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Economic Equilibrium Theorems of Shafer–Sonnenschein Version and Nonempty Intersection Theorems in Interval Spaces*

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In this paper some equilibrium existence theorems for an abstract economy (or generalized game) of Shafer–Sonnenschein version and some nonempty interaction theorems for sets with W-convex sections in interval spaces are obtained. The results presented in this paper all are new and they are comparable to corresponding results in the relevant literature.

1. Introduction and Preliminaries

In recent years great progress has been made in the theory and applications for the study of the abstract economic equilibrium problems and

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nonempty intersection theorems for sets with convex section (see [1, 4, 5, 7–13, 18, 19] and the references therein). But the study of equilibrium problems for an abstract economy used to be nonexistent in interval spaces without any linear structure.

The purpose of this paper is to study the Shafer–Sonnenschein type of the abstract economic equilibrium problems and the nonempty intersection problems for sets with W-convex section in interval spaces without any linear structure. The results presented in this paper are new, even in the case of topological vector spaces.

For the sake of convenience, we first give some definitions, notations, and terminologies.

DEFINITION 1.1 [17]. A topological space X is called an interval space, if there exists a mapping $[\cdot, \cdot]: X \times X \to 2^X$ such that for any $x_1, x_2 \in X$, $[x_1, x_2] = [x_2, x_1]$ is a connected set containing $\{x_1, x_2\}$.

DEFINITION 1.2 [17]. Let X be an interval space. A subset K of X is called to be W-convex, if for any $x_1, x_2 \in K$ we have $[x_1, x_2] \subset K$.

DEFINITION 1.3 [17]. An interval space X is called Dedekind complete, if for any $x_1, x_2 \in X$ and for any W-convex sets $H_1, H_2 \subset X$, if $x_j \in H_j$, $j = 1, 2$, and $[x_1, x_2] \subset H_1 \cup H_2$, then $k \in \{1, 2\}$ and an $x \in H_k$ must exist such that $[x_i, x) := [x_i, x] \backslash \{x\} \subset H_i$, $i \in \{1, 2\} \backslash \{k\}$.

DEFINITION 1.4. Let X be a Dedekind complete interval space. X is called strongly Dedekind complete, if for any $x_1, x_2 \in X$ and for any finite subset $\{u_1, ..., u_n\} \subset [x_2, x_1)$, we have $\bigcap_{i=1}^n [u_i, x_1) \neq \varnothing$.

Remark. It was pointed out in [3] that if Y is a convex set in a Hausdorff topological vector space, for any $y_1, y_2 \in Y$, letting $[y_1, y_2] :=$ co $\{x_1, x_2\}$, then Y is a strongly Dedekind complete interval space. Moreover, if K is a W-convex set, then K must be a connected subset.

Let $\{X_i : i \in I\}$ be a family of interval spaces, $X := \prod_{i \in I} X_i$. For any x, $y \in X$, letting $[x, y] := \prod_{i \in I} [x_i, y_i]$, where x_i and y_i are the projections of x and y onto X_i, respectively, then X is an interval space and it is called the product interval space of $\{X_i : i \in I\}$.

In the sequel we always denote $X_{-i} := \prod_{k \in I, k \neq i} X_k$ and denote by π_i and π_{-i} the projections of X onto X_i and X_{-i}, respectively. For any $x \in X$, denote $x_{-i} := \pi_{-i}(x)$, $x_i = \pi_i(x)$, and $x = x_i \otimes x_{-i}$. Letting $A \subset X$, for any $x_i \in X_i$ and for any $y_{-i} \in X_{-i}$ we denote

$$A(x_i) = \{x_{-i} \in X_{-i} : x_i \otimes x_{-i} \in A\}$$
$$A(y_{-i}) = \{y_i \in X_i : y_i \otimes y_{-i} \in A\}.$$

DEFINITION 1.5. Let Y be an interval space and let B be a subset of Y. The intersection of all W-convex closed subsets in Y containing B is called the W-closed convex hull of B and we denote it by $W\text{-}\overline{\text{co}}\, B$.

