

STRUCTURE SPACES AND DECOMPOSITION IN JB*-TRIPLES

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1. Introduction

Complex Banach spaces for which the group of biholomorphic automorphisms of the open unit ball acts transitively, alias JB*-triples, possess a ternary algebraic structure uniquely determined by the holomorphic properties of the open unit ball [21]. A large and important class of these spaces is comprised of the JC*-triples of [17] (known also as J^* -algebras) which are up to isometry the norm closed subspaces of $B(H, K)$, where H and K are complex Hilbert spaces, that are closed under the ternary product

$$\{xyz\} = \frac{1}{2}(xy^*z + zy^*x).$$

Hence, C^* -algebras are JC*-triples. On the other hand, the range of contractive projection on a C^* -algebra is a JC*-triple [13, 22, 27] but not necessarily a C^* -algebra. An “exceptional” class of JB*-triples involves certain subspaces of three by three matrices with complex Cayley numbers entries.

A detailed survey of JB*-triples recording recent developments including applications to quantum mechanics, complex holomorphy and operator algebras is to be found in [26].

Representation theory in terms of appropriate “irreducible” factors is a basic concept in algebra. In JB*-triples, for any integer $n \geq 1$, there are an infinite number of (appropriately “irreducible”) Cartan factors at rank n . An additional complexity is the existence of six distinct generic types of Cartan factors.

The purpose of this paper is to investigate Cartan representation theory of JB*-triples. To this end we study the *structure space* of primitive M -ideals in some detail and we devise and apply techniques for decomposing JB*-triples into others with a simpler Cartan representation structure.

1. Notation and preliminaries

A JB*-triple is a complex Banach space A with a ternary product $A^3 \rightarrow A$ given by $(a, b, c) \mapsto \{abc\}$ which, where $D(a, b)$ denotes the multiplication operator $x \mapsto \{abx\}$, satisfies

- (i) $\{abc\}$ is symmetric and complex linear in a, c and conjugate linear in b ,
- (ii) $[D(a, b), D(c, d)] = D(\{abc\}, d) - D(c, \{dab\})$,
- (iii) $D(a, a)$ is hermitian with positive spectrum,
- (iv) $\|\{aaa\}\| = \|a\|^3$.

The conjugate linear operator $x \mapsto \{axb\}$ is denoted by $Q_{a,b}$. We write $Q_a = Q_{a,a}$. The elements a, b are said to be *orthogonal* if $D(a, b) = 0$ (equivalently $D(b, a) = 0$)

A subspace I of A is said to be an *ideal* of A if $\{AIA\} + \{AAI\} \subset I$ and to be an *inner ideal* of A if $\{IAI\} \subset I$. If I is a norm closed subspace, it is an ideal of A if $\{AII\} \subset I$ [6]. The annihilator, $I^\perp = \{x : \{xIA\} = 0\}$ of an ideal of A is also a norm closed ideal. By [2], the norm closed ideals of A are precisely the M -ideals.

A JBW*-triple is a JB*-triple with a (unique) predual [2, 18]. Frequent and tacit use shall be made of the facts [9, 2] that the second dual A^{**} of a JB*-triple A is a JBW*-triple containing A as a JB*-subtriple and that the triple product is separately weak* continuous in each variable in a JBW*-triple.

Associated with a tripotent e (i.e. $e = \{eee\}$) in A are the *Peirce projections*

$$P_2(e) = Q_e^2, \quad P_1(e) = 2(D(e, e) - Q_e^2), \quad P_0(e) = I - 2D(e, e) + Q_e^2$$

which are mutually orthogonal with sum I and ranges

$$P_j(e)(A) = A_j(e) = \left\{ x : \{eex\} = \frac{j}{2}x \right\}$$

giving $A = A_2(e) \oplus A_1(e) \oplus A_0(e)$.

JB*-algebras and their hermitian parts, JB -algebras, appear naturally as, for a tripotent e , the Peirce space $A_2(e)$ is a JB*-algebra with the identity e , product $x \circ y = \{xey\}$ and involution $x \mapsto \{exe\}$. If A is a JBW*-triple, then $A_2(e)$ is a JBW*-algebra. We refer to [15, 29] for the theory of JB-algebras and JB*-algebras.

The tripotent e of A is said to be *complete* if $A_0(e) = 0$ and to be *minimal* if $e \neq 0$ and $A_2(e) = Ce$. For $\rho \in \partial_e(A_1^*)$ (extreme points of the dual ball) there is a unique minimal tripotent e of A^{**} for which $\rho(e) = 1$, called the *support* $s(\rho)$ of ρ . The map $\rho \mapsto s(\rho)$ is a bijection from $\partial_e(A_1^*)$ onto the set of minimal tripotents of a JBW*-triple A^{**} [12].

A linear bijection between JB*-triples is an isometry if and only if it is a triple homomorphism (i.e. preserves the triple product). The JBW*-triples

containing a minimal tripotent but without proper weak* closed ideals are called *Cartan factors* [8, 19] which, up to isometry, are as follows. For arbitrary Hilbert spaces and conjugation $j : H \rightarrow H$, the JB*-triples $B(H, K)$, $\{x \in B(H); x = -jx^*j\}$ and $\{x \in B(H); x = jx^*j\}$ characterize three families of Cartan factors. A fourth is given by the complex spin factors. The remaining two exceptional Cartan factors are the 1×2 matrices over the complex octonions \mathbb{O} and the self-adjoint 3×3 matrices over \mathbb{O} .

A Cartan factor M is said to have infinite rank if it contains an infinite orthogonal family of tripotents. Otherwise, each maximal orthogonal family of minimal tripotents has the same finite cardinality, the rank of M . Apart from infinite dimensional spin factors and $B(H, C^n)$, where $n < \infty$ and H is infinite dimensional, all other finite rank Cartan factors have finite dimension.

For unmentioned and further details of JB*-triples we refer to [26, 29].

2. Functional calculus and ideals

In this section we show that a JB*-triple is inundated with inner ideals that are naturally JB*-algebras and we describe certain other properties of inner ideals needed later. We begin with a description of triple functional calculus.

Given an element x of a JB*-triple A , we shall use A_x to denote the JB*-subtriple generated by x . If A is a C^* -algebra and $x \geq 0$, then A_x equals the C^* -algebra generated by x [17, Lemma 5.7.].

On the other hand it follows from [21] that for an arbitrary JB*-triple and $x \in A$ there exists a surjective linear isometry (hence a triple isomorphism) $\varphi : A_x \rightarrow C$ onto a commutative C^* -algebra generated by $\varphi(x) \geq 0$. Let $\tilde{\varphi} : A_x^{**} \rightarrow C^{**}$ be the bitransposed extension of φ . In these circumstances, we shall write

$$S(x) = \sigma(\varphi(x)) \setminus \{0\}, \quad f_i(x) = \varphi^{-1}f(\varphi(x)) \quad \text{if } f \in C_0(S(x)), \quad e(x) = \tilde{\varphi}^{-1}(1),$$

and we note that this is unambiguous. For if $\psi : A_x \rightarrow D$ is another surjective linear isometry onto a commutative C^* -algebra D generated by $\psi(x) \geq 0$, then $\psi\varphi^{-1} : C \rightarrow D$ is a positive isometry and hence a *-automorphism sending $\varphi(x)$ to $\psi(x)$. So, $(\psi\varphi^{-1})(f(\varphi(x))) = f(\psi(x))$ if $f \in C_0(S(x))$. Similarly, $\psi\tilde{\varphi}^{-1}(1) = 1$. In particular,

$$A_x = \{f_i(x) : f \in C_0(S(x))\}.$$

Let $A(x)$ denote the norm closure of $\{xAx\}$. Then $A(x)$ is an inner ideal of A , as follows from the triple identity $Q_{\{aba\}} = Q_aQ_bQ_a$. With $y = \{xxx\}$ we have that $\varphi(y) = \varphi(x)^3$ also generates C and the functional calculus gives

$A_x = A_y \subset A(x)$. In particular, $A(x)$ is the smallest norm closed inner ideal of A containing x and $A(x)$ is weak* dense in $(A^{**})_2(e(x))$.

