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Further Generalizations of Minimax Inequalities for Mixed Concave–Convex Functions and Applications*

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In this paper we obtain a minimax inequality for generalized mixed concave–convex functions which contain the corresponding results from the literature as special cases and improves the corresponding result in (B. L. Lin and X. C. Quan, *J. Math. Anal. Appl.* **161** (1991), 587–590). As applications, in Sections 3 and 4, we utilize the results presented in this paper to study the abstract variational inequality problem and the coincidence point problem for set-valued mappings and for saddle problems.

1. INTRODUCTION AND PRELIMINARIES

The minimax problem is a very important problem in nonlinear analysis which plays a significant role in game theory and mathematical economics.

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Since 1928, when Von Neumann first gave a minimax theorem for mixed concave–convex functions, the minimax problem has been extensively studied by many authors (see [1–9]). The purpose of this paper is to obtain a minimax inequality theorem for generalized mixed concave–convex functions which generalizes the minimax theorem of Neumann type and contains the corresponding minimax theorems for mixed concave–convex functions in [1–4, 6–9] as its special cases. As applications, in Sections 3 and 4 we use the results presented in this paper to study abstract variational inequalities, the coincidence point problem for set-valued mappings and the saddle problem.

For the sake of convenience, we first give some definitions, notations, and some propositions.

Throughout this paper, we denote $R = (-\infty, +\infty)$. In this paper the topological space means the Hausdorff topological space. We denote by $\mathscr{C}(X, Y)$ the set of all continuous functions from X to Y and we denote by $\mathscr{F}(X)$ the family of all finite subsets of X.

DEFINITION 1.1. Let Z be a totally ordered space. Z is called a complete totally ordered space if each subset has a least upper bound. If, in addition, for any $z_1, z_2 \in Z$, $z_1 < z_2$, there exists $z_3 \in Z$ such that $z_1 < z_3 < z_2$, then Z is called a complete dense totally ordered space.

DEFINITION 1.2 [7]. A topological space X is called an interval space if there exists a mapping $[\cdot, \cdot]: X \times X \to C(X)$, where $C(X)$ is the family of all connected subsets of X, such that for any $x, y \in X$,

$$x, y \in [x, y] = [y, x].$$

DEFINITION 1.3 [7]. Let X be an interval space. A subset K of X is called W-convex if for any $x, y \in K$ we have $[x, y] \subset K$. A mapping $f: X \to Z$, where Z is a totally ordered space, is called quasi-convex (or quasi-concave), if for any $z \in Z$, the set

$$\{x \in X: f(x) \le z\} \qquad (\text{or } \{x \in X: f(x) \ge z\})$$

is W-convex in X.

DEFINITION 1.4 [4]. Let X be a topological space and Z be a totally ordered space. A mapping $f: X \to Z$ is called upper semi-continuous (or lower semi-continuous) if the set $\{x \in X: f(x) \geqslant z\}$ (or $\{x \in X: f(x) \leqslant z\}$) is a closed set in X for all $z \in Z$.

PROPOSITION 1.1 [7]. *If X is an interval space, then any W-convex set in X is connected set.*

PROPOSITION 1.2 [7]. *Let X be an interval space and Z be a complete dense totally ordered space and $f: X \to Z$ be a mapping. Then*

(i) *f is quasi-convex if and only if for any $x, y \in X$ and for any $z \in [x, y]$ the following holds:*

$$f(z) \leqslant \max\{f(x), f(y)\};$$

(ii) *f is quasi-concave if and only if for any $x, y \in X$ and for any $z \in [x, y]$ the following holds:*

$$f(z) \geqslant \min\{f(x), f(y)\}.$$

DEFINITION 1.5 [3]. A topological space X is called a strong interval space, if there exists a mapping $[\cdot, \cdot]: X \times X \to CP(X)$, where $CP(X)$ is the family of all path-connected subsets of X, such that for all $x, y \in X$, $x, y \in [x, y] = [y, x]$.

