10. K. Gopalsamy, X. Z. He, and L. Wen, Global attractivity and oscillations in a periodic logistic integrodifferential equation, *Houston J. Math.* **17**, No. 2 (1991), 157–177.
11. Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics," Academic Press, New York, 1993.
12. R. M. May, Time delays versus stability in population models with two or three trophic levels, *Ecology* **54**, No. 2 (1973), 315–325.
13. K. N. Murty and M. A. S. Srinivas, Convergence of ecological competition between two species, *J. Math. Anal. Appl.* **158** (1991), 333–341.
14. W. Wang and Z. Ma, Harmless delays for uniform persistence, *J. Math. Anal. Appl.* **158** (1991), 256–268.

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Minimax Inequalities for Vector-Valued Mappings on W-Spaces*

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The topological KKM theorem of Chang *et al.* is slightly modified. By using our topological KKM theorem, we obtain a generalized section theorem and a generalized fixed point theorem on W-spaces which do not have any linear structure, and establish minimax inequalities for vector-valued mappings in Hausdorff topological vector spaces with closed pointed convex cones.

1. INTRODUCTION

In [13], the concept of H-space was firstly introduced by Horvath, and later some important results on H-spaces were obtained by several authors [2, 3, 6, 7]. Recently, Chen [10] proved a generalized Fan's section theorem

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and a generalized Browder's fixed point theorem for set-valued mappings on H-spaces, and using these results he obtained a minimax inequality theorem for vector-valued mappings.

Most recently, Chang *et al.* in [8, 9] introduced the concept of a W-space which is a topological space equipped with the family of its nonempty connected subsets, proved a new and more general version of the topological KKM theorem on a W-space, and obtained some minimax theorems as its applications.

In this paper, sightly modifying the topological KKM theorem of Chang *et al.* [8, 9], we obtain a generalized section theorem and a generalized fixed point theorem on W-spaces, and establish some minimax inequalities for vector-valued mappings in Hausdorff topological vector spaces with closed pointed convex cones.

2. MODIFIED TOPOLOGICAL KKM THEOREM

First, we give some definitions needed in this section.

DEFINITION 2.1 [8]. Let X be a Hausdorff topological space and $\{C_A\}$ a family of nonempty connected subsets of X indexed by finite subsets A of X such that $A \subset C_A$, then we call $(X, \{C_A\})$ a W-space.

DEFINITION 2.2 [8]. Let $(X, \{C_A\})$ be a W-space. Then a subset $D \subset X$ is called W-convex if for any finite subset A of D, $C_A \subset D$.

Remark. Note that Hausdorff topological vector spaces, convex spaces, contractible spaces, and connected spaces are special cases of W-spaces.

Remark. Let $(X, \{\Gamma_A\})$ be an H-space; that is, X is a topological space and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subsets of X such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$ [2, 3, 13]. If for any finite subset A of X, $A \subset \Gamma_A$, then $(X, \{\Gamma_A\})$ is a W-space.

DEFINITION 2.3. Let X and Y be two topological spaces; then a multifunction $F: X \to 2^Y$ is said to be upper semicontinuous if for any $x \in X$ and for any neighborhood V of $F(x)$ in Y, there exists a neighborhood U of x such that $F(u) \subset V$ for all $u \in U$.

LEMMA 2.1 [1]. *Let X and Y be two Hausdorff topological spaces, and $F: X \to 2^Y$ a multifunction.*

(1) *If F is upper semicontinuous with nonempty compact values, then the graph of F (that is, $\{(x, y) \in X \times Y: y \in F(x)\}$) is closed.*

(2) *If Y is compact and the graph of F is closed, then F is upper semicontinuous.*

In the sequel, we denote the graph of F by Graph(F).

By slightly modifying the arguments of Chang *et al.* [8, 9], we can obtain the following topological KKM theorem. For the completeness, we prove our theorem.

