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$\mathbb{B}$-CONVEXITY

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Given a homeomorphism $\Phi : X \to \mathbb{R}^n$ one can define on the topological space X a set operator through the formula $Co^{\Phi}(A) = \Phi^{-1}(Co(\Phi(A))$. Such a convexity on X has all the topological, geometric and algebraic properties of the usual convexity on $\mathbb{R}^n$; up to a change of variable, it is a linear convexity. In the context of convex analysis and optimization theory such operators were considered by Avriel (1972) and Ben Tal (1977). We consider a sequence on homeomorphisms $\Phi_r : \mathbb{R}^n \to \mathbb{R}^n$ and we study the abstract convexity which is associated to the limit, in the appropriate sense, of the sequence of set operators $A \mapsto Co^{\Phi_r}(A)$; we call the limit-convexity $\mathbb{B}$-convexity. On $\mathbb{R}^n_+$ one can loosely say that this $\mathbb{B}$-convexity is obtained from the usual linear convexity through the formal substitution $+ \mapsto \max$. We end this article with some simple applications to duality and "max-programming".

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1 INTRODUCTION

A bijection $\Phi : Z \longrightarrow \mathbb{R}^n$ induces a a vector space structure on Z with the sum and the scalar multiplication defined by $z_1 \overset{\Phi}{+} z_2 = \Phi^{-1}(\Phi(z_1) + \Phi(z_2))$ and for $\lambda \in \mathbb{R}$, $\lambda \overset{\phi}{\cdot} z = \Phi^{-1}(\lambda \cdot \Phi(z))$; furthermore, if we declare $U \subset Z$ open if $\Phi(U)$ is open in $\mathbb{R}^n$ then $\Phi : Z \longrightarrow \mathbb{R}^n$ becomes a linear homeomorphism. One can even modify the field of scalars; given a bijection $\varphi : K \longrightarrow \mathbb{R}$ from a set K to $\mathbb{R}$, we can induce a field structure on K for which φ becomes a field isomorphism (K is therefore $\mathbb{R}$, granted a change of notation, since there is a unique field isomorphism from $\mathbb{R}$ to itself, the identity); given this change of notation via φ and Φ we can define a K-vector space structure on Z by: $k \overset{\phi}{\cdot} z = \Phi^{-1}(\varphi(k) \cdot \Phi(z))$ and $z_1 \overset{\Phi}{+} z_2 = \Phi^{-1}(\Phi(z_1) + \Phi(z_2))$; we call these two operations the indexed scalar product and the indexed sum (indexed by φ of course). One can also transport on Z all the topological and geometric constructions which make sense on $\mathbb{R}^n$. A familiar example is provided by taking $K =]0, +\infty[$ and φ the natural log function, $Z = K^n$ and $\Phi : K^n \longrightarrow \mathbb{R}^n$ to be $(r_1, \ldots, r_n) \longrightarrow (\log r_1, \ldots, \log r_n)$. More abstractly, one can consider an arbitrary

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bijection $\varphi : K \longrightarrow \mathbb{R}$ and then take $Z = K^n$ and $\Phi(k_1, \dots, k_n) = (\varphi(k_1), \dots, \varphi(k_n))$; this is basically the approach of Ben-Tal [3] and Avriel [1]. We give a brief description of the general construction.

Let

$$k_1 \overset{\varphi}{\cdot} k_2 = \varphi^{-1}(\varphi(k_1) \cdot \varphi(k_2)) \text{ and } k_1 \overset{\varphi}{+} k_2 = \varphi^{-1}(\varphi(k_1) + \varphi(k_2))$$

Moreover for $k \in K$ and $x, y \in K^n$ we put:

$$k \overset{\varphi}{\cdot} x = \Phi^{-1}(\varphi(k) \cdot \Phi(x)) \quad \text{and} \quad x \overset{\varphi}{+} y = \Phi^{-1}(\Phi(x) + \Phi(y))$$

where $\Phi(x) = (\varphi(x_1), \dots, \varphi(x_n))$. The φ-sum – denoted $\sum^{\varphi}$ – of $A \subset K^n$, where A is a finite nonempty set, is defined by

$$\sum_{a \in A}^{\varphi} a = \Phi^{-1}\left(\sum_{a \in A} \Phi(a)\right)$$

and the Φ-convex hull of a finite set $A \subset K^n$ is defined by:

$$Co^{\Phi}(A) = \left\{ \sum_{a \in A}^{\varphi} k_a \overset{\phi}{\cdot} a \colon \sum_{a \in A}^{\varphi} k_a = \varphi^{-1}(1) \text{ and } \forall a \in A \; \varphi(k_a) \geq 0 \right\}$$

A simple calculation shows that:

$$Co^{\Phi}(A) = \Phi^{-1}(Co(\Phi(A)))$$

In this article we consider a sequence $\varphi_r : \mathbb{R} \longrightarrow \mathbb{R}$ of homeomorphisms, $r \in \mathbb{N}$; following the procedure outlined above, each of these homeomorphisms induces a hull operator on $\mathbb{R}^n$, which we will note Co^r instead of Co^{Φ_r}, given by $Co^r(A) = \Phi_r^{-1}(Co(\Phi_r(A)))$; taking into account that each Φ_r is a homeomorphism, we see that, for all (nonempty) finite subset $A \subset \mathbb{R}^n$, the set $Co^r(A)$ is (nonempty) compact.

We define the limit hull of a finite subset $A \subset \mathbb{R}^n$ as the Kuratowski–Painlevé upper limit of the sequence of compact sets $\{Co^r(A)\}_{r \in \mathbb{N}}$.[1] This limit hull will be denoted by $Co^{\infty}(A)$; it is the subject matter of this article.

To be more precise, we will concentrate our attention on a particular example: $\varphi_r(x) = x^{2r+1}$. That sequence $\{\varphi_r\}_{r \in \mathbb{N}}$ has the following properties:

(a) $\Phi_r(0) = 0$
(b) if $x \in \mathbb{R}^n_+$, then $\Phi_r(x) \in \mathbb{R}^n_+$; more generally, if $x \leq y$ then $\Phi_r(x) \leq \Phi_r(y)$
(c) $\Phi_0 = Id$
(d) $\Phi_r \circ \Phi_{r'} = \Phi_{r'} \circ \Phi_r = \Phi_{2r+2r'+4rr'}$

$\{\Phi_r \colon r \in \mathbb{N}\}$ is therefore an abelian semigroup of order preserving transformation of $\mathbb{R}^n$ (with respect to the partial order defined by the positive cone $\mathbb{R}^n_+$). We can expect that the limit hull $Co^{\infty}(A)$ will be related to the hull associated to order convexity on

[1]The Kuratowski–Painlevé upper limit of the sequence of sets $\{A_n\}$ is $\cap_n \overline{\cup_k A_{n+k}}$; it is also the set points p for which there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ and points $p_{n_k} \in A_{n_k}$ such that $p = \lim_{k \to \infty} p_{n_k}$.

$\mathbb{R}^n$; as we will see, this is the case, but $Co^{\infty}(A)$ will generally be much smaller, or not even comparable to the order convex hull. Computational aspects and applications to the "existence of convex analysis without linearity" to borrow the expression of Pallascke and Rolewicz in [7], Preface XI, will be briefly considered at the end of this article.

2 𝔹-CONVEX SETS

For all $r \in \mathbb{N}$ the map $x \mapsto \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}$ to itself; $x = (x_1, \dots, x_n) \mapsto \Phi_r(x) = (\varphi_r(x_1), \dots, \varphi_r(x_n))$ is a homeomorphism from $\mathbb{R}^n$ to itself. Let us explicitly write down the indexed sums and scalar multiplications associated to the the homeomorphisms $\varphi_r(x)$; the index here will be the natural number r instead of the homeomorphism $\varphi_r(x)$, as in the Introduction. For $k_1, k_2 \in \mathbb{R}$ the indexed sum and the indexed product are simply

$$k_1 \overset{r}{+} k_2 = \left(k_1^{2r+1} + k_2^{2r+1}\right)^{1/(2r+1)}$$

and

$$k_1 \overset{r}{\cdot} k_2 = k_1 \cdot k_2$$

For $k_i \in \mathbb{R}$ and $x_i = (x_{i,1}, \dots, x_{i,n}) \in \mathbb{R}^n$, we have:

$$\begin{aligned}(k_1 \overset{r}{\cdot} x_1) \overset{r}{+} \cdots \overset{r}{+} (k_m \overset{r}{\cdot} x_m) &= \Phi_r^{-1}\left(\sum_{i=1}^{m} k_i^{2r+1} \Phi_r(x_i)\right) \\ &= \Big(\left[(k_1 x_{1,1})^{2r+1} + \cdots + (k_m x_{m,1})^{2r+1}\right]^{1/(2r+1)}, \dots, \\ &\qquad \left[(k_1 x_{1,n})^{2r+1} + \cdots + (k_m x_{m,n})^{2r+1}\right]^{1/(2r+1)}\Big)\end{aligned}$$

Let $v_j = (k_1 x_{1,j}, \dots, k_m x_{m,j}) \in \mathbb{R}^n$, $j = 1, \dots, n$; if $x_i \in \mathbb{R}_+^n$ and $k_i \in \mathbb{R}_+$ for all $i = 1, \dots, n$, then

$$(k_1 \overset{r}{\cdot} x_1) \overset{r}{+} \cdots \overset{r}{+} (k_m \overset{r}{\cdot} x_m) = (\|v_1\|_{2r+1}, \dots, \|v_n\|_{2r+1})$$

where for $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, $\|a\|_r = (\sum_{i=1}^k |a_i|^r)^{1/r}$, and $\|a\|_\infty = \max_{i=1}^k \{|a_i|\}$. For a finite non empty set $A = \{x_1, \dots, x_m\} \subset \mathbb{R}^n$, the r-convex hull of A, which we denote $Co^r(A)$ instead of $Co^{\Phi_r}(A)$ as in the Introduction, is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{a \in A} t_a \Phi_r(a)\right) : t_a \geq 0, \sum_{a \in A} t_a = 1\right\}$$

Since any $t \in [0, 1]$ is of the form ρ^{2r+1} for a unique ρ, one can also say that a vector $u = (u_1, \dots, u_n)$ belongs to $Co^r(A)$ if and only if there exist $\rho_i \geq 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \rho_i^{2r+1} = 1$ and, for all $j = 1, \dots, n, u_j = (\sum_{i=1}^m \rho_i^{2r+1} x_{i,j}^{2r+1})^{1/(2r+1)}$; if $A \subset \mathbb{R}_+^n$ the $u_j = \|(\rho_1 x_{1,j}, \dots, \rho_m x_{m,j})\|_{2r+1}$.

