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Images of contractive projections on operator algebras

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Abstract

It is shown that if P is a weak*-continuous contractive projection on a JBW*-triple M, then $P(M)$ is of type I or semifinite, respectively, if M is of the corresponding type. We also show that $P(M)$ has no infinite spin part if M is a type I von Neumann algebra.

0. Introduction

JW*-triples, that is, weak*-closed subspaces of $B(H)$ that are also closed under $x \mapsto xx^*x$, arise as images of contractive (i.e., norm one) projections on von Neumann algebras. Their generalisations, JBW*-triples, are those complex Banach dual spaces whose open unit ball is a bounded symmetric domain. The holomorphy of such spaces induces a ternary Jordan algebraic structure determined by a certain "triple product" $\{a, b, c\}$ [18]. If $P : M \to M$ is a weak*-continuous contractive projection on a JBW*-triple M, then $P(M)$ is a JBW*-

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triple with a triple product given by $\{a,b,c\}_P := P\{a,b,c\}$ by [19,21], and by [9,11] if M is a JW*-triple. The interesting special cases that occur when P is positive unital acting on von Neumann algebra or a JBW*-algebra were studied earlier in [4,7,20].

Suppose $P: M \to M$ is a weak*-continuous contractive projection on a JBW*-triple M. In this paper we study the stability of $P(M)$ with respect to the type theory of [15–17]. We show that if M is of type I or semifinite, respectively, then $P(M)$ is of the corresponding type. This extends the classical results of [27] when M is a von Neumann algebra and $P(M)$ is a subalgebra. We remark that in general $P(M)$ is not a subtriple of M. Using recent results on properties of the predual of a type I von Neumann algebra we deduce that $P(M)$ cannot be isometric to an infinite-dimensional spin factor whenever M is a type I von Neumann algebra.

Section 1 of this paper contains preliminary results on JBW*-algebras. This is continued in Section 2 where we study the fixed point JW*-algebra, W^α, of an involution α on a von Neumann algebra W. A principal aim here is to show that a faithful weak*-continuous contractive projection from W^α onto a continuous JW*-subalgebra induces a weak*-continuous contractive projection from W onto a continuous von Neumann subalgebra. This allows us to apply [27] to obtain our main results in Section 4. The formulation of type theory of JBW*-triples contained in Section 3 is extracted from [15–17] and is included for completeness.

For later reference we shall recall some of the fundamentals of JBW*-triples used in this paper. A JBW*-triple can be realised [18] as a complex Banach space M with predual M_* and continuous ternary triple product $(a,b,c) \mapsto \{a,b,c\}$ that is conjugate linear in b and symmetric bilinear in a, c such that $\|\{a,a,a\}\| = \|a\|^3$ and such that the operator $x \mapsto \{a,a,x\}$, denoted by $D(a,a)$, is Hermitian with non-negative spectrum and satisfies

$$D(a,a)\big(\{x,y,z\}\big) = \big\{D(a,a)x, y, z\big\} - \big\{x, D(a,a)y, z\big\} \\ + \big\{x, y, D(a,a)z\big\}.$$

The predual is unique and the triple product is separately weak*-continuous [2, 15]. The surjective linear isometries between JBW*-triples are the triple product preserving bijections (triple isomorphisms) [18]. A von Neumann algebra is a JBW*-triple with triple product $\{a,b,c\} = (1/2)(ab^*c + cb^*a)$. The weak*-closed subtriples of von Neumann algebras are the JW*-triples. By [16,17] most JBW*-triples are of this form. See Section 3 for further details.

An element u in a JBW*-triple M satisfying $\{u,u,u\} = u$ is called a *tripotent*, when M is a JW*-triple these are precisely the partial isometries of M. Associated with a tripotent u are the mutually orthogonal *Peirce* projections $P_2(u)$, $P_1(u)$, and $P_0(u)$. We have

$$P_2(u)(x) = \big\{u, \{u,x,u\}, u\big\} \quad \text{for all } x,$$
$$P_1(u) = 2\big(D(u,u) - P_2(u)\big) \quad \text{and} \quad P_2(u) + P_1(u) + P_0(u) = i$$

