}$$

If $y \notin \bigcup_{i=1}^{2} T(x_i)$, then $f(x_i, y) \leqslant r_1$, $i = 1, 2$, and so $\{x_1, x_2\} \subset \{x \in X : f(x, y) \leqslant r_1\}$. By the condition (iii), we know that $[x_1, x_2] \subset \{x \in X : f(x, y) \leqslant r_1\}$. Hence for any $\widehat{x} \in [x_1, x_2]$ we have $f(\widehat{x}, y) \leqslant r_1$, that is, $y \notin T(\widehat{x})$. This implies that $T([x_1, x_2]) \subset \bigcup_{i=1}^{2} T(x_i)$ and so $H \cap T([x_1, x_2]) \subset \bigcup_{i=1}^{2} H \cap T(x_i)$. Thus for any $\widehat{x} \in [x_1, x_2]$,

$$H \cap T(\widehat{x}) \subset \bigcup_{i=1}^{2} H \cap T(x_i).$$

Next we prove that for any $\widehat{x} \in [x_1, x_2]$,

$$H \cap T(\widehat{x}) \subset H \cap T(x_1) \quad \text{or} \quad H \cap T(\widehat{x}) \subset H \cap T(x_2). \tag{2.3}$$

In fact, if there exist $y_1, y_2 \in H \cap T(\widehat{x})$ such that $y_1 \in H \cap T(x_1)$ and $y_2 \in H \cap T(x_2)$, since $H \cap T(\widehat{x})$ is chainable, for any positive integer n, there exists a $(1/n, 1/n)$-chain joining y_1 and y_2. Hence there exists $p_n \in H \cap T(x_1)$ and $q_n \in H \cap T(x_2)$ such that $\widetilde{F}_{p_n, q_n}(1/n) > 1 - 1/n$. Since $\overline{H \cap T(x_i)}$ is compact, $i = 1, 2$, without loss of generality, we assume $p_n \to p \in \overline{H \cap T(x_1)}$ and $q_n \to q \in \overline{H \cap T(x_2)}$. By Proposition 1.2, $p = q$ and so we have

$$p = q \in \overline{(H \cap T(x_1))} \cap \overline{(H \cap T(x_2))},$$

which contradicts (2.2). Thus (2.3) is true.

Letting

$$E_1 = \{x \in [x_1, x_2] : H \cap T(x) \subset H \cap T(x_1)\},$$
$$E_2 = \{x \in [x_1, x_2] : H \cap T(x) \subset H \cap T(x_2)\},$$

it is obvious that $x_i \in E_i$, $i = 1, 2$, $E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = [x_1, x_2]$. Hence E_1 or E_2 is not a relatively closed set in $[x_1, x_2]$. Otherwise, since $[x_1, x_2]$ is chainable and $x_1, x_2 \in [x_1, x_2]$, for any n, there exists a $(1/n, 1/n)$-chain joining x_1 and x_2 in

$[x_1, x_2]$. It follows from $E_1 \cap E_2 = \phi$ and $x_i \in E_i$, $i = 1, 2$, that there exist $a_n \in E_1$ and $b_n \in E_2$ such that

$$F_{a_n, b_n}(1/n) > 1 - 1/n, \quad n = 1, 2, \cdots$$

Since $[x_1, x_2]$ is compact, we may assume that $a_n \to a$ and $b_n \to b$. By Proposition 1.2, we know that $a = b \in E_1 \cap E_2$, which is a contradiction.

Without loss of generality, we assume that E_2 is not a relatively closed set in $[x_1, x_2]$. Then there exists $x_0 \in (\overline{E}_2 \backslash E_2) \cap E_1$. Hence we have $H \cap T(x_0) \subset H \cap T(x_1)$ and there exists a net $\{x_\alpha\}_{\alpha \in I} \subset E_2$ such that $x_\alpha \to x_0$. By the definition of E_2, we have $H \cap T(x_\alpha) \subset H \cap T(x_2)$ for all $\alpha \in I$.