2. MINIMAX INEQUALITIES

THEOREM 2.1. *Let X be an interval space, Y be a topological space, and Z be a complete dense totally ordered space. If $f: X \times Y \to Z$ satisfies the following conditions:*

(i) *$f(x, \cdot)$ is upper semi-continuous for all $x \in X$;*

(ii) *for all $A \in \mathscr{F}(X)$ and for all $z \in Z$, the set*

$$\bigcap_{x \in A} \{y \in Y: f(x, y) > z\}$$

is connected;

(iii) *$f(\cdot, y)$ is quasi-convex for all $y \in Y$ and is lower semi-continuous on any connected set of X;*

(iv) *there exist $x_0 \in X$, $z_0 < \inf_{x \in X} \sup_{y \in Y} f(x, y)$ and a compact subset $L \subset Y$ such that*

$$f(x_0, y) < z_0 \qquad \textit{for all } y \in Y \backslash L.$$

Then

$$z_* = \sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y) = z^*.$$

Proof. By the completeness of Z, both z^* and z_* are defined and are in Z. It is obvious that $z^* \geqslant z_*$. Now we prove that $z_* \geqslant z^*$.

In fact, for any $z \in Z$ with $z < z^*$ and for any $x \in X$, let

$$F(x, z) = \{y \in Y: f(x, y) \geqslant z\},$$

$$G(x, z) = \{y \in Y: f(x, y) > z\}.$$

By the definition of z^*, $G(x, z) \neq \varnothing$ for all $x \in X$ and $z < z^*$. By condition (i) we know that $F(x, z)$ is closed, and it is obvious that $G(x, z) \subset F(x, z)$.

Next we prove that the family $\{F(x, z): x \in X, z < z^*\}$ has the finite intersection property.

It is obvious that for all $x \in X$ and $z < z^*$, $F(x, z) \neq \varnothing$. Suppose that for any n elements of $\{F(x, z): x \in X, z < z^*\}$, $n \geqslant 2$, their intersection is nonempty, now we prove that for any $n + 1$ elements of $\{F(x, z): x \in X, z < z^*\}$ their intersection is also nonempty. Suppose the contrary, then there exist subset $\{x_1, \ldots, x_{n+1}\} \subset X$ and $\{z_1, \ldots, z_{n+1}\} \subset Z$ satisfying $z_i < z^*$, $i = 1, 2, \ldots, n + 1$, such that $\cap_{i=1}^{n+1} F(x_i, z_i) = \varnothing$. Because Z is totally ordered space, without loss of generality, we can assume that $z^* > z_1 \geqslant z_2 \geqslant \ldots \geqslant z_{n+1}$. Since Z is dense, there exists a $\bar{z} \in Z$ such that $z^* > \bar{z} > z_1$.

Now we define two set-valued mappings $T, U: X \to 2^Y$ by

$$T(x) = \{y \in Y: f(x, y) > z_1\} = G(x, z_1), \qquad \text{for all } x \in X,$$
$$U(x) = \{y \in Y: f(x, y) \geqslant z_1\} = F(x, z_1), \qquad \text{for all } x \in X.$$

Letting $H = \cap_{i=3}^{n+1} T(x_i)$, then for all $x \in X$, by the assumption of induction we know that

$$H \cap T(x) = \left(\bigcap_{i=3}^{n+1} G(x_i, z_1)\right) \bigcap G(x, z_1)$$
$$\supset \left(\bigcap_{i=3}^{n+1} F(x_i, \bar{z})\right) \bigcap F(x, \bar{z}) \neq \varnothing.$$

Hence, by condition (ii) we know that $H \cap T(x)$ is a nonempty connected set.

Next by condition (i), we have

$$\overline{(H \cap T(x_1))} \cap \overline{(H \cap T(x_2))} = \overline{H \cap T(x_1) \cap T(x_2)} \subset \bigcap_{i=1}^{n+1} F(x_i, z_i) = \varnothing.$$

This implies that $H \cap T(x_1)$ and $H \cap T(x_2)$ are a pair of separating sets.