Letting $F, G: X \to 2^Y$ be two set-valued mappings, we denote

$$(F \cap G)(x) := F(x) \cap G(x).$$

2. LEMMAS

LEMMA 2.1 [17]. *Let X be an interval space, Y a topological space, and $F: X \to 2^Y$ a set-valued mapping with nonempty compact values. If the following conditions are satisfied:*

(i) *for any $x_1, x_2 \in X$, $F(x) \subset \bigcup_{i=1}^2 F(x_i)$ for all $x \in [x_1, x_2]$;*

(ii) *for any $x_1, ..., x_n \in X$, $\bigcap_{i=1}^n F(x_i)$ is connected;*

(iii) *the graph of F is closed;*

then $\bigcap_{x \in X} F(x) \neq \varnothing$.

From Lemma 2.1 we can obtain the following result immediately:

LEMMA 2.2. *Let X be an interval space, $D \subset X$ a nonempty subset, and $F: X \to 2^D$ a set-valued mapping with nonempty compact W-convex values and satisfying the following conditions:*

(i) *for any $y \in D$, $X \backslash F^{-1}(y)$ is W-convex;*

(ii) *the graph of F is closed.*

Then F has a fixed point in D.

Proof. Since F is a mapping with W-convex values, we know that F satisfies condition (ii) in Lemma 2.1. Furthermore, for any $x_1, x_2 \in X$ and for any $y \in D$, if $y \notin \bigcup_{i=1}^2 F(x_i)$, then $\{x_1, x_2\} \subset X \backslash F^{-1}(y)$. By condition (i), $[x_1, x_2] \subset X \backslash F^{-1}(y)$, and so for all $x \in [x_1, x_2]$ we have $y \notin F(x)$. Therefore we have

$$F(x) \subset \bigcup_{i=1}^{2} F(x_i) \qquad \text{for all } x \in [x_1, x_2].$$

This implies that F satisfies condition (i) in Lemma 2.1. Therefore F satisfies all conditions in Lemma 2.1. By Lemma 2.1 we know

$$\bigcap_{x \in X} F(x) \neq \varnothing.$$

Hence there exists a $y \in D$ such that $y \in F(x)$ for all $x \in X$, and so $y \in F(y)$, i.e., y is a fixed point of F in D.

LEMMA 2.3. [3]. *Let X be a strongly Dedekind complete Hausdorff interval space, Y an interval space, and $F: X \to 2^Y$ a set-valued mapping with nonempty compact values. If the following conditions are satisfied:*

(i) *for any finite subset $A \subset X$, $\cap_{x \in A} F(x)$ is connected;*

(ii) *for any $x_1, x_2 \in X$, $F(x) \subset F(x_1) \cup F(x_2)$ for all $x \in [x_1, x_2]$;*

(iii) *for any $y \in Y$, $F^{-1}(y)$ is closed;*

then $\cap_{x \in X} F(x) \neq \phi$.

By Lemma 2.3 and by using the same method as in the proof of Lemma 2.2 we can prove the following.

LEMMA 2.4. *Let X be a strongly Dedekind complete Hausdorff interval space, D a nonempty W-convex subset of X, and $F: X \to 2^D$ a set-valued mapping with nonempty compact W-convex values. If the following conditions are satisfied:*

(i) *for any $y \in D$, $X \backslash F^{-1}(y)$ is W-convex;*

(ii) *for any $y \in D$, $F^{-1}(y)$ is closed.*

then F has a fixed point in D.

3. ECONOMIC EQUILIBRIUM THEOREMS OF SHAFER–SONNENSCHEIN VERSION

DEFINITION 3.1. An abstract economy (or generalized game) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i), where I is a finite or an infinite set of agents, X_i is a nonempty topological space (a choice set), A_i, B_i: $\prod_{k \in I} X_k \to 2^{X_i}$ are constraint correspondences, and P_i: $\prod_{k \in I} X_k \to 2^{X_i}$ is a preference correspondence. An equilibrium for Γ is a point $\bar{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\bar{x}_i \in \bar{B}_i(\bar{x})$ and $P_i(\bar{x}) \cap A_i(\bar{x}) = \phi$. When $A_i = B_i$ for each $i \in I$ and X_i is a topological vector space, $i \in I$, then the definitions of an abstract economy and an equilibrium coincide with the standard definition of Shafer–Sonnenschein [14].