On the other hand, since $H \cap T(x_0) \neq \phi$, there exists $y_0 \in H \cap T(x_0) \subset H \cap T(x_1)$. Hence we have $y_0 \notin H \cap T(x_2)$ and so $y_0 \notin H \cap T(x_\alpha)$ for all $\alpha \in I$. This implies that $y_0 \notin T(x_\alpha)$ for all $\alpha \in I$, that is, $f(x_\alpha, y_0) \leqslant r_1$ for all $\alpha \in I$. By the condition (iii), we have $f(x_0, y_0) \leqslant r_1$, that is, $y_0 \notin T(x_0)$, which contradicts the choice of y_0. This shows that $\{\widehat{M}(x, r) : x \in X,\ r < r^*\}$ has the finite intersection property. Since Y is compact, we have

$$\bigcap_{x \in X, r < r^*} \widehat{M}(x, r) \neq \phi.$$

Hence there exists $\widehat{y} \in \widehat{M}(x, r)$ for all $x \in X$ and for all $r < r^*$, and so we have

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) \geqslant r.$$

By the density of R, we have

$$r_* = \sup_{y \in Y} \inf_{x \in X} f(x, y) \geqslant r^*.$$

This completes the proof. $\square$

Theorem 2.2. *Let $(X, \mathcal{F}, T)$ be a probabilistic interval space with a continuous t-norm T and $\left(Y, \widetilde{\mathcal{F}}, \widetilde{T}\right)$ be a compact Menger PM-space with a continuous t-norm $\widetilde{T}$. Let $f : X \times Y \to R$ be a function satisfying the following conditions:*

(i) *$y \mapsto f(x, \cdot)$ is upper semi-continuous,*
(ii) *for any finite set $A \subset X$ and for any $r \in R$, $\bigcap_{x \in A} \{y \in Y : f(x, y) \geqslant r\}$ is chainable,*
(iii) *$x \mapsto f(\cdot, y)$ is probabilistic quasi-convex and lower semi-contiuous on any probabilistic interval of X.*

Then

$$\sup_{y\in Y}\inf_{x\in X} f(x,y) = \inf_{x\in X}\sup_{y\in Y} f(x,y).$$

PROOF: Let

$$r_* = \sup_{y\in Y}\inf_{x\in X} f(x,y)$$

and

$$r^* = \inf_{x\in X}\sup_{y\in Y} f(x,y).$$

First we prove that for any finite set $A \subset X$ and for any $r \in R$, the set $\bigcap_{x\in A}\{y \in Y : f(x,y) > r\}$ is chainable. In fact, since

$$\bigcap_{x\in A}\{y\in Y : f(x,y) > r\} = \bigcup_{\varepsilon>0}\bigcap_{x\in A}\{y \in Y : f(x,y) \geqslant r+\varepsilon\},$$

if for any $\varepsilon > 0$, $\bigcap_{x\in A}\{y \in Y : f(x,y) \geqslant r+\varepsilon\} = \phi$, then $\bigcap_{x\in A}\{y\in Y : f(x,y) > r\} = \phi$ is chainable. Therefore, without loss of generality, we may assume that there exists an $\varepsilon_0 > 0$ such that $\bigcap_{x\in A}\{y \in Y : f(x,y) \geqslant r+\varepsilon_0\} \neq \phi$. Note that if $\varepsilon > \varepsilon_0$, then

$$\bigcap_{x\in A}\{y\in Y : f(x,y) \geqslant r+\varepsilon_0\} \supset \bigcap_{x\in A}\{y\in Y : f(x,y) \geqslant r+\varepsilon\}$$

and so we have

$$\bigcap_{x\in A}\{y\in Y : f(x,y) > r\} = \bigcup_{0<\varepsilon\leqslant\varepsilon_0}\bigcap_{x\in A}\{y\in Y : f(x,y) \geqslant r+\varepsilon\}.$$