Now we prove that $T([x_1, x_2]) \subset \cup_{i=1}^{2} T(x_i)$. In fact, if $y \notin \cup_{i=1}^{2} T(x_i)$, then $f(x_i, y) \leqslant z_1$, $i = 1, 2$, i.e., $\{x_1, x_2\} \subset \{x \in X: f(x, y) \leqslant z_1\}$. Since $f(\cdot, y)$ is quasi-convex, we know that

$$[x_1, x_2] \subset \{x \in X: f(x, y) \leqslant z_1\}.$$

This implies that $y \notin T(x)$ for all $x \in [x_1, x_2]$; therefore, the desired conclusion is proved. Hence we have

$$H \cap T([x_1, x_2]) \subset \bigcup_{i=1}^{2} H \cap T(x_i).$$

Thus for any $x \in [x_1, x_2]$, we have $H \cap T(x) \subset \cup_{i=1}^{2} H \cap T(x_i)$. By the connectedness of $H \cap T(x)$ and the separating property of $H \cap T(x_1)$ and $H \cap T(x_2)$, we know that $H \cap T(x) \subset H \cap T(x_1)$ or $H \cap T(x) \subset H \cap T(x_2)$ for all $x\ [x_1, x_2]$.

Letting

$$E_1 = \{x \in [x_1, x_2]: H \cap T(x) \subset H \cap T(x_1)\},$$
$$E_2 = \{x \in [x_1, x_2]: H \cap T(x) \subset H \cap T(x_2)\},$$

we know that $E_i \neq \varnothing$, $i = 1, 2$ (because $x_i \in E_i$) and $[x_1, x_2] = E_1 \cup E_2$. Since $[x_1, x_2]$ is nonempty and connected, we know that $E_1 \cap \overline{E}_2$ or $\overline{E}_1 \cap E_2$ is nonempty. Without loss of generality, we can assume $E_1 \cap \overline{E}_2 \neq \varnothing$. Hence there exist $u_0 \in E_1 \cap \overline{E}_2$ (therefore $H \cap T(u_0) \subset H \cap T(x_1)$) and a net $\{x_\alpha\}_{\alpha\in J} \subset E_2$ such that $x_\alpha \to u_0$. By the definition of E_2, $H \cap T(x_\alpha) \subset H \cap T(x_2)$ for all $\alpha \in J$. Since $H \cap T(u_0) \neq \varnothing$, taking $y_0 \in H \cap T(u_0)$, we know $y_0 \in H \cap T(x_1)$. Thus $y_0 \notin H \cap T(x_2)$, and so $y_0 \notin T(x_\alpha)$ for all $\alpha \in J$, i.e.,

$$\{x_\alpha\}_{\alpha\in J} \subset X\backslash T^{-1}(y_0) = \{x \in X: f(x, y_0) \leq z_1\}.$$

However, since $\{x_\alpha\}_{\alpha\in J} \subset E_2 \subset [x_1, x_2]$, we have

$$\{x_\alpha\}_{\alpha\in J} \subset \{x \in [x_1, x_2]: f(x, y_0) \leq z_1\}.$$

In view of $x_\alpha \to u_0 \in [x_1, x_2]$ and condition (iii), we have $f(u_0, y_0) \leqslant z_1$, i.e., $y_0 \notin T(u_0)$. This contradicts the choice of y_0. Therefore the family $\{F(x, z): x \in X, z < z^*\}$ of sets has the finite intersection property. By condition (iv), there exist $z_0 < z^*$, $x_0 \in X$ and a compact subset L in Y such that $F(x_0, z_0) \subset L$. Since $F(x_0, z_0)$ is closed, $F(x_0, z_0)$ is compact. Therefore $\cap_{x\in X, z<z^*} F(x, z) \neq \varnothing$, and so there exists $\bar{y} \in Y$ such that

$$\bar{y} \in F(x, z) \qquad \text{for all } x \in X \text{ and for all } z < z^*;$$

i.e., $f(x, \bar{y}) \geqslant z$ for all $x \in X$ and for all $z < z^*$. Hence we have

$$z \leqslant \inf_{x\in X} f(x, \bar{y}) \qquad \text{for all } z < z^*.$$

Thus we have $\sup_{y\in Y} \inf_{x\in X} f(x, y) \geqslant z$ for all $z < z^*$. By the density of Z we have

$$\sup_{y\in Y} \inf_{x\in X} f(x, y) \geqslant z^*.$$

This completes the proof.

Remark. Since strong interval space must be a interval space, Theorem 2.1 contains the main results in [3] as a special case, and so it also contains the main results in Brezis *et al.* [1], Komornik [4], and Geraghty and Lin [6] as special cases.

