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A TWO FUNCTIONS, NONCOMPACT TOPOLOGICAL MINIMAX THEOREM

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Following the approach of Joó [5], Stachó [8] introduced the interval spaces. A topological space X is called an interval space [8] if there is a mapping $[\cdot,\cdot]$ from $X \times X$ to the connected subsets of X such that $x_1, x_2 \in [x_1, x_2] = [x_2, x_1]$ for all x_1, x_2 in X. If, in addition $[x_1, x_2]$ is pathwise connected for all x_1, x_2 in X, then X is called a strong interval space. A subset K of an interval space is convex if $[x_1, x_2] \subset K$ for all x_1, x_2 in K. Let X be an interval space and let Z be an order complete order dense linear space. A mapping $f: X \to Z$ is called quasi-convex (quasi-concave) if the set $\{x \in X : f(x) \leqq z\}$ $(\{x \in X : f(x) \geqq z\})$ is convex in X for all $z \in Z$. It is easy to see that if f is quasi-convex then for any x_1, x_2 in X and for any $x \in [x_1, x_2]$, $f(x) \leqq f(x_1) \vee f(x_2)$ and if $f_i, i = 1, 2, \dots, n$ are quasi-concave then the set $\bigcap_{i=1}^{n} \{x \in X : f_i(x) \geqq z\}$ is connected or empty for any z in Z. We assume all topological spaces to be Hausdorff. The following result is proved in [3].

THEOREM 1. *Let X be a topological space, Y a strong interval space and Z an order complete, order dense linear space. Let $f: X \times Y \to Z$ be a mapping such that*

(1) *for any $y_1, y_2, \dots, y_n$ in Y and for any $z \in Z$, the set $\bigcap_{i=1}^{n} \{x \in X : f(x, y_i) > z\}$ is connected or empty;*

(2) (a) *for all y in Y, $f(\cdot, y)$ is upper semicontinuous (usc) on X;*

(b) *for all x in X, $f(x, \cdot)$ is lower semicontinuous (lsc) on every interval in Y;*

(3) *for all x in X, $f(x, \cdot)$ is quasi-convex on Y;*

(4) *there exist y_0 in Y and z_0 in Z with $z_0 < \inf_Y \sup_X f(x, y)$ such that the set $\{x \in X : f(x, y_0) \geqq z_0\}$ is compact.*

Then

$$\sup_X \inf_Y f(x, y) = \inf_Y \sup_X f(x, y).$$

The following result is a consequence of the theorem proved in [7].

THEOREM 2. *Let X and Y be topological spaces and let $f, g: X \times Y \to \mathbf{R}$. Assume that*

(0) *$f(x,y) \leqq g(x,y)$ for all $(x,y) \in X \times Y$;*

(1) *for all $y_1, y_2, \ldots, y_n$ in Y and for all $\beta \in \mathbf{R}$, the set $\{x \in X : g(x,y_i) > \beta,\ i = 1,2,\ldots,n\}$ is either connected or empty;*

(2) (a) *for all y in Y, $f(\cdot,y)$ and $g(\cdot,y)$ are usc on X;*

(b) *for all x in X, $f(x,\cdot)$ and $g(x,\cdot)$ are lsc on Y;*

(3) *for all y_1, y_2 in Y there exists a connected set C containing y_1, y_2 and for all y in C and all x in X,*

$$g(x,y) \leqq f(x,y_1) \vee g(x,y_2)\ ;$$

(4) *X is compact.*

Then

$$\sup_X \inf_Y g(x,y) \geqq \inf_Y \sup_X f(x,y).$$

In this note, we give a generalization for both results.

THEOREM. *Let X be a topological space, Y an interval space and Z an order complete, order dense linear space. Let $f, g : X \times Y \to Z$ be mappings such that*

(0) *$f(x,y) \leqq g(x,y)$ for all $(x,y) \in X \times Y$;*

(1) *for all $y_1, y_2, \ldots, y_n$ in Y and for all z in Z, the set $\bigcap_{i=1}^{n} \{x \in X : g(x,y_i) > z\}$ is either connected or empty;*

(2) (a) *for all y in Y, $f(\cdot,y)$ and $g(\cdot,y)$ are usc on X;*

(b) *for all x in X, $f(x,\cdot)$ is lsc on every interval of Y;*

(3) *for all y_1, y_2 in Y, for all $y \in [y_1, y_2]$ and for all x in X,*

$$g(x,y) \leqq f(x,y_1) \vee g(x,y_2)\ ;$$

(4) *there exist $\bar{y}$ in Y and $\bar{z}$ in Z with $\bar{z} \leqq \sup_X \inf_Y g(x,y)$ such that the set $\{x \in X : g(x,\bar{y}) \geqq \bar{z}\}$ is compact.*