Since

$$\bigcap_{0<\varepsilon\leqslant\varepsilon_0}\left(\bigcap_{x\in A}\{y\in Y : f(x,y) \geqslant r+\varepsilon\}\right) = \bigcap_{x\in A}\{y\in Y : f(x,y) \geqslant r+\varepsilon_0\} \neq \phi,$$

there exists $y_0 \in \bigcap_{x\in A}\{y \in Y : f(x,y) \geqslant r+\varepsilon\}$ for all ε with $0 < \varepsilon \leqslant \varepsilon_0$. For any $y_1, y_2 \in \bigcap_{x\in A}\{y \in Y : f(x,y) > r\}$, there exist $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_i \leqslant \varepsilon_0$, $i = 1,2$, such that

$$y_i \in \bigcap_{x\in A}\{y\in Y : f(x,y) \geqslant r+\varepsilon_i\}, \quad i = 1,2.$$

On the other hand, for any $\eta > 0$ and $\lambda \in (0,1]$, by the condition (ii), there exists an (η,λ)-chain joining y_1 and y_0 in $\bigcap_{x\in A}\{y \in Y : f(x,y) \geqslant r+\varepsilon_1\}$ and there exists an (η,λ)-chain joining y_0 and y_2 in $\bigcap\limits_{x\in A}\{y\in Y : f(x,y)\geqslant r+\varepsilon_2\}$. Hence there exists an (η,λ)-chain joining y_1,y_2 in $\bigcap\limits_{x\in A}\{y\in Y : f(x,y)>r\}$. This implies that $\bigcap\limits_{x\in A}\{y\in Y : f(x,y)>r\}$ is chainable. Hence all the conditions in Theorem 2.1 are satisfied. Thus, this theorem follows from Theorem 2.1. This completes the proof. $\square$

Corollary 2.3. *Let $(X,\mathcal{F},T)$ be a probabilistic interval space with a continuous t-norm T and $\left(Y,\widetilde{\mathcal{F}},\widetilde{T}\right)$ be a compact probabilistic interval space with a continuous t-norm $\widetilde{T}$. Let $f : X\times Y\to R$ be a function satisfying the following conditions:*

(i) *$y\mapsto f(x,y)$ is upper semi-contiuous and probabilistic quasi-concave,*
(ii) *$x\mapsto f(x,y)$ is probabilistic quasi-convex and lower semi-contiuous on any probabilistic interval of X.*

Then

$$\sup_{y\in Y}\inf_{x\in X} f(x,y)=\inf_{x\in X}\sup_{y\in Y} f(x,y).$$

Proof: Noting that the intersection of any number of W-chainable subsets is also W-chainable and every W-chainable subset is chainable, then the condition (ii) in Theorem 2.2 is satisfied. Thus, by Theorem 2.2, the conclusion follows. $\square$

Corollary 2.4. *Let $(X,\mathcal{F},T)$ and $\left(Y,\widetilde{\mathcal{F}},\widetilde{T}\right)$ be two probabilistic interval spaces with continuous t-norms T and $\widetilde{T}$, respectively, and $\left(Y,\widetilde{\mathcal{F}},\widetilde{T}\right)$ be compact. Let $f : X\times Y\to R$ be a function satisfying the following conditions:*

(i) *$y\mapsto f(x,y)$ is upper semi-continuous,*
(ii) *for any $r\in R$ and for any $x\in X$, $\{y\in Y : f(x,y)>r\}$ is W-chainable,*
(iii) *$x\mapsto f(x,y)$ is probabilistic quasi-convex and lower semi-contiuous on any probabilistic interval of X.*

Then

$$\inf_{x\in X}\sup_{y\in Y} f(x,y)=\sup_{y\in Y}\inf_{x\in X} f(x,y).$$

Proof: By the same method as stated in Corollary 2.3 and using Theorem 2.1, the conclusion is obtained. $\square$

Remark. The results presented in this section improve and extend the corresponding results in [**1**, **3**, **4**, **5**, **6**] and [**7**, **8**, **9**], and generalise them to the cases of probabilistic metric spaces.

3. Abstract Variational Inequalities in PM-spaces

In this section, we shall apply the results presented in the section 2 to show the existence problems of solutions for abstract variational inequalities in probabilistic interval spaces.