(where i is the identity map). A tripotent u of M is said to be *complete* (or maximal) if $P_0(u) = 0$, to be unitary if $P_2(u) = i$ and to be minimal if $P_2(u)(M) = \mathbb{C}u$. We recall (see [5, Corollary 4.8], for example) that the complete tripotents of M are the extreme points of the closed unit ball of M. A crucial simplifying property of JBW*-triples is that for a tripotent u of M the Peirce-2 subspace $P_2(u)(M)$ is a JBW*-algebra with product $a \circ b = \{a, u, b\}$ and involution $a^* = \{u, a, u\}$. For further properties of JBW*-triples we refer to the papers [5,6,9,15–18] and the book [29]. Since JBW*-algebras are just the complexifications of JW-algebras we refer to [14] for their theory.

1. Positive unital projections on JBW*-algebras

Let M be a JBW*-algebra. Writing

$$[a, b, c] := (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b,$$

M is a JBW*-triple with triple product given by $\{a, b, c\} := [a, b^*, c]$. The Peirce-2 projection, $P_2(e)$, associated with a projection e of M satisfies $P_2(e)(x) = [e, x, e]$ for all x in M.

Elements a and b of M are said to *operator commute* in M if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in M. Self-adjoint elements a and b in M generate a JBW*-subalgebra that can be realised as a JW*-subalgebra of some $B(H)$ [30] and, in this realisation, a and b commute in the usual sense if they operator commute in M [28, Proposition 1]. By the same references, self-adjoint elements a and b of M operator commute if and only if $a^2 \circ b = [a, b, a]$ $(= \{a, b, a\})$. If N is a JBW*-subalgebra of M we use $M \cap N'$ to denote the set of elements in M that operator commute with every element of N. (This corresponds to the usual notation when M is a von Neumann algebra.) The *centre* of M is $M \cap M'$ which we also denote by $Z(M)$.

Let P be a unital (i.e., $P(1) = 1$) weak*-continuous contractive projection on a JBW*-algebra M. Then P is positive and therefore is invariant on the self-adjoint part. Such projections were studied in [7,20]. Suppose now that $P(M)$ is a JBW*-subalgebra N of M. Then, by [7, Lemma 1.5] or [20, Lemma 1.5] we have $P(a \circ x) = a \circ P(x)$ for all $a \in N$ and $x \in M$. Further, if e denotes the support projection of P in M (i.e., the least projection in M sent to 1 by P) then $P = PP_2(e)$ and, by a slight extension of [7, Lemma 1.2(2)], $e \in M \cap N'$. Moreover, if $x \geqslant 0$ and $P(x) = 0$ then $P_2(e)(x) = 0$. If $e = 1$, P is said to be faithful.

Lemma 1.1. *Let $P : M \to M$ be a weak*-continuous unital contractive projection from a JBW*-algebra M onto a JBW*-subalgebra N. Let e be the support projection of P. Then $P_2(e)P$ is a faithful weak*-continuous unital projection from $P_2(e)(M)$ onto $N \circ e$. Moreover, N is isomorphic to $N \circ e$.*

Proof. Suppose $x \in P_2(e)(M)$ such that $x \geqslant 0$ and $P_2(e)P(x) = 0$. Then $P(x) = PP_2(e)P(x) = 0$ so that $x = P_2(e)(x) = 0$. Together with the above remarks this proves the first statement.

Since $e \in M \cap N'$ multiplication by e induces a (Jordan) homomorphism, π, from N onto $N \circ e$. Let a in N such that $a \geqslant 0$ and $a \circ e = 0$; then $a = P(a \circ e) = 0$. It follows that π is injective. $\square$

Lemma 1.2. *Let $P : M \to M$ be a weak*-continuous unital contractive projection from a JBW*-algebra onto a JBW*-subalgebra N. Let e be any non-zero projection in $M \cap N'$. Suppose that P is faithful. Then there exists a faithful weak*-continuous unital contractive projection from $P_2(e)(M)$ onto $N \circ e$.*