The structure of $\mathbb{B}$-convex sets, which we have not yet defined, will involve the order structure, with respect to the positive cone of $\mathbb{R}^n$; we denote by $\vee_{i=1}^m x_i$ the least upper bound of $x_1, \ldots, x_m \in \mathbb{R}^n$, that is:

$$\bigvee_{i=1}^{m} x_i = \big(\max\{x_{1,1}, \ldots, x_{m,1}\}, \ldots, \max\{x_{1,n}, \ldots, x_{m,n}\}\big)$$

For future reference, we gather in the lemma below some elementary facts.

LEMMA 2.0.1

(a) *For all increasing sequence of natural number* $\{r_k\}_{k\in\mathbb{N}}$ *and for all* $a \in \mathbb{R}^m$, $\|a\|_\infty = \lim_{k\longrightarrow\infty} \|a\|_{r_k}$.
(b) *If* $\{a_k\}_{k\in\mathbb{N}}$ *is a sequence in* $\mathbb{R}^m$ *which converges to* $a \in \mathbb{R}^m$ *and if* $\{r_k\}_{k\in\mathbb{N}}$ *is an increasing sequence of natural numbers then* $\|a\|_\infty = \lim_{k\longrightarrow\infty} \|a_k\|_{r_k}$.
(c) *If* $x_1, \ldots, x_m \in \mathbb{R}^n_+$, *then given convergent sequences of positive real number* $\{\rho_{k,i}\}_{k\in\mathbb{N}}$, $i = 1, \ldots, m$, *whose limits are respectively* $\rho_1, \ldots, \rho_m$ *and an increasing sequence of natural number* $\{r_k\}_{k\in\mathbb{N}}$, *the sequence* $\{\rho_{k,1}x_1 \overset{r_k}{+} \cdots \overset{r_k}{+} \rho_{k,m}x_m\}_{k\in\mathbb{N}}$ *converges in* $\mathbb{R}^n$ *to* $\vee_{i=1}^m \rho_i x_i$.

Proof Assertions (a) and (b) follow from:

$$\max_{1\le i\le m} \{|a_{k,i}|\} \le \left(\sum_{i=1}^{m} |a_{k,i}|^{r_k}\right)^{1/r_k} \le m^{1/r_k} \max_{1\le i\le m} \{|a_{k,i}|\}$$

for all $k \in \mathbb{N}$ and $(a_{k,1}, \ldots, a_{k,m}) \in \mathbb{R}^m$.

For (c), we have $\rho_{k,1}x_1 \overset{r_k}{+} \cdots \overset{r_k}{+} \rho_{k,m}x_m = (u_{k,1}, \ldots, u_{k,n})$, where for $j = 1, \ldots, n$, $u_{k,j} = (\sum_{i=1}^m \rho_{k,i}^{2r_k+1} x_{i,j}^{2r_k+1})^{1/2r_k+1}$. By (b),

$$\lim_{k\longrightarrow\infty} u_{k,j} = \lim_{r_k\to\infty} \|(\rho_{k,1}x_{1,j}, \ldots, \rho_{k,m}x_{m,j})\|_{2r_k+1} = \max_{i=1,\ldots,m} \{\rho_i x_{i,j}\}$$

■

Following the program outlined in the introduction, we define the limit hull of a finite set A as the Kuratowski–Painlevé upper limit of the sequence of sets $\{Co^r(A)\}_{r\in\mathbb{N}}$; that is the set of point $x^* \in \mathbb{R}^n$ for which there exist an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ and points $x_{n_k} \in Co^{n_k}(A)$ such that $x^* = \lim_{k\to\infty} x_{n_k}$.

2.1 $\mathbb{B}$-Polytopes

The Kuratowski–Painlevé upper limit of the sequence of sets $(Co^r(A))_{r\in\mathbb{N}}$, where A is finite, will be denoted by $Co^\infty(A)$. By definition, a $\mathbb{B}$-polytope is a set of the form $Co^\infty(A)$ for some finite subset of $\mathbb{R}^n$.

We will see that in $\mathbb{R}^n_+$ the upper-limit is in fact a limit and that elements of $Co^\infty(A)$ have a simple analytic description.[2] Our first result, Theorem 2.1.1, gives a simple

[2]The Kuratowski–Painlevé lower limit of the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, denoted $Li_{n\to\infty}A_n$, is the set of points p for which there exists a sequence $\{p_n\}$ of points such that $p_n \in A_n$ for all n and $p = \lim_{n\to\infty} p_n$; a sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^m$ is said to converge, in the Kuratowski–Painlevé sense, to a set A if $Ls_{n\to\infty}A_n = A = Li_{n\to\infty}A_n$, in which case we write $A = Lim_{n\to\infty}A_n$.

algebraic description of $Co^{\infty}(A)$; it will be later extended to compact sets, without much difficulty, and then to arbitrary sets.

Theorem 2.1.1 *For all nonempty finite subset $A \subset \mathbb{R}^n_+$ we have $Co^{\infty}(A) = Lim_{r\to\infty} Co^r(A) = \{\vee_{x\in A} t_x x \colon t_x \in [0,1],\ \max_{x\in A}\{t_x\} = 1\}$.*

Proof Let $A = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n_+$; we first establish that

$$\left\{ \bigvee_{i=1,\ldots,m} t_i x_i \colon\ t_i \in [0,1],\ \max_{1\le i\le m}\{t_i\} = 1 \right\} \subset Li_{r\longrightarrow\infty} Co^r(A)$$

Let $x = \rho_1 x_1 \vee \cdots \vee \rho_m x_m$ with $\rho_1, \ldots, \rho_m \in [0,1]$ and $\max_{1\le i\le m}\{\rho_i\} = 1$. Define $y_r \in Co^r(A)$ by

$$y_r = \frac{1}{\rho_1 \overset{r}{+} \cdots \overset{r}{+} \rho_m} \left(\rho_1 \overset{r}{\cdot} x_1 \overset{r}{+} \cdots \overset{r}{+} \rho_m \overset{r}{\cdot} x_m \right)$$

Since $x_1, \ldots, x_m \in \mathbb{R}^n_+$ and

$$\lim_{r\longrightarrow\infty} \left(\rho_1 \overset{r}{+} \cdots \overset{r}{+} \rho_m \right) = \max_{1\le i\le m}\{\rho_i\} = 1$$

we deduce that

$$\lim_{r\to\infty} y_r = \lim_{r\longrightarrow\infty} \left(\rho_1 \overset{r}{\cdot} x_1 \overset{r}{+} \cdots \overset{r}{+} \rho_m \overset{r}{\cdot} x_m \right) = \rho_1 x_1 \vee \cdots \vee \rho_m x_m = x.$$

This completes the first part of the proof.

Next, we establish that

$$Ls_{r\longrightarrow\infty} Co^r(A) \subset \left\{ \bigvee_{i=1,\ldots,m} t_i x_i \colon\ t_i \in [0,1],\ \max_{1\le i\le m}\{t_i\} = 1 \right\}.$$

Take $x \in Ls_{r\longrightarrow\infty} Co^r(A)$; there is an increasing sequence $\{r_k\}_{k\in\mathbb{N}}$ and a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ such that $p_k \in Co^{r_k}(A)$ and $\lim_{k\longrightarrow\infty} p_k = x$. Each p_k being in $Co^{r_k}(A)$, we can write

$$p_k = (p_{k,1}, \ldots, p_{k,n}) = \rho_{k,1} \overset{r_k}{\cdot} x_1 \overset{r_k}{+} \cdots \overset{r_k}{+} \rho_{k,m} \overset{r_k}{\cdot} x_m$$

or, more explicitly,

$$p_k = \left(\left(\sum_{i=1}^{m} \rho_{k,i}^{2r_k+1} x_{i,1}^{2r_k+1} \right)^{1/(2r_k+1)}, \ldots, \left(\sum_{i=1}^{m} \rho_{k,i}^{2r_k+1} x_{i,n}^{2r_k+1} \right)^{1/(2r_k+1)} \right)$$

from which we see that

$$p_{k,j} = \|(\rho_{k,1}x_{1,j}, \dots, \rho_{k,m}x_{m,j})\|_{2r_k+1}$$

Since $\rho_k = (\rho_{k,1}, \dots, \rho_{k,m}) \in [0,1]^m$ we can assume that the sequence $(\rho_k)_{k\in\mathbb{N}}$ converges to a point $\rho^* = (\rho_1^*, \dots, \rho_m^*) \in [0,1]^m$. Furthermore, by Lemma 2.0.1,

$$\lim_{k\to\infty} \|(\rho_{k,1}, \dots, \rho_{k,m})\|_{2r_k+1} = \|(\rho_1^*, \dots, \rho_m^*)\|_\infty$$

and, from,

$$\|(\rho_{k,1}, \dots, \rho_{k,m})\|_{2r_k+1} = \left(\sum_{i=1}^m \rho_{k,i}^{2r_k+1}\right)^{1/(2r_k+1)} = 1$$

we have

$$\|(\rho_1^*, \dots, \rho_m^*)\|_\infty = 1 = \max_{1\le i\le m}\{\rho_i^*\}$$

Finally,

$$\begin{aligned}\lim_{k\to\infty} p_{k,j} &= \lim_{k\to\infty} \|(\rho_{k,1}x_{1,j}, \dots, \rho_{k,m}x_{m,j})\|_{2r_k+1}\\ &= \|(\rho_1^* x_{1,j}, \dots, \rho_m^* x_{m,j})\|_\infty\end{aligned}$$

We have shown that $x = \vee_{i=1}^m \rho_i^* x_i$ with $\max_{1\le i\le m}\{\rho_i^*\} = 1$.

The first and the second part of the proof show that

$$Ls_{r\longrightarrow\infty}Co^r(A) \subset \left\{\bigvee_{i=1,\dots,m} t_i x_i\colon\ t_i \in [0,1],\ \max_{1\le i\le m}\{t_i\} = 1\right\} \subset Li_{r\longrightarrow\infty}Co^r(A)$$

and this completes the proof since we always have $Li_{r\longrightarrow\infty}Co^r(A) \subset Ls_{r\longrightarrow\infty}Co^r(A)$ ■

That such an analytic description for $Co^\infty(A)$ does not hold generally can be seen from the following example: with $A = \{(1,1),(1,-1)\}$ we have

$$\{(\max\{t_1,t_2\},\ \max\{t_1,-t_2\})\colon\ \max\{t_1,t_2\} = 1\} = \{(1,t)\colon\ 0 \le t \le 1\}$$

That set does not contain A while $Co^\infty(A)$ does; they are therefore different.