Theorem 3.1. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T and $f : X \times X \to R$ be a function with $f(x,x) \geqslant 0$ for all $x \in X$. If the following conditions are satisfied:*

(i) *f is lower-upper semi-continuous,*
(ii) *for any finite set $A \subset X$ and for any $r \in R$, $\bigcap_{x\in A} \{y \in X : f(x,y) > r\}$ is chainable,*
(iii) *$x \mapsto f(x,y)$ is probabilistic quasi-convex,*

then there exists a $\widehat{y} \in X$ such that $f(x,\widehat{y}) \geqslant 0$ for all $x \in X$.

Proof: By Theorem 2.1, we have

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) = \inf_{x\in X} \sup_{y\in X} f(x,y).$$

By the condition (i), since $y \mapsto f(x,y)$ is upper semi-continuous, [**2**, Proposition 1.4.6] $\inf_{x\in X} f(x,y)$ is upper semi-continuous in y. Since X is compact, there exists an $\widehat{y} \in Y$ such that $\sup_{y\in X} \inf_{x\in X} f(x,y) = \inf_{x\in X} f(x,\widehat{y})$ and so

$$\inf_{x\in X} f(x,\widehat{y}) = \inf_{x\in X} \sup_{y\in X} f(x,y) \geqslant \inf_{x\in X} f(x,x) \geqslant 0.$$

Therefore, we have $f(x,\widehat{y}) \geqslant 0$ for all $x \in X$. This completes the proof. □

Theorem 3.2. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T and $f : X \times X \to R$ a function with $f(x,x) \leqslant 0$ for all $x \in X$. If the following conditions are satisfied:*

(i) *f is lower-upper semi-continuous,*
(ii) *for any finite set $A \subset X$ and for any $r \in R$, $\bigcap_{x\in A} \{y \in X : f(x,y) > r\}$ is chainable,*
(iii) *$x \mapsto f(x,y)$ is probabilistic quasi-convex,*

then there exists $\widehat{x} \in X$ such that $f(\widehat{x},y) \leqslant 0$ for all $y \in X$.

Proof: The proof is similar to Theorem 3.1, we omit it here. □

Corollary 3.3. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T. Let $\varphi : X \times X \to R$ and $h : X \to R$ be two functions satisfying $\varphi(x,x) \geqslant 0$ for all $x \in X$ and the following conditions:*

(i) *φ is lower-upper semi-continuous and h is upper semi-continuous,*

(ii) *for any finite set* $A \subset X$ *and for any* $r \in R$, $\bigcap_{x \in A} \{y \in X : \varphi(x,y) + h(y) > r + h(x)\}$ *is chainable (or* $\bigcap_{x \in A} \{y \in X : \varphi(x,y) + h(y) \geqslant r + h(x)\}$ *is chainable),*

(iii) $\varphi(x,y) - h(x)$ *is probabilistic quasi-convex in* x.

Then there exists a $\widehat{y} \in X$ *such that* $\varphi(x,\widehat{y}) \geqslant h(x) - h(\widehat{y})$ *for all* $x \in X$.

PROOF: Letting $f(x,y) = \varphi(x,y) - h(x) + h(y)$, it is easy to prove that f satisfies all the conditions in Theorem 3.1. Thus, the conclusion is obtained from Theorem 3.1 immediately. □

Using Theorem 3.2, we can prove the following:

COROLLARY 3.4. *Let* $(X,\mathcal{F},T)$ *be a compact probabilistic interval space with a continuous t-norm* T. *Let* $\varphi : X \times X \to R$ *and* $h : X \to R$ *be two functions satisfying* $\varphi(x,x) \leqslant 0$ *for all* $x \in X$ *and the following conditions:*

(i) φ *is lower-upper semi-ccontinuous and* h *is upper semi-continuous,*

(ii) *for all finite set* $A \subset X$ *and for any* $r \in R$, $\bigcap_{x \in A} \{y \in X : \varphi(x,y) + h(y) > r\}$ *is chainable (or* $\bigcap_{x \in A} \{y \in X : \varphi(x,y) + h(y) \geqslant r + h(x)\}$ *is chainable),*

(iii) $\varphi(x,y) - h(x)$ *is probabilistic quasi-convex in* x,

then there exists an $\widehat{x} \in X$ *such that* $\varphi(\widehat{x},y) \leqslant h(\widehat{x}) - h(y)$ *for all* $y \in X$.