COROLLARY 2.2. *Let X be an interval space, Y a topological space and $f: X \times Y \to R$ satisfy the following conditions:*

(i) *$f(x, \cdot)$ is upper semi-continuous for any given $x \in X$;*

(ii) *for any $A \in \mathscr{F}(X)$ and for all $r \in R$, the set $\cap_{x\in A} \{y \in Y: f(x, y) \geqslant r\}$ is connected;*

(iii) *for any given $y \in Y$, $f(\cdot, y)$ is quasi-convex, and is lower semi-continuous on any connected set of X;*

(iv) *there exist $x_0 \in X$, $z_0 < \inf_{x\in X} \sup_{y\in Y} f(x, y)$ and a compact set L of Y such that $f(x_0, y) < z_0$ for all $y \in Y\backslash L$.*

Then

$$\sup_{y\in Y} \inf_{x\in X} f(x, y) = \inf_{x\in X} \sup_{y\in Y} f(x, y).$$

Proof. It is sufficient to prove that for all $A \in \mathscr{F}(X)$ and for all $\alpha \in R$ the set $\cap_{x\in A}\{y \in Y: f(x, y) > \alpha\}$ is connected. In fact, because

$$\bigcap_{x\in A} \{y \in Y: f(x, y) > \alpha\} = \bigcup_{\varepsilon>0} \bigcap_{x\in A} \{y \in Y: f(x, y) \geqslant \alpha + \varepsilon\},$$

if for all $\varepsilon > 0$, $\cap_{x\in A} \{y \in Y: f(x, y) \geqslant \alpha + \varepsilon\} = \varnothing$, then

$$\bigcap_{x\in A} \{y \in Y: f(x, y) > \alpha\} = \varnothing$$

is connected. Therefore without loss of generality we can assume that there exists $\varepsilon_0 > 0$ such that $\cap_{x\in A}\{y \in Y: f(x, y) \geqslant \alpha + \varepsilon_0\} \neq \varnothing$. Noting that when $\varepsilon > \varepsilon_0$,

$$\bigcap_{x\in A} \{y \in Y: f(x, y) \geqslant \alpha + \varepsilon_0\} \supset \bigcap_{x\in A} \{y \in Y: f(x, y) \geqslant \alpha + \varepsilon\},$$

we have

$$\bigcap_{x\in A}\{y\in Y: f(x,y)>\alpha\}=\bigcup_{0<\varepsilon\le\varepsilon_0}\bigcap_{x\in A}\{y\in Y: f(x,y)\geqslant\alpha+\varepsilon\}.$$

Besides, because

$$\bigcap_{0<\varepsilon\le\varepsilon_0}\left(\bigcap_{x\in A}\{y\in Y: f(x,y)\geqslant\alpha+\varepsilon\}\right)$$
$$=\bigcap_{x\in A}\{y\in Y: f(x,y)\geqslant\alpha+\varepsilon_0\}\neq\varnothing,$$

$\bigcap_{x\in A}\{y\in Y: f(x,y)>\alpha\}$ is connected.

COROLLARY 2.3. *Let X and Y be two topological spaces, and f, g: $X\times Y\to R$ be two functions satisfying the following conditions:*

(i) *$f(x,y)\leqslant g(x,y)$ for all $x\in X$ and for all $y\in Y$;*

(ii) *$g(x,\cdot)$ is lower semi-continuous and $g(\cdot,y)$ is upper semi-continuous;*

(iii) (a) *for all $B\in\mathscr{F}(Y)$ and for all $r\in R$, $\bigcap_{y\in B}\{x\in X: g(x,y)\geqslant r\}$ is connected;*

(b) *for all $y_1,y_2\in Y$, there exists a connected subset $C_{\{y_1,y_2\}}$ containing y_1 and y_2 such that*

$$g(x,y)\leqslant\max\{f(x,y_1),g(x,y_2)\},\qquad \textit{for all } y\in C_{\{y_1,y_2\}} \textit{ and all } x\in X;$$

(iv) *there exist $y_0\in Y$, $r_0<\inf_{y\in Y}\sup_{x\in X} g(x,y)$ and a compact subset L of X such that $g(x,y_0)<r_0$ for all $x\in X\backslash L$.*

Then

$$\sup_{x\in X}\inf_{y\in Y} g(x,y)\geqslant\inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Proof. For any $y_1,y_2\in Y$, taking $[y_1,y_2]=C_{\{y_1,y_2\}}$ then Y is an interval space. By condition (iii), we know that $g(x,\cdot)$ is quasi-convex. Therefore the conclusion can be obtained from Theorem 2.1.