For all finite and nonempty set A contained in $\mathbb{R}^n$, $Co^r(A)$ belongs to $\mathcal{K}(\mathbb{R}^n)$, the space of nonempty compact subsets of $\mathbb{R}^n$, which is metrizable by the the Hausdorff metric

$$D_H(K_1,K_2) = \inf\left\{\varepsilon > 0\colon\ K_1 \subset \bigcup_{x\in K_2} B(x,\varepsilon),\ \text{and}\ K_2 \subset \bigcup_{x\in K_1} B(x,\varepsilon)\right\}.$$

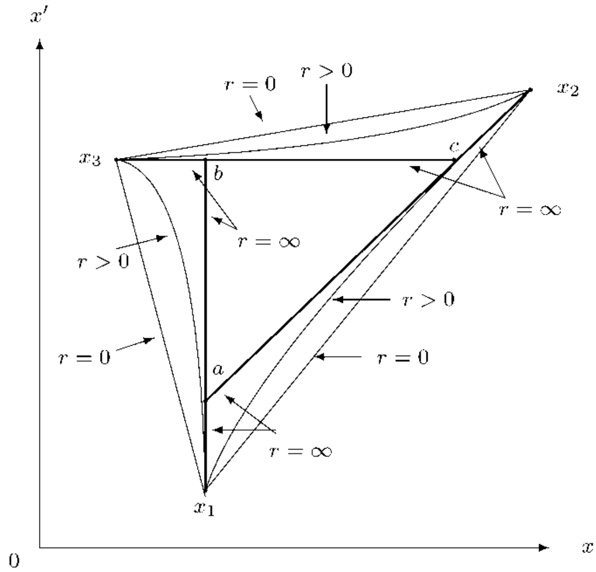

FIGURE 1 Limit sets.

Corollary 2.1.2 *For all finite nonempty subsets A of $\mathbb{R}^n_+$, the sequence $\{Co^r(A)\}_{r\in\mathbb{N}}$ converges to $Co^\infty(A)$ in $\mathcal{K}(\mathbb{R}^n)$, with respect to the Hausdorff metric.*

Proof Choose $\delta > 0$ such that $A \subset [0,\delta]^n$; we have $\Phi_r(A) \subset [0,\delta^{2r+1}]^n$, and therefore also $Co(\Phi_r(A)) \subset [0,\delta^{2r+1}]^n$. Taking the inverse image by Φ_r yields $Co^r(A) \subset [0,\delta]^n$; all the terms of the sequence $\{Co^r(A)\}_{r\in\mathbb{N}}$ are contained in the compact set $[0,\delta]^n$. To conclude, recall that on compact metric spaces, Kuratowski–Painlevé convergence of a sequence of compact sets implies convergence in the Hausdorff metric. ■

The convergence process is illustrated in Fig. 1.

The limit hull $Co^\infty(\{x_1,x_2\})$ is the broken line $[x_1,a,x_2]$; $Co^\infty(\{x^1,x^3\})$ is the broken line $[x_1,b,x_3]$. $Co^\infty(\{x_2,x_3\})$ is the broken line $[x_2,c,x_3]$. The intermediary strings corresponding to $r=1$, $1<r<\infty$ are represented in Fig. 1.

A more general case is illustrated in Fig. 2.

The Fig. 2 represents the 𝔹-polytope spanned by five points.

The finiteness condition in Corollary 2.1.2 will be removed in Section 2.2.

2.2 𝔹-Convex Sets

Definition 2.2.1 A subset L of $\mathbb{R}^n$ is 𝔹-convex if for all finite subset $A \subset L$ the 𝔹-polytope $Co^\infty(A)$ is contained in L.

The proof of following obvious, but nonetheless important propositions, is left to the reader.

Proposition 2.2.2

(a) *The emptyset, $\mathbb{R}^n$, as well as all the singletons are 𝔹-convex;*
(b) *if $\{L_\lambda\colon \lambda\in\Lambda\}$ is an arbitrary family of 𝔹-convex sets then $\cap_\lambda L_\lambda$ is 𝔹-convex;*
(c) *if $\{L_\lambda\colon \lambda\in\Lambda\}$ is a family of 𝔹-convex sets such that $\forall\lambda_1,\lambda_2\in\Lambda\ \ \exists\lambda_3\in\Lambda$ such that $L_{\lambda_1}\cup L_{\lambda_2}\subset L_{\lambda_3}$ then $\cup_\lambda L_\lambda$ is 𝔹-convex.*

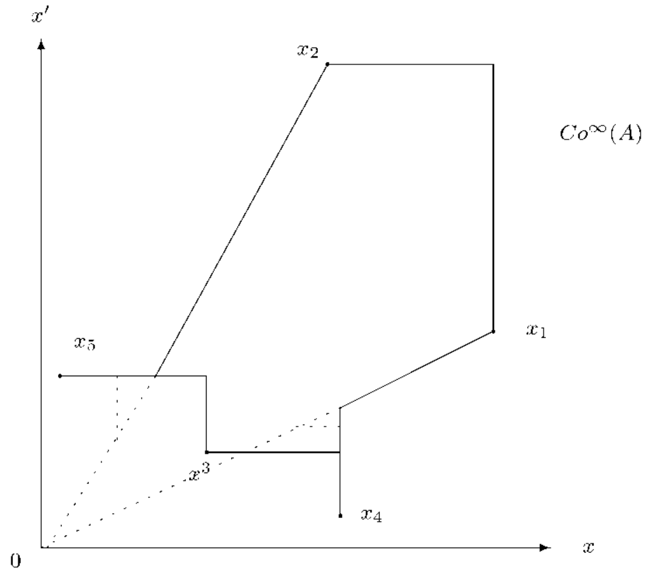

FIGURE 2 The limit set $Co^\infty(A)$.

Given a set $S \subset \mathbb{R}^n$ there is, according to (a) above, a $\mathbb{B}$-convex set which contains S; by (b) the intersection of all such $\mathbb{B}$-convex sets is $\mathbb{B}$-convex; we call it the $\mathbb{B}$-convex hull of S and we write $\mathbb{B}[S]$ for the $\mathbb{B}$-convex hull of S.

PROPOSITION 2.2.3 *The following properties hold:*

(a) $\mathbb{B}[\emptyset] = \emptyset$, $\mathbb{B}[\mathbb{R}^n] = \mathbb{R}^n$, *for all* $x \in \mathbb{R}^n$, $\mathbb{B}[\{x\}] = \{x\}$;
(b) *for all* $S \subset \mathbb{R}^n$, $S \subset \mathbb{B}[S]$ *and* $\mathbb{B}[\,[\mathbb{B}[S]\,]\,] = \mathbb{B}[S]$;
(c) *for all* $S_1, S_2 \subset \mathbb{R}^n$, *if* $S_1 \subset S_2$ *then* $\mathbb{B}[S_1] \subset \mathbb{B}[S_2]$;
(d) *for all* $S \subset \mathbb{R}^n$, $\mathbb{B}[S] = \cup\{\mathbb{B}[A]$: *A is a finite subset of* $S\}$;
(e) *a subset* $L \subset \mathbb{R}^n$ *is* $\mathbb{B}$-*convex if and only if, for all finite subset A of* L, $\mathbb{B}[A] \subset L$.

Propositions 2.2.2 and 2.2.3 are rather standard in the context of generalized convexities, they imply, among other things, that the family of $\mathbb{B}$-convex subsets of $\mathbb{R}^n$ is a complete lattice; the greatest lower bound of a family $\{L_\lambda\colon \lambda \in \Lambda\}$ of $\mathbb{B}$-convex sets is $\cap_\lambda L_\lambda$ and the least upper bound is $\mathbb{B}[\cup_\lambda L_\lambda]$.

A set of the form $\prod_{i=1}^n [x_i, y_i]$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^n$; if $A \subset \prod_{i=1}^n [x_i, y_i]$ then $\Phi_r(A) \subset \prod_{i=1}^n [x_i^{2r+1}, y_i^{2r+1}]$, from the convexity of a product of intervals we obtain, after taking the inverse image by Φ_r, $Co^r(A) \subset \prod_{i=1}^n [x_i, y_i]$ and therefore $Co^\infty(A) \subset \prod_{i=1}^n [x_i, y_i]$.

It is not clear from the definition that, for an arbitrary subset A of $\mathbb{R}^n$, $Co^\infty(A)$ is $\mathbb{B}$-convex. First, we will see that it is the case for finite subsets of $\mathbb{R}^n_+$. This will establish that $\mathbb{B}$-polytopes, that is $\mathbb{B}$-convex hull of finite sets, are upper limits of sequences of sets; general $\mathbb{B}$-convex sets are sets which contain all the polytopes spanned by their finite subsets. An upper limit is always closed, consequently, for an arbitrary set L, $\mathbb{B}[L]$ will be different from $Co^\infty(L)$. But if L is compact, we will see that $\mathbb{B}[L] = Co^\infty(L)$; and that, for an arbitrary subset L of $\mathbb{R}^n_+$, $Co^\infty(L)$ is the closed $\mathbb{B}$-convex hull of L, that is the smallest closed $\mathbb{B}$-convex set containing L.

PROPOSITION 2.2.4

(a) *A subset L of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if, for all $x_1, x_2 \in L$; $Co^\infty(\{x_1, x_2\}) \subset L$ whenever x_1 and x_2 belong to L.*
(b) *if A is a finite subset of $\mathbb{R}^n_+$ then $Co^\infty(A)$ is $\mathbb{B}$-convex.*

Proof

(a) Since $L \subset \mathbb{R}^n_+$ we have, by hypothesis and by Theorem 2.1.1,

$$\forall x_1, x_2 \in L \quad \{\rho_1 x_1 \vee \rho_2 x_2\colon \rho_1, \rho_2 \geq 0 \text{ and } \max\{\rho_1, \rho_2\} = 1\} \subset L$$

We show by induction on the cardinality of A that $Co^\infty(A) \subset L$ for all $A \subset L$. Assume that the property holds for $k = 1, \ldots, m-1$ where $m \geq 3$ and let $A = \{x_1, \ldots, x_m\} \subset L$. If $(\rho_1, \ldots, \rho_m) \in [0,1]^m$ with $\max\{\rho_1, \ldots, \rho_m\} = 1$ at least one of the ρ_i is 1; we can assume that it is ρ_1. By Theorem 2.1.1 and by the induction hypothesis we have $\vee_{i=1}^{m-1} \rho_i x_i \in L$. Let $y_1 = \vee_{i=1}^{m-1} \rho_i x_i$, $y_2 = x_m$, $\mu_1 = 1$ and $\mu_2 = \rho_m$; then $\vee_{i=1}^{m} \rho_i x_i = \mu_1 y_1 \vee \mu_2 y_2 \in L$.