Proof. For each self-adjoint $a \in N$ we have

$$a^2 \circ P(e) = P\big(a^2 \circ e\big) = P\big(\{a, e, a\}\big) = \big\{a, P(e), a\big\}$$

and so, by the previous remark, $P(e) \in Z(N)$. Therefore the range projection $r(P(e)) \in Z(N)$. Denote $r(P(e))$ by h. The ideal of N, $N \circ P(e) = P(N \circ e)$, is weak*-closed and so equals $N \circ h$. It follows that $P(e)$ is invertible in $N \circ h$ with inverse b, say, in $Z(N) \circ h$. Define $Q : P_2(e)(M) \to P_2(e)(M)$ by $Q(x) = (P(x) \circ b) \circ e$. Let $a \in N$ where $a \geqslant 0$. By operator commutivity we have $(a \circ (1 - h)) \circ e \geqslant 0$ and

$$P\big(\big(a \circ (1 - h)\big) \circ e\big) = \big(a \circ (1 - h)\big) \circ P(e) = a \circ \big((1 - h) \circ P(e)\big) = 0.$$

Since P is faithful, $a \circ e = (a \circ h) \circ e$, and so

$$Q(a \circ e) = \big(P(a \circ e) \circ b\big) \circ e = \big(\big(a \circ P(e)\big) \circ b\big) \circ e = (a \circ h) \circ e = a \circ e,$$

implying that Q is a unital projection onto $N \circ e$. To see that Q is faithful let $x \in P_2(e)(M)$ such that $x \geqslant 0$ and $(P(x) \circ b) \circ e = 0$. By the above,

$$P(x) \circ e = \big(P(x) \circ h\big) \circ e = \big(\big(P(x) \circ b\big) \circ e\big) \circ P(e) = 0.$$

Therefore, $P(x) \circ P(e) = P(P(x) \circ e) = 0$. But $P(x) \leqslant \|x\| P(e)$. Hence, $P(x) = 0$ and so $x = 0$ because P is faithful. $\square$

2. Involutory * antiautomorphisms

Following [24] by an involution α on a von Neumann algebra we shall mean an involutory * antiautomorphism on the algebra. Let α be an involution on a von Neumann algebra W. We shall write $R(W) := \{x \in W\colon\ \alpha(x) = x^*\}$ and $W^\alpha := \{x \in W\colon\ \alpha(x) = x\}$. (The latter notation is different from that used in [24], where it stands for the Hermitian part.) Then $R(W)$ is a weak*-closed real *-subalgebra of W with $R(W) \cap iR(W) = \{0\}$ and $W = R(W) + iR(W)$. We have $W^\alpha = R(W)_{sa} + iR(W)_{sa}$ and, for $a, b \in R(W)$, we have $\alpha(a + ib) = a^* + ib^*$.

Lemma 2.1. *Let α be an involution on a von Neumann algebra W and suppose e is a central projection in W such that $e + \alpha(e) = 1$. Then $eW^\alpha = eW$ and W^α is (Jordan) isomorphic to eW via $x \mapsto ex$.*

Proof. For each x in W, $ex + (1-e)\alpha(x) \in W^\alpha$ and every element of W^α is of this form. Thus $eW^\alpha = eW$ and W^α is isomorphic to eW in the way stated. □

Lemma 2.2. *Let α be an involution on a von Neumann algebra W and suppose that e is a projection in W with $e + \alpha(e) = 1$. Then we have the following:*

(i) *There is a faithful weak*-continuous unital contractive projection, $P : W^\alpha \to W^\alpha$, such that $P(W^\alpha)$ is a JW*-subalgebra isomorphic to eWe (and to $(1-e)W(1-e)$).*
(ii) *If W^α generates W as a von Neumann algebra and $eW^\alpha\alpha(e) = 0$, then $e \in Z(W)$.*

Proof. (i) Let V denote the von Neumann algebra $eWe + (1-e)W(1-e)$. Define $P : W \to W$ by $P(x) := exe + (1-e)x(1-e)$. Then $P(W) = V = \alpha(V)$. If s denotes the symmetry $2e - 1$ we see that $P(x) = (1/2)(x + sxs)$. Since $\alpha(s) = -s$, we have $\alpha P = P\alpha$ from which we deduce that $P(W^\alpha) = V^\alpha$. Since e lies in the centre of V, Lemma 2.1 implies that V^α is isomorphic to $eV = eWe$. It is clear that P satisfies (i).