THEOREM 2.1. *Let $(X, \{C_A\})$ be a W-space, Y a Hausdorff topological space and $F: X \to 2^Y$ a multifunction satisfying the following conditions:*

(i) *F is upper semicontinuous with nonempty closed values;*

(ii) *for any finite set $A \subset X$, $\bigcap_{x \in A} F(x)$ is connected;*

(iii) *for any $x_1, x_2 \in X$,*

$$F(C_{\{x_1, x_2\}}) \subset F(x_1) \cup F(x_2),$$

where $F(C_{\{x_1, x_2\}}) = \bigcup_{x \in C_{\{x_1, x_2\}}} F(x)$;

(iv) *Y is compact.*

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

Proof. First, we prove by induction that the family $\{F(x): x \in X\}$ has the finite intersection property. By condition (i), $F(x)$ is nonempty for each $x \in X$. Suppose that for any n elements of $\{F(x): x \in X\}$, $n \geq 2$, their intersection is nonempty and now we prove that for any $n+1$ elements of $\{F(x): x \in X\}$ their intersection is also nonempty. Suppose that this is not the case, then there exists some subset $\{x_1, \ldots, x_n, x_{n+1}\}$ in X such that

$$\bigcap_{i=1}^{n+1} F(x_i) = \varnothing.$$

Letting $H = \bigcap_{i=3}^{n+1} F(x_i)$, by the assumption of induction and condition (ii), $H \cap F(x_i)$, $i = 1, 2$, is a nonempty connected set and

$$(H \cap F(x_1)) \cap (H \cap F(x_2)) = \varnothing. \tag{1.1}$$

In view of condition (iii), for x_1, x_2, we have

$$H \cap F(C_{\{x_1, x_2\}}) \subset (H \cap F(x_1)) \cup (H \cap F(x_2)). \tag{1.2}$$

Letting

$$E_1 = \{x \in C_{\{x_1, x_2\}}: H \cap F(x) \subset H \cap F(x_1)\} \quad \text{and}$$

$$E_2 = \{x \in C_{\{x_1, x_2\}}: H \cap F(x) \subset H \cap F(x_2)\},$$

then both E_1 and E_2 are nonempty by the fact that $x_1 \in E_1$ and $x_2 \in E_2$. By (1.1) and (1.2), we have $C_{\{x_1, x_2\}} = E_1 \cup E_2$. Since $C_{\{x_1, x_2\}}$ is connected and $E_1 \cap E_2 = \varnothing$, we know that either E_1 or E_2 must not be a closed set. Without loss of generality, we can assume that E_2 is not closed. Taking $x_0 \in (\overline{E}_2 \setminus E_2) \cap E_1$, there exists a net $\{x_\alpha\}_{\alpha \in I} \subset E_2$ such that $x_\alpha \to x_0$. Since $x_0 \in E_1$ and $x_\alpha \in E_2$ ($\alpha \in I$), we have

$$H \cap F(x_0) \subset H \cap F(x_1) \quad \text{and}$$
$$H \cap F(x_\alpha) \subset H \cap F(x_2) \quad \text{for all } \alpha \in I.$$

Taking $y_\alpha \in H \cap F(x_\alpha)$ for each $\alpha \in I$, then we have $y_\alpha \in H \cap F(x_2)$ for all $\alpha \in I$. Since $H \cap F(x_2)$ is compact, we may assume that $y_\alpha \to y_0 \in H \cap F(x_2)$. On the other hand, since $x_\alpha \to x_0$ and F is upper semicontinuous with nonempty closed values, by Lemma 2.1, $y_0 \in F(x_0)$ and hence $y_0 \in H \cap F(x_1)$. Thus $(H \cap F(x_1)) \cap (H \cap F(x_2)) \neq \varnothing$, which contradicts (1.1). Therefore the family $\{F(x): x \in X\}$ of sets has the finite intersection property. By conditions (i) and (iv), we have $\bigcap_{x \in X} F(x) \neq \varnothing$.

Remark. We can find the essentially same result as the above Theorem 2.1 in [15].

3. GENERALIZED SECTION THEOREM

Now we give a generalized section theorem on W-spaces as follows;

THEOREM 3.1. *Let $(X, \{C_A\})$ be a W-space, Y a compact Hausdorff topological space, and G a nonempty subset of $X \times Y$ such that*

(1) *G is closed in $X \times Y$;*

(2) *for any finite subset A of X, $\bigcap_{x \in A}\{y \in Y \mid (x, y) \in G\}$ is connected;*

(3) *for each $y \in Y$, $B_y = \{x \in X \mid (x, y) \notin G\}$ is W-convex or empty.*

Then there exists $y_0 \in Y$ such that $X \times \{y_0\} \subset G$.