(b) Let $A = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n_+$, $x = \vee_{j=1}^m \rho_j x_j$, $y = \vee_{j=1}^m \eta_j x_j$ with $(\rho_1, \ldots, \rho_m)$, $(\eta_1, \ldots, \eta_m) \in [0,1]^m$ and $\max\{\rho_1, \ldots, \rho_m\} = \max\{\eta_1, \ldots, \eta_m\} = 1$; both x and y are two elements of $Co^\infty(A)$. We have to see that $Co^\infty(\{x,y\}) \subset Co^\infty(A)$. Let $u \in Co^\infty\{x,y\}$; there exists $(\mu_1, \mu_2) \in [0,1]^2$ with $\max\{\mu_1, \mu_2\} = 1$ such that $u = \mu_1 x \vee \mu_2 y$.

$$u = \mu_1\left(\bigvee_{j=1}^m \rho_j x_j\right) \vee \mu_2\left(\bigvee_{j=1}^m \eta_j x_j\right) = \bigvee_{j=1}^m \max\{\mu_1\rho_j, \mu_2\eta_j\} x_j$$

To conclude the proof, just notice that $\max_{1 \leq j \leq m}\big\{\max\{\mu_1\rho_j, \mu_2\eta_j\}\big\} = 1$. ■

COROLLARY 2.2.5 *Let $L \subset \mathbb{R}^n_+$ and denote by $\langle L \rangle$ be the family of nonempty finite subsets of L, then*

$$\mathbb{B}[L] = \bigcup_{A \in \langle L \rangle} Co^\infty(A)$$

Proof Clearly, from (d) of Proposition 2.2.3 we have $\mathbb{B}(L) = \cup\{\mathbb{B}(A)\colon A \in \langle L \rangle\}$ and we have shown above that $\mathbb{B}(A) = Co^\infty(A)$ for $A \subset \mathbb{R}^n_+$ ■

We denote by $\langle S \rangle_m$, the family of nonempty subsets of S of cardinality at most m.

THEOREM 2.2.6 (Carathéodory in $\mathbb{B}$-convexity) *If L is a compact subset of $\mathbb{R}^n_+$ then*

$$Co^\infty(L) = \bigcup_{A \in \langle L \rangle_{n+1}} Co^\infty(A).$$

Consequently, for all subsets S of $\mathbb{R}^n_+$,

$$\mathbb{B}[S] = \bigcup_{A \in \langle S \rangle_{n+1}} \mathbb{B}[A] = \bigcup_{A \in \langle S \rangle_{n+1}} Co^\infty(A);$$

and, if S is compact, $\mathbb{B}[S] = Co^\infty(S)$.

Proof If $x \in Co^{\infty}(L)$ then there is a sequence $(x_{r_k})_{r_k \in \mathbb{N}}$ with $x_{r_k} \in Co^{r_k}(L)$, $\forall k \in \mathbb{N}$ which converges to x. But from Carathéodory's theorem, there is, for each k, a set of points $x_k^1, \ldots, x_k^{n+1}$ in L and a set of numbers $\rho_k^1, \ldots, \rho_k^{n+1}$ in [0, 1] such that

$$\sum_{j=1}^{n+1} (\rho_k^j)^{2r_k+1} = 1$$

and

$$\Phi_{r_k}(x_{r_k}) = \sum_{j=1}^{n+1} (\rho_r^j)^{2r_k+1} \Phi_{r_k}(x_k^j)$$

or, for $i = 1, \ldots, n$,

$$x_{r_k,i} = \left(\sum_{j=1}^{n+1} \left(\rho_k^j x_{r_k,i}^j \right)^{2r_k+1} \right)^{1/(2r_k+1)}$$

Since L is compact we can without loss of generality assume that each of the sequences $(x_k^j)_{k \in \mathbb{N}}$, $j = 1, \ldots, n+1$ converges in L to a point x^j, and also that each of the sequences $\rho_k^j, j = 1, \ldots, n+1$ converges in L to a point ρ^j in [0, 1]. Taking into account that all the numbers involved are positive we have

$$\lim_{k \to \infty} \left(\sum_{j=1}^{n+1} \left(\rho_k^j x_{r_k,i}^j \right)^{2r_k+1} \right)^{1/(2r_k+1)} = \max_{1 \leq j \leq n+1} \{\rho^j x_i^j\}$$

moreover

$$\max_{1 \leq j \leq n+1} \{\rho^j\} = 1.$$

Taking the limit componentwise we obtain $x = \vee_{j=1}^{n+1} \rho^j x^j$, with $\rho^j \geq 0$ for all j and $\max_{1 \leq j \leq n+1}\{\rho^j\} = 1$. We have shown that $x \in Co^{\infty}(A)$ with $A = \{x_1, \ldots, x_{n+1}\} \subset L$. The last formula follows from $\mathbb{B}[A] = Co^{\infty}(A)$ for all finite sets A, $\mathbb{B}[S] = \cup_{A \in \langle S \rangle} Co^{\infty}(A)$ and the first part applied to the finite sets $A \in \langle S \rangle$. ■

COROLLARY 2.2.7 *If S is a compact subset of $\mathbb{R}_+^n$ then $\mathbb{B}[S]$ is compact.*

Proof If $S \subset \prod_{i=1}^n [a_i, b_i]$ then $Co^{\infty}(S) \subset \prod_{i=1}^n [a_i, b_i]$; $Co^{\infty}(S)$ is therefore compact. The equality $\mathbb{B}[S] = Co^{\infty}(S)$ concludes the proof. ■

COROLLARY 2.2.8 *If $L \subset \mathbb{R}_+^n$ is a compact $\mathbb{B}$-convex set then*

$$L = Co^{\infty}(L) = Lim_{r \to \infty} Co^r(L) = \bigcap_{r \in \mathbb{N}} Co^r(L).$$

L is also the limit for the Hausdorff metric of the sequence of compact sets $\{Co^r(L)\}_{r \in \mathbb{N}}$.

Proof Since $L \subset Co^r(L)$ for all r we have $L \subset \cap_{r\in\mathbb{N}} Co^r(L)$; the string of inclusions

$$\bigcap_{r\in\mathbb{N}} Co^r(L) \subset Li_{r\to\infty} Co^r(L) \subset Ls_{r\to\infty} Co^r(L) = Co^\infty(L)$$

is always valid and, finally, we have shown that $\mathbb{B}[L] = Co^\infty(L)$ and we have $\mathbb{B}[L] = L$ by hypothesis. Since L is compact, it is contained in a cube, and therefore all the $Co^r(L)$ are contained in that same cube; convergence for the Hausdorff metric follows from the first part of the proof. ■

Finally, we show that the $\mathbb{B}$-hull of a compact subset of $\mathbb{R}^n_+$ is the limit of its Co^r hulls.

THEOREM 2.2.9 *If S is a compact subset of $\mathbb{R}^n_+$ then $\mathbb{B}[S]$ is the limit, in the Kuratowski–Painlevé sense, and also in the sense of the Hausdorff metric, of the sequence $\{Co^r(S)\}_{r\in\mathbb{N}}$.*

Proof We have already shown that

$$\mathbb{B}[S] = Co^\infty(S)$$

and also that $Co^\infty(A) = Lim_{r\to\infty} Co^r(A)$ for all finite subsets A. If A is a finite subset of S then $Co^\infty(A) = Li_{r\to\infty} Co^r(A) \subset Li_{r\to\infty} Co^r(S)$, this shows that

$$\bigcup_{A\in\langle S\rangle} Co^\infty(A) \subset Li_{r\to\infty} Co^r(S);$$

the left hand side is $\mathbb{B}[S]$ and therefore

$$\mathbb{B}[S] \subset Li_{r\to\infty} Co^r(S) \subset Ls_{r\to\infty} Co^r(S) = Co^\infty(S) = \mathbb{B}[S].$$

■

We have two set operators, $S \mapsto Co^\infty(S)$ and $S \mapsto \mathbb{B}[S]$ that coincide on compact sets; we will now see that for arbitrary sets $S \in \mathbb{R}^n_+$, $Co^\infty(S)$ is the $\mathbb{B}$ closed convex hull of S. We start with a convergence result which extends Corollary 2.2.8 to arbitrary closed $\mathbb{B}$-convex sets.

THEOREM 2.2.10 *If L is a closed $\mathbb{B}$-convex set of $\mathbb{R}^n_+$ then*

$$L = Co^\infty(L) = Lim_{r\to\infty} Co^r(L) = \bigcap_{r\in\mathbb{N}} Co^r(L).$$

Proof First, we establish that $L = Co^\infty(L)$. Let $y = \lim_{k\to\infty} x_k$ with $y_k \in Co^{r_k}(L)$; we have to see that $y \in L$. Now from Carathéodory's Theorem, we can find, for each k, $n+1$ points $x_1^{(k)}, \ldots, x_{n+1}^{(k)} \in L$ and numbers $\rho_1^{(k)}, \ldots, \rho_{n+1}^{(k)} \in [0, 1]$ such that

$$y_{k,i} = \left(\sum_{j=1}^{n+1}\left(\rho_j^{(k)} x_{j,i}^{(k)}\right)^{2r_k+1}\right)^{1/(2r_k+1)}.$$

But, from $\lim_{k\to\infty} y_{k,i} = y_i$ and $y_{k,i} \geq \max_{1\leq j\leq n+1}\{\rho_j^{(k)} x_{j,i}^{(k)}\}$ we see that the sequence $\{\max_{1\leq j\leq n+1}\{\rho_j^{(k)} x_{j,i}^{(k)}\}\}_{k\in\mathbb{N}}$ is bounded for all $i = 1, \ldots, n$ and all $j = 1, \ldots, n+1$, we can therefore extract convergent subsequences from all the finite sequences involved, including the sequences $\{\rho_j^{(k)}\}_{k\in\mathbb{N}}$; and without loss of generality, we can assume that the original sequences themselves were convergent. In other words, we can assume that there are positive numbers $z_{j,i}$ and ρ_j such that, for all i and j,

$$\lim_{k\to\infty} \rho_j^{(k)} x_{j,i}^{(k)} = z_{j,i} \quad \text{and} \quad \lim_{k\to\infty} \rho_j^{(k)} = \rho_j.$$