PROPOSITION 2.1. *Let A be a JB*-triple and let $x \in A$. Then $A(x)$ is a JB*-subalgebra of the JBW*-algebra $(A^{**})_2(e(x))$ and contains x as a positive element.*

PROOF. Let $\varphi : A_x \rightarrow C$ and its bitransition $\tilde{\varphi} : A_x^{**} \rightarrow C^{**}$ be as given above. Let $e = e(x)$ and put $y = \{xex\}$. Then $y \in A_x^{**}$ and $\tilde{\varphi}(y) = \varphi(x)^2$ lies in C and generates it both as a C^* -algebra and as a JB*-triple. Hence, $y \in \tilde{\varphi}^{-1}(C) = A_x = A_y$. In particular, $x \in A_y \subset A(x)$. So, $A(y) = A(x)$.

Now let $a \in A$ and put $z = \{xax\}$. Then

$$\{zez\} = Q_x Q_a Q_x(e) = Q_x Q_a(y) \subset A(x)$$

and it follows that $A(x)$ is a norm closed Jordan subalgebra of $(A^{**})_2(e)$. To see that $A(x)$ is closed under involution $a \mapsto \{eae\}$, note first that $x = \{exe\}$ so that

$$Q_e(Q_y(A)) = Q_e Q_x Q_e(Q_x(A)) = Q_x^2(A) \subset A(x)$$

which gives $Q_e(A(x)) = Q_e(A(y)) \subset A(x)$ and proves that $A(x)$ is a JB*-subalgebra of $(A^{**})_2(e)$. With $f(\lambda) = \lambda^{\frac{1}{2}}, \lambda \geq 0$, we have $f_i(x) = \{ef_i(x)e\}$ and $x = \{f_i(x)ef_i(x)\}$. So, $x \in A(x)_+$.

REMARK 2.2. (a) Let $\pi : A \rightarrow B$ be a triple homomorphism between JB*-triples. Let $x \in A$ and put $y = \pi(x)$. Then it follows from the above proposition that the restriction $\pi : A(x) \rightarrow B(y)$ is a Jordan homomorphism of JB*-algebras. Further, $\pi(f_i(x)) = f_i(y)$ for all $f \in C_0(S(x))$.

(b) Let A be a weak* dense JB*-subtriple of a JBW*-triple M and let $x \in A$. Let $\pi : A^{**} \rightarrow M$ denote the weak* continuous projection onto M . Put $f = \pi(e(x))$. As π projects $(A^{**})_2(e(x))$ onto $M_2(f)$ and acts identically on $A(x)$, $A(x)$ is seen to be a weak* dense JB*-subalgebra of the JBW*-algebra $M_2(f)$ in the obvious way.

Next we describe some relevant ideal theory of inner ideals. If I is a norm closed inner ideal of a JB*-triple A , $T(I)$ shall denote the norm closed triple ideal of A generated by I .

LEMMA 2.3. *Let I be a norm closed inner ideal of a JB*-triple A and let J be a norm closed inner ideal of I . Then J is an inner ideal of A .*

PROOF. Let $x \in J$. By functional calculus, choose $y \in J$ such that $x = \{yyy\}$. Then

$$Q_x(A) = Q_y^2(Q_y(A)) \subset Q_y^2(I) \subset J.$$

LEMMA 2.4. *Let A be a JB*-triple. Let I be a norm closed inner ideal of A and let J be a norm closed triple ideal of A . Then*

- (i) $I \cap J = \{III\}$,
- (ii) $T(I \cap J) = T(I) \cap J$.

PROOF. (i) Given $x \in I \cap J$ take $y \in i \cap J$ with $x = \{y\!y\!y\}$. Then $x \in \{III\}$. This gives one inclusion and the other is clear.

(ii) Suppose first that $I \cap J = 0$. Given $x \in I$ and $y \in J$ we have $Q_x Q_y = 0$ so that $Q_{\{y\!y\!y\}} = Q_y Q_x Q_y = 0$ implying that $\{JII\} = 0$. By the fundamental identity

$$\{J\{JII\}J\} \subset \{\{JJJ\}IJ\} - \{JJ\{JII\}\} = 0$$

which gives $\{JII\} = 0$. In turn, we have

$$\{\{JIA\}JJ\} = \{IJ\{AJJ\}\} - \{AJ\{IJJ\}J\} = 0.$$

So, $\{JIA\} = 0$ giving $I \subset J^\perp$ and so $T(I) \subset J^\perp$. Hence, $T(I) \cap J = 0$.

Reverting to the general case, the canonical surjection $\pi : A \rightarrow A/T(I \cap J)$ gives $\pi(T(I)) = T(\pi(I))$ and $\{\pi(I)\pi(J)\pi(I)\} = \pi(\{III\}) = 0$. Therefore, by (i) together with the first part of the proof of (ii),

$$\pi(T(I) \cap J) = T(\pi(I)) \cap \pi(J) = 0$$

Hence, $T(I) \cap J \subset T(I \cap J)$, as required.

PROPOSITION 2.5. *Let I be a norm closed inner ideal of a JB*-triple A and let J be a norm closed triple ideal of I . Then $J = T(J) \cap I$.*

PROOF. Let f be a complete tripotent of I^{**} and, via [18, (4.2)], let e be a complete tripotent of A^{**} such that f is a projection of the JBW*-algebra $M = (A^{**})_2(e)$. Now, $J^{**} \cap N$ is a weak* closed Jordan ideal of the hereditary JBW*-subalgebra $N = \{fMf\}$ of M . Thus, by [11, Theorem], there is a central projection z of M such that $J^{**} \cap N = z \circ N$, where \circ denotes the Jordan product in M . In particular, $J^{**} \cap N$ is contained in the weak* closed triple ideal of A^{**} , $K = (A^{**})_2(z) + (A^{**})_1(z)$, and so lies in $(K \cap I^{**}) \cap N$. By [18, (4.2)] applied to I^{**} this gives $J^{**} \subset K \cap I^{**}$ from which it follows that $T(J)^{**} \subset K$. Hence,

$$T(J)^{**} \cap N \subset K \cap N = (z \circ M) \cap N \subset z \circ N = J^{**} \cap N.$$

But then $(T(J)^{**} \cap I^{**}) \cap N = J^{**} \cap N$ so that, as before, [17, (4.2)] gives $T(J)^{**} \cap I^{**} = J^{**}$. Intersecting both sides of which with A results in $T(J) \cap I = J$.

3. The structure space

The structure space of primitive M -ideals of a Banach space was introduced and investigated in [1]. A particularly important and very comprehensive reference on M -ideals is given by [16] to which we refer, together with [4], for any unmentioned details or M -structure in Banach spaces.

