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A MINIMAX THEOREM INVOLVING WEAKLY DOWNWARD FUNCTIONS

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Abstract. A minimax theorem involving weakly downward functions is proved.

1. Introduction

Let X and Y be nonempty sets and let f be a real-valued function defined on $X \times Y$. f is said to be *downward on* X if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x_1, x_2 in X, there exists an element x_0 in X with the properties that for all y in Y,

$$(*) \qquad f(x_0, y) \geqq \min\left\{ f(x_1, y), f(x_2, y)\right\}$$

and

$$\left| f(x_1, y) - f(x_2, y)\right| \geqq \varepsilon \Rightarrow f(x_0, y) \geqq \min\left\{ f(x_1, y), f(x_2, y)\right\} + \delta.$$

f is said to be *upward on* Y if $-f$ is downward on Y.

S. Simons [7] introduced the upward-downward functions to include t-convex and s-concave functions considered in [3]. Recall that for $s, t \in (0,1)$, f is said to be *s-concave on* X if for all x_1, x_2 in X there exists x_0 in X such that for all y in Y,

$$f(x_0, y) \geqq s \max\left\{ f(x_1, y), f(x_2, y)\right\} + (1-s) \min\left\{ f(x_1, y), f(x_2, y)\right\}.$$

f is said to be *t-convex on* Y if $-f$ is $(1-t)$-concave on Y. Simons has proved the following result.

THEOREM 1 [8]. *Let X be a compact (Hausdorff) space and let Y be a nonempty set. Suppose $f : X \times Y \to \mathbf{R}$ such that $f(\cdot, y)$ is upper semicontinuous (usc) on X for all y in Y. If f is downward on X and is upward on Y, then*

$$(**) \qquad \inf_Y \sup_X f(x,y) = \sup_X \inf_Y f(x,y).$$

For other minimax theorems involving upward-downward functions, see [7].

Recently, A. Domokos [2] introduced the following functions with certain properties. Let us call them weakly downward and weakly upward functions.

DEFINITION [2]. Let X and Y be nonempty sets and let f be a real-valued function defined on $X \times Y$. f is said to be *weakly downward on* X if for all x_1, x_2 in X, there exists x_0 in X such that for all y in Y, $(*)$ holds and

$$f(x_1, y) \neq f(x_2, y) \Rightarrow f(x_0, y) > \min\{f(x_1, y), f(x_2, y)\}.$$

f is said to be *weakly upward on* Y if $-f$ is weakly downward on Y.

A. Domokos [2] proved that in the case when both spaces X and Y are compact, and f is a real-valued continuous function on $X \times Y$, then $(**)$ holds provided f is weakly downward on X and is weakly upward on Y.

The result was independently extended by J. Kindler and by Cao-Zong Cheng, Bor-Luh Lin and Feng-Shu Yu [1] by requiring $f(\cdot, y)$ to be usc on X for all y in Y and $f(x, \cdot)$ to be lsc on Y for all x in X. In fact, Kindler proved that the following theorem holds. The authors wish to thank Professor Kindler for sending us the preprint [6].

THEOREM 2 [6]. *Let X be a compact space, Y be a countably compact space and let f be a real valued function on $X \times Y$ such that $f(\cdot, y)$ is usc on X for all y in Y and $f(x, \cdot)$ is lsc on Y for all x in X. Suppose that f is finitely weakly downward on X and f is weakly upward on Y. Then $(**)$ holds.*

f is said to be *finitely weakly downward on* X if for all finite subsets B in Y and for all x_1, x_2 in X, there exists x_0 in X such that for all y in B, the condition $(*)$ holds.

It is easy to see that if f is s-concave on X then f is weakly downward on X and if f is t-convex on Y then f is weakly upward on Y. Furthermore, it is interesting to compare the weakly upward functions with the submaximum functions studied in [4]. The concept of submaximum functions is based on the result of W. Wu [11]. Submaximum functions may be looked at as the limit case of t-convex functions when t approaches zero with a certain additional condition. Also, weakly upward-downward functions may be considered as the limit case of upward-downward functions when $\varepsilon = \delta = 0$. In this paper, we prove a minimax theorem involving weakly downward functions. The result is interesting in comparison with the result in [5].