Let $J = \{j\colon \rho_j > 0\}$, since $\{\rho_j^{(k)}\}_{k\in\mathbb{N}}$ converges to a strictly positive number and $\{\rho_j^{(k)} x_{j,i}^{(k)}\}_{k\in\mathbb{N}}$ converges, the sequence $\{x_{j,i}^{(k)}\}_{k\in\mathbb{N}}$ is bounded. Taking again subsequences we can assume that $\{x_{j,i}^{(k)}\}_{k\in\mathbb{N}}$ converges to some $x_{j,i}$; since L is closed and $x_j^{(k)} \in L$ we have $x_j = (x_{j,1}, \ldots, x_{j,n}) \in L$, for $j \in J$. Write

$$y_{k,i} = \left(\sum_{j\in J}\left(\rho_j^{(k)} x_{j,i}^{(k)}\right)^{2r_k+1} + \sum_{j\notin J}\left(\rho_j^{(k)} x_{j,i}^{(k)}\right)^{2r_k+1}\right)^{1/(2r_k+1)}$$

and take the limit as k goes to infinity to obtain

$$y = \lim_{k\to\infty} y_k = \left(\bigvee_{j\in J} \rho_j x_j\right) \bigvee \left(\bigvee_{j\notin J} z_j\right)$$

Let $w_k = (\vee_{j\in J}\rho_j x_j) \vee (\vee_{j\notin J}\rho_j^{(k)} x_j^{(k)})$; we also have $y = \lim_{k\to\infty} w_k$. Since L is $\mathbb{B}$-convex we will have $w_k \in L$ if we can show that $\max\{\max_{j\in J}\{\rho_j\}, \max_{j\notin J}\{\rho_j^{(k)}\}\} = 1$; since L is closed, we will also have $y \in L$, which is what we want to prove. We have, for all k,

$$\sum_{j\in J}(\rho_j^{(k)})^{2r_k+1} + \sum_{j\notin J}(\rho_j^{(k)})^{2r_k+1} = 1;$$

we raise both sides to the power $(1/2r_k + 1)$ and we take the limit as k goes to infinity; since $\{\sum_{j\in J}(\rho_j^{(k)})^{2r_k+1}\}_{k\in\mathbb{N}}$ converges to $\max\{\rho_j\colon j \in J\}$ and $\{\sum_{j\notin J}(\rho_j^{(k)})^{2r_k+1}\}_{k\in\mathbb{N}}$ converges to 0 the first part of the proof is done. As previously, to conclude, we simply notice that

$$L \subset \bigcap_{r\in\mathbb{N}} Co^r(L) \subset Li_{r\to\infty} Co^r(L) \subset Ls_{r\to\infty} Co^r(L) = Co^\infty(L) = L.$$

■

Lemmas 2.2.11 and 2.2.12 will be used to show that $Co^\infty(S)$ is $\mathbb{B}$-convex for all subsets S of $\mathbb{R}^n_+$.

LEMMA 2.2.11 *If $\{S_r\}_{r\in\mathbb{N}} \subset \mathbb{R}^n_+$ is a sequence of sets such that, for all $r \in \mathbb{N}$, $S_r = Co^r(S_r)$ then $Li_{r\longrightarrow\infty} S_r$ is $\mathbb{B}$-convex. In particular, $Li_{r\longrightarrow\infty} Co^r(S)$ is $\mathbb{B}$-convex, and closed, for all subsets S of $\mathbb{R}^n_+$.*

Proof If $x, y \in Li_{r\longrightarrow\infty} S_r$ then there exist sequences $\{x_r\}_{r\in\mathbb{N}}$ and $\{y_r\}_{r\in\mathbb{N}}$ such that $\lim_{r\longrightarrow\infty} x_r = x$, $\lim_{r\longrightarrow\infty} y_r = y$, and, for all r in $\mathbb{N}$, $x_r, y_r \in S_r$. Let $\rho_1, \rho_2 \in [0, 1]$

with $\max\{\rho_1, \rho_2\} = 1$; we show that $\rho_1 x \vee \rho_2 y \in Li_{r\longrightarrow\infty} S_r$. Now, from $S_r = Co^r(S_r)$ we have $1/(\rho_1 \overset{r}{+} \rho_2)(\rho_1 \overset{r}{\cdot} x_r \overset{r}{+} \rho_2 \overset{r}{\cdot} y_r) = z_r \in S_r$. Moreover, from Lemma 2.0.1, $\lim_{r\longrightarrow\infty}(\rho_1 \overset{r}{+} \rho_2) = \max\{\rho_1, \rho_2\} = 1$, and, since $x, y \in \mathbb{R}^n_+$, we deduce that $\rho_1 x \vee \rho_2 y = \lim_{r\longrightarrow\infty}(\rho_1 \overset{r}{\cdot} x_r \overset{r}{+} \rho_2 \overset{r}{\cdot} y_r) \in Li_{r\longrightarrow\infty} S_r$ ■

LEMMA 2.2.12 *The closure of a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex.*

Proof Let L be a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$; if $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ with $x_n, y_n \in L$ for all n, then, for all $t \in [0, 1]$, $tx \vee y = \lim_{n\to\infty} tx_n \vee y_n$, and $tx_n \vee y_n \in L$ for all n. ■

THEOREM 2.2.13 *For all subsets S of $\mathbb{R}^n_+$, $Co^\infty(S)$ is the smallest closed $\mathbb{B}$-convex set containing S; it is the closure of $\mathbb{B}[S]$. Furthermore,*

$$Co^\infty(S) = Lim_{r\to\infty} Co^r(S).$$

Proof Let L be the intersection of all the closed $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ containing S. Moreover, from $S \subset L$ and from Theorem 2.2.10 we have $Co^\infty(S) \subset L$, and from Lemma 2.2.11 we have $L \subset Li_{r\longrightarrow\infty} Co^r(S)$; all together, we have $L = Lim_{r\to\infty} Co^r(S)$. The closure of a $\mathbb{B}$-convex set is convex and $S \subset \mathbb{B}[S] \subset L$, from the definitions of $\mathbb{B}[S]$ and L; from the minimality of L we have $L = \overline{\mathbb{B}[S]}$. ■

We have seen that Carathéodory's Theorem holds in $\mathbb{B}$-convexity, at least in $\mathbb{R}^n_+$; we close this section with $\mathbb{B}$-convex versions of two more classical results, Helly's Theorem and Radon's Theorem.

THEOREM 2.2.14 (Helly's Theorem) *If $\{L_\lambda: \lambda \in \Lambda\}$ is a family of closed $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ such that any $n+1$ members have a point in common then $\cap_{\lambda\in F} L_\lambda \neq \emptyset$ for all finite subset F of Λ; furthermore, if L_{λ_0} is compact for at least one $\lambda_0 \in \Lambda$ then $\cap_{\lambda\in\Lambda} L_\lambda \neq \emptyset$.*

Proof Let $F \subset \Lambda$ be a subset of cardinality at least $n+2$, otherwise there is nothing to prove; for each subset $A \subset F$ of cardinality $n+1$ choose a point $x_A \in \cap_{\lambda\in A} L_\lambda$ and let B be the $\mathbb{B}$-hull of all these points, it is compact. For each $\lambda \in F$ set $B_\lambda = \mathbb{B}[\{x_A: \lambda \in A\}]$, it is contained in B, compact and $\mathbb{B}$-convex. By construction, if $A \subset F$ is of cardinality $n+1$ then $x_A \in \cap_{\lambda\in A} B_\lambda$, and therefore $\cap_{\lambda\in A} \Phi_r(B_\lambda) \neq \emptyset$ for all $r \in \mathbb{N}$. Now, from the usual Helly's Theorem we have $\cap_{\lambda\in F} Co(\Phi_r(B_\lambda)) \neq \emptyset$; taking the inverse image by Φ_r gives $\cap_{\lambda\in F} Co^r(B_\lambda) \neq \emptyset$. Fix a cube $\prod_{i=0}^n [a_i, b_i]$ such that $B \subset \prod_{i=0}^n [a_i, b_i]$; we also have, for all r and all $\lambda \in F$, $B_\lambda \subset B$ and $Co^r(B_\lambda) \subset Co^r(B) \subset Co^r(\prod_{i=0}^n [a_i, b_i]) = \prod_{i=0}^n [a_i, b_i]$. Next, for each r choose a point $x_r \in \cap_{\lambda\in F} Co^r(B_\lambda)$; by compactness, there exists a subsequence $\{x_{r_k}\}_{k\in\mathbb{N}}$ which converges to a point $x^* \in \prod_{i=0}^n [a_i, b_i]$. But from $x_{r_k} \in Co^{r_k}(B_\lambda)$ for all $\lambda \in F$ we have $x^* \in Ls_{r\to\infty} Co^r(B_\lambda) = Co^\infty(B_\lambda)$ for all $\lambda \in F$. We have shown that $x^* \in \cap_{\lambda\in F} Co^\infty(B_\lambda)$. But B_λ is compact and $\mathbb{B}$-convex, therefore $B_\lambda = Co^\infty(B_\lambda)$; finally, we have, by construction, $B_\lambda \subset L_\lambda$ for all $\lambda \in F$, and therefore $\cap_{\lambda\in F} L_\lambda \neq \emptyset$. We have shown that the family $\{L_\lambda: \lambda \in \Lambda\}$ has the finite intersection property; if all the L_λ are closed and one of them is compact then $\cap_{\lambda\in\Lambda} L_\lambda \neq \emptyset$. ■

THEOREM 2.2.15 (Radon's Theorem) *If $S \subset \mathbb{R}^n_+$ is a finite set of cardinality at least $n+2$ then there is a partition $S = A_1 \cup A_2$, in nonempty subsets, such that $Co^\infty(A_1) \cap Co^\infty(A_2) \neq \emptyset$.*

Proof We can apply Radon's Theorem to each of the operators $A \mapsto Co^r(A)$; for each r there is a partition $(A_{r,1}, A_{r,2})$ of S such that $Co^r(A_{r,1}) \cap Co^r(A_{r,2}) \neq \emptyset$. Since the number of partitions of S is finite, there is a partition (A_1, A_2) and a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that $(A_{r_k,1}, A_{r_k,2}) = (A_1, A_2)$ for all $k \in \mathbb{N}$. For each k choose a point $x_{r_k} \in Co^{r_k}(A_{r_k,1}) \cap Co^{r_k}(A_{r_k,2}) = Co^{r_k}(A_1) \cap Co^{r_k}(A_2)$; take a cube B such that $S \subset B$, we then have $Co^{r_k}(A_1) \cup Co^{r_k}(A_2) \subset Co^{r_k}(S) \subset B$ for all k. By compactness, we can assume that $\{x_{r_k}\}_{k\in\mathbb{N}}$ converges to a point $x^\star$; we then have $x^\star \in Co^\infty(A_1) \cap Co^\infty(A_2)$. ■

2.3 On the Convergence Rate of $Co^r(A)$ to $Co^\infty(A)$

We denote by h_∞ the Hausdorff metric on the space of nonempty compact subsets of R^n_+ associated to the distance $(x, y) \mapsto \|x - y\|_\infty$.