It was shown in [2] that the M -ideals of a JB^* -triple A are the norm closed ideals. By a *primitive ideal* of A we shall mean primitive M -ideal. Thus P is primitive ideal of A if for some $\rho \in \partial_e(A_1^*)$ is the largest norm closed ideal of A contained in $\ker \rho$. Let $\text{Prim}(A)$ denote the set of all primitive ideals of A and given $X \subset A$, $S \subset \text{Prim}(A)$ write

$$h(X) = \{P \in \text{Prim}(A) : X \subset P\}, \quad k(S) = \bigcap \{P \in \text{Prim}(A) : P \in S\}.$$

Primitive ideals are prime ideals (in the usual sense) and there is a unique topology on $\text{Prim}(A)$, the *structure topology*, for which $hk(S)$ is the closure of S . Endowed with this structure topology, $\text{Prim}(A)$ is referred to as the *structure space* of A . There is a bijective correspondance, $J \rightarrow h(J)$, between the norm closed ideals of A and the closed sets of $\text{Prim}(A)$ and we have the homeomorphisms

$$h(J) \rightarrow \text{Prim}(A/J)$$

and

$$\text{Prim}(A) \setminus h(J) \rightarrow \text{Prim}(J) \quad (P \mapsto P \cap J)$$

for each norm closed ideal J of A .

A triple homomorphism, $\pi : A \rightarrow M$, into a JBW^* -triple M has unique weak* continuous extension, $\tilde{\pi} : A^{**} \rightarrow M$, with $\tilde{\pi} : (A^{**}) = \overline{\pi(A)}$ [3], where here and later the bar refers to weak* closure. If M is a Cartan factor and $\overline{\pi(A)} = M$, then π is said to be a *Cartan factor* representation. The set of all Cartan factor representations of A is denoted by $C(A)$.

Given $\rho \in \partial_e(A_1^*)$, let A_ρ^{**} be the weak* closed ideal of A^{**} generated by the (minimal) support tripotent $s(\rho)$ [12]. Then A_ρ^{**} is a complemented Cartan factor in A^{**} [8, 18] and the restriction, $\pi_\rho : A \rightarrow A_\rho^{**}$, of the natural weak* continuous projection, $P_\rho : A^{**} \rightarrow A_\rho^{**}$, is a Cartan factor representation of A .

The following is contained in detail of [2, Theorem 3.6].

LEMMA 3.1. *Let A be a JB^* -triple and let ρ be an extreme point of the dual ball. Then $\ker \pi_\rho$ is the largest norm closed ideal of A in $\ker \rho$. Hence, $\text{Prim}(A) = \{\ker \pi_\rho : \rho \in \partial_e(A_1^*)\}$.*

LEMMA 3.2. *Let $\pi : A \rightarrow M$ be a Cartan factor representation of a JB^* -triple*

A. Then there exists $\rho \in \partial_e(A_1^*)$ and a surjective isometry $\varphi : A_\rho^{**} \rightarrow M$ such that $\varphi\pi_\rho = \pi$. Hence, $\text{Prim}(A) = \{\ker \pi : \pi \in C(A)\}$.

PROOF. Let $J = \ker \tilde{\pi}$ where $\tilde{\pi} : A^{**} \rightarrow M$ is a weak* continuous extension of π onto M . Then the complement of J in A^{**} , $J^\perp \approx A^{**}/J \approx M$. Choose a minimal tripotent e of A^{**} contained in J^\perp and let $\rho \in \partial_e(A_1^*)$ with $s(\rho) = e$, using [12, Proposition 4]. It follows that $A_\rho^{**} = J^\perp$ and that $\varphi\pi_\rho = \pi$.

PROPOSITION 3.3. Let I be a norm closed inner ideal of a JB*-triple A . Then

$$\beta : \text{Prim}(A) \setminus h(I) \rightarrow \text{Prim}(I) \quad (P \mapsto P \cap I)$$

is a homeomorphism.

PROOF. As a weak* closed inner ideal of a Cartan factor is a Cartan factor, it follows that a Cartan factor representation of A which fails to kill I restricts to a Cartan factor representation of I . It follows from Lemma 3.2 that β is well-defined.

On the other hand, given $\rho \in \partial_e(I_1^*)$ let $\bar{\rho} \in \partial_e(A_1^*)$ extend ρ . As $s(\rho)$ is minimal in the weak* closed inner ideal I^{**} it is also minimal in A^{**} . So $s(\rho) = s(\bar{\rho})$ and hence, $I_\rho^{**} \subset A_{\bar{\rho}}^{**}$. Let G be the complementary ideal of I_ρ^{**} in I^{**} . Let J be the norm closed ideal generated by G in A^{**} . Then $G \subset J^\perp$ by Lemma 2.4. Hence, $G \subset A_{\bar{\rho}}^{**}$ because $J^\perp = (\bar{J})^\perp = A_{\bar{\rho}}^{**}$. It follows that $P_{\bar{\rho}} : A^{**} \rightarrow A_{\bar{\rho}}^{**}$ restricts to $P_\rho : I^{**} \rightarrow I_\rho^{**}$ so that $\pi_{\bar{\rho}}$ restricts to π_ρ and hence $\ker \pi_\rho = I \cap \ker \pi_{\bar{\rho}}$. Therefore, β is surjective by Lemma 3.1.

By Lemma 2.4 together with primeness of primitive ideals, β is injective and for each norm closed ideal J of A , $\beta(h(J) \setminus h(I)) = h(I \cap J)$ (taken in I) so that β is a closed map. By Proposition 2.5, the right hand side of the equation runs through all closed sets of $\text{Prim}(I)$ and so β is continuous.

REMARK 3.4. Let A be a JB*-algebra, $\pi : A \rightarrow M$ a Cartan factor representation and $\tilde{\pi} : A^{**} \rightarrow M$ its weak* continuous extension. Then with $e = \tilde{\pi}(1)$, π is reconstituted as a * Jordan Cartan factor representation, $\pi : A \rightarrow M_2(e)(= M)$ and induces by restriction (in the sense of 15, p.133) a Jordan type I factor representation of the JB-algebra A_{sa} . Thus, by restriction and by complexification in the opposite direction (cf. [29]) the structure space of A is naturally identified with the usual structure space [6] of the JC-algebra A_{sa} . We shall make frequent and often tacit use of this fact.

LEMMA 3.5. Let A be a JB*-triple.

- (i) $\hat{x} : \text{Prim}(A) \mapsto [0, \infty)$ ($P \mapsto \|x + P\|$) is lower semicontinuous for all x .
- (ii) The sets $\{P \in \text{Prim}(A) : \|x + P\| \geq \alpha\}$, where $\alpha > .0$ and $x \in A$, form a basis of quasi-compact sets for $\text{Prim}(A)$.
- (iii) $\text{Prim}(A)$ is Hausdorff if \hat{x} and only if is continuous for all $x \in A$.

PROOF. When A is a JB*-algebra and “ $x \in A$ ” is replaced by “ $x \in A_+$ ”, (i), (ii) and (iii) follow as for C^* -algebras (cf. [10, (3.3)], [25, (4.4)]).

(i): Let $x \in A$. Via the triple and hence isometric embedding $A(x)/P \cap A(x) \rightarrow A/P$ we have $\|x + P\| = \|x + P \cap A(x)\|$ for each $P \in \text{Prim}(A)$. Consider the open set $U = \text{Prim}(A) \setminus h(A(x)) \approx \text{Prim}(A(x))$. By Proposition 2.1., $A(x)$ can be realised as a JB*-algebra such that x is positive there. Therefore, the opening remark together with Proposition 3.3 imply that $\hat{x} : U \rightarrow [0, \infty)$ is lower semicontinuous. But for $\alpha \geq 0$, $\hat{x}^{-1}((\alpha, \infty))$ is contained in U by Proposition 3.3. Hence, \hat{x} is lower semicontinuous on $\text{Prim}(A)$.

(ii), (iii): Via the opening remark these follow by similar use of Proposition 2.1 and Proposition 3.3.