Proof. Let $F(x) = \{y \in Y \mid (x, y) \in G\}$ for each $x \in X$. Since Graph$(F) = G$, by (1) Graph(F) is closed. Since Y is compact, by Lemma 2.1, F is upper semicontinuous with closed values. By (2), for any finite subset A of X, $\bigcap_{x \in A} F(x)$ is connected. Now we prove that for any $x_1, x_2 \in X$, $F(C_{\{x_1, x_2\}}) \subset F(x_1) \cup F(x_2)$. Suppose to the contrary that there exist $x_1, x_2 \in X$ such that $F(C_{\{x_1, x_2\}}) \not\subset F(x_1) \cup F(x_2)$. Then there exists $y_* \in F(C_{\{x_1, x_2\}})$ such that $y_* \notin F(x_1)$ and $y_* \notin F(x_2)$. Thus there exists $x_* \in C_{\{x_1, x_2\}}$ such that $y_* \in F(x_*)$, $y_* \notin F(x_1)$ and $y_* \notin F(x_2)$. Hence we have $(x_*, y_*) \in G$, $(x_1, y_*) \notin G$ and $(x_2, y_*) \notin G$. Since $(x_1, y_*) \notin G$ and $(x_2, y_*) \notin G$, then $x_1, x_2 \in B_{y_*}$. Since by (3) B_{y_*} is W-convex, $C_{\{x_1, x_2\}} \subset B_{y_*}$, hence $(x_*, y_*) \notin G$, which contradicts the fact that $(x_*, y_*) \in G$. Therefore F satisfies all the assumptions in Theorem 2.1. Thus we have $\bigcap_{x \in X} F(x) \neq \varnothing$. Hence there exists a $y_0 \in Y$ such that $X \times \{y_0\} \subset G$.

As an application of Theorem 3.1, we can obtain the following generalized fixed point theorem on a compact W-space which is closely related to the Fan–Browder fixed point theorem in [5].

THEOREM 3.2. *Let $(X, \{C_A\})$ be a compact W-space and $P: X \to 2^X$ a multifunction such that*

(1) *for any $x \in X$, $P(x) \neq \varnothing$ and Graph(P) is open;*

(2) *for any finite subset A of X, $\bigcap_{y \in A} [X \setminus P^{-1}(y)]$ is connected;*

(3) *for each $x \in X$, $P(x)$ is W-convex.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in P(\bar{x})$.

Proof. Suppose to the contrary that there is no fixed point of P in X. Thus for any $x \in X$, $x \notin P(x)$. Consider the set $G = \{(x, y) \in X \times X \mid y \notin P(x)\}$. Since for each $x \in X$ $(x, x) \in G$, G is a nonempty subset of $X \times X$. By (1), G is closed in $X \times X$. By (2), for any finite subset A of X,

$$\bigcap_{y \in A} [X \setminus P^{-1}(y)] = \bigcap_{y \in A} [X \setminus \{x \in X: (x, y) \notin G\}]$$
$$= \bigcap_{y \in A} \{x \in X: (x, y) \in G\},$$

which is connected. By (3), for each $x \in X$ $\{y \in X \mid (x, y) \notin G\} = P(x)$ is W-convex or empty. By Theorem 3.1, there exists $x_0 \in X$ such that $\{x_0\} \times X \subset G$. Hence for any $x \in X$, $x \notin P(x_0)$, i.e., $P(x_0) = \varnothing$, which contradicts the fact that $P(x_0) \neq \varnothing$. Thus, there exists $\bar{x} \in X$ such that $\bar{x} \in P(\bar{x})$.

4. MINIMAX INEQUALITIES FOR VECTOR-VALUED MAPPINGS

In 1961, Fan [11] proved the following minimax inequality for real-valued mappings.

THEOREM (K. Fan). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f: X \times X \to \mathbb{R}$ such that*

(i) *for each $y \in X$, $f(x, y)$ is lower semicontinuous in x;*

(ii) *for each* $x \in X$, $f(x, y)$ *is quasi-concave in* y.
Then we have

$$\min_{x\in X} \sup_{y\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

Now we give some minimax inequalities for vector-valued mappings in topological vector spaces with closed pointed convex cones.

DEFINITION 4.1. Let Y be a Hausdorff topological vector space with a closed pointed convex cone S such that the interior of S, int S is nonempty and C be a nonempty subset of Y.