PROPOSITION 2.3.1 Let L be a compact $\mathbb{B}$-convex set $\mathbb{R}^n_+$ and choose $\delta > 0$ such that $L \subset [0, \delta]^n$. Then,

$$h_\infty(Co^r(L), L) \leq \left((n+1)^{1/(2r+1)} - \frac{1}{(n+1)^{1/(2r+1)}}\right)\delta.$$

Proof Let u be an arbitrary, but fixed, point of $Co^r(L)$. Now, from Carathéodory's Theorem there exist $x_1, \ldots, x_{n+1} \in L$ and $\rho = (\rho_1, \ldots, \rho_{n+1}) \in [0, 1]^{n+1}$ such that $\rho_1^{2r+1} + \cdots + \rho_{n+1}^{2r+1} = 1$ and $u = \sum^{\varphi_r}_{j=1,\ldots,n+1} \rho_j x^j$. Let $\eta_j = (\rho_j/\mu)$ where $\mu = \max_{1 \leq l \leq n+1}\{\rho_l\}$, and set $x = \vee_{j=1}^{n+1} \eta_j x^j$; notice that $x \in L$ and $0 \leq \mu \leq 1$. Now, from

$$u_i = \left(\sum_{j=1}^{n+1} (\mu\eta_j)^{(2r+1)} x_{j,i}^{(2r+1)}\right)^{1/(2r+1)} \quad \text{and} \quad x_i = \max_{1 \leq j \leq n+1}\{\eta_j x_{j,i}\}$$

we have $\mu x_i \leq u_i \leq \mu(n+1)^{1/(2r+1)} x_i$ and therefore

$$\| u - \mu x \|_\infty \leq \mu((n+1)^{1/(2r+1)} - 1)\|x\|_\infty \leq ((n+1)^{1/(2r+1)} - 1)\|x\|_\infty.$$

But from $\rho_1^{(2r+1)} + \cdots + \rho_{n+1}^{(2r+1)} = 1$ we have $1 \leq \mu(n+1)^{1/(2r+1)}$ and from this

$$\| x - \mu x \|_\infty \leq (1 - \mu)\| x \|_\infty \leq \left(1 - \frac{1}{(n+1)^{1/(2r+1)}}\right)\| x \|_\infty.$$

Finally, from $\| x \|_\infty \leq \delta$ we obtain

$$\sup_{u \in Co^r(L)} d(u, L) \leq \left((n+1)^{1/(2r+1)} - \frac{1}{(n+1)^{1/(2r+1)}}\right)\delta$$

and the general conclusion follows from $L \subset Co^r(L)$ for all r. ■

Proposition 2.3.2 *If A is a finite subset $\mathbb{R}^n_+$ of cardinality $m > 0$ then*

$$h_\infty(Co^r(A), Co^\infty(A)) \le \left(m^{1/(2r+1)} - \frac{1}{m^{1/(2r+1)}}\right)\left\|\bigvee_{x\in A} x\right\|_\infty$$

Proof Let $A = \{x^1, \dots, x^m\}$ and $x = \vee_{j=1}^m \rho_j x^j \in Co^\infty(A)$ where $\max_{1\le j\le m}\{\rho_j\} = 1$ and $\rho = (\rho_1, \dots, \rho_m) \ge 0$. Let

$$\mu = \left(\sum_{j=1}^m \rho_j^{(2r+1)}\right)^{1/(2r+1)} \quad \text{and} \quad u_i = \frac{1}{\mu}\left(\sum_{j=1}^m \rho_j^{(2r+1)} x_{i,j}^{(2r+1)}\right)^{1/(2r+1)}$$

that is, in terms of indexed sums,

$$u = \frac{1}{\sum_{j=1,\dots,m}^{\varphi_r} \rho_j}\left(\sum_{j=1,\dots,m}^{\varphi_r} \rho_j x^j\right).$$

By construction, $u \in Co^r(A)$. Proceeding as in Proposition 2.3.1 and taking into account that

$$\sup_{x\in Co^\infty(A)} \|x\|_\infty \le \sup_{x\in A} \|x\|_\infty$$

we get

$$d(x, Co^r(A)) \le \left(m^{1/(2r+1)} - \frac{1}{m^{1/(2r+1)}}\right)\left\|\bigvee_{x\in A} x\right\|_\infty$$

and from here the conclusion is easily reached. ■

2.4 On the Topology of 𝔹-Convex Sets

This section presents some basic topological properties of 𝔹-convex sets. We recall that a topological space X is contractible if there exists a continuous map $h : X \times [0, 1] \to X$, such that $h(\cdot, 0)$ is a constant map and $h(\cdot, 1)$ is the identity map. A contractible space in path-connected, and in particular, connected.

Proposition 2.4.1 *An arbitrary 𝔹-convex subset of $\mathbb{R}^n$ is connected. A nonempty 𝔹-convex subset of $\mathbb{R}^n_+$ is contractible.*

Proof Let L be a 𝔹-convex subset of $\mathbb{R}^n$, and x_1, x_2 be two points of L. Since $\mathbb{B}[\{x_1, x_2\}] \subset L$ we have to prove that $\mathbb{B}[\{x_1, x_2\}]$ is connected; we prove that the 𝔹-hull of a compact set in connected. Let K be a nonempty compact subset of $\mathbb{R}^n$; for all r, the set $Co^r(K)$ is compact and connected, since it is homeomorphic to the usual convex hull of the compact set $\Phi_r(K)$. The lower limit $Li_{r\to\infty} Co^r(K)$ is not empty, since K is contained in all the $Co^r(K)$. Since K is compact, there is a cube $\prod_{i=1}^n [a_i, b_i]$ such that $K \subset \prod_{i=1}^n [a_i, b_i]$, and therefore $Co^r(K) \subset \prod_{i=1}^n [a_i, b_i]$ for all r.

Finally, recall that a sequence of subcontinua – that is compact and connected spaces – of a compact metric space whose lower limit is not empty has an upper limit which is a continuum, Kuratowski [6], Chapter V. Since, by definition, $\mathbb{B}[K] = Ls_{r\to\infty}Co^r(K)$, we conclude that $\mathbb{B}[K]$ is a continuum. For $(x_1, x_2, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times [0,1]$, let $H(x_1, x_2, t) = \psi_1(t)x_1 \vee \psi_2(t)x_2$ where

$$\psi_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \dfrac{1}{2} \\ 2-2t & \text{if } \dfrac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$\psi_2(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \dfrac{1}{2} \\ 1 & \text{if } \dfrac{1}{2} \leq t \leq 1 \end{cases}$$

We have, $\forall x_1, x_2 \in \mathbb{R}^n_+$ and for all $t \in [0,1]$, $\psi_1(t)x_1 \vee \psi_2(t)x_2 \in Co^\infty(\{x_1, x_2\})$. If L is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ then $Co^\infty(\{x_1, x_2\}) \subset L$ for all $x_1, x_2 \in L$, and consequently, $H(L \times L \times [0,1]) \subset L$. To see that L is contractible, just fix an arbitrary point x_0 of L and let $h(x,t) = H(x_0, x, t)$. ■

The proof shows that a $\mathbb{B}$-convex set is much more than only contractible. Indeed H is continuous in all three variables, $H(x_0, x_1, 0) = x_0$, $H(x_1, x_0, 1) = x_1$, $H(x_0, x_1, t) = H(x_1, x_0, 1-t)$, $H(x, x, t) = x$ and $L \subset \mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if $H(L \times L \times [0,1]) \subset L$.

We have already seen that the closure of a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex; this follows also from the next proposition which will be of some importance in future investigations. Let us denote by $U_\infty(S;\delta)$ the δ neighborhood of a set $S \subset \mathbb{R}^n_+$ with respect to the norm, that is, $x \in U_\infty(S;\delta)$ if there exists $x' \in S$ such that $\|x - x'\|_\infty < \delta$.

Proposition 2.4.2 *If L is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ then $U_\infty(L;\delta)$ is also $\mathbb{B}$-convex.*

Proof Let $y, y' \in U_\infty(L;\delta)$ and $x, x' \in L$ such that $\|y - x\|_\infty < \delta$ and $\|y' - x'\|_\infty < \delta$. Fix $t \in [0,1]$; we have to see that $ty \vee y' \in U_\infty(L;\delta)$.

We show that $\|ty \vee y' - tx \vee x'\|_\infty < \delta$; in other words, we have to see that $|\max\{ty_i, y'_i\} - \max\{tx_i, x'_i\}| < \delta$ for all i. There are four possible ways to remove both max, two of which give us trivially the conclusion we want, the other two are symmetrical; there is only one case to check:

$$\max\{ty_i, y'_i\} = ty_i \quad \text{and} \quad \max\{tx_i, x'_i\} = x'_i.$$

We can assume $y'_i < ty_i$ and $tx_i < x'_i$, otherwise we are back at one of the easy cases, and we also set $t > 0$. But from $tx_i - \delta < x'_i - \delta < y'_i \leq ty_i$ we obtain $ty_i - x'_i + \delta < ty_i - tx_i + \delta$ and therefore, $ty_i - x'_i < t|y_i - x_i| < t\delta \leq \delta$. Now, from $ty_i < tx_i + t\delta$ we have $ty_i - (t+1)\delta < tx_i - \delta < x'_i - \delta$, and, from this, $(x'_i - \delta) - (ty_i - (t+1)\delta) < (tx_i - \delta) - (ty_i - (t+1)\delta)$ or, $x'_i - ty_i < tx_i - ty_i \leq t|x_i - y_i| < \delta$.

We have shown that $|ty_i - x'_i| < \delta$. ■

If the interior of a closed (linear) convex set of $\mathbb{R}^n$ is not empty then the closure of the interior is the convex set itself; such a property cannot hold in $\mathbb{B}$-convexity, since a closed $\mathbb{B}$-convex set can have "spikes" as one can see from the previous examples. Nonetheless, something can be saved:

Proposition 2.4.3 *The interior of a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex.*

Proof Let L be a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ with nonempty interior; if $y_1, \dots, y_m$ are in $\operatorname{int}(L)$ there are open sets $W_1, \dots, W_m$ of $\mathbb{R}^n_+$ such that $y_i \in W_i \subset L$. Fix $(\rho_1, \dots, \rho_m) \in [0,1]^m$ such that $\max\{\rho_1, \dots, \rho_m\} = 1$ and let

$$\bigvee_{i=1}^{m} \rho_i W_i = \left\{ \bigvee_{i=1}^{m} \rho_i z_i \colon z_i \in W_i \right\}$$

We have $\vee_{i=1}^m \rho_i y_i \in \vee_{i=1}^m \rho_i W_i \subset L$; we show that $\vee_{i=1}^m \rho_i W_i$ is open by induction on m.