THEOREM 3.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy satisfying the following conditions:*

(i) *for any $i \in I$, X_i is a Hausdorff interval space and their product interval space $X := \prod_{i \in I} X_i$ is strongly Dedekind complete;*

(ii) *for any $i \in I$, D_i is a nonempty compact W-convex subset of X_i;*

the constraint correspondences A_i, B_i: $X \to 2^{D_i}$ and the preference correspondence $P_i: X \to 2^{D_i}$ satisfy the following conditions:

(a) *for all $x \in X$, $\phi \neq A_i(x) \subset B_i(x)$ is W-convex closed;*

(b) *the set $M_i := \{x \in X: A_i(x) \cap P_i(x) \neq \phi\}$ is open, $B_i^{-1}(y_i)$ and $(W\text{-}co(A_i \cap P_i))^{-1}(y_i)$ are both closed for all $y_i \in D_i$;*

(c) *for any $y \in D := \prod_{i \in I} D_i$, $y_i \notin W\text{-}\overline{co}(A_i \cap P_i)(y)$;*

(iii) *for any $y \in D$, for any $x_1, x_2 \in X$, and for any $x \in (x_1, x_2) := [x_1, x_2] \backslash \{x_1, x_2\}$, there exists an $i_0 \in I$ such that $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ and $y_{i_0} \notin W\text{-}\overline{co}(A_{i_0} \cap P_{i_0})(x)$.*

Then Γ has an equilibrium $\bar{x}$ in D, i.e., for any $i \in I$

$$\bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \phi.$$

Proof. For each $i \in I$ we define a mapping $T_i: X \to 2^{D_i}$ as follows:

$$T_i(x) = \begin{cases} W\text{-}\overline{co}(A_i \cap P_i)(x), & \text{if } x \in M_i; \\ B_i(x), & \text{if } x \notin M_i. \end{cases}$$

It is easy to see that T_i is a set-valued mapping with nonempty compact W-convex values. By condition (ii)(a), for any $y_i \in D_i$

$$\begin{aligned} T_i^{-1}(y_i) &= \{x \in X: y_i \in T_i(x)\} \\ &= \{x \in M_i: y_i \in T_i(x)\} \cup \{x \in X \backslash M_i: y_i \in T_i(x)\} \\ &= \{x \in X: y_i \in W\text{-}\overline{co}(A_i \cap P_i)(x)\} \cup ((X \backslash M_i) \cap B_i^{-1}(y_i)) \\ &= (W\text{-}\overline{co}(A_i \cap P_I))^{-1}(y_i) \cup ((X \backslash M_i) \cap B_i^{-1}(y_i)). \end{aligned}$$

By condition (ii)(b), $T_i^{-1}(y_i)$ is closed for all $y_i \in D_i$. Denote

$$T(x) := \prod_{i \in I} T_i(x), \qquad x \in X$$

then $T: X \to 2^D$ is a set-valued mapping with nonempty compact W-convex values and for any $y \in D$

$$T^{-1}(y) = \{x \in X: y \in Tx\} = \bigcap_{i \in I} T_i^{-1}(y_i)$$

is a closed set in X.

Next we prove that for each $y \in D$, $X \backslash T^{-1}(y)$ is W-convex.

In fact, if there exists a $y \in D$ such that $X \backslash T^{-1}(y)$ is not W-convex, then there exist $x_1, x_2 \in X \backslash T^{-1}(y)$ and $x \in (x_1, x_2)$ such that $x \notin X \backslash T^{-1}(y)$, i.e., $y \in Tx$. Hence for any $i \in I$, $y_i \in T_i(x)$. On the other hand, it follows from condition (iii) that there exists an $i_0 \in I$ such that $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ and $y_{i_0} \notin W\text{-}\overline{\text{co}}(A_{i_0} \cap P_{i_0})(x)$, i.e., $y_{i_0} \notin T_{i_0}(x)$, a contradiction. Therefore the conclusion is true.

Summing up the above arguments, we know that T satisfies all the conditions in Lemma 2.4. By Lemma 2.4, there exists an $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$. Hence for all $i \in I$, $\bar{x}_i \in T_i(\bar{x})$. Again by condition (ii)(c) we know that $\bar{x} \notin M_i$ and for all $i \in I$, $\bar{x}_i \in B_i(\bar{x})$. This means that for all $i \in I$ we have $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \phi$. This completes the proof.