(1) A point $y_0 \in C$ is called a minimal point of C if $C \cap (y_0 - S) = \{y_0\}$, by Min C we denote the set of all minimal points of C;

(2) A point $y_0 \in C$ is called a weakly minimal point of C if $C \cap (y_0 - \text{int } S) = \varnothing$, by $\text{Min}_w C$ we denote the set of all weakly minimal points of C.

(3) A point $y_0 \in C$ is called a maximal point of C if $C \cap (y_0 + S) = \{y_0\}$, by Max C we denote the set of all maximal points of C.

(4) A point $y_0 \in C$ is called a weakly maximal point of C if $C \cap (y_0 + \text{int } S) = \varnothing$, by $\text{Max}_w C$ we denote the set of all weakly maximal points of C.

LEMMA 4.1 [4, 14]. *If C is a nonempty compact subset of Y. Then we have* (a) Min $C \neq \varnothing$; (b) Max $C \neq \varnothing$

Remark. Min $C \subset \text{Min}_w C$ and Max $C \subset \text{Max}_w C$.

The following lemma is a special case of Lemma 5.5 in [16].

LEMMA 4.2. *Let X be a nonempty compact subset of a Hausdorff topological space, and Y a topological vector space, and $S \subset Y$ a closed pointed convex cone with* int $S \neq \varnothing$. *If $f\colon X \times X \to Y$ is continuous, then*

$$\varnothing \neq \bigcup_{t\in X} \text{Max}_w f(X,t) \subset \text{Min} \bigcup_{t\in X} \text{Max}_w f(X,t) + S.$$

DEFINITION 4.2. Let $(X, \{C_A\})$ be a W-space. Then a subset D of X is called weakly W-convex if for any finite subset A of D, $C_A \cap D$ is connected, i.e., $(D, \{C_{A\cap D} \cap D\})$ is a W-space.

THEOREM 4.1. *Let $(U, \{C_A\})$ be a W-space and Y a Hausdorff topological vector space with a closed pointed convex cone S such that* int S *is nonempty. Let X be a compact weakly W-convex subset of U and $f\colon X \times X \to Y$ a vector-valued mapping such that*

(1) *given* $m \in \text{Max}_w \bigcup_{t\in X} f(t,t)$, $\bigcap_{x\in A}\{u \in X \mid f(x,u) - m \notin \text{int } S\}$ *is connected for any finite subset A of X;*

(2) *given* $m \in \text{Max}_w \bigcup_{t\in X} f(t,t)$, $B_u = \{x \in X \mid f(x,u) - m \in \text{int } S\}$ *is W-convex or empty for each* $u \in X$;

(3) *f is continuous.*

Then for every $m \in \text{Max}_w \bigcup_{t\in X} f(t,t)$ *there exists*

$$z \in \text{Min} \bigcup_{t\in X} \text{Max}_w f(X,t)$$

such that $z - m \notin \text{int } S$. *Moreover, we have*

$$\text{Max} \bigcup_{t\in X} f(t,t) \subset \text{Min} \bigcup_{t\in X} \text{Max}_w f(X,t) + K, \quad \text{where } K = Y \setminus (-\text{int } S).$$

Proof. Note that $(X, \{C_{A\cap X} \cap X\})$ is a W-space. Since $\{(x,x) \mid x \in X\}$ is compact and f is continuous, by Lemma 4.1 Max $\bigcup_{t\in X} f(t,t) \neq \varnothing$ and hence $\text{Max}_w \bigcup_{t\in X} f(t,t) \neq \varnothing$. Let $m \in \text{Max}_w \bigcup_{t\in X} f(t,t)$, then there exists $t_0 \in X$ such that $m = f(t_0, t_0)$. Let $G = \{(x,u) \in X \times X \mid m \in f(x,u) + K\}$. Since $m \in \text{Max}_w \bigcup_{t\in X} f(t,t)$, by the weak maximality of m, $(x,x) \in G$ for any $x \in X$, and hence G is nonempty. Since $m - K$ is closed and f is continuous, G is also closed in $X \times X$. By (1), for any finite subset A of X, $\bigcap_{x\in A}\{u \in X \mid (x,u) \in G\} = \bigcap_{x\in A}\{u \in X \mid f(x,u) - m \notin \text{int } S\}$ is connected. By (2), for each $u \in X$, $\{x \in X \mid (x,u) \notin G\} = \{x \in X \mid f(x,u) - m \in \text{int } S\}$ is W-convex or empty. By Theorem 3.1, there exists $u_0 \in X$ such that $X \times \{u_0\} \subset G$, that is, for any $x \in X$, $m \in f(x, u_0) + K$. Since $\text{Max}_w f(X, u_0)$ is nonempty, let $x_0 \in X$ such that $f(x_0, u_0) \in \text{Max}_w f(X, u_0)$. Thus $m \in f(x_0, u_0) + K$. By Lemma 4.2,

$$\varnothing \neq \bigcup_{t\in X} \text{Max}_w f(X,t) \subset \text{Min} \bigcup_{t\in X} \text{Max}_w f(X,t) + S.$$