4. SADDLE POINT THEOREMS IN PM-SPACES

In this section, by using Theorem 2.1, we give some saddle point theorems in probabilistic interval spaces.

THEOREM 4.1. *Let* $(X,\mathcal{F},T)$ *be a probabilistic interval space with a continuous t-norm* T *and* $\left(Y,\widetilde{\mathcal{F}},\widetilde{T}\right)$ *be a compact Menger PM-space with a continuous t-norm* $\widetilde{T}$. *If* $f : X \times Y \to R$ *is a function satisfying the following conditions:*

(i) f *is lower-upper semicontonuous,*

(ii) *for any finite set* $A \subset X$ *and for any* $r \in R$, $\bigcap_{x \in A} \{y \in Y : f(x,y) > r\}$ *(or* $\bigcap_{x \in A} \{y \in Y : f(x,y) \geqslant r\}$*) is chainable,*

(iii) $x \mapsto f(x,y)$ *is probabilistic quasi-convex and there exists a* $y_1 \in Y$ *such that* $x \mapsto f(x,y_1)$ *is lower-compact.*

Then there exists $(\widehat{x},\widehat{y}) \in X \times Y$ *such that* $f(\widehat{x},y) \leqslant f(\widehat{x},\widehat{y}) \leqslant f(x,\widehat{y})$ *for all* $x \in X$ *and* $y \in Y$.

PROOF: Let

$$\varphi(x) = \sup_{y \in Y} f(x,y), \quad \psi(y) = \inf_{x \in X} f(x,y).$$

By [2, Proposition 1.4.6], φ is lower semi-continuous and ψ is upper semi-continuous. Since Y is compact and $y \mapsto f(x,y)$ is upper semi-continuous, φ is a function from X to R and ψ is a function from Y to $R \cup \{-\infty\}$. Besides, by the condition (iii), since $x \mapsto f(x, y_1)$ is lower-compact, for any $x_1 \in X$, letting

$$L_1 = \{x \in X : f(x,y_1) \leqslant f(x_1,y_1)\},$$

then L_1 is a nonempty compact subset of X and

$$\inf_{x \in X} f(x,y_1) = \inf_{x \in L_1} f(x,y_1).$$

It follows from the lower semi-continuity of $f(\cdot, y_1)$ and the compactness of L_1 that there exists $x_* \in L_1$ such that

$$\inf_{x \in X} f(x,y_1) = \inf_{x \in L_1} f(x,y_1) = f(x_*,y_1).$$

Therefore, $\psi(y_1) \in R$ and so $\psi \not\equiv -\infty$.

Again by the upper semi-coninuity of ψ and the compactness of Y, there exists a $\widehat{y} \in Y$ such that

$$\psi(\widehat{y}) = \max_{y \in Y} \psi(y) = \sup_{y \in Y} \psi(y).$$

Letting

$$Q_1 = \{x \in X : \varphi(x) \leqslant \varphi(x_1)\},$$

by the lower semi-continuity of φ, we know that Q_1 is a nonempty closed set and

$$Q_1 \subset \{x \in X : f(x,y_1) \leqslant \varphi(x_1)\} := D_1.$$

Since $f(\cdot, y_1)$ is lower-compact, D_1 is compact and so Q_1 is compact. Besides, it is obvious that

$$\inf_{x \in X} \varphi(x) = \inf_{x \in Q_1} \varphi(x).$$

Hence there exists an $\widehat{x} \in Q_1$ such that

$$\inf_{x \in X} \varphi(x) = \inf_{x \in Q_1} \varphi(x) = \varphi(\widehat{x}).$$

However, from Theorem 2.1, we have

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) = \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

Therefore, we have $\psi(\widehat{y}) = \varphi(\widehat{x})$, which implies that for all $x \in X$ and $y \in Y$

$$f(\widehat{x}, y) \leqslant \sup_{y \in Y} f(\widehat{x}, y) = \varphi(\widehat{x}) = \psi(\widehat{y}) = \inf_{x \in X} f(x, \widehat{y}) \leqslant f(x, \widehat{y}).$$