(ii) Suppose $eW^\alpha\alpha(e) = 0$. Then for $x \in W^\alpha$ we have

$$x = exe + (1-e)x(1-e)$$

so that $ex = exe$. Passing to the self-adjoint part we see that e commutes with all elements of W^α and so lies in the centre of W if W is the von Neumann algebra generated by W^α. □

Lemma 2.3. *Let α be an involution in a non-Abelian von Neumann algebra W. Then there is a non-zero projection e in W with $e\alpha(e) = 0$.*

Proof. We have $R(W)_{sa} \neq R(W)$; otherwise α is the identity map on W and therefore W is Abelian. Choose a in $R(W)$ such that $a \neq a^*$ and let $a - a^* = b$. Let V denote the von Neumann subalgebra of W generated by b. We have that V is Abelian, that $\alpha(b) = -b$ and $\alpha(V) = V$. Since α is not the identity map on V, by [14, 7.3.4] there is a non-zero projection $e \in V$ such that $e\alpha(e) = 0$. □

Proposition 2.4. *Let α be an involution on a von Neumann algebra W and suppose that W^α has no type I part. Then there is a projection e in W and a faithful weak*-continuous unital contractive projection from W^α onto a JW*-subalgebra M such that $e \in W \cap M'$ and Me is a W*-algebra isomorphic to M.*

Moreover, if W^α is of type II_1, II_∞ or III, respectively, then M is of the corresponding type.

Proof. Let (e_i) be a family of projections in W maximal subject to the condition that $(e_i + \alpha(e_i))$ is a mutually orthogonal family of projections. Put $e = \sum_i e_i$. Then $e\alpha(e) = 0$. Let $f = 1 - e - \alpha(e)$. Then $\alpha(fWf) = fWf$ and it follows from Lemma 2.3, and maximality, that fWf is Abelian and hence that $fW^\alpha f$ is Abelian. By assumption, we must have $f = 0$. Lemma 2.2(i) now gives the first statement. Since W^α generates W, by [13, Theorem 2.8], the second statement follows from [1, Theorem 8] together with Lemma 2.2(i). □

Proposition 2.5. *Let α be an involution on a von Neumann algebra W, and let M denote W^α. Suppose there is a faithful weak*-continuous unital contractive projection, P, from M onto a JW*-subalgebra N. If N is continuous* (*respectively, of type III*) *then there is a weak*-continuous contractive projection from W onto a continuous* (*respectively, type III*) *W*-subalgebra.*

Proof. Let V be the von Neumann subalgebra of W generated by N and let R be the weak*-closed real *-subalgebra of W generated by V_{sa}. We have $\alpha(V) = V$ since α fixes each element of N, and $R \cap iR = \{0\}$ since $R \subset R(W)$. Suppose N is continuous (respectively, of type III). Then $N_{sa} = R_{sa}$, using [14, 7.3.3], so that $V = R + iR$, by [22, Theorem 2.4]. Hence, $V^\alpha = R_{sa} + iR_{sa} = N$. By Proposition 2.4 there exists a faithful weak*-continuous unital contractive projection, $Q : N \to N$, onto a continuous (respectively, type III) JW*-subalgebra K together with a projection $e \in W \cap K'$ such that Ke is a W*-algebra isomorphic to K. If E denotes the (faithful) canonical projection $(1/2)(i + \alpha) : W \to M$, then the proof is completed by application of Lemma 1.2 to the projection $QPE : W \to K$. □

We recall [24] that an involution α is said to be a *central* involution if it fixes every element in $Z(W)$.

Lemma 2.6. *Let α be a central involution on a continuous von Neumann algebra W. Let u be a partial isometry of W^α such that $(1 - uu^*)W^\alpha(1 - u^*u) = 0$. Then $u^*u = uu^* = 1$.*

Proof. Let e denote $1 - uu^*$. Then $\alpha(e) = 1 - u^*u$. Put $p = e + \alpha(e)$. Then α is a central involution on pWp. By [13, Theorem 2.8] or [24, Proposition 3.2] $(pWp)^\alpha$ $(= pW^\alpha p)$ generates pWp. Hence by Lemma 2.2(ii), $e \in Z(pWp) = Z(W)p$ so that $\alpha(e) = e$, whence the result. □

3. Types of JBW*-triples

The aim of this short section, which contains no new results, is to collate existing theory into a form easy to use subsequently.