Hence there exists

$$z \in \text{Min} \bigcup_{t\in X} \text{Max}_w f(X,t) \qquad \text{such that } f(x_0, u_0) \in z + S.$$

Thus

$$m \in f(x_0, u_0) + K \in z + S + K = z + K,$$

i.e., $z - m \notin \text{int } S$. Furthermore, by the arbitrariness of $m \in$

$\mathrm{Max}_w \bigcup_{t \in X} f(t,t)$ we have $\mathrm{Max}_w \bigcup_{t \in X} f(t,t) \subset \mathrm{Min} \bigcup_{t \in X} \mathrm{Max}_w f(X,t) + K$. Since S is pointed, $\mathrm{Max} \bigcup_{t \in X} f(t,t) \subset \mathrm{Max}_w \bigcup_{t \in X} f(t,t)$. Thus,

$$\mathrm{Max} \bigcup_{t \in X} f(t,t) \subset \mathrm{Min} \bigcup_{t \in X} \mathrm{Max}_w f(X,t) + K.$$

This completes the proof.

THEOREM 4.2. *Let $(X, \{C_A\})$ be a W-space, Y be a Hausdorff topological space, and Z a Hausdorff topological vector space with a closed pointed convex cone S such that* int S *is nonempty. Let $f\colon X \times Y \to Z$ be a mapping satisfying the following conditions*:

(i) *f is continuous*;

(ii) (a) *for any finite subset A of X, the set $\{y \in Y \mid f(x,y) \notin \operatorname{int} S, \forall x \in A\}$ is connected.*

(b) *for any $x_1, x_2 \in X$, we have*

$$f(x,y) - f(x_1,y) \in S \text{ or } f(x,y) - f(x_2,y) \in S$$
$$\text{for all } x \in C_{\{x_1,x_2\}} \text{ and all } y \in Y;$$

(iii) *Y is compact.*

Then one of the following conclusions holds;

(1) *there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \in \operatorname{int} S$ for any $y \in Y$.*

(2) *there exists $\bar{y} \in Y$ such that $f(x, \bar{y}) \notin \operatorname{int} S$ for any $x \in X$.*

Proof. Define $F\colon X \to 2^Y$ by $F(x) = \{y \in Y \mid f(x,y) \notin \operatorname{int} S\}$ for any $x \in X$. If there exists $\bar{x} \in X$ such that $F(\bar{x}) = \emptyset$, then the conclusion (1) holds. Suppose that $F(x) \neq \emptyset$ for any $x \in X$. Since f is continuous, $\mathrm{Graph}(F) \equiv \{(x,y) \mid f(x,y) \in Z \setminus \operatorname{int} S\}$ is closed. Since Y is compact, by Lemma 2.1 F is upper semicontinuous with closed values. By (ii)(a), assumption (ii) of Theorem 2.1 holds. By (ii)(b), for any $x_1, x_2 \in X$, we have

$$f(x_1,y) \in f(x,y) - S \text{ or } f(x_2,y) \in f(x,y) - S$$
$$\text{for all } x \in C_{\{x_1,x_2\}} \text{ and all } y \in Y.$$

Let $z \in C_{\{x_1,x_2\}}$ and $y \in F(z)$, then $f(z,y) \notin \operatorname{int} S$. Since $f(x_1,y) \in f(z,y) - S$ or $f(x_2,y) \in f(z,y) - S$,

$$f(x_1,y) \in (Z \setminus \operatorname{int} S) - S = Z \setminus \operatorname{int} S \quad \text{or}$$
$$f(x_2,y) \in (Z \setminus \operatorname{int} S) - S = Z \setminus \operatorname{int} S.$$

Hence $y \in F(x_1) \cup F(x_2)$. Thus assumption (iii) of Theorem 2.1 holds. By Theorem 2.1, $\bigcap_{x \in X} F(x) \neq \emptyset$. Hence there exists $\bar{y} \in Y$ such that $f(x, \bar{y}) \notin \operatorname{int} S$ for any $x \in X$.