THEOREM 3 [5]. *Let X be a compact space and let Y be a topological space. Let f be a real valued function on $X \times Y$ with the following properties:*

(1) $f(\cdot, y)$ usc on X for all y in Y;

(2) f is submaximum on Y, i.e., for any y_1, y_2 in Y, there exists a continuous function $S : [0,1] \to Y$ such that $S(0) = y_1$, $S(1) = y_2$ and for any $0 \leqq a \leqq c \leqq b \leqq 1$,

$$f\big(x, S(c)\big) \leqq \max\big\{ f\big(x, S(a)\big), f\big(x, S(b)\big) \big\} \qquad \textit{for all } x \textit{ in } X;$$

(3) *f is s-concave on X for some $s \in (0,1)$.*
Then $(**)$ holds.

The main result of this paper is the following theorem.

THEOREM. *Let X be a compact space and let Y be an interval space. Suppose that $f : X \times Y \to \mathbf{R}$ satisfies the following conditions:*
(i) *$f(\cdot, y)$ is usc on X for all y in Y and $f(x, \cdot)$ is lsc on all intervals in Y;*
(ii) *for all y_1, y_2 in Y and for all $y \in [y_1, y_2]$,*

$$f(x,y) \leqq \max \{ f(x,y_1), f(x,y_2) \}$$

holds for all x in X;
(iii) *f is weakly downward on X.*
Then $(**)$ holds.

Recall that a topological space X is called an *interval space* [10] if there is a mapping $[\cdot, \cdot]$ from $X \times X$ to the connected subsets of X such that $x_1, x_2 \in [x_1, x_2] = [x_2, x_1]$ for all x_1, x_2 in X. $[x_1, x_2]$ is called an *interval in X*.

2. Proof of the Theorem

A family $\mathcal{H}$ of subsets of a set X is said to be *pseudoconnected* [9] if for any H_0, H_1, H_2 in $\mathcal{H}$, $H_0 \cap H_1 \neq \emptyset \neq H_0 \cap H_2$ and $H_0 \subset H_1 \cup H_2$ imply that $H_1 \cap H_2 \neq \emptyset$.

Let $f : X \times Y \to \mathbf{R}$. For $A \subset Y$ and $r \in \mathbf{R}$, we denote $U^r(A) = \{ x : x \in X,\ f(x,y) \geqq r, \text{ for all } y \in A \}$. The proof of the theorem is based on the following lemma.

LEMMA. *Let X be a compact space and let Y be a nonempty set. Suppose $f : X \times Y \to \mathbf{R}$ satisfies the following conditions:*
(1) *$f(\cdot, y)$ is usc on X for all $y \in Y$;*
(2) *f is weakly downward on X.*
Then for any finite subset A in Y and for any $r \in \mathbf{R}$, the family $\mathcal{H} = \{ U^r(y) \cap U^r(A) : y \in Y \}$ is pseudoconnected.

PROOF. Suppose there exist y_0, y_1, y_2 in Y such that

$$U^r(y_0) \cap U^r(A) \subset U^r(y_1) \cup U^r(y_2), \quad U^r(y_0) \cap U^r(y_1) \cap U^r(A) \neq \emptyset,$$

$$U^r(y_0) \cap U^r(y_2) \cap U^r(A) \neq \emptyset \quad \text{and} \quad U^r(y_1) \cap U^r(y_2) \cap U^r(A) = \emptyset.$$

Let $D = U^r(y_0) \cap U^r(A)$ and $D_i = D \cap U^r(y_i)$, $i = 1, 2$. Then

$$D \subset D_1 \cup D_2, \quad D_i \neq \emptyset, \quad i = 1, 2 \tag{1}$$

and

$$D_1 \cap D_2 = \emptyset. \tag{2}$$

Since $f(\cdot, y)$ is usc on the compact space X for all $y \in Y$, D_i, $i = 1, 2$ are compact subsets in X. Hence there exist $x_i \in D_i$, $i = 1, 2$ such that $f(x_1, y_2) = \max_{x \in D_1} f(x, y_2)$ and $f(x_2, y_1) = \max_{x \in D_2} f(x, y_1)$.