COROLLARY 3.2. *Let* $\Gamma = (A_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy satisfying the following conditions*:

(i) *for any* $i \in I$, X_i *is a nonempty convex subset of a Hausdorff topological vector space* E_i *and* $X := \prod_{i \in I} X_i$;

(ii) *for any* $i \in I$, D_i *is a nonempty compact convex subset of* X_i *and the constraint correspondences* $A_i, B_i : X \to 2^{D_i}$ *and the preference correspondence* $P_i : X \to 2^{D_i}$ *satisfy the following conditions*:

(a) *for each* $x \in X$, $B_i(x) \supset A_i(x) \neq \phi$ *and* $B_i(x)$ *is closed convex*;

(b) *the set* $M_i := \{x \in X: A_i(x) \cap P_i(x) \neq \phi\}$ *is open and for all* $y_i \in D_i$, $B_i^{-1}(y_i)$ *and* $(\overline{\text{co}}(A_i \cap P_i))^{-1}(y_i)$ *are both closed*;

(c) *for each* $y \in D := \prod_{i \in I} D_i$, $y_i \notin \overline{\text{co}}(A_i \cap P_i)(y)$;

(iii) *for any* $y \in D$, *for any* $x_1, x_2 \in X$, *and for any* $x \in co\{x_1, x_2\} \backslash \{x_1, x_2\}$ *there exists an* $i_0 \in I$ *such that*

$$A_{i_0}(x) \cap P_{i_0}(x) \neq \phi \quad \text{and} \quad y_{i_0} \notin \overline{\text{co}}(A_{i_0} \cap P_{i_0})(x).$$

Then Γ *has an equilibrium* $\bar{x}$ in D.

COROLLARY 3.3. *Let* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy satisfying the following conditions*:

(i) *for each* $i \in I$, X_i *is a Hausdorff interval space and their product interval space* $X := \prod_{i \in I} X_i$ *is strongly Dedekind complete*;

(ii) *for each* $i \in I$, $D_i \subset X_i$ *is a nonempty compact W-convex subset*; *the constraint correspondence* $A_i : X \to 2^{D_i}$ *has nonempty W-convex closed values, the preference correspondence* $P_i : X \to 2^{D_i}$ *has W-convex closed values, and the following conditions are satisfied*:

(a) *the set* $M_i := \{x \in X: A_i(x) \cap P_i(x) \neq \phi\}$ *is open and for each* $y_i \in D_i$, $A_i^{-1}(y_i)$ *and* $P_i^{-1}(y_i)$ *are both closed*;

(b) *for each* $y \in D := \prod_{i \in I} D_i$, $y_i \in (A_i \cap P_i)(y)$;

(iii) *for any* $y \in D$, *for any* $x_1, x_2 \in X$, *and for all* $x \in (x_1, x_2)$, *there exists an* $i_0 \in I$ *such that* $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ *and* $y_{i_0} \notin A_{i_0}(x) \cap P_{i_0}(x)$.

Then Γ *has an equilibrium* x *in* D.

THEOREM 3.4. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy satisfying the following conditions*:

(i) *for each* $i \in I$, X_i *is a Hausdorff interval space*, D_i *is a nonempty compact subset of* X_i, *and* $X := \prod_{i \in I} X_i$ *is the product interval space*;

(ii) *for each* $i \in I$ *the constraint correspondences* $A_i, B_i: X \to 2^{D_i}$ *and the preference correspondence* $P_i : X \to 2^{D_i}$ *satisfy the following conditions*:

(a) *for each* $x \in X$, $\phi \neq A_i(x) \subset B_i(x)$ *and* $B_i(x)$ *is W-convex closed*;

(b) $M_i = \{x \in X: (A_i \cap P_i)(x) \neq \phi\}$ *is open*;

(c) *for all* $x \in D := \prod_{i \in I} D_i$, $x_i \notin W\text{-}\overline{\text{co}}(A_i \cap P_i)(x)$;

(d) B_i *and* $W\text{-}\overline{\text{co}}(A_i \cap P_i): X \to 2^{D_i}$ *are both upper semi-continuous*;

(iii) *for any* $y \in D$, *for any* $x_1, x_2 \in X$, *and for any* $x \in (x_1, x_2)$ *there exists an* $i_0 \in I$ *such that* $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ *and* $y_{i_0} \notin W\text{-}\overline{\text{co}}(A_{i_0} \cap P_{i_0})(x)$.