4. Rank and collinear systems

Let A be a JB*-triple. The *rank*, $\text{rank}(\pi)$, of a Cartan factor representation, $\pi : A \rightarrow M$, is the rank of M . If for fixed n , where $1 \leq n < \infty$, $\text{rank}(\pi) = n$ for all Cartan representations, then A is said to be of *constant rank* n . The JB*-triple is said to be of *bounded rank* if $\{\text{rank}(\pi) : \pi \in C(A)\}$ is bounded.

In the Cartan factor M , the JB*-subtriple generated by all minimal tripotents is a simple norm closed ideal, $K(M)$, of M such that its second dual is isometric to M [7]. We have,

M has finite rank if and only if M is reflexive if and only if $K(M) = M$.

As seen in the proof of Proposition 3.3., given $\pi \in C(A)$ and $x \in A$ with $\pi(x) \neq 0$, π restricts to a Cartan factor representation of $A(x)$. In the following this induced representation is denoted by π_x .

LEMMA 4.1. *Let A be a JB*-triple and $\pi : A \rightarrow M$ a Cartan representation of A .*

(i) *If $\text{rank}(\pi) < \infty$, then there exists $x \in A$ with $\pi(x) \neq 0$ such that $\text{rank}(\pi) = \text{rank}(\pi_x)$.*

(ii) *If for all $x \in A$ with $\pi(x) \neq 0$ we have $\text{rank}(\pi_x) < \infty$, then $\text{rank}(\pi) < \infty$.*

PROOF. (i): Suppose that the Cartan factor has a finite rank. Then M is reflexive so that $\pi(A) = M$. Choose $x \in A$ such that $\pi(x) = e$ is a complete tripotent of M . Then $\pi_x(A(x)) = M_2(e)$. Hence, $\text{rank}(\pi_x) = \text{rank}(\pi)$.

(ii): Let $x \in A$ be such that $\pi(x) \neq 0$ and $\text{rank}(\pi_x) < \infty$. Then $\pi(A(x)) = \pi_x(A(x))$ is a reflexive, so weak* closed, inner ideal of M which implies that $\pi(A(x)) \subset K(M)$. Hence, given that the stated condition is satisfied, $\pi(A) \subset K(M)$. Now, the natural projection $Q : K(M)^{**} \rightarrow M$ is an

isometry onto M and Q maps $\pi(A)^{**}$ onto $\overline{\pi(A)} = M$. Therefore, $\pi(A)^{**} = K(M)^{**}$ and so $\pi(A) = K(M)$. It follows that π has a finite rank. Otherwise, there is an infinite sequence (e_n) of orthogonal minimal tripotents in M . In this case $y = \sum \frac{e_n}{2^n} \in K(M)$ and, choosing $x \in A$ with $\pi(x) = y$ and putting $e = \sum e_n$, we obtain that $\pi_x : A_x \rightarrow M_2(e)$ is a Cartan factor representation of infinite rank. This contradiction concludes the proof.

For a JB*-triple A and natural number n we denote by $\text{Prim}_n(A)$ the set of those primitive ideals $\ker \pi$ for which $\text{rank}(\pi) \leq n$.

PROPOSITION 4.2. *Let A be a JB*-triple and n a natural number. Then $\text{Prim}_n(A)$ is closed in $\text{Prim}(A)$.*

PROOF. Take $\pi \in C(A)$ such that $\ker \pi \notin \text{Prim}_n(A)$. By Lemma 4.1 there exists $x \in A$ such that $\pi(x) \neq 0$ and $\ker \pi \cap A(x) = \ker \pi_x \notin \text{Prim}_n(A(x)) = F$, which is closed in $\text{Prim}(A(x))$ as follows from Proposition 2.1 together with [5, Lemma 6]. Now $U = \beta^{-1}(\text{Prim}(A(x)) \setminus F)$, where β is the homeomorphism of Proposition 3.3, satisfies $U \cap \text{Prim}_n(A) = \emptyset$, and U is an open neighbourhood of $\ker \pi$. This proves that $\text{Prim}_n(A)$ is closed.

REMARK 4.3. Given a JB*-algebra A consider the functions $T_x : \text{Prim}(A) \rightarrow [0, \infty]$, $x \in A_+^{**}$, given by $T_x(\ker \pi) = \text{Tr}(\tilde{\pi}(x))$ where $\pi : A \rightarrow M$ and $\tilde{\pi} : A^{**} \rightarrow M$ is its weak* continuous extension and Tr is the Jordan trace on M . The functions T_x are lower semicontinuous for all $x \in A_+$ (cf. [5, Lemma 6]). Hence, T_x is lower semicontinuous whenever $x \in A^{**}$ is the strong limit of an increasing net in A_+ . If A has constant rank n , then it follows as for C^* -algebras (cf. [25, 4.4.10]) that T_x is continuous for all $x \in A_+$ and that $\text{Prim}(A)$ is Hausdorff.

LEMMA 4.4. *Let A be a JB*-triple of constant finite rank n . Then $\text{Prim}(A)$ is Hausdorff.*

PROOF. Let $P_1, P_2 \in \text{Prim}(A)$ with $P_1 \neq P_2$. By assumption, the canonical maps $\pi_i : A \rightarrow A/P_i = M_i$ belong to $C(A)$ $i = 1, 2$. For $i = 1, 2$, choose $x_i \in A$ such that $\pi_i(x_i) = e_i$ is a complete tripotent of M_i and let $a_i \in P_i$ such that $x_i - x_2 = a_1 + a_2$, which is possible because $P_1 + P_2 = A$. For $x = x_1 - a_1 = x_2 + a_2$ we have $\pi_1(x) = e_1$, $\pi_2(x) = e_2$. Hence, for $i = 1, 2$, the $Q_i = P_i \cap A(x)$ are, by Proposition 3.3, distinct elements of

$$\text{Prim}_n(A(x)) \setminus \text{Prim}_{n-1}(A(x)) = \text{Prim}(A(x)) \setminus h(J) \approx \text{Prim}(J),$$

where J is the closed ideal of $A(x)$ with hull equal to $\text{Prim}_{n-1}(A(x))$ (where we let $J = 0$ if $n = 1$). But then J is a JB*-algebra of constant rank (using Proposition 2.1) so that $\text{Prim}(J)$ is Hausdorff by Remark 4.3. Now Proposi-

tion 3.3 implies that P_1, P_2 are separated by open sets in $\text{Prim}(A) \setminus h(A(x))$. Hence $\text{Prim}(A)$ is Hausdorff.

In the following, which is inspired by [28, pp. 506–507], we let h be the continuous function on \mathbb{R} satisfying

$$h((-\infty, \frac{1}{4})) = \{0\}, h([\frac{3}{4}, \infty)) = \{1\} \text{ and } h \text{ is linear on } [\frac{1}{4}, \frac{3}{4}].$$

Recall that $h_t(x)$ refers to the element of A_x given by the triple functional calculus (see §2). We shall also use the following: given tripotents e and f in a JBW*-triple such that e is minimal and $\|e - f\| < 1$, it follows that f is minimal too. For, indeed, $P_2(f)(e) = \alpha u$ where u is a minimal tripotent and $\alpha \in \mathbb{C}$ [12, Proposition 6] so that $\|f - \alpha u\| = \|P_2(f)(f - e)\| < 1$ which implies that u is invertible in the JBW*-algebra $M_2(f)$. Hence, $M_2(f) = M_2(u) \simeq \mathbb{C}$.