For x_1, x_2 in X, since f is weakly downward on X, there exists $x_0 \in X$ such that for all $y \in Y$,

$$f(x_0, y) \geqq \min \left\{ f(x_1, y), f(x_2, y) \right\} \tag{3}$$

and

$$f(x_1, y) \neq f(x_2, y) \Rightarrow f(x_0, y) > \min \left\{ f(x_1, y), f(x_2, y) \right\}. \tag{4}$$

Since $x_1, x_2 \in D$, hence $x_0 \in D \subset D_1 \cup D_2$ by (3) and (1). If $x_0 \in D_1$, then since $x_2 \in D_2$ and by (2), $x_1 \notin D_2$, we have $f(x_2, y_2) \geqq r > f(x_1, y_2)$.

By (4),

$$f(x_0, y_2) > \min \left\{ f(x_1, y_2), f(x_2, y_2) \right\} = f(x_1, y_2).$$

This contradicts the maximality of $f(x_1, y_2)$. Similarly, if $x_0 \in D_2$, we get a contradiction with the maximality of $f(x_2, y_1)$. Therefore $U^r(y_1) \cap U^r(y_2) \cap U^r(A) \neq \emptyset$. This completes the proof of Lemma. $\square$

Proof of Theorem. Since X is compact and $f(\cdot, y)$ is usc on X for all y in Y, it suffices to show that for any $\alpha < \inf_Y \sup_X f(x_1, y)$, $f_*(X, A) \equiv \sup_X \inf_A f(x, y) \geqq \alpha$ for all finite subsets A in Y.

Fix $\alpha < \inf_Y \sup_X f(x, y)$. We shall prove $f_*(X, A) \geqq \alpha$ by induction on the cardinality $|A|$ of any finite subset A in Y.

If $|A| = 1$, obviously, $f_*(X, A) \geqq \inf_Y \sup_X f(x, y) > \alpha$.

Suppose that $f_*(X, A) \geqq \alpha$ for all $A \subset Y$ with $|A| \leqq n$, but for some $B \subset Y$, $|B| = n + 1$ and $f_*(X, B) < \alpha$. Choose $r \in \mathbf{R}$, $f_*(X, B) < r < \alpha$. Let $B = A \cup \{y_1, y_2\}$, $y_i \notin A$, $i = 1, 2$ and $y_1 \neq y_2$. Since $f_*(X, B) < r < \alpha$,

$$U^r(B) = \emptyset \tag{5}$$

and for all y in Y,

$$\begin{aligned} \emptyset &\neq U^\alpha(y) \cap U^\alpha(A) \\ &\subset \left\{ x : x \in X, f(x, y) > r \right\} \cap U^r(A) \subset U^r(y) \cap U^r(A). \end{aligned} \tag{6}$$

From (ii), for all $y \in [y_1, y_2]$, we have $U^r(y) \subset U^r(y_1) \cup U^r(y_2)$ which implies that

$$U^r(y) \cap U^r(A) \subset \left[U^r(y_1) \cap U^r(A)\right] \cup \left[U^r(y_2) \cap U^r(A)\right]. \tag{7}$$

For $i = 1, 2$, let

$$I_i = \left\{ y \,:\, y \in [y_1, y_2], U^r(y) \cap U^r(A) \subset U^r(y_i) \right\}.$$