Then Γ *has an equilibrium* $\bar{x}$ *in* D.

Proof. For each $i \in I$, define a set-valued mapping $T_i : X \to 2^{D_i}$ by

$$T_i(x) := \begin{cases} W\text{-}\overline{\text{co}}(A_i \cap P_i)(x), & \text{if } x \in M_i, \\ B_i(x), & \text{if } x \notin M_i. \end{cases}$$

By condition (ii)(a) and the definition of M_i, T_i has the W-convex closed values and for any $x \in X$, $T_i(x) \subset B_i(x)$. Letting $T(x) := \prod_{i \in I} T_i(x)$, $x \in X$, then T: $X \to 2^D$ is a set-valued mapping with nonempty W-convex closed values.

Next we prove that for any $i \in I$, $T_i : X \to 2^{D_i}$ is an upper semi-continuous mapping. In fact, for any $x \in X$, if V is a relatively open set in D_i which contains $T_i(x)$.

(I) If $x \notin M_i$, then $T_i(x) = B_i(x) \subset V$. By the upper semi-continuity of B_i, there exists an open neighborhood U of x such that for any $y \in U$, we have $B_i(y) \subset V$. Hence for any $y \in U$ we have

$$T_i(y) \subset B_i(y) \subset V.$$

(II) If $x \in M_i$, then $T_i(x) = W\text{-}\overline{\text{co}}(A_i \cap P_i)(x) \subset V$. Since $W\text{-}\overline{\text{co}}(A_i \cap P_i)$: $X \to 2^{D_i}$ is upper semi-continuous, there exists an open neighborhood N of x such that for any $y \in N$ we have $W\text{-}\overline{\text{co}}(A_i \cap P_i)(y) \subset V$. Again since M_i is open, there exists an open neighborhood $N' \subset M_i$ of x. Taking $U = N \cap N'$, then U is an open neighborhood of x and

$$T_i(y)\ W\text{-}\overline{\text{co}}(A_i \cap P_i)(y) \cap V \qquad \text{for all } y \in U.$$

Combining (I) and (II) we know that $T_i: X \to 2^{D_i}$ is upper semi-continuous. In view of Lemma 3 in Ky Fan [6] we know that $T: X \to 2^D$ is upper semi-continuous. Since D is compact and T has the closed values, T has the compact values. By Proposition 5.1.2 in [2], T has a closed graph.

Finally we prove that for any $y \in D$, $X \backslash T^{-1}(y)$ is W-convex. In fact, if there exists $y \in D$ such that $X \backslash T^{-1}(y)$ is not W convex, then there exist $x_1, x_2 \in X \backslash T^{-1}(y)$ and $x \in (x_1, x_2)$ such that $x \notin X \backslash T^{-1}(y)$, i.e., $y \in T(x)$. Hence for any $i \in I$, we have $y_i \in T_i(x)$.

On the other hand, by condition (iii), there exists $i_0 \in I$ such that $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ and $y_{i_0} \notin W\text{-}\overline{\text{co}}(A_{i_0} \cap P_{i_0})(x)$, i.e., $y_{i_0} \notin T_{i_0}(x)$, a contradiction. Hence the conclusion is true. By Lemma 2.2, there exists $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$, and so for all $i \in I$, $\bar{x}_i \in T_i(\bar{x})$. By condition (ii)(c), we have $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \phi$, i.e., $\bar{x}$ is an equilibrium of Γ. This completes the proof.