If $m = 1$ there is nothing to prove. Let $m = k + 1$; without loss of generality, we can assume that $\rho_1 = 1$. Then $\vee_{i=1}^k \rho_i W_i$ is open, by the induction hypothesis, call it U. If $\rho_{k+1} = 0$ then $\vee_{i=1}^{k+1} \rho_i W_i = U$; if $\rho_{k+1} \neq 0$ then $\rho_{k+1} W_{k+1}$ is open in $\mathbb{R}^n_+$, since $x \mapsto \rho_{k+1} x$ is a homeomorphism of $\mathbb{R}^n_+$ onto itself, let $\rho_{k+1} W_{k+1} = W$. We have reduced the general proof to the proof of the following statement: if U and W are open subsets of $\mathbb{R}^n_+$ then $U \vee W$ is also open in $\mathbb{R}^n_+$. Let us show that this is the case. Let $y = x \vee z$ with $x \in U$ and $z \in W$; there exists $\delta > 0$ such that $U_\infty(x, \delta) \subset U$ and $U_\infty(z, \delta) \subset W$; we have to find $\eta > 0$ such that $U_\infty(y, \eta) \subset U \vee W$. For all $i = 1, \dots, n$ we have $y_i = \max\{x_i, z_i\}$; we distinguish two cases (three by symmetry):

(1) $y_i = x_i = z_i$. If $|y_i' - y_i| < \delta$ we can find x_i' and z_i' such that $|x_i' - x_i| < \delta$, $|z_i' - z_i| < \delta$ and $y_i' = \max\{x_i', z_i'\}$; simply take $y_i' = x_i' = z_i'$.
(2) $y_i = x_i > z_i$. If $|y_i' - y_i| < \min\{\delta, 2^{-1}(x_i - z_i)\}$ then, with $x_i' = y_i'$ and $z_i' = z_i$ we have $y_i' = \max\{x_i', z_i'\}$, $|x_i' - x_i| < \delta$ and $|z_i' - z_i| < \delta$.

Put $J(x) = \{i\colon x_i > z_i\}$, $J(z) = \{i\colon z_i > x_i\}$ and $\eta = \min\{\delta, 2^{-1}(x_i - z_i), 2^{-1}(z_j - x_j)\colon (i,j) \in J(x) \times J(z)\}$; we have shown that $U_\infty(y, \eta) \subset U_\infty(x, \delta) \vee U_\infty(z, \delta)$. ■

3 𝔹-CONVEX MAPS

Generalized convexities, that is convexities which are not associated to a linear structure, are defined by their basic objects, "convex sets", the same way topologies are defined by their basic objects, closed, or open, sets, and measurable spaces are defined by measurable sets; rarely are convexities defined by combinatorial structures – like taking convex sums of points. Accordingly, convex maps cannot, generally, be defined by simple algebraic properties; they have to be defined with respect to the basic "geometric objects", the convex sets of the structure. This short discussion justifies the following definition: a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{B}$-measurable if for all $\mathbb{B}$-convex set C of $\mathbb{R}^m$ the inverse image $f^{-1}(C)$ is a $\mathbb{B}$-convex set in $\mathbb{R}^n$.[3] We will restrict our attention to $\mathbb{R}^n_+$, where, as we have seen, $\mathbb{B}$-convexity is characterize by algebraic properties.

[3]We have taken some liberty here with the accepted standard terminology; such maps are usually called *convexity preserving maps*.

Before proceeding further with $\mathbb{B}$-convexity let us see what this definition gives when applied to the standard linear convexity and maps from $\mathbb{R}^n$ to $\mathbb{R}$. Convex sets of $\mathbb{R}$ are intervals, therefore a map $f : \mathbb{R}^n \to \mathbb{R}$ will be measurable with respect to the usual linear convexity if, for all $t \in \mathbb{R}$, $\{x: f(x) \leq t\}$ and $\{x: f(x) \geq t\}$ are convex; in other words, f is simultaneously quasiconvex and quasiconcave.

Proposition 3.0.1 *For a map $f : \mathbb{R}^n \to \mathbb{R}$ the following properties are equivalent*:

(1) *f is $\mathbb{B}$-measurable*;
(2) *for all $t \in \mathbb{R}$ the sets $\{x: f(x) \leq t\}$ and $\{x: f(x) \geq t\}$ are $\mathbb{B}$-convex*;
(3) *for all interval $I \subset \mathbb{R}$, the inverse image $f^{-1}(I)$ is $\mathbb{B}$-convex.*

Proof Unions and intersections of sets are preserved by inverse images, intersections of $\mathbb{B}$-convex sets are $\mathbb{B}$-convex and unions of increasing sequences of $\mathbb{B}$-convex sets are $\mathbb{B}$-convex. ∎

A more appropriate characterization is given by the the following proposition.

Proposition 3.0.2 *A map $f : \mathbb{R}^n_+ \to \mathbb{R}$ is $\mathbb{B}$-measurable if and only if, for all $x_1, x_2 \in \mathbb{R}^n_+$ and for all $t \in [0, 1]$,*

$$(\star) \qquad \min\{f(x_1), f(x_2)\} \leq f(tx_1 \vee x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Proof First, assume that $(\star)$ holds; we have to show the $\mathbb{B}$-convexity of $\{x \in \mathbb{R}^n_+: f(x) \leq t\}$ and $\{x \in \mathbb{R}^n_+: f(x) \geq t\}$ for all t. Since we are in $\mathbb{R}^n_+$, a subset is $\mathbb{B}$-convex if and only if it contains $tx_1 \vee x_2$, whenever it contains x_1 and x_2 and $0 \leq t \leq 1$; it is now clear that $\{x \in \mathbb{R}^n_+: f(x) \leq t\}$ and $\{x \in \mathbb{R}^n_+: f(x) \geq t\}$ are $\mathbb{B}$-convex for all t. If f is $\mathbb{B}$-measurable then the sets $\{x \in \mathbb{R}^n_+: f(x) \leq \max\{f(x_1), f(x_2)\}\}$ and $\{x \in \mathbb{R}^n_+: f(x) \geq \min\{f(x_1), f(x_2)\}\}$ are $\mathbb{B}$-convex; property $(\star)$ follows easily from this. ∎

Given a $\mathbb{B}$-convex subset L of $\mathbb{R}^n_+$ let us say that a map $f : L \to \overline{\mathbb{R}}$ is $\mathbb{B}$-quasiconvex if

$$\forall\, x_1, x_2 \in L \;\; \sup\{f(x): x \in Co^\infty(\{x_1, x_2\})\} \leq \max\{f(x_1), f(x_2)\}.$$

Using the characterization of $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ we see that $f : L \to \overline{\mathbb{R}}$ is $\mathbb{B}$-quasiconvex if, for all $t \in \mathbb{R}$ the set $\{x \in L: f(x) \leq t\}$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$. $\mathbb{B}$-quasiconcave maps are similarly defined. $\mathbb{B}$ measurable maps form a very large class which does not take into account the specific algebraic description of $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$; Proposition 3.0.3 singles out a subclass which is to $\mathbb{B}$ convexity on $\mathbb{R}^n_+$ what the class of affine maps is to linear convexity.

Proposition 3.0.3

(A) *For a map $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ to be $\mathbb{B}$-measurable it is sufficient that* (1) *and* (2) *below hold*:

   (1) $\forall x, y \in \mathbb{R}^n_+, f(x \vee y) = \max\{f(x), f(y)\}$ *and*
   (2) $\forall x \in \mathbb{R}^n_+, \forall t \in [0, 1], f(tx) = max\{tf(x), f(0)\}$;
   furthermore, if $f(0) = 0$ then (1) *and* (2) *imply*

   (2)′ $\forall x \in \mathbb{R}^n_+, \forall t \in \mathbb{R}_+, f(tx) = tf(x)$.

(B) (1) *and* (2) *hold if and only if $f(tx_1 \vee x_2) = \max\{tf(x_1), f(x_2)\}$ for all x_1, x_2 and $t \in [0, 1]$.*

(C) *In particular, the canonical projections* $(x_1,\dots,x_n)\mapsto x_j$, $j=1,\dots,n$, *are* $\mathbb{B}$*-measurable on* $\mathbb{R}^n_+$, *and more generally, given* $(a_1,\dots,a_n)\in\mathbb{R}^n_+$ *the map* $(x_1,\dots,x_n)\mapsto \max_{1\le i\le n}\{a_ix_i\}$ *is* $\mathbb{B}$*-measurable. All the maps* $f:\mathbb{R}^n_+\to\mathbb{R}_+$ *for which* (1) *and* (2) *hold are of the form* $x=(x_1,\dots,x_n)\mapsto \max_{1\le i\le n}\{a_ix_i,a_0\}$ *where* $(a_0,a_1,\dots,a_n)\in\mathbb{R}^{n+1}_+$.

Proof The equivalence between (A) and (B) is easy to prove. If (B) holds then $\{x\in\mathbb{R}^n_+ : f(x)\le\lambda\}$ and $\{x\in\mathbb{R}^n_+ : f(x)\ge\lambda\}$ are $\mathbb{B}$-convex for all $\lambda\in\mathbb{R}^n_+$. First, for all $x\in\mathbb{R}^n_+$ we have $x=x\vee 0$, and therefore $f(x)=\max\{f(x),\ f(0)\}$; in other words, $f(x)\ge f(0)$. If $f(0)=0$ then $f(tx)=tf(x)$ for all $t\in[0,1]$; if $t>1$ then $f(x)=f((1/t)(tx))=(1/t)f(tx)$. Finally, in $\mathbb{R}^n_+$ we have $(x_1,\dots,x_n)=\vee x_iE_i$ where $E_1,\dots,E_n$ are the vectors of the canonical basis of $\mathbb{R}^n$; from (1) and (2) we get $f(x_1,\dots,x_n)=\max\{f(E_i)x_i,\ f(0)\}$. ■

Proposition 3.0.3 suggests other natural classes of maps: those maps $\mathbb{R}^n_+\to\mathbb{R}_+$ such that

(1.1) $f(tx_1\vee x_2)\le\max\{tf(x_1),f(x_2)\}$ for all $x_1,\ x_2\in\mathbb{R}^n_+$ and for all $t\in[0,1]$. For these maps the sublevel sets $\{x\in\mathbb{R}^n_+ : f(x)\le\lambda\}$ are $\mathbb{B}$-convex.

If we impose

(1.2) $\max\{tf(x_1),f(x_2)\}\le f(tx_1\vee x_2)$ for all $x_1,\ x_2\in\mathbb{R}^n_+$ and for all $t\in[0,1]$ then all the upper level sets $\{x\in\mathbb{R}^n_+ : \lambda\le f(x)\}$ are $\mathbb{B}$-convex.

Recall that a map $f:\mathbb{R}^n_+\to\mathbb{R}_+$ is ICR in the sense of Rubinov [9, page 77] if

(a) $tf(x)\le f(tx)$ for all $t\in[0,1]$ and
(b) $f(x_1)\le f(x_2)$ if $x_1\le x_2$.