LEMMA 4.5. *Let A be a JB*-triple of constant finite rank n . Let $P_0 \in \text{Prim}(A)$ and let $x \in A$ such that $x + P_0$ is a nonzero tripotent.*

(i) *$h_t(x) + P_0 = x + P_0$ and $h_t(x) + P$ is a nonzero tripotent for all P in some neighbourhood V of P_0 .*

(ii) *If $x + P_0$ is minimal, then $h_t(x) + P$ is a minimal tripotent for all P in some neighbourhood W of P_0 .*

PROOF. (i) Regarding $A(x)$ as a JB*-algebra and $x \in A(x)_+$ by Proposition 2.1, we have that $x + Q_0$ is a non-zero projection in $A(x)/Q_0$ where $Q_0 = P_0 \cap A(x)$. As $\text{Prim}(A)$ is Hausdorff by Lemma 4.4, so $\text{Prim}(A(x))$ is Hausdorff by Proposition 3.3, and the argument on page 506 of [28] gives an open neighbourhood V of Q_0 such that $h(x) + Q = x + Q$ and $h(x) + Q$ is a non-zero projection for all $Q \in V$. Now Proposition 3.3 together with triple functional calculus gives (i).

(ii) Let $\pi_0 : A \rightarrow A/P_0 = M$ be the quotient map and let $\pi_0(x) = e$ be a minimal tripotent of M . Choose a complete tripotent u of M such that the Type I_n JBW*-factor $M_2(u)$ contains e as a minimal projection (cf [18]). Choose $y \in A$ with $\pi_0(y) = u$ and let I be the norm closed ideal of the JB*-algebra $B = A(y)$ corresponding to $\text{Prim}(B) \setminus \text{Prim}_{n-1}(B)$. Then J is a JB*-algebra of constant rank n and $P_0 \cap J \in \text{Prim}(J)$. Choose with $z \in J$ with $z \geq 0$ and $\pi_0(z) = e$. Transparent modification to the argument on page 507 of [28] now gives that $h(z) + Q$ is a minimal projection in J/Q for all Q in a neighbourhood of $P_0 \cap J$ in $\text{Prim}(J)$. Via Proposition 3.3., this gives rise to a neighbourhood U of P_0 in $\text{Prim}(A)$ such that $h_t(x) + P$ is a minimal tripotent for all $P \in U$. We may suppose that $U \subset V$, where V is given in (i). Now put $W = \{P \in U : \|h_t(x) - h_t(z) + P\| < 1\}$. Then $P_0 \in W$ and W is open by

Lemma 3.5 (iv) and Lemma 4.4. Finally, by the remark immediately preceding the statement, $h_t(x) + P$ is minimal for all $P \in W$.

Tripotents e and f in a JB*-triple A are said to be *collinear* if $e \in A_1(f)$ and $f \in A_1(e)$. If e_1, \dots, e_n are minimal and mutually collinear in A , we say that they form a *collinear system of length n* .

LEMMA 4.6. *Let A be a JB*-triple.*

- (i) *If e and f are minimal tripotents in A and $e \in A_1(f)$, then $f \in A_1(e)$.*
- (ii) *If e_1, \dots, e_n is a collinear system in A and*

$$T = \left(1 - \sum_{i=1}^n P_2(e_i)\right) (I - P_0(e_1)) \dots (I - P_0(e_n)),$$

then $T(A) \subset \bigcap_{i=1}^n A_1(e_i)$.

PROOF. (i) This follows from [12, Lemma 2.1].

(ii) Let e_1, \dots, e_n be mutually collinear minimal tripotents. Put $a = y - \sum_{i=1}^n P_2(e_i)y$ where $y = (I - P_0(e_1)) \dots (I - P_0(e_n))(x)$ and $x \in A$. The Peirce projections $P_k(e_i)$, $k = 0, 1, 2$, $i = 1, \dots, n$, commute by [18, (1.10)], so $P_0(e_j)y = 0$, $j = 1, \dots, n$ and

$$\begin{aligned} 2D(e_j, e_j)(a) &= 2D(e_j, e_j)y - \sum_{i \neq j} P_2(e_i)y - 2P_2(e_j)y \\ &= (2D(e_j, e_j) - P_2(e_j) - I)y + \left(y - \sum_{i=1}^n P_2(e_i)y\right) \\ &= -P_0(e_j)(y) + a \\ &= a. \end{aligned}$$

PROPOSITION 4.7. *Let A be a JB*-triple of constant rank and m a natural number. Then the set*

$$S = \{P \in \text{Prim}(A) : A/P \text{ contains a collinear system of length } > m\}$$

is open in $\text{Prim}(A)$.

PROOF. Let $P_0 \in S$. By assumption, A/P_0 contains minimal and mutually collinear tripotents e_1, \dots, e_n , where $n > m$. Choose $x_1, \dots, x_n \in A$ such that $x_i + P_0 = e_i$, $i = 1, \dots, n$.

We shall show, by induction, that for all P in some neighbourhood of P_0 , A/P contains a collinear system of length n . To this end we make the following induction hypothesis.

Let $1 \leq k < n$ and suppose that we have $y_1, \dots, y_k \in A$ and a neighbourhood U of P_0 such that $y_i + P_0 = e_i$, $i = 1, \dots, k$, and $\{y_1 + P, \dots, y_k + P\}$ is a collinear system for all $P \in U$. By Lemma 4.5, we note that this hypothesis holds for $k = 1$. Put

$$y = \left(I - \sum_{i=1}^k Q(y_i)^2 \right) (2D(y_1, y_1) - Q(y_1)^2) \dots (2D(y_k, y_k) - Q(y_k)^2)(x_{k+1})$$

and $y_{k+1} = h_t(y)$, where h is the function defined prior to Lemma 4.5. Then,

$$y + P_0 = (I - \sum P_2(e_i))(I - P_0(e_i)) \dots (I - P_0(e_k))(e_{k+1}) = e_{k+1}.$$

Hence, by Lemma 4.5, $y_{k+1} + P_0 = e_{k+1}$ and there is a neighbourhood V of P_0 such that

$$(\alpha) \quad y_{k+1} + P \text{ is a minimal tripotent for all } P \in V.$$

Also, by Lemma 4.6(ii) we have, for all $P \in U \cap V = W$,

$$y + P \in \bigcap_{i=1}^k (A/P)_1(y_i + P) \text{ so that } y_{k+1} + P \in \bigcap_{i=1}^k (A/P)_1(y_i + P),$$

(as latter is a JB*-subtriple of A/P) and hence $\{y_1 + P, \dots, y_{k+1} + P\}$ is a collinear system by (α) together with Lemma 4.6(i). This completes the proof.

Elements a, b in a JC*-algebra are said to be *J-orthogonal* if $L_a(b) + \frac{1}{2}(ab + ba) = 0$. Let V_α be the spin factor that, when realised as a JC*-algebra, contains a maximal *J-orthogonal* family of symmetries $\{s_i\}_{i \in I}$ with $\text{card}(I) = \alpha$ (cf. [15, Chapter 6]).

LEMMA 4.8. *Let V be a spin factor realised as a JC*-algebra.*

- (i) *If s, t are J-orthogonal symmetries, then $L_s^2 L_t^2 = L_t^2 L_s^2$.*
- (ii) *If s_1, \dots, s_n are mutually J-orthogonal symmetries in V , then $(I - L_{s_1}^2) \dots (I - L_{s_n}^2)(V)$ is elementwise J-orthogonal to s_i for all $i = 1, \dots, n$.*
- (iii) *Let $x^* = x \in V$ be nonzero and J-orthogonal to a symmetry in V . Then each nontrivial symmetry in the JC-algebra generated by x (there are two) is a scalar multiple of x .*

PROOF. (i) is routine and (ii) then follows from the rule $L_{s_i}^3 = L_{s_i}$.