Remark. Corollary 2.3 generalizes and improves Theorem 1 in Lin and Quan [5].

COROLLARY 2.4. *Let X and Y be two interval spaces, Z be a complete dense totally ordered space and f: $X\times Y\to Z$ be a mapping satisfying the following conditions:*

(i) *$f(x,\cdot)$ is upper semi-continuous and quasi-concave for all $y\in Y$;*

(ii) *$f(\cdot,y)$ is quasi-convex and is lower semi-continuous on any connected set of X;*

(iii) *there exist $x_0\in X$, $z_0<\inf_{x\in X}\sup_{y\in Y} f(x,y)$ and a compact subset $L\subset Y$ such that $f(x_0,y)<z_0$ for all $y\in Y\backslash L$.*

Then

$$\sup_{y\in Y}\inf_{x\in X} f(x,y)=\inf_{x\in X}\sup_{y\in Y} f(x,y).$$

Remark. Corollary 2.4 contains the main results in Sion [8] and Stachó [7] as its special cases and weakens the constraint conditions on space Y.

COROLLARY 2.5. *Let X be a compact topological space, Y a path-connected space and f: $X\times Y\to R$ be a function satisfying the following conditions:*

(i) *$f(\cdot,y)$ is lower semi-continuous and $f(x,\cdot)$ is upper semi-continuous;*

(ii) *for any $y_0,y_1\in Y$, there exists a continuous mapping h: $[0,1]=I\to Y$ such that $h(0)=y_0$, $h(1)=y_1$ and $\{t\in[0,1]: f(x,h(t))\geq\alpha\}$ is connected for all $x\in X$ and for all $\alpha\in R$;*

(iii) *for all $B\in\mathscr{F}(Y)$ and for all $\alpha\in R$, the set*

$$\bigcap_{y\in B}\{x\in X: f(x,y)<\alpha\}$$

is connected.

Then

$$\sup_{y\in Y}\inf_{x\in X} f(x,y=\inf_{x\in X}\sup_{y\in Y} f(x,y).$$

Proof. Let $g(y,x)=-f(x,y)$, for $x\in X$ and $y\in Y$. Then $g(y,\cdot)$ is upper semi-continuous and $g(\cdot,x)$ is lower semi-continuous. By condition (iii), for all $B\in\mathscr{F}(Y)$ and for all $r\in R$ the set $\bigcap_{y\in B}\{x\in X: g(y,x)>r\}$ is connected. By condition (ii), for all $y_0,y_1\in Y$, there exists a continuous mapping $h: I\to Y$ such that $h(0)=y_0$, $h(1)=y_1$, and $\{t\in I: f(x,h(t))\geqslant r\}$ is connected for all $x\in X$ and for $r\in R$. Letting $[y_0,y_1]=h(I)$, then we know Y is an interval space.

Besides, for any $y_0,y_1\in Y$ and for any $y\in[y_0,y_1]$ and any $x\in X$, there exists $t_0\in I$ such that $y=h(t_0)$. Now we take

$$r=-\max\{g(y_0,x),g(y_1,x)\}=-\max\{-f(x,y_0),-f(x,y_1)\}$$
$$=\min\{f(x,y_0),f(x,y_1)\}$$

Since $\{t \in I: f(x, h(t)) \geqslant r\}$ is connected and it is obvious that $\{0, 1\} \subset \{t \in I: f(x, h(t)) \geqslant r\}$, thus $I = \{t \in I: f(x, h(t)) \geqslant r\}$, and so $f(x, h(t_0) \geqslant r$. Hence

$$g(y, x) \leqslant -r = \max\{g(y_0, x), g(y_1, x)\}.$$

This implies that $g(\cdot, x)$ is quasi-convex. By Theorem 2.1, we have

$$\sup_{x \in X} \inf_{y \in Y} g(y, x) = \inf_{y \in Y} \sup_{x \in X} g(y, x).$$

Therefore, we have

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y).$$

Remark. Corollary 2.5 contains the main results of Wu [9] as its special case.