Taking $x = \widehat{x}$ and $y = \widehat{y}$ in the preceding expression, we have

$$f(\widehat{x}, \widehat{y}) = \varphi(\widehat{x}) = \psi(\widehat{y}).$$

Hence we have

$$f(\widehat{x}, y) \leqslant \sup_{y \in Y} f(\widehat{x}, y) = f(\widehat{x}, \widehat{y}) = \inf_{x \in X} f(x, \widehat{y}) \leqslant f(x, \widehat{y})$$

for all $x \in X$ and $y \in Y$. This means that $(\widehat{x}, \widehat{y}) \in X \times Y$ is a saddle point of f. This completes the proof. □

From Theorem 4.1, we can obtain the following:

COROLLARY 4.2. *Let $(X, \mathcal{F}, T)$ be a probabilistic interval space with a continuous t-norm T and $\left(Y, \widetilde{\mathcal{F}}, \widetilde{T}\right)$ be a compact probabilistic interval space with a continuous t-norm $\widetilde{T}$. If $f : X \times Y \to R$ is a function satisfying the followig conditions:*

(i) *f is lower-upper semi-continuous,*

(ii) *f is probabilistic quasi-convex-concave,*

(iii) *there exists $y_1 \in Y$ such that $f(\cdot, y_1)$ is lower-compact.*

Then f has a saddle point $(\widehat{x}, \widehat{y}) \in X \times Y$.

REMARK. Corollary 4.2 generalises the famous von Neumann saddle point theorem to the case of probabilistic metric spaces.

5. Coincidence Point Theorems in PM-spaces

THEOREM 5.1. *Let $(X, \mathcal{F}, T)$ be a compact probabilistic interval space with a continuous t-norm T and Y be a topological space. Let $S : X \to Y$ be a continuous mapping and $F : X \to 2^Y$ be a mapping with nonempty closed values satisfying the following conditions:*

(i) *for any finite set $A \subset X$, $\bigcap_{x \in A} S^{-1}F(x)$ is chainable,*

(ii) *for all $x \in X$, $X \backslash F^{-1}(S(x))$ is a closed W-chainable subset of X,*

(iii) *for any $x \in X$, $S^{-1}F(x) \neq \phi$.*

Then there exists an $\widehat{x} \in X$ such that $S(\widehat{x}) \in F(\widehat{x})$.

PROOF: Define a function $f : X \times X \to R$ by

$$f(x, z) = \begin{cases} 0, & \text{if } S(z) \notin F(x), \\ 1, & \text{if } S(z) \in F(x). \end{cases}$$

If F and S have no coincidence point in X, then for any $x \in X$, $S(x) \notin F(x)$ and so $f(x,x) = 0$.

Next we prove that all the conditions in Theorem 3.2 are satisfied. In fact,

(I) For any $\alpha \in R$ and for any $x \in X$, we have

$$M = \{z \in X : f(x,z) \geqslant \alpha\} = \begin{cases} X, & \text{if } \alpha \leqslant 0, \\ \phi, & \text{if } \alpha > 1, \\ S^{-1}F(x), & \text{if } 0 < \alpha \leqslant 1. \end{cases}$$

Since F has closed values and S is continuous, M is a closed set in X. Hence $y \mapsto f(x,y)$ is upper semi-continuous.

(II) For any $\alpha \in R$ and for any finite set $A \subset X$, we have

$$\bigcap_{x \in A} \{z \in X : f(x,z) > \alpha\} = \begin{cases} X, & \text{if } \alpha < 0, \\ \phi, & \text{if } \alpha \geqslant 1, \\ \bigcap_{x \in A} S^{-1}F(x), & \text{if } 0 \leqslant \alpha < 1. \end{cases}$$

By the condition (i), it is chainable.