(iii) Let $t \neq \pm 1$ be a symmetry J-orthogonal to $x = \alpha \cdot 1 + \beta s$ where $\alpha, \beta \in \mathbb{R}$ and s is a nontrivial symmetry. We have $L_t(s) \in \mathbb{R}$ so that $\alpha = 0$. The JC-algebra generated by x is $\{\lambda \cdot 1 + \mu s : \lambda, \mu \in \mathbb{R}\}$, the only nontrivial symmetries in which are s and $-s$

For a JB*triple A and integer $n \geq 2$, let

$$S^n(A) = \{P \in \text{Prim}(A) : A/P \approx V_m, m > n\}.$$

LEMMA 4.9. *Let A be a JB*-algebra for which all Cartan factor representations have rank 1 or 2 and let n be an integer with $n \geq 2$. Then $S^n(A)$ is open in $\text{Prim}(A)$.*

PROOF. As $S^n(A) \subset \text{Prim}(A) \setminus \text{Prim}_1(A)$ we may suppose that A has constant rank 2. Let h be the real function given prior to Lemma 4.5 and let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(\lambda) = \frac{\lambda(\lambda+1)}{2}, g(\lambda) = h(f(\lambda)) - h[(1 - h(f(\lambda))) \cdot (f(-\lambda))]$.

Let $P_0 \in \text{Prim}(A)$. Let $x \in A_{\text{sa}}$ such that $x + P_0 = s$ is a non-trivial symmetry in A/P_0 . Then with $x_1 = f(x)$ and $x_2 = f(-x)$ we have $x_1 + P_0 = e_1, x_2 + P_0 = e_2$ are orthogonal minimal projections in A/P_0 with sum unity. So (cf. [27, pages 506–507] or Lemma 4.5) with $y_1 = h(x_1), y_2 = h((1 - y_1)x_2)$ we have that, $y_1 + P_0 = e_1, y_2 + P_0 = e_2$ and $y_1 + P, y_2 + P$ are orthogonal minimal projections in A/P for all P in a neighbourhood U of P_0 . Note that $g(x) = y_1 - y_2$. Hence, $g(x) + P_0 = s$, and $g(x) + P$ is a non-trivial symmetry for all $P \in U$.

Now suppose that $P_0 \in S^n(A)$. Then, for some $m > n$, there exist x_1, \dots, x_m in A_{sa} such that $x_1 + P_0 = s_1, \dots, x_m + P_0 = s_m$ are mutually J -orthogonal symmetries in A/P_0 . We proceed by induction to show that there exist $y_1, \dots, y_m \in A_{\text{sa}}$ such that $y_1 + P, \dots, y_m + P$ are mutually J -orthogonal symmetries in some neighbourhood of P_0 .

Let $1 \leq k < m$. Suppose that $y_1, \dots, y_k \in A_{\text{sa}}$ have been chosen so that $y_i + P_0 = s_i, i = 1, \dots, k$ and $y_1 + P, \dots, y_k + P$, are mutually orthogonal symmetries for all P in a neighbourhood V of P_0 .

Put $y = (I - L_{y_1}^2) \dots (I - L_{y_k}^2)(x_{k+1})$ and put $y_{k+1} = g(y)$. Then, by the first part of the proof, $y_{k+1} + P_0 = s_{k+1}$ and $y_{k+1} + P$ is a non-trivial symmetry in A/P for all P in a neighbourhood W of P_0 . It follows from Lemma 4.8 (ii, iii), that $y_1 + P, \dots, y_{k+1} + P$ are mutually J -orthogonal symmetries in A/P for all $P \in V \cap W$. Hence, y_1, \dots, y_{k+1} satisfy the inductive hypothesis and the result follows.

5. Decompositions of JB*-triples

We apply the structure space techniques developed earlier to study decomposition in JB*-triples. We are mostly interested in JB*-triples of bounded rank. Some results are more general. Relevant features and notation of finite rank Cartan factors are listed below for convenience. There are six generic types (cf. [17, 24]).

- (1) Rectangular: $M_{n,\alpha} = B(H, K), 1 \leq n = \dim(K) \leq \alpha = \dim(H), n < \infty$ ($n \times \alpha$ matrices)
- (2) Symplectic: $A_n, 4 \leq n < \infty$ (antisymmetric $n \times n$ matrices)

- (3) Hermitian: $S_n, 2 \leq n < \infty$ (symmetric $n \times n$ matrices)
- (4) Spin: $V_\lambda, 2 \leq \lambda$ ($\dim(V_\lambda) = \lambda + 1$)

In (1) and (4), the cardinals α and λ can be infinite. $M_{n,\alpha}$, for $n \geq 1$, and A_{2n}, A_{2n+1}, S_n for $n \geq 2$ are all of rank n . Spin factors have rank 2 and are, together with A_{2n} and S_n for $n \geq 2$, isometric to JC*-algebras. We have the isomorphisms $S_2 \approx V_2, M_{2,2} \approx V_3, A_4 \approx V_5$ and, for $n \geq 2$, $A_{2n} \approx M_n(\mathbb{H})_{\text{sa}} \otimes_{\mathbb{R}} \mathbb{C}, S_n \approx M_n(\mathbb{R})_{\text{sa}} \otimes_{\mathbb{R}} \mathbb{C}$. The factors $M_{1,\alpha}$ are the α -dimensional Hilbert spaces.

There are two exceptional factors.

- (5) $B_{1,2}$: (1×2 matrices over the complex Cayley numbers)
- (6) M_3^8 : (self-adjoint 3×3 matrices over the complex Cayley numbers)

Let A be a JB*-triple. If for all $P \in \text{Prim}(A), A/P$ is a finite rank rectangular Cartan factor, then A is said to be of *rectangular type*.

The appellations *symplectic, hermitian, spin* and *exceptional* are employed correspondingly.

If for a fixed finite rank Cartan factor M we have $A/P \approx M$ for all primitive ideals P , then A is said to be of *type M*. By convention, the zero triple is considered to be of every type.

We recall that by the Gelfand-Naimark theorem of [14] in a JB*-triple A there is a unique norm closed ideal J such that A/J is a JC*-triple and J is exceptional.

THEOREM 5.1. *Let A be a JB*-triple and let $n \in \mathbb{N}$. There is a (unique) norm closed ideal J of A such that $\text{rank}(\pi) \leq n$ for all $\pi \in C(A/J)$ and $\text{rank}(\pi) > n$ for all $\pi \in C(J)$.*

PROOF. This is the algebraic translation of Proposition 4.2.

COROLLARY 5.2. *Let A be a JB*-triple for which all non-exceptional Cartan factor representations have rank greater than 3. Then the exceptional ideal of A is a direct summand.*

PROOF. Let J be the exceptional ideal of A . Then $h(J) = \{\ker(\pi) : \pi \in C(A), \pi \text{ is non-exceptional}\} = \{\ker(\pi) : \pi \in C(A), \text{rank}(\pi) > 3\}$ is both open and closed in $\text{Prim}(A)$. It follows that J is a direct summand.

PROPOSITION 5.3. *Let A be a JB*-triple of bounded rank and let $\{\text{rank}(A/P) : P \in \text{Prim}(A)\} = \{n_i\}_{i=1}^k$ where $1 \leq n_1 < n_2 < \dots < n_k$. Then there is a finite composition series of norm closed ideals, $0 = J_0 \subset J_1 \subset \dots \subset J_{k-1} \subset J_k = A$ such that for $r = 0, \dots, k - 1, J_{r+1}/J_r$ is non-trivial of constant rank n_{k-r} with Hausdorff structure space.*

PROOF. This follows from Theorem 5.1 together with Lemma 4.4.