COROLLARY 3.5. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy satisfying the following conditions*:

(i) *for any $i \in I$, X_i is a Hausdorff interval space, D_i is a nonempty compact subset of X_i, and $X := \prod_{i\in I} X_i$ is a product interval space*;

(ii) *for any $i \in I$, the constraint correspondence $A_i: X \to 2^{D_i}$ is upper semi-continuous and has nonempty W-convex closed values, and the preference correspondence $P_i: X \to 2^{D_i}$ has the W-convex closed values*;

(iii) *for any $i \in I$, $M_i := \{x \in X: A_i(x) \cap P_i(x) \neq \phi\}$ is open*;

(iv) *for any $x \in D := \prod_{i\in I} D_i$, $x_i \notin A_i(x) \cap P_i(x)$ for all $i \in I$*;

(v) *for any $y \in D$, for any $x_1, x_2 \in X$, and for any $x \in (x_1, x_2)$, there exists an $i_0 \in I$ such that $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ and $y_{i_0} \notin A_{i_0}(x) \cap P_{i_0}(x)$.*

Then Γ has an equilibrium $\bar{x}$, i.e., for all $i \in I$

$$\bar{x}_i \in A_i(\bar{x}) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \phi.$$

Taking $B_i = A_i$ for all $i \in I$, and by using condition (ii), it follows from Lemma 2 in [20] that $A_i \cap P_i: X \to 2^{D_i}$ is a upper semi-continuous mapping and it is easy to prove that all conditions in Theorem 3.4 are satisfied. Therefore the conclusion follows from Theorem 3.4 immediately.

COROLLARY 3.6. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy satisfying the following conditions*:

(i) *for any $i \in I$, E_i is a Hausdorff topological vector space, X_i is a convex subset of E_i, D_i is a nonempty compact subset of X_i, and $X := \prod_{i\in I} X_i$*;

(ii) *for any $i \in I$, the constraint correspondence $A_i: X \to 2^{D_i}$ is upper semi-continuous and has nonempty convex closed values and the preference correspondence $P_i: x \to 2^{D_i}$ has closed convex values*;

(iii) *for any $i \in I$, the set $M_i := \{x \in X: A_i(x) \cap P_i(x) \neq \phi\}$ is open*;

(iv) *for any $x \in D := \prod_{i\in I} D_i$, $x_i \notin A_i(x) \cap P_i(x)$ for all $i \in I$*;

(v) *for any $y \in D$, for all $x_1, x_2 \in X$, and for any $x \in \text{co}\{x_1, x_2\} \backslash \{x_1, x_2\}$, there exists an $i_0 \in I$ such that $A_{i_0}(x) \cap P_{i_0}(x) \neq \phi$ and $y_{i_0} \notin A_{i_0}(x) \cap P_{i_0}(x)$.*

Then there exists an equilibrium $\bar{x}$ of Γ in D such that for any $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \phi.$$

Remark. All theorems and corollaries given in this section are new. They may be compared with the economic equilibrium theorems in [1, 4, 5, 11–13, 18–20] and the commodity space (or choice set) is not required to have any linear structure or contractible structure.

4. INTERSECTION THEOREMS FOR SETS WITH W-CONVEX SECTION

THEOREM 4.1. *Let $\{X_i : i \in I\}$ be a family of Hausdorff compact interval spaces and let their product interval space $X := \prod_{i\in I} X_i$ be strongly Dedekind complete. Let $\{A_i : i \in I\}$, $\{B_i : i \in I\}$ be two families of subsets in X. If the following conditions are satisfied:*

(i) *for each $i \in I$ and for any $y_i \in X_i$, the set $\{x \in X: y_i \in W\text{-}\overline{\text{co}}(A_i(x_{-1}))\}$ is closed*;

(ii) *for any $i \in I$ and for any $x_{-i} \in X_{-i}$, $A_i(x_{-i}) \neq \phi$ and*

$$W\text{-}\overline{\text{co}}(A_i(x_{-i})) \subset B_i(x_{-i});$$

(iii) *for any $y \in X$, for any $x_1, x_2 \in X$, and for all $x \in (x_1, x_2)$ there exists an $i_0 \in I$ such that $y_{i_0} \notin W\text{-}\overline{\text{co}}(A_{i_0}(x_{-i_0}))$.*

then $\bigcap_{i\in I} B_i \neq \phi$.