(III) For any $\alpha \in R$ and for any $z \in X$, we have

$$P = \{x \in X : f(x,z) \leqslant \alpha\} = \begin{cases} \phi, & \text{if } \alpha < 0, \\ X, & \text{if } \alpha \geqslant 1, \\ X \backslash F^{-1}(S(z)), & \text{if } 0 \leqslant \alpha < 1. \end{cases}$$

By the condition (ii), we know that P is a closed W-chainable subset. Hence $f(\cdot, z)$ is a lower semi-continuous and probabilistic quasi-convex function. By Theorem 3.2, there exists an $\widehat{x} \in X$ such that $f(\widehat{x}, z) \leqslant 0$ for all $z \in X$. Since $f(\widehat{x}, z) \geqslant 0$ for all $z \in X$, we have $f(\widehat{x}, z) = 0$. Hence for all $z \in X$, we have $S(z) \notin F(\widehat{x})$, that is, $S^{-1}F(\widehat{x}) = \phi$. This contradicts the condition (iii). Therefore, S and F have a coincidence point, that is, there exists a $x^* \in X$ such that $S(x^*) \in F(x^*)$. This completes the proof. $\square$

Corollary 5.2. *Under the conditions in Theorem 5.1, but with condition (i) replaced by the following conditon:*

(i') *for all $x \in X$, $S^{-1}F(x)$ is W-chainable,*

then S and F have a coincidence point in X.

Taking $Y = X$ and $S = I_X$ (the identity mapping on X) in Theorem 5.1 and Corollary 5.2, we have the following:

COROLLARY 5.3. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T. Let $F : X \to 2^X$ be a set-valued mapping with nonempty closed values satisfying the following conditions:*

(i) *for any finite set $A \subset X$, $\bigcap_{x\in A} F(x)$ is chainable (or for any $x \in X$, $F(x)$ is a W-chainable subset in X),*

(ii) *for any $x \in X$, $X \backslash F^{-1}(x)$ is a closed W-chainable subset.*

Then F has a fixed point in X.

6. Implicit Variational Inequalities in PM-spaces

In this section, by using some results from the previous sections, we show the existence of solutions of variational inequalities in PM-spaces.

THEOREM 6.1. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T. Let $\psi : X \times X \times X \to R$, $g : X \times X \to R$ be two functions satisfying $\psi(z,x,x) \geqslant 0$ for all $z,x \in X$ and the following conditions:*

(i) *$\psi(z,\cdot,y)$ is lower semi-continuous, $\psi(z,x,\cdot)$ is upper semi-continuous and $g(z,\cdot)$ is upper semi-continuous,*

(ii) *$\psi(z,x,y) + g(z,y)$ is probabilistic quasi-concave in y,*

(iii) *$\psi(z,x,y) - g(z,x)$ is probabilistic quasi-convex in x,*

(iv) *for any $x \in X$, $\{z \in X : \sup_{y\in X}[\psi(z,x,y) + g(z,y)] > g(z,x)\}$ is a closed W-chainable subset in X.*

Then there exists a $\widehat{z} \in X$ such that $\psi(\widehat{z},\widehat{z},y) \leqslant g(\widehat{z},\widehat{z}) - g(\widehat{z},y)$ for all $y \in X$.

PROOF: For any $z \in X$, let

$$\varphi(x,y) = \psi(z,x,y), \quad h(x) = g(z,x).$$

It is easy to check that all the conditions in Corollary 3.4 are satisfied. Hence, by Corollary 3.4, it follows that

$$\varphi(x,y) \leqslant h(x) - h(y) \tag{6.1}$$

for all $y \in X$ has a solution in X. Let $S(z)$ be the set of solutions of the variational inequality (6.1). Then $S : X \to 2^X$ is a multi-valued mapping. Let $\{x_\alpha\}_{\alpha\in J}$ be any net in $S(z)$ and $x_\alpha \to \widehat{x}$. Since $\{x_\alpha\}_{\alpha\in J} \subset S(z)$, we have $\varphi(x_\alpha,y) \leqslant h(x_\alpha) - h(y)$ for all $y \in X$ and $\alpha \in J$, that is, $\psi(z,x_\alpha,y) \leqslant g(z,x_\alpha) - g(z,y)$. By the condition (i), we know

$$\psi(z,\widehat{x},y) - g(z,\widehat{x}) \leqslant \varliminf_{\alpha}[\psi(z,x_\alpha,y) - g(z,x_\alpha)] \leqslant -g(z,y), \quad y \in X.$$

Therefore, we have

$$\psi(z,\hat{x},y) \leqslant g(z,\hat{x}) - g(z,y), \quad y \in X,$$

that is, $\hat{x} \in S(z)$. This implies that $S(z)$ is a nonempty closed set in X.