Then

$$\sup_X \inf_Y g(x,y) \geqq \inf_Y \sup_X f(x,y).$$

LEMMA. *Under the conditions (0)–(3) of the Theorem, if for any y_1, y_2 in Y and any z in Z with $\sup_X \big(f(x,y_1) \wedge g(x,y_2)\big) < z$, then there exists y_0 in Y such that $\sup_X f(x,y_0) < z$.*

PROOF. Suppose there are y_1, y_2 in Y and z in Z such that $\sup_X \big(f(x,y_1) \wedge g(x,y_2)\big) < z$ and $\sup_X f(x,y) \geqq z$ for all y in Y. Choose z_0 with $\sup_X \big(f(x,y_1) \wedge g(x,y_2)\big) < z_0 < z$. Let $A = \{x \in X : f(x,y_1) \geqq z_0\}$ and $B = \{x \in X :$

$g(x, y_2) \geqq z_0\}$. Then A, B are closed, non-empty and $A \cap B = \emptyset$. Let $K_g(y) = \{x \in X : g(x,y) > z_0\}$. Then $K_g(y)$ is a nonempty connected set for all y in Y. By (3), $K_g(y) \subset A \cup B$ for all y in $[y_1, y_2]$. Hence either $K_g(y) \subset A$ or $K_g(y) \subset B$ for all $y \in [y_1, y_2]$. Let $K_f(y) = \{x \in X : f(x,y) > z_0\}$. Then $K_f(y) \subset K_g(y)$. Let $I = \{y \in [y_1, y_2] : K_f(y) \subset A\}$ and $J = \{y \in [y_1, y_2] : K_f(y) \subset B\}$. Then I, J are nonempty, $I \cap J = \emptyset$ and $I \cup J = [y_1, y_2]$. To get a contradiction, it suffices to show that I and J are closed in $[y_1, y_2]$.

Let $\{y_i\}$ be a net in I and $\lim y_i = y$ in $[y_1, y_2]$. Let $K = \bigcup_i K_f(y_i)$. Then $K \subset A$. Suppose that $K_f(y) \cap A = \emptyset$. Then $K_f(y) \cap K = \emptyset$. Thus for all $x \in K_f(y)$, $f(x, y_i) \leqq z_0$ for all i. Since $f(x, \cdot)$ is lower semicontinuous and $\lim_i y_i = y$, it follows that $f(x,y) \leqq z_0$ for all $x \in K_f(y)$ which is a contradiction since $K_f(y)$ is nonempty and $f(x,y) > z_0$ for $x \in K_f(y)$. Therefore I is closed. Similarly, J is closed. This completes the proof. □

PROOF OF THEOREM. Let z in Z with $\sup_X \inf_Y g(x,y) < z$. Choose z_0 in Z such that $\sup_X \inf_Y g(x,y) < z_0 < z$. Let $L_g(y) = \{x \in X : g(x,y) \geqq z_0\}$ for each $y \in Y$. Then $L_g(y)$ is closed for all y in Y and $\bigcap_{y \in Y} L_g(y) = \emptyset$. By (4), since $z_0 > \bar{z}$, it follows that $L_g(\bar{y})$ is compact. Thus there are $y_1, y_2, \dots, y_n$ in Y such that $\bigcap_{i=1}^n L_g(y_i) \cap L_g(\bar{y}) = \emptyset$. That is, $\sup_X \min\{g(x, y_1), \dots, g(x, y_n), g(x, \bar{y})\} \leqq z_0$. It remains to use induction to find y_0 in Y such that $\sup_X f(x, y_0) < z$.

If $n = 1$, applying the Lemma to y_1 and $\bar{y}$, there exists y_0 in Y such that $\sup_X f(x, y_0) < z_0 < z$.