COROLLARY 2.6. *Let X be a compact interval space, Y be a compact topological space and $f: X \times Y \to R$ be a function satisfying the following conditions:*

(i) *$f(x, \cdot)$ is upper semi-continuous for all $x \in X$;*

(ii) *for all $A \in \mathscr{F}(X)$ and for all $r \in R$, the set*

$$\bigcap_{x \in A} \{y \in Y: f(x, y) > r\}$$

is connected;

(iii) *$f(\cdot, y)$ is quasi-convex and is lower semi-continuous on X.*

Then there exists saddle point $(\bar{x}, \bar{y}) \in X \times Y$ of f.

Proof. Using our Theorem 2.1 along with Proposition 1.4.6 and Theorem 3.10.4 in [2], we can obtain the conclusion of Corollary 2.6 immediately.

Remark. Corollary 2.6 contains the famous Neumann Theorem as its special case. As is known to all, the Neumann Theorem plays an important role in the theory of mathematical economics and game theory.

3. APPLICATIONS TO VARIATIONAL INEQUALITIES

In this section we shall use the results presented in Section 2 to study the variational inequality problems. We have the following

THEOREM 3.1. *Let X be a compact interval space and $\varphi: X \times X \to R$ a function satisfying $\varphi(x, x) \leq 0$, for all $x \in X$ and the following conditions:*

(i) *$\varphi(x, \cdot)$ is upper semi-continuous;*

(ii) *for all $A \in \mathscr{F}(X)$ and for all $r \in R$ the set*

$$\bigcap_{x \in A} \{y \in X: \varphi(x, y) > r\}$$

is connected;

(iii) *$\varphi(\cdot, y)$ is quasi-convex and lower semi-continuous.*

Then there exists an $x_0 \in X$ such that $\varphi(x_0, y) \leqslant 0$ for all $y \in X$.

Proof. By Theorem 2.1, we have

$$\sup_{y \in X} \inf_{x \in X} \varphi(x, y) = \inf_{x \in X} \sup_{y \in X} \varphi(x, y)$$

Since $\varphi(\cdot, y)$ is lower semi-continuous, by using Proposition 1.4.6 in [2], we know that $\sup_{y \in X} \varphi(x, y)$ is lower semi-continuous in x; thus, by the compactness of X, we know that there exists $x^* \in X$ such that

$$\sup_{y \in X} \inf_{x \in X} \varphi(x, y) = \inf_{x \in X} \sup_{y \in X} \varphi(x, y) = \sup_{y \in X} \varphi(x^*, y).$$

Since $\inf_{x \in X} \sup_{y \in X} \varphi(x, y) \leqslant \sup_{y \in X} \varphi(y, y) \leqslant 0$, we have $\varphi(x^*, y) \leqslant 0$ for all y $\in$ X.

COROLLARY 3.2. *Let X be a compact interval space, $\varphi: X \times X \to R$ a function with $\varphi(x, x) \leqslant 0$ for all $x \in X$ and $h: X \to R$ be an upper semi-continuous function. If the following conditions are satisfied*

(i) *$\varphi(x, \cdot)$ is upper semi-continuous and $\varphi(\cdot, y)$ is lower semi-continuous;*

(ii) *for all $A \in \mathscr{F}(X)$ and for all $r \in R$ the set*

$$\bigcap_{x \in A} \{y \in Y: \varphi(x, y) + h(y) > r\}$$

is connected;

(iii) *$\varphi(x, y) - h(x)$ is quasi-convex in x.*

Then there exists an $x^* \in X$ such that

$$\varphi(x^*, y) \leq h(x^*) - h(y) \qquad \text{for all } y \in X.$$

Proof. Letting $\psi(x, y) = \varphi(x, y) - h(x) + h(y)$, it is easy to prove that ψ satisfies all the conditions in Theorem 3.1. Therefore the conclusion follows from Theorem 3.1 immediately.