On the other hand, for any finite set $\{z_1, \cdots, z_n\} \subset X$ we have

$$\begin{aligned}\bigcap_{i=1}^{n} S(z_i) &= \bigcap_{i=1}^{n} \{x \in X : \psi(z_i,x,y) \leqslant g(z_i,x) - g(z_i,y) \text{ for all } y \in X\} \\ &= \bigcap_{i=1}^{n} \bigcap_{y\in X} \{x \in X : \psi(z_i,x,y) \leqslant g(z_i,x) - g(z_i,y)\}.\end{aligned}$$

By the condition (iii), $\bigcap_{i=1}^{n} S(z_i)$ is W-chainable. Hence $\bigcap_{i=1}^{n} S(z_i)$ is chainable. On the other hand, for any $x \in X$, we have

$$\begin{aligned}X \backslash S^{-1}(x) &= \{z \in X : z \notin S^{-1}(x)\} = \{z \in X : x \notin S(z)\} \\ &= \{z \in X : \sup_{y\in X}[\psi(z,x,y) + g(z,y)] > g(z,x)\}.\end{aligned}$$

By the condition (iv), $X\backslash S^{-1}(x)$ is a closed W-chainable subset. It follows from Corollary 5.3 that there exists a $\hat{z} \in S(\hat{z})$, that is, $\psi(\hat{z},\hat{z},y) \leqslant g(\hat{z},\hat{z}) - g(\hat{z},y)$ for all $y \in X$. This completes the proof. □

By Corollaries 3.3 and 5.3 and using the similar method as in the proof of Theorem 6.1, we can obtain the following:

THEOREM 6.2. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T. Let $\psi : X \times X \times X \to R$ and $g : X \times X \to R$ be two functions satisfying $\psi(z,x,x) \geqslant 0$ for all $z,x \in X$ and the following conditions:*

(i) *$\psi(z,\cdot,y)$ is lower semi-continuous, $\psi(z,x,\cdot)$ is upper semi-continuous and $g(z,\cdot)$ is upper semi-continuous,*

(ii) *$\psi(z,x,y) + g(z,y)$ is probabilistic quasi-concave in y,*

(iii) *$\psi(z,x,y) - g(z,x)$ is probabilistic quasi-convex in x,*

(iv) *for any $y \in X$, $\{z \in X : \inf_{x\in X}[\psi(z,x,y) - g(z,x)] < -g(z,y)\}$ is a closed W-chainable subset in X.*

Then there exists a $\hat{z} \in X$ such that $\psi(\hat{z},x,\hat{z}) \geqslant g(\hat{z},x) - g(\hat{z},\hat{z})$ for all $x \in X$.

COROLLARY 6.3. *Let $(X,\mathcal{F},T)$ be a compact probabilistic interval space with a continuous t-norm T. Let $\psi : X \times X \times X \to R$ be a function satisfing $\psi(z,x,x) \leqslant 0$ for all $z,x \in X$ (respectively, $\psi(z,x,x) \geqslant 0$ for all $z,x \in X$) and the following conditions:*

(i) *$\psi(z,\cdot,y)$ is lower semi-continuous and probabilistic quasi-convex,*

(ii) *$\psi(z,x,\cdot)$ is upper semi-continuous and probabilistic quasi-concave,*

(iii) *for any $x \in X$, the set $\{z \in X : \sup_{y\in X} \psi(z,x,y) > 0\}$ (respectively, the set $\{z \in X : \inf_{x\in X} \psi(z,x,y) < 0\}$ for all $y \in X$) is a closed W-chainable subset.*

Then the implicit variational inequality of Ky Fan type

$$\psi(z,z,y) \leqslant 0$$

for all $y \in X$ (respectively, $\psi(z,x,z) \geqslant 0$ for all $x \in X$) has a solution in X.

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