By (5), we have

$$I_1 = \left\{ y \,:\, y \in [y_1, y_2], U^r(y) \cap U^r(y_2) \cap U^r(A) = \emptyset \right\}$$

and

$$I_2 = \left\{ y \,:\, y \in [y_1, y_2], U^r(y) \cap U^r(y_1) \cap U^r(A) = \emptyset \right\}.$$

It is clear that $y_i \in I_i$, $i = 1, 2$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 \subset [y_1, y_2]$. If there exists $y_0 \in [y_1, y_2]$ and $y_0 \notin I_1 \cup I_2$, then

$$U^r(y_0) \cap U^r(y_i) \cap U^r(A) \neq \emptyset, \quad i = 1, 2. \tag{8}$$

From (iii) and the Lemma, $\left\{ U^r(y) \cap U^r(A) \right\}_{y \in Y}$ is pseudoconnected. It follows from (7) and (8) that $U^r(B) \neq \emptyset$. Hence we have that $I_1 \cup I_2 = [y_1, y_2]$.

We next show that I_i, $i = 1, 2$ are closed subsets of $[y_1, y_2]$.

Let $\{\xi_k\} \subset I_1$, $\xi_0 \in [y_1, y_2]$ and $\xi_k \to \xi_0$. Then for all k, $U^r(\xi_k) \cap U^r(A) \subset U^r(y_1)$. Suppose $\xi_0 \notin I_1$, then $\xi_0 \in I_2$. Hence $U^r(\xi_0) \cap U^r(A) \cap U^r(y_1) = \emptyset$ and

$$\left\{ x \,:\, x \in X, f(x, \xi_0) > r \right\} \cap U^r(A) \cap U^r(y_1) = \emptyset.$$

By (6), we can choose $x_0 \in \left\{ x \,:\, x \in X, f(x, \xi_0) > r \right\} \cap U^r(A)$. Then $x_0 \notin U^r(y_1)$. It follows that $x_0 \notin U^r(\xi_k)$ for all k. Thus $f(x_0, \xi_0) \geqq r$ by the fact that f is lsc on $[y_1, y_2]$. This contradiction shows that $\xi_0 \in I_1$ and so I_1 is closed in $[y_1, y_2]$. Similarly, I_2 is closed in $[y_1, y_2]$.

Thus $[y_1, y_2]$ is connected. We conclude that $f_*(X, A) \geqq \alpha$ for all finite subsets A in Y. □

REMARKS. (1) The Theorem can be proved by using Lemma and the results in [9].

(2) The Theorem remains true if (i) is replaced by the following: f is usc on $X \times [y_1, y_2]$ for all y_1, y_2 in Y. The proof is similar to that of the Theorem. We only need to prove that I_i, $i = 1, 2$ are closed in $[y_1, y_2]$ in this case.

(3) The Theorem fails even if X and Y are compact connected sets in $\mathbf{R}$ when (ii) is replaced by the following weaker condition:

(ii′) For all y_1, y_2 in Y, there exists y_0 in Y such that

$$f(x,y_0) \leqq \max\{f(x,y_1), f(x,y_2)\}.$$

EXAMPLE. Let $X = Y = [-1,1]$. Define $f : X \times Y \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} -x-4y+3 & \text{if } (x,y) \in [0,1]\times[0,1] \\ -(3y+1)x-4y+3 & \text{if } (x,y) \in [-1,0)\times[0,1] \\ (y+2)x-2y-1 & \text{if } (x,y) \in [0,1]\times[-1,0) \\ (1-y)x-2y-1 & \text{if } (x,y) \in [-1,0)\times[-1,0). \end{cases}$$

Obviously, $f(\cdot, y)$ is continuous on X for all $y \in Y$ and $f(x,\cdot)$ is lsc on Y for all $x \in X$. Condition (ii′) is satisfied by taking $y_0 = y_1$ or $y_0 = y_2$. For any x_1, x_2 in X, let $x_0 = \frac{x_1+x_2}{2}$. Then (iii) is satisfied. However $\inf_Y \sup_X f(x,y) = 1$ and $\sup_X \inf_Y f(x,y) = -1$.

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