Remark. If condition (ii) in Corollary 3.1 is replaced by

(ii)′ $\varphi(x, y) + h(y)$ is quasi-concave in y,

then the conclusion of Corollary 3.2 still holds because condition (ii)′ implies condition (ii).

4. Applications to Coincidence Point Problems

In this section we use the results presented in Section 2 to study coincidence problems. We have the following results.

THEOREM 4.1. *Let X be a compact interval space, Y be a topological space and $F: X \to 2^Y$ be a mapping with nonempty closed values, $S \in \mathscr{C}(X, Y)$. If $S^{-1}F(x) \neq \varnothing$ for all $x \in X$ and satisfies the following conditions:*

(i) *for all $A \in \mathscr{F}(X)$, $\bigcap_{x \in A} S^{-1}F(x)$ is connected;*

(ii) *for all $x \in X$, $F^{-1}(S(x))$ is open and $X \backslash F^{-1}(S(x))$ is W-convex.*

Then F and S have a coincidence point x^* in X, i.e., $S(x^*) \in F(x^*)$.

Proof. Define a mapping $f: X \times X \to R$ as follows:

$$f(x, z) = \begin{cases} 0, & \text{if } S(z) \notin F(x), \\ 1, & \text{if } S(z) \in F(x). \end{cases}$$

If F and S have no coincidence point in X, then for all $x \in X$, we have $S(x) \notin F(x)$, and so $f(x, x) = 0$. For any $\alpha \in R$, since

$$M = \{z \in X: f(x, z) \geq \alpha\} = \begin{cases} S^{-1}F(x), & \text{if } 0 < \alpha \leq 1, \\ X, & \text{if } \alpha \leq 0, \\ \varnothing, & \text{if } \alpha > 1, \end{cases}$$

and F is a closed valued mapping and S is continuous, we know that M is closed. Therefore $f(x, \cdot)$ is upper semi-continuous.

Furthermore, for any $\alpha \in R$ and for any $A \in \mathscr{F}(X)$ we have

$$\bigcap_{x \in A} \{z \in X: f(x, z) > \alpha\} = \begin{cases} \varnothing, & \text{if } \alpha \geq 1, \\ \bigcap_{x \in A} \{S^{-1}F(x)\}, & \text{if } 0 \leq \alpha < 1, \\ X, & \text{if } \alpha < 0. \end{cases}$$

By condition (i), for all $\alpha \in R$ and for all $A \in \mathscr{F}(X)$, the set

$$\bigcap_{x \in A} \{z \in X: f(x, z) > \alpha\}$$

is connected because for all $\alpha \in R$ and for all $z \in X$

$$\{x \in X: f(x, z) \leq \alpha\} = \begin{cases} \varnothing, & \text{if } \alpha < 0, \\ X, & \text{if } \alpha \geq 1, \\ X \backslash F^{-1}(S(z)), & \text{if } 0 \leq \alpha < 1. \end{cases}$$

Thus by condition (ii) we know that $f(\cdot, z)$ is lower semi-continuous and quasi-convex. Hence by Theorem 3.1 we know that there exists an x^* such that

$$f(x^*, z) \leq 0 \qquad \text{for all } z \in X.$$

However, since $f(x^*, z) \geq 0$ for all $z \in X$, we have $f(x^*, z) = 0$ for all $z \in X$, i.e., $S(z) \notin F(x^*)$ for all $z \in X$. Therefore for all $z \in X$, we have $z \notin S^{-1}F(x^*)$, and so $S^{-1}F(x^*) = \varnothing$. This contradicts $S^{-1}F(x) \neq \varnothing$ for all $x \in X$. Thus F and S have a coincidence point in X.

COROLLARY 4.2. *Let X be a compact interval space and $F: X \to 2^X$ be a mapping with nonempty closed values. If the following conditions are satisfied;*

(i) *for any $A \in \mathscr{F}(X)$, $\bigcap_{x \in A} F(x)$ is connected;*

(ii) *for any $y \in X$, $F^{-1}(y)$ is open and $X \backslash F^{-1}(y)$ is W-convex;*

then F has a fixed point in X.

Proof. Taking $Y = X$ and $S = I$ (identity mapping) in Theorem 4.1, the conclusion follows from Theorem 4.1.

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