Cartan factors. Of the six kinds of Cartan factors (up to linear isometry), three are of the form $pB(H)$, $\{x \in B(H)\colon\ x = jx^*j\}$ and $\{x \in B(H)\colon\ x = -jx^*j\}$, where H is a complex Hilbert space, p is a projection in $B(H)$ and $j : H \to H$ is a conjugation. These are referred to as *rectangular*, *Hermitian* and *symplectic* Cartan factors, respectively. Hermitian factors are type I JW*-algebra factors and, if H is even or infinite-dimensional, symplectic factors are linearly isometric to type I JW*-algebra factors. *Spin* factors (complexifications of real spin factors) comprise a fourth kind. The remaining two *exceptional* Cartan factors can be realised as the 3×3 Hermitian matrices and the 1×2 matrices, respectively, over the complex Cayley numbers.

Type I JBW-triples.* In view of [15, 4.14] a JBW*-triple M is said to be of *type I* if there is a complete tripotent u of M such that $P_2(u)(M)$ is a type I JBW*-algebra. By the type I classification theorem [16, 1.7] the type I JBW*-triples are precisely the ℓ_∞-sums of JBW*-triples of the form:

(i) $A \mathbin{\overline{\otimes}} C$, where A is an Abelian von Neumann algebra and C is a Cartan factor realised as a JW*-subtriple of some $B(H)$, the bar denoting the weak*-closure in the usual von Neumann tensor product $A \mathbin{\overline{\otimes}} B(H)$, and
(ii) $A \otimes C$ (algebraic tensor product), where A is as before and C is an exceptional Cartan factor.
(Of course, $A \mathbin{\overline{\otimes}} C = A \otimes C$ whenever C is a finite-dimensional non-exceptional Cartan factor.)

Let e be a tripotent in a type I JBW*-triple M. A known consequence of the type I classification theorem is that $P_2(e)(M)$ is of type I. We include an argument for completeness and want of a precise reference.

We may suppose that M is of the form (i) or (ii) above. In the latter case it is clear that $P_2(e)(M)$ is of type I since every subfactor of it must have rank less than 4. Thus we may assume that we are in the case (i) and, consequently, that we are working in $A \mathbin{\overline{\otimes}} B(H)$.

Let u be a non-zero (we assume $e \neq 0$) in a weak*-closed ideal J of $P_2(e)(M)$.

Since $\{u, (A \otimes B(H)), u\} = (A \otimes 1)\{u, (1 \otimes B(H)), u\}$ and $B(H)$ is the weak*-closed linear span of its minimal tripotents, $\{u, (1 \otimes v), u\} \neq 0$ for some minimal tripotent v. We have $\{(1 \otimes v), M, (1 \otimes v)\} = A \otimes v$ so that with $x = \{u, (1 \otimes v), u\}$ ($\in P_2(u)(M)$) we have $\{x, M, x\} \subset (A \otimes 1)x$. Since $A \otimes 1$ commutes elementwise with x, $(A \otimes 1)x$ generates an Abelian subtriple in the sense of [15, 1.4]. But, as follows from [5, Lemma 3.1], the weak*-closure of $\{x, M, x\}$ equals $P_2(w)(M)$, for some tripotent w, and so is Abelian. Since $w \in J$, $P_2(e)(M)$ is of type I, by [15, 4.14 (2) $\Rightarrow$ (1)].

Continuous JBW-triples.* A JBW*-triple M is said to be *continuous* if it has no type I ℓ_∞-summand. In that case, up to isometry, M is a JW*-triple with unique decomposition, $M = W^\alpha \oplus pV$, where W and V are continuous von Neumann algebras, p is a projection in V and α is a *central* involution on W [17, 2.1 and 4.8]. It is implicit in [17] that every complete tripotent of W^α is a unitary tripotent. An alternative proof of this fact is provided by Lemma 2.6. Thus, by [17, 5.1–5.7], for every complete tripotent u in M, $P_2(u)(M)$ is isometric to $W^\alpha \oplus pWp$. We define M to be of type II_1, II_∞ or III, respectively, if both W and pWp are of the corresponding type. M is said to be *semifinite* if it has no type III ℓ_∞-summand.