For $n > 1$, let $X_n = \{x \in X : g(x, y_n) \geqq z_0\}$. Then X_n, Y and Z satisfy the hypotheses of the Theorem and $\sup_{X_n} \min\{g(x, y_1), \dots, g(x, y_{n-1}), g(x, \bar{y})\} \leqq z_0 < z$. By induction hypothesis, there is y in Y with $\sup_{X_n} f(x,y) < z$. Since $\sup_X \min\{f(x,y), g(x, y_n)\} < z$, by the Lemma, there is y_0 in Y such that $\sup_X f(x, y_0) < z$. Thus $\inf_Y \sup_X f(x,y) < z$. Therefore $\inf_Y \sup_X f(x,y) \leqq \sup_X \inf_Y g(x,y)$. □

For $f = g$, we obtain

COROLLARY. *Let X be a topological space, Y an interval space and Z an order complete order dense linear space. Let $f : X \times Y \to Z$ be a function such that*

(1) *for all* $y_1, y_2, \dots, y_n$ *in* Y *and for all* z *in* Z, *the set* $\bigcap_{i=1}^{n} \{x \in X : f(x, y_i) > z\}$ *is connected or empty;*

(2) a) *for all* y *in* Y, $f(\cdot, y)$ *is usc on* X;

b) *for all* x *in* X, $f(x, \cdot)$ *is lsc on any interval of* Y;

(3) *for any* y_1, y_2 *in* Y *and any* y *in* $[y_1, y_2]$, $f(x,y) \leqq f(x,y_1) \vee f(x,y_2)$.

(4) *there exist* $\bar{y}$ *in* Y *and* $\bar{z}$ *in* Z *with* $\bar{z} \leqq \sup_X \inf_Y f(x,y)$ *such that the set* $\{x \in X : f(x, \bar{y}) \geqq \bar{z}\}$ *is compact.*

Then

$$\inf_Y \sup_X f(x,y) = \sup_X \inf_Y f(x,y).$$

Since a strong interval space is an interval space, Theorem 1 is a consequence of the Corollary. It is clear that Theorems 1 and 2 in [8] are consequences of the Corollary. As observed in [3] and is also easy to see, the results of [1], [4], [6] and [9] are consequences of Theorem 1. We omit the details.

Next we give an example that the conditions in the Corollary are satisfied but the theorems in [3] and [8] cannot be applied.

Let X be any nonempty compact connected topological space and let $Y = A \cup B$ where

$$A = \left\{(s,t) : t = \sin \frac{1}{s}, \quad 0 < s \leqq \frac{1}{\pi}\right\},$$

and

$$B = \{(0,t) : -1 \leqq t \leqq 1\}$$

are subsets of $\mathbf{R}^2$. Topologize Y with the relative topology of $\mathbf{R}^2$. Y is an interval space with the intervals $[y_1, y_2] = [y_2, y_1]$ defined as follows:

(1) if y_1, and y_2 are in A, then $[y_1, y_2]$ is the graph of the sine curve joining y_1 and y_2;

(2) if y_1 and y_2 are in B, then $[y_1, y_2]$ is the segment joining y_1 and y_2;

(3) if $y_1 = (s_1, t_1) \in A$ and $y_2 = (s_2, t_2) \in B$ then $[y_1, y_2] = B \cup \left\{(s,t) : t = \sin \frac{1}{s}, 0 < s \leqq s_1\right\}$.

Define

$$f : X \times Y \longrightarrow \mathbf{R}, \quad f(x,y) = \begin{cases} 1 & \text{if } x \in X \text{ and } y \in A \\ 0 & \text{if } x \in X \text{ and } y \in B. \end{cases}$$

It is easy to see that all the conditions of the Corollary are satisfied and $\min_Y \max_X f(x,y) = 0 = \max_X \min_Y f(x,y)$. Since Y is not pathwise connected, it is clear that Y is not a strong interval space, hence the theorem in [3] cannot be applied. To see that Y is not a Dedekind complete interval space, consider the convex subsets $H_1 = A$ and $H_2 = B$ in Y. Then for any $y_i \in H_i$, $i = 1, 2, [y_1, y_2] \subset H_1 \cup H_2$ but there is no z in H_1 such that $[z, y_2] \backslash \{z\} \subset H_2$ and there is no z in H_2 such that $[y_1, z] \backslash \{z\} \subset H_1$. Hence Theorem 2 in [8] cannot be applied since it requires that Y is Dedekind complete. Theorem 1 in [8] cannot be applied since it requires both X and Y to be compact.

REMARK. It is interesting to compare the result in this paper with those in [2]. In [2] similar conditions are considered, except on certain continuity conditions and in particular, that they impose most of the conditions on f only. For example, instead of (3) in the Theorem, they considered the condition: for all y_1, y_2 in Y, for all y in $[y_1, y_2]$ and for all x in X, $f(x, y) \leqq f(x, y_1) \vee f(x, y_2)$.

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