COROLLARY 5.4. *Let A be an exceptional JB*-triple. Then there is a norm closed ideal J of A such that J is of type M_3^8 and A/J is type $B_{1,2}$.*

REMARK 5.5. Let (e_{ij}) be the canonical matrix units of $M_{n,\alpha}$, where $n \leq \alpha < \infty$.

(a) The tripotents $e_{11}, \dots, e_{1\alpha}$ form a collinear system (see Section 4) in $M_{n,\alpha}$. Moreover, any collinear system S has cardinality $\leq \alpha$. Indeed, as two minimal tripotents are exchanged by some automorphism (see [24, §5]) we may suppose that $e_{11} \in S$. Then the collinearity and minimality of the elements of S implies by straightforward calculation that S is contained either in the linear span of $e_{11}, \dots, e_{1\alpha}$ or S is contained in the linear span of e_{11}, \dots, e_{n1} . By [8, Lemma on page 306] it follows that $\text{card}(S) \leq \alpha$.

(b) Let $n \geq 4$ and $f_{ij} = e_{ij} - e_{ji}$, $1 \leq i, j \leq n$. Then $\{f_{12}, \dots, f_{1n}\}$ is a collinear system in $A - n$. Let S be any collinear system in A_n . The claim now is that $\text{card}(S) \leq n - 1$. As before, by [24, §5], we may suppose that $f_{12} \in S$. In this case calculation shows that $S \setminus \{f_{12}\}$ is contained in the image of the injective triple homomorphism $\pi : M_{2,n-2} \rightarrow A_n$ given by $\pi(x) = \begin{bmatrix} 0 & x \\ -x^T & 0 \end{bmatrix}$. So, the claim follows from (a).

By the above results the study of JB*-triples of bounded rank reduces to that of constant rank. We shall now proceed to analyse JB*-triples of constant rank.

The main decomposition result is Theorem 5.8. The JB*-algebra version is known (cf. [6]), of which we shall make essential use (in Lemma 5.7). In order to treat JB*-triples we need to come to grips with and synthesise phenomena that do not arise in JB*-algebras.

LEMMA 5.6. *Let A be a JB*-triple of constant rank n .*

(i) *If A is rectangular and $1 \leq n \leq \alpha < \infty$, then there is a norm closed ideal J of A such that all primitive quotients of J and A/J are respectively of the form $M_{n,\beta}$ where $\alpha < \beta \leq \infty$ and $M_{n,\beta}$ where $n \leq \beta \leq \alpha$.*

(ii) *If (up to isometry) $\{A/P : P \in \text{Prim}(A)\} = \{M_{n,\alpha_i}\}_{i=1}^k$ where $1 \leq n \leq \alpha_1 < \dots < \alpha_k < \infty$, then there are norm closed ideals in $A, 0 = J_0 \subset J_1 \subset \dots \subset J_k \subset J_{k+1} = A$, such that J_{r+1}/J_r is non-trivial type M_{n,α_k-r} for $r = 0, \dots, k$.*

(iii) *If $n \geq 2$ and A is symplectic, then there is a norm closed ideal J of A such that J is type A_{2n+1} and A/J is type A_{2n} .*

PROOF. (i) If A is rectangular, then the set

$$C_\alpha(A) = \{P \in \text{prim}(A) : A/P \approx M_{n,\beta}, \beta \leq \alpha\}$$

is closed in $\text{Prim}(A)$ by Proposition 4.7 together with Remark 5.5 (a).

Thus $k(C_\alpha(A))$ is the required ideal.

(ii) This follows from (i) by repeated application.

(iii) In this case, by Proposition 4.7 and Remark 5.5 (b),

$$S = \{P \in \text{Prim}(A) : A/P \approx A_{2n}\}$$

is closed in $\text{Prim}(A)$ and $J = k(S)$ is the required ideal.

LEMMA 5.7. *Let A be a JC*-algebra of constant rank n where $3 \leq n < \infty$. Then there are norm closed ideals of $A, J_1 \subset J_2$ such that J_1 is of type $A_{2n}, J_2/J_1$ is type $M_{n,n}$ and A/J_2 is type S_n .*

PROOF. As all Type I factor representations (in the sense of [15], page 133) of the JC-algebra A_{sa} must be of Type I_n , this follows from [6, §5] because $A_{2n} \approx M_n(\mathbb{H})_{\text{sa}} \otimes_{\mathbb{R}} \mathbb{C}$ and $S_{2n} \approx M_n(\mathbb{R})_{\text{sa}} \otimes_{\mathbb{R}} \mathbb{C}$.

THEOREM 5.8. *Let A be a JC*-triple of constant rank n , where $3 \leq n < \infty$. Then there are norm closed ideals of $A, J_1 \subset J_2 \subset J_3$ such that*

- (i) J_1 is type A_{2n+1} ;
- (ii) J_2/J_1 is type A_{2n} ;
- (iii) J_3/J_2 is rectangular;
- (iv) A/J_3 is type S_n .

PROOF. Let $P_0 \in \text{Prim}(A)$. Let $x \in A$ and e be complete tripotent of $M = A/P_0$ such that $x + P_0 = e$. Then by Section 2, $A(x)/P_0 \cap A(x) \approx M_2(e)$ as JC*-algebras. Let I be the norm closed ideal of the JC*-algebra $A(x)$ such that

$$V = \text{Prim}(A(x)) \setminus h(I) = \{Q : \text{rank}(A(x)/Q) = n\}.$$

Then $P_0 \cap A(x) \in V$ and I is a JC*-algebra of constant rank n .

Now suppose that $P_0 \in S = \{P \in \text{Prim}(A) : A/P \text{ is symplectic}\}$. Then $M \approx A_{2n}$ or A_{2n+1} so that $M_2(e) \approx A_{2n}$, and by Lemma 5.7 there is a non-zero norm closed ideal J of I all primitive quotients of which are isometric to A_{2n} . Then $P_0 \cap J \neq 0$. Let $K = T(J)$ be the norm closed ideal of A generated by J and let $P \in \text{Prim}(A)$ such that $P \cap K \neq 0$. Then $P \cap J \neq 0$, by Lemma 2.4. Hence, $A_{2n} \approx J/P \cap J$ imbeds as a subtriple into A/P . As A_{2n} cannot be so embedded into $M_{n,\alpha}$ nor into S_n , we must have $A/P \approx A_{2n}$ or A_{2n+1} . Therefore, $P \in \text{Prim}(A) \setminus h(K) \subset S$, which proves that S is open in $\text{Prim}(A)$. Hence, there is a norm closed ideal J_2 of A such that J_2 is symplectic and A/J_2 has no symplectic primitive quotients. The required ideal $J_1 \subset J_2$ comes from Lemma 5.6 (iii).

Passing to A/J_2 we may assume that $J_2 = 0$ and emulate the above argument for $P_0 \in R = \{P \in \text{Prim}(A) : A/P \text{ is rectangular}\}$. In this case, in the notations of the first paragraph of the proof, $A(x)/P_0 \cap A(x) \approx M_2(e) \approx$

$M_{n,n}$ as JC*-algebras. Applying Lemma 5.7 together with the fact that $M_{n,n}$ is not embeddable in S_n , we obtain by the same argument an ideal J_3 of A such that J_3 is rectangular and A/J_3 has no rectangular primitive quotients and so is type S_n .

It remains to deal with the general constant rank 2 case (Lemma 5.6(i) takes care of the general constant rank 1 case). Let $V \subset M$, where V is a spin factor and M is a JBW*-triple factor of rank 2. For convenience we tabulate the possible structure of M determined by $V = V_\alpha$, $\alpha \geq 3$.