Lemma 3.1 below summarizes the above. The second statement is a consequence of the fact that every tripotent in a JBW*-triple M is a projection in $P_2(u)(M)$ for some complete tripotent u [15, 3.12].

Lemma 3.1. *A JBW*-triple M is of type I, II_1, II_∞, III or is semifinite, respectively, if and only if $P_2(u)(M)$ is of the corresponding type for some, and hence every, complete tripotent u of M. If M is of type I, II_1, III or is semifinite, respectively, then so is $P_2(u)(M)$ for every tripotent u of M.*

We shall say that a JBW*-triple has no *infinite spin part* if it has no ℓ_∞-summands of the form $A \overline{\otimes} C$, where A is an Abelian von Neumann algebra and C is an infinite-dimensional spin factor.

4. Contractive projections on JBW*-triples

By [19] and [21] the image of a weak*-continuous contractive projection, $P: M \to M$, on a JBW*-triple M is again a JBW*-triple with triple product $\{x, y, z\}_P := P(\{x, y, z\})$ for x, y, z in $P(M)$ and

$$P\{P(x), y, P(z)\} = P\{P(x), P(y), P(z)\}$$

for all x, y, z in M. The image, $P(M)$, need not be a JBW*-subtriple of M. However, as is made explicit in [6, Lemma 5.3] and its proof, we do have the following:

Lemma 4.1 [6, Lemma 5.3]. *If $P: M \to M$ is a weak*-continuous contractive projection on a JBW*-triple M, there exists a JBW*-subtriple C of M such that C is linearly isometric to $P(M)$ and such that C is the image of a weak*-continuous projection on M.*

We are now in a position to prove our first main result. We freely use Lemma 3.1 throughout.

Theorem 4.2. *Let $P : M \to M$ be a weak*-continuous contractive projection on a JBW*-triple M. If M is of type I* (*respectively, semifinite*) *then $P(M)$ is of type I* (*respectively, semifinite*).

Proof. Let M be of type I (respectively, semifinite). By Lemma 4.1 we may suppose $P(M)$ to be a JBW*-subtriple, N, of M. Let u be a complete tripotent of N. By the above formula, P restricts to a unital projection from $P_2(u)(M)$ to $P_2(u)(N)$.

By this fact, together with Lemma 1.1, we may suppose P to be faithful, M to be a JBW*-algebra and N to be a JBW*-subalgebra.

Let $M \circ z$ be the type I finite part of M, where z is a central projection of M. Then $N \circ z$ is type I finite, being a subalgebra of $M \circ z$, and it remains only to show that $N \circ (1 - z)$ is of type I (respectively, semifinite). Since, by Lemma 1.2, $N \circ (1 - z)$ is the image of some faithful weak*-continuous unital contractive projection on $M \circ (1 - z)$, it can be supposed that $z = 0$. In that case, by [14, 7.2.7 and 7.3.3], we may suppose that $M = W^\alpha$, where α is an involution on a von Neumann algebra W. Since W^α generates W [13, Theorem 2.8], W is of type I (respectively, semifinite) by [14, 7.4.2] and [1, Theorem 8].

In order to obtain a contradiction, suppose now that N has a non-zero continuous (respectively, type III) part, $N \circ e$, where e is a central projection of N. Now, α is an involution on eWe with $(eWe)^\alpha = eMe$. Applying Proposition 2.5 to $P : eMe \to N \circ e$, which is surjective, we obtain a weak*-continuous projection from the type I (respectively, semifinite) W*-algebra eWe onto a continuous (respectively, type III) W*-subalgebra. This contradicts [27, Theorem 3 (respectively, Theorem 4)] and so completes the proof. □

In order to prove a refinement of part of Theorem 4.2, we first recall a Banach space property introduced in [8].

Definition. A Banach space E is said to have the DP1 if whenever a sequence $x_n \to x$ weakly in E with $\|x_n\| = \|x\| = 1$ for all n, and (ρ_n) is a weakly null sequence in E^*, then $\rho_n(x_n) \to 0$.

We write M_* for the predual of a JBW*-triple M and we note that if $P : M \to M$ is a weak*-continuous contractive projection then the dual projection restricts to a contractive projection on M_* and that $P(M)_*$ is linearly isometric to $P^*(M_*)$ via $\tau \mapsto \tau \circ P$. It follows that if M_* has the DP1 then so does $P(M)_*$.