Proof. For any $i \in I$ and for any $x \in X$, letting $T_i(x) := W\text{-}\overline{\text{co}}(A_i(x_{-i}))$, by condition (ii), $T_i: X \to 2^{X_i}$ is a set-valued mapping with nonempty compact W-convex values. By condition (i), for all $y_i \in X_i$, $T_i^{-1}(y_i)$ is closed. Letting $T(x) := \prod_{i\in I} T_i(x)$, $x \in X$, then $T: X \to 2^X$ has the nonempty compact W-convex values and for any $y \in X$

$$\begin{aligned} T^{-1}(y) &= \{x \in X: y \in Tx\} \\ &= \{x \in X: y_i \in T_i(x) \quad \text{for all } i \in I\} \\ &= \bigcap_{i\in I} T_i^{-1}(y_i) \end{aligned}$$

is a closed set in X.

Next we prove that for each $y \in X$, $X \backslash T^{-1}(y)$ is W-convex.

In fact, if there exists an $y \in X$ such that $X \backslash T^{-1}(y)$ is not W-convex, then there exist $x_1, x_2 \in X \backslash T^{-1}(y)$ such that $(x_1, x_2) \not\subset X \backslash T^{-1}(y)$, and so there exists $x \in (x_1, x_2)$ and $x \in T^{-1}(y)$, i.e., $y \in T(x)$. Hence for all $i \in I$, $y_i \in T_i(x)$. On the other hand, by condition (iii), there exists an $i_0 \in I$ such that $y_{i_0} \notin W\text{-}\overline{\text{co}}(A_{i_0}(x_{-i_0}))$, i.e., $y_{i_0} \notin T_{i_0}(x)$, a contradiction. Hence the conclusion is true.

Summing up the above arguments we know that all conditions in Lemma 2.4 are satisfied. By Lemma 2.4, there exists an $\bar{x} \in T(\bar{x})$, i.e., for all $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) = W\text{-}\overline{\text{co}}(A_i(\bar{x}_{-i})) \subset B_i(\bar{x}_{-i}).$$

Hence for all $i \in I$, $\bar{x} \in B_i$, and so $\bigcap_{i \in I} B_i \neq \phi$.

COROLLARY 4.2. *Let X_i be a nonempty compact convex subset of a Hausdorff topological vector space for all $i \in I$. Let $\{A_i : i \in I\}$, $\{B_i : i \in I\}$ be two families of subsets of $X := \prod_{i \in I} X_i$. If the following conditions are satisfied*

(i) *for any $i \in I$ and for any $y_i \in X_i$ the set $\{x \in X : y_i \in \overline{\text{co}}(A_i(x_{-i}))\}$ is closed;*

(ii) *for any $i \in I$ and for any $x_{-i} \in X_{-i}$, $A_i(x_{-i}) \neq \phi$ and*

$$\overline{\text{co}}(A_i(x_{-i})) \subset B_i(x_{-i});$$

(iii) *for all $y \in X$ and for all $x_1, x_2 \in X$ and for any $x \in \text{co}\{x_1, x_2\} \backslash \{x_1, x_2\}$, there exists an $i_0 \in I$ such that $y_{i_0} \notin \overline{\text{co}} A_{i_0}(x_{-i_0})$;*

then $\bigcap_{i \in I} B_i \neq \phi$.

COROLLARY 4.3. *Let $\{X : i \in I\}$ be a family of compact Hausdorff interval spaces and let their product interval space $X := \prod_{i \in I} X_i$ be strongly Dedekind complete. Let $\{A_i : i \in I\}$ be a family of subsets of X satisfying the following conditions:*

(i) *for any $i \in I$ and for any $x_{-i} \in X_{-i}$, the section $A_i(x_{-i})$ is nonempty W-convex closed;*

(ii) *for any $i \in I$ and for any $y_i \in X_i$, the set $\{x \in X : y_i \in A_i(x_{-i})\}$ is closed;*

(iii) *for any $y \in X$, for any $x_1, x_2 \in X$, and for any $x \in (_1, x_2)$, there exists an $i_0 \in I$ such that $y_{i_0} \notin A_{i_0}(x_{-i_0})$.*

Then $\bigcap_{i \in I} A_i \neq \phi$.