V	V_3	V_4	V_5	$V_{\alpha>5}$
M	$M_{2,\alpha}, A_5, V_{\gamma \geq 3}$	$A_5, V_{\gamma \geq 4}$	$A_5, V_{\gamma \geq 5}$	$V_{\gamma > 5}$

THEOREM 5.9. *Let A be a JC*-triple of constant rank 2. If A is a spin type and $2 \leq \gamma < \infty$, then $S^\gamma(A) = \{P \in \text{Prim}(A) : A/P \approx V_\lambda, \lambda > \gamma \text{ is open in } \text{Prim}(A)\}$.*

In general, there are ideals $J_1 \subset J_2 \subset J_3 \subset J_4 \subset J_5 \subset A$ such that

- (i) J_1 is spin type with $\text{Prim}(J_1) = S^5$;
- (ii) J_2/J_1 is type A_5 ;
- (iii) J_3/J_2 is type V_5 ;
- (iv) J_3/J_2 is type V_4 ;
- (v) J_5/J_4 is rectangular;
- (vi) A/J_5 is type V_2 .

PROOF. Let $P_0 \in (A)$. As in the proof of Theorem 5.8, for a complete tripotent $e \in M$ and $x \in A$ we have $A(x)/P_0 \cap A(x) \approx M_2(e)$ as JC*-algebras. As M is rank 2, $M_2(e)$ is a spin factor.

Assume that A is of spin type. Suppose that $P_0 \cap A(x) \in S^\gamma(A(x))$ which is open in $\text{Prim}(A)$ by Lemma 4.9. Thus, by Proposition 3.3 and its notation $U = \beta^{-1}(S^\gamma(A(x)))$ is open neighbourhood of P_0 and $U \subset S^\gamma(A)$ by the table above. It follows that $S^\gamma(A)$ is open.

Reverting to the general case, the same argument shows that $S^5(A)$ is open. This gives the ideal J_1 . Passing to A/J_1 we may assume that $S^5(A) = \emptyset$. In this case, suppose that $P_0 \in \{Q \in \text{Prim}(A) : A/Q \text{ is symplectic}\} = S$. Then $M \approx A_5$ or $M \approx V_5 \approx A_4$ so that $M_2(e) \approx V_5$ and $P_0 \cap A(x) \in S^4(A(x))$. Hence, by Proposition 3.3 and Lemma 4.9 together with the above table, there is a neighbourhood of P_0 contained in S which is therefore open in $\text{Prim}(A)$. By Lemma 5.6(iii), the corresponding ideal, J_3 contains the ideal J_2 as stated.

Proceeding, we now assume that S and $S^5(A)$ are empty to find, in the same way, that $S^3(A)$ is now open. This gives the ideal J_4 .

Finally assume that $S^3(A)$ is empty and let

$$P_0 \in R = \{P \in \text{Prim}(A) : A/P \text{ is rectangular}\}.$$

Then $M_2(e) = V_3$ and we find $P_0 \in \beta^{-1}(S^2(A(x))) \cap R$ from the first column of the table. It follows that R is open in $\text{Prim}(A)$, which gives the ideal J_5 .

COROLLARY 5.10. *Let A be a JB^* -triple of bounded rank such that all primitive quotients are finite dimensional and let, up to isometry, $\{A/P : P \in \text{Prim}(A)\} = \{M_i\}_{i=1}^K$. Then there is a permutation π of $\{1, \dots, k\}$ and norm closed ideals of A , $0 = I_0 \subset I_1 \subset \dots \subset I_k \subset I_{k+1} = A$ such that I_{r+1}/I_r is non-trivial type $M_{\pi(r)}$, for $r = 0, \dots, k$.*

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REFERENCES

1. E.M. Alfsen and E.G. Effros, *Structure in real Banach spaces I, II*, Ann. of Math. 96 (1972), 98–128 and 129–173.
2. T. Barton and R. Timoney, *Weak* continuity of Jordan triple product and applications*, Math Scand. 59 (1986), 177–191.
3. T. Barton, T. Dang and G. Horn, *Normal representations of Banach triple systems*, Proc. Amer. Math. Soc. 102 (1987), 551–555.
4. E. Behrens, *M-structure and the Banach-Stone Theorem*, Lecture Notes in Math. 736, 1979.
5. H. Behncke and W. Bos, *JB-algebras with an exceptional ideal*, Math. Scand. 42 (1978), 306–312.
6. L.J. Bunce *Structure of representations and ideals of homogeneous type in Jordan algebras*, Quart. J. Math. Oxford 37 (1986), 1–10.
7. L.J. Bunce and C.H. Chu, *Compact operations, multipliers and the Radon- Nikodym property*, Pacific J. Math 153 (1992), 249–265.
8. T. Dang and Y. Friedman, *Classification of JBW^* -triple factors and applications*, Math. Scand. 61 (1987), 292–330.
9. S. Dineen, *The second dual of a JB^* -triple system*, Complex Analysis, Functional Analysis and Approximation theory, vol. 125, North Holland Math. Studies, 1986, pp. 67–69.
10. J. Dixmier, *C^* -algebras*, North Holland, 1969.
11. C.M. Edwards, *On the centres of hereditary JBW subalgebras of a JBW algebra*, Math. Proc. Cambridge Philos. Soc. 85 (1979), 317–324.
12. Y. Friedman and B. Russo, *Structure of the predual of a JBW^* -triple*, J. Reine Angew. Math. 356 (1985), 67–84.
13. Y. Friedman and B. Russo, *Solution of the contractive projection problem*, J. Funct. Anal. 60 (1985), 56–79.
14. Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for JB^* -triples*, Duke Math. J. 53 (1986), 139–148.
15. H. Hanche-Olsen and E. Stormer, *Jordan Operator Algebras*, Pitman, London 1984.
16. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach Spaces and Banach algebras*. Lecture Notes in Math. 1547, 1993.
17. L.A. Harris, *A generalization of C^* -algebras*, Proc. London Math. Soc. 42 (1982), 331–361.
18. G. Horn, *Characterization of the predual and ideal structure of a JBW^* -triple*, Math. Scand. 61 (1987), 117–133.
19. G. Horn, *Classification of JBW^* -triples of type I*, Math. Z. 196 (1987), 271–291.

20. E. Horn and E. Neher, *Classification of JBW^* -triples*, Trans. Amer. Math. Soc. 306 (1988), 553–578.
21. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 138 (1983), 503–529.
22. W. Kaup, *Contractive projections on Jordan C^* -algebras and generalizations*, Math. Scand. 54 (1985), 95–100.
23. W. Kaup, *On JB^* -triples defined by fibre bundles*, Manuscripta Math. 87 (1987), 379–403.
24. O. Loos, *Bounded Symmetric Domains and Jordan Pairs* (1977), Irvine Lecture Notes.
25. G.K. Pedersen, *C^* -algebras and their automorphism groups*, North Holland, 1979.
26. B. Russo, *Structure of JB^* -triples*, Jordan Algebras (W. Kaup, K. McCrimmon, H.P. Pettersen, eds.), de Gruyter, 1994.
27. L.L. Stacho, *A projection principle concerning biholomorphic automorphisms of Banach spaces*, Acta Sci. Math. (Szeged) 46 (1982).
28. J. Tomiyama and M. Takesaki, *Application of fibre bundles to the certain class of C^* -algebras*, Tohoku Math. J. 13 (1961), 498–522.
29. H. Upmeyer, *Symmetric Banach Manifolds and Jordan C^* -algebras*, North Holland, 1985.
30. J.D.M Wright, *Jordan C^* -algebras*, Michigan Math. J. 24 (1977), 291–302.

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