Recently, the authors characterised the von Neumann algebras whose predual has the DP1.

Lemma 4.3 [3, Theorem 6]. *A von Neumann algebra is of type I if and only if its predual has the DP1.*

For properties of (real) spin factors used in the next proof, see [14, Section 6].

Lemma 4.4. *Let C be an infinite-dimensional spin factor. Then C_* does not have the DP1.*

Proof. The argument is similar to that in [3, Proposition 5]. Let τ denote the tracial state of C and let R be the real Banach space generated by the non-trivial symmetries in C. Then R is isometric to an infinite-dimensional real Hilbert space and $\tau(R) = \{0\}$. Let (s_n) be an infinite orthogonal sequence in the Hilbert space R. Then $(s_n) \to 0$ weakly in R and hence in C. Moreover, each s_n is a non-trivial symmetry. For each n, let e_n denote the projection $(1/2)(1 + s_n)$ and let τ_n denote the normal state $2\tau(e_n \cdot e_n)$. For all n, $e_n s_n e_n = e_n$ so that $\tau(s_n) = 1$. However, $\tau_n \to \tau$ weakly in C_*, since $\tau_n(x) = 2\tau_n(e_n \circ x)$, for all x and n. Therefore, C_* does not have the DP1. □

One immediate consequence of Lemma 4.4 is that if A is an Abelian von Neumann algebra and C is an infinite-dimensional spin factor then (in the notation of Section 3) $(A \overline{\otimes} C)_*$ cannot have the DP1 because of the canonical (weak*-continuous) contractive projection $A \overline{\otimes} C \to C$.

Theorem 4.5. *Let M be a JBW*-triple. Then M_* has the DP1 if and only if M is of type I without infinite spin part.*

Proof. Suppose M_* has the DP1. Then the predual of every ℓ_∞-summand of M has the DP1. Thus by Proposition 2.4 and Lemma 4.3, M cannot have a non-zero ℓ_∞-summand of the form W^α where α is an involution on a continuous von Neumann algebra W, nor of the form pV where p is a non-zero projection in a continuous von Neumann algebra V. (In the latter case because of the natural projection $pV \to pVp$.) Therefore, M is of type I and, by the remark prior to the statement of the theorem, has no infinite spin part.

On the other hand, consider an Abelian von Neumann algebra A and a Cartan factor C. If C is finite-dimensional then $A \otimes C$ has the Dunford–Pettis property because A does, and so $(A \otimes C)_*$ has the Dunford–Pettis property and therefore it has the DP1. Suppose C is infinite-dimensional. If C is (rectangular) of the form $pB(H)$ for a projection $p \in B(H)$, then $A \overline{\otimes} C = (1 \otimes p)A \overline{\otimes} C$ and is clearly the image of a weak*-continuous projection on $A \overline{\otimes} B(H)$, implying that $(A \overline{\otimes} C)_*$ has the DP1, by Lemma 4.3. If C is Hermitian or symplectic then $A \overline{\otimes} C$ can be realised as W^α where α is an involution on a type I von Neumann algebra W, by [14, 7.3.3]. Since W^α is the image of the weak*-continuous contractive projection $(1/2)(i + \alpha)$ on W, Lemma 4.3 again gives that $(A \overline{\otimes} C)_*$ has the DP1. Thus, if M is of type I with no infinite spin part, M_* has the DP1 by [8, 1.10] together with [16, 1.7]. □

This leads to the following refinement of Theorem 4.2. the proof is immediate from Theorem 4.5.

Theorem 4.6. *Let $P : M \to M$ be a weak*-continuous contractive projection on a JBW*-triple M where M is of type I with no infinite spin part. Then $P(M)$ is of type I with no infinite spin part.*

For every spin factor C acting on a complex Hilbert space H there is a positive unital projection from $B(H)$ onto C [7, Lemma 2.3]. Since a von Neumann algebra never has infinite spin part, Theorem 4.6 gives:

Corollary 4.7. *There is no weak*-continuous contractive projection from a type I von Neumann algebra onto an infinite-dimensional spin factor.*

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