THEOREM 4.4. *Let $\{X_i : i \in I\}$ be a family of compact Hausdorff interval spaces, let $X := \prod_{i \in I} X_i$ be their product interval space, and let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be two families of subsets of X. If the following conditions are satisfied:*

(i) *for any $i \in I$, the set $\{(x, y_i) \in X \times X_i : y_i \in W\text{-}\overline{\text{co}}(A_i(x_{-i}))\}$ is closed;*

(ii) *for any $i \in I$ and for any $x_{-i} \in X_{-i}$, the section $A_i(x_{-i}) \neq \phi$ and $W\text{-}\overline{\text{co}}(A_i(x_{-i})) \subset B_i(x_{-i})$;*

(iii) *for any $y \in X$, for any $x_1, x_2 \in X$, and for any $x \in (x_1, x_2)$, there exists an $i_0 \in I$ such that $y_{i_0} \notin W\text{-}\overline{\text{co}}(A_{i_0}(x_{-i_0}))$;*

then $\bigcap_{i \in I} B_i \neq \phi$.

Proof. For any $i \in I$ and for any $x \in X$, letting $T_i(x) = W\text{-}\overline{\text{co}}(A_i(x_{-i}))$, by condition (ii), we know that $T_i : X \to 2^{X_i}$ is a set-valued mapping with nonempty compact W-convex values. By condition (i), T_i has a closed graph. By using Lemma 2 in Ky Fan [6] we know that T_i is upper semi-continuous. Letting

$$T(x) := \prod_{i \in I} T_i(x) \qquad \text{for all } x \in X$$

by Ky Fan [6, Lemma 3], $T : X \to 2^X$ is also upper semi-continuous. Since T_i has the nonempty compact W-convex values, T also has nonempty compact W-convex values. By Chang [2, Proposition 5.1.2] we know that T has a closed graph. Furthermore, by using condition (iii) and by using the same methods as in the proof of Theorem 4.1, we can prove that for any $y \in X$, $X \backslash T^{-1}(y)$ is W-convex. Hence from Lemma 2.2, there exists an $\bar{x} \in X$ such that $\bar{x} \in T\bar{x}$. By the same methods as in the proof of Theorem 4.1, we can prove that $\bar{x} \in \bigcap_{i \in I} B_i$. Hence $\bigcap_{i \in I} B_i \neq \phi$.

COROLLARY 4.5. *Let X_i, $i \in I$, be a nonempty compact convex subset of a Hausdorff topological vector space E_i, let $X = \prod_{i \in I} X_i$, and let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be two families of subsets in X. If the following conditions are satisfied:*

(i) *for any $i \in I$, the set $\{(x, x_i) \in X \times X_i : y_i \in \overline{\text{co}}(A_i(x_{-i}))\}$ is closed;*

(ii) *for any $i \in I$ and for any $x_{-i} \in X_{-i}$, the section $A_i(x_{-i}) \neq \phi$ and $\overline{\text{co}}(A_i(x_{-i})) \subset B_i(x_{-i})$;*

(iii) *for any $y \in X$, for any $x_1, x_2 \in X$, and for any $x \in \text{co}\{x_1, x_2\} \backslash \{x_1, x_2\}$ there exists an $i_0 \in I$ such that $y_{i_0} \notin \overline{\text{co}}(A_{i_0}(x_{-i_0}))$;*

then $\bigcap_{i \in I} B_i \neq \phi$.

COROLLARY 4.6. *Let $\{X_i : i \in I\}$ be a family of compact Hausdorff interval spaces, let $X := \prod_{i \in I} X_i$ be the product interval space, and let*

$\{A_i : i \in I\}$ *be a family of subsets in* X. *If the following conditions are satisfied*:

(i) *for any* $i \in I$ *and for any* $x_{-i} \in X_{-i}$, *the section* $A_i(x_{-i})$ *is nonempty W-convex closed*;

(ii) *for any* $i \in I$, *the set* $\{(x, y_i) \in X \times X_i : y_i \in A_i(x_{-i})\}$ *is closed*;

(iii) *for any* $y \in X$, *for any* $x_1, x_2 \in X$, *and for any* $x \in (x_1, x_2)$, *there exists an* $i_0 \in I$ *such that* $y_{i_0} \notin A_{i_0}(x_{-i_0})$;

then $\bigcap_{i \in I} A_i \neq \phi$.

Remark. The results presented in this section all are new. The conditions are rather simple and totally different from the conditions of results in [7–10, 15, 16].

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