This completes the proof.

Remark. If X is a Hausdorff topological vector space, then condition (ii)(b) becomes the definition of the properly quasi S-concavity, which appeared in [12, 16].

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REFERENCES

1. J. P. Aubin, "Mathematical Methods of Game and Economic Theory," North-Holland, Amsterdam, 1979.
2. C. Bardaro and R. Ceppitelli, Some further generalization of Knaster–Kuratowski–Mazurkiewicz theorem and minimax inequalities, *J. Math. Anal. Appl.* **132** (1988), 484–490.
3. C. Bardaro and R. Ceppitelli, Applications of the generalized Knaster–Kuratowski–Mazurkiewicz theorem to variational inequalities, *J. Math. Anal. Appl.* **137** (1989), 46–58.
4. J. M. Borwein, On the existence of Pareto efficient points, *Math. Oper. Res.* **8** (1983), 64–73.
5. F. E. Browder, The fixed point theorem of multi-valued mappings in topological vector spaces, *Math. Ann.* **117** (1968), 283–301.
6. S. S. Chang and Y. H. Ma, Generalized KKM theorem on *H*-space with applications, *J. Math. Anal. Appl.* **163** (1992), 406–421.
7. S. S. Chang, "Variational Inequality and Complementarity Problem Theory with Applications," Shanghai Scientific and Technological Literature Publishing House, Shanghai, 1991.
8. S. S. Chang, Y. J. Cho, X. Wu, and Y. Zhang, Topological versions of KKM theorem and Fan's matching theorem with applications, *Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center*, **1** (1993), 231–245.
9. S. S. Chang, Xian Wu, and Shu-wen Xiang, A topological KKM theorem and minimax theorem, *J. Math. Anal. Appl.* **182** (1994), 756–767.
10. G. Y. Chen, A generalized section theorem and a minimax inequality for a vector-valued mapping, *Optimization* **22** (1991), 745–754.
11. K. Fan, A minimax inequality and applications, *in* "Inequalities III" (O. Shisha, Ed.), pp. 103–113, Academic Press, New York, 1972.
12. F. Ferro, A minimax theorem for vector-valued functions, *J. Optim. Theory Appl.* **60** (1989), 19–31.
13. C. Horvath, Some results on multivalued mappings and inequalities without convexity, *in* "Nonlinear and Convex Analysis," Lecture Notes in Pure and Appl. Math., Vol. 107, Springer-Verlag, Berlin/New York, 1987.

14. J. Jahn, "Mathematical Vector Optimization in Partially Ordered Linear Spaces," Verlag Peter Lang, Frankfurt am Main, 1986.
15. L. L. Stacho, Minimax theorems beyond topological vector spaces, *Acta Sci. Math.* (*Szeged*) **42** (1980), 157–164.
16. T. Tanaka, Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions, *J. Optim. Theory Appl.* **81** (1994), 355–377.

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Stability and Existence of Periodic Solutions of a Functional Differential Equation

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1. INTRODUCTION

In this paper, equations of the type

$$\dot{x}(t) = -\lambda x(t) + \lambda f(x(t-1)) \tag{1.1}$$

f: $R^n \to R^n$, $\lambda > 0$ are considered.

Táboas, in [5], has studied the planar delay differential equation $\dot{x}(t) = -x(t) + \alpha f(x(t-1))$ for $\alpha > 0$, and the elements of the diagonal of the Jacobian Matrix of f at $(0,0)$, i.e., $Jf(0,0)$ are all zeros. An existence theorem for nonconstant periodic solutions is achieved for some value of $\alpha > \alpha_0$, for some α_0. The method used consists of finding a cone in the phase space which is mapped into itself under a certain operator defined by the flow. A fixed point of this operator corresponds to a periodic solution.

We treat the case where the diagonal of $Jf(0,0)$ is not null. Another aspect of this paper is to take the functions in a set K in such a way that it becomes an equicontinuous set. Since f is continuous and $f(0,0) = 0$ and the work is done in a neighbourhood of zero, it is easy to show that the solutions of (1.1) have the same properties when they return to the set K.