Let us see that (1.2) holds if and only if $f:\mathbb{R}^n_+\to\mathbb{R}_+$ is ICR.

(i) If (1.2) holds and $x_1\le x_2$ then $f(x_1)\le\max\{f(x_1),f(x_2)\}\le f(x_1\vee x_2)=f(x_2)$; this shows that (b) holds. Furthermore, $tf(x)\le\max\{tf(x),f(tx)\}\le f(tx\vee tx)=f(tx)$; (a) holds.
(ii) Assume that $f:\mathbb{R}^n_+\to\mathbb{R}_+$ is ICR. We have $tf(x_1)\le f(tx_1)\le f(tx_1\vee x_2)$ and $f(x_2)\le f(tx_1\vee x_2)$, and therefore (1.2) holds.

The epigraph of a map $f:\mathbb{R}^n_+\to\mathbb{R}_+$ is a subset of $\mathbb{R}^{n+1}_+$; if $(x,s),(x',s')\in\operatorname{epi}(f)$ and $t\in[0,1]$then $t(x,s)\vee(x',s')$ belongs to epi (f) if and only if $f(tx\vee x')\le\max\{ts,s'\}$; this shows that the maps for which (1.1) holds are exactly the maps with $\mathbb{B}$-convex epigraph. Similarly, the maps for which (1.2) holds are exactly the maps with $\mathbb{B}$-convex hypograph.

In the last section we briefly outline some applications to "convex analysis without convexity" and to "linear programming without linearity".

4 SOME APPLICATIONS

4.1 Duality

An interval space is a pair $(Y,[\![\cdot,\cdot]\!])$ where Y is a topological space and $[\![\cdot,\cdot]\!]$ assigns to each pair $(x_0,x_1)\subset Y\times Y$ a connected subset $[\![\{x_0,x_1\}]\!]$ of Y which contains $\{x_0,x_1\}$.

There are some obvious examples of interval spaces: (1) a vector space with the finite topology – a set is closed if its intersection with all finite dimensional affine subspace is closed in that subspace – and $[\![\{x_0, x_1\}]\!]$ is the convex hull of $\{x_0, x_1\}$; a topological space Y endowed with a continuous map $H : Y \times Y \times [0,1] \to Y$ such that, for all $x_0, x_1 \in Y$, $H(x_0, x_1, 0) = H(x_1, x_0, 1) = x_0$ and $[\![\{x_0, x_1\}]\!] = H(\{x_0\} \times \{x_1\} \times [0,1])$; $\mathbb{R}^n$ with $[\![\{x_0, x_1\}]\!] = Co^{\infty}(\{x_0, x_1\})$. Interval spaces were introduced by Stachó in [10] and used in the context of minimax theorems by Kindler and Trost in [5].

Theorem 4.1.1, which we present without an explicit proof, since it follows from more general considerations, indicates that we can expect a nice duality theory within the framework of $\mathbb{B}$-convexity. A map $f : X \times Y \to \overline{\mathbb{R}}$, where Y is a topological space, is inf-compact on Y if, for all $t \in \mathbb{R}$ there exists $x_0 \in X$ such that $\{y \in Y : f(x_0, y) \le t\}$ is compact.

THEOREM 4.1.1 *Let X and Y be $\mathbb{B}$-convex subsets of, respectively, $\mathbb{R}^m$ and $\mathbb{R}^n$, and $\mathcal{L} : X \times Y \to \overline{\mathbb{R}}$ a map which is inf-compact on Y. Assume that the following conditions hold*:

(1) $\forall x \in X \; y \mapsto \mathcal{L}(x, y)$ *is upper-semicontinuous and $\mathbb{B}$-quasiconcave on Y*;
(2) $\forall y \in Y \; x \mapsto \mathcal{L}(x, y)$ *is lower-semicontinuous and $\mathbb{B}$-quasiconvex on X*; *then* $\sup_y \inf_x \mathcal{L}(x, y) = \inf_x \sup_y \mathcal{L}(x, y)$

Theorem 4.1.1, which is a $\mathbb{B}$-convex version of the Sion–von Neumann Theorem, is a straightforward consequence of Theorem 4 in [4] – which is itself a topological version of Passy–Prisman's Theorem [8] – from which one can deduce as an easy corollary that $\sup_y \inf_x \mathcal{L}(x, y) = \inf_x \sup_y \mathcal{L}(x, y)$ holds if $\mathcal{L}$ is inf-compact on Y and, X and Y are interval spaces such (1) and (2) hold, where, in the context of interval spaces, quasi-concavity with respect to y means

$$\min\{\mathcal{L}(x, y_0), \mathcal{L}(x, y_1)\} \le \inf_{y \in [\![\{y_0, y_1\}]\!]} \mathcal{L}(x, y)$$

for all $x \in X$, and similarly for quasiconvexity on X.

As a simple application, we derive a saddle point theorem for "*mixed strategies without expectations*". Since von Neumann's 1928 Minimax Theorem one knows that a map $F : S_1 \times S_2 \to \mathbb{R}$, defined on the product of two finite sets, S_1 and S_2 of cardinalities $m_1 + 1$ and $m_2 + 1$, of "*pure strategies*", has an equilibrium point in "*mixed strategies*", that is a pair $(\overline{x}, \overline{y}) \in \Delta_{m_1} \times \Delta_{m_2}$ such that, for all $(x, y) \in \Delta_{m_1} \times \Delta_{m_2}$[4]

$$\sum_{(s_1, s_2) \in S_1 \times S_2} \overline{x}_{s_1} F(s_1, s_2) y_{s_2} \le \sum_{(s_1, s_2) \in S_1 \times S_2} \overline{x}_{s_1} F(s_1, s_2) \overline{y}_{s_2} \le \sum_{(s_1, s_2) \in S_1 \times S_2} x_{s_1} F(s_1, s_2) \overline{y}_{s_2}$$

and these mixed strategies have a well-known and canonical interpretation in terms of optimal expected values.

Using $\mathbb{B}$-convexity we propose a different version of the idea of "*mixed strategies*" which does not involve expected values. Let $\mathcal{B}_k = \{(t_0, \ldots, t_k) \in [0,1]^{k+1} : \max_{0 \le i \le k}\{t_i\} = 1\}$, it is a compact $\mathbb{B}$-convex subset of $\mathbb{R}_+^{k+1}$; given two sets of "*pure strategies*"

[4] Δ_{m_i} is the standard simplex of dimension m_i, interpreted as the set of probability distributions on S_i.

$S_1 = \{a_0, \ldots, a_m\}$, $S_2 = \{b_0, \ldots, b_n\}$ and a "*pay-off*" function $F : S_1 \times S_2 \to \mathbb{R}_+$ we define $F^\beta : \mathcal{B}_m \times \mathcal{B}_n \to \mathbb{R}_+$ by

$$F^\beta((t_0, \ldots, t_m), (s_0, \ldots, s_n)) = \max\{t_i F(a_i, b_j) s_j \colon 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

Theorem 4.1.2 *$F^\beta : \mathcal{B}_m \times \mathcal{B}_n \to \mathbb{R}_+$ has a saddle point.*

Proof To $t = (t_0, \ldots, t_m) \in \mathcal{B}_m$ we associate

$$x(t) = \left(\max_i \{ t_i F(a_i, b_0) \}, \ldots, \max_i \{ t_i F(a_i, b_n) \} \right) \in \mathbb{R}_+^{n+1}.$$

We have seen that $s = (s_0, \ldots, s_n) \mapsto \max_j\{x(t)_j s_j\}$ is 𝔹-measurable. Taking into account that the numbers involved are positive we have $\max_i\{t_i F(a_i, b_j)\} s_j = \max_i\{t_i F(a_i, b_j) s_j\}$ and therefore

$$\max_j\{x(t)_j s_j\} = F^\beta(t, s).$$

This shows that $s \mapsto F^\beta(t, s)$ is 𝔹-measurable on $\mathcal{B}_n$ for all $t \in \mathcal{B}_m$; similarly, $t \mapsto F^\beta(t, s)$ is 𝔹-measurable on $\mathcal{B}_m$ for all $s \in \mathcal{B}_n$. Now, from Theorem 4.1.1, and from the continuity of $(t, s) \mapsto F^\beta(t, s)$ and the compactness of the sets $\mathcal{B}_k$ we conclude that there is $(\bar{t}, \bar{s}) \in \mathcal{B}_m \times \mathcal{B}_n$ such that

$$\min_{t \in \mathcal{B}_m} \max_{s \in \mathcal{B}_n} F^\beta(t, s) = F^\beta(\bar{t}, \bar{s}) = \max_{s \in \mathcal{B}_n} \min_{t \in \mathcal{B}_m} F^\beta(t, s). \qquad \blacksquare$$

4.2 Max-programming

Lemma 4.2.1 If $L \subset \mathbb{R}_+^n$ is 𝔹-*convex and if* $f : L \mapsto \mathbb{R}_+$ is 𝔹-*quasiconvex then*

(1) *for all* $(x_1, \ldots, x_m) \in L^m$ *and all* $(t_1, \ldots, t_m) \in [0, 1]^m$ *such that* $\max\{t_1, \ldots, t_m\} = 1$ *we have*

$$f(t_1 x_1 \vee \cdots \vee t_m x_m) \leq \max\{f(x_1), \ldots, f(x_m)\}$$

(2) *for all nonempty finite subset* $A \subset L$,

$$\max\{f(x) \colon x \in Co^\infty(A)\} = \max\{f(x) \colon x \in A\}.$$

Proof Since $\{x \in L \colon f(x) \leq t\}$ is 𝔹-convex for all $t \in \mathbb{R}$, one obtains (1) by taking $t = \max\{f(x_1), \ldots, f(x_m)\}$. (2) follows from (1) and the characterization of $Co^\infty(A)$ for a finite subset of $\mathbb{R}_+^n$. $\blacksquare$

There is an obvious version of Lemma 4.2.1 for 𝔹-quasiconcave maps.

Now, from Lemma 4.2.1, and taking into account that for $a \in \mathbb{R}_+^n$ the map – defined on $\mathbb{R}_+^n$ – $x \mapsto \langle a, x\rangle^\infty = \max_{1 \leq i \leq n}\{a_i x_i\}$ is 𝔹-measurable, we have

Proposition 4.2.2 *If A is a finite nonempty subset of $\mathbb{R}_+^n$ and if $a \in \mathbb{R}_+^n$ then*

$$\max\{\langle a, x\rangle^\infty \colon x \in Co^\infty(A)\} = \max\{\langle a, x\rangle^\infty \colon x \in A\}$$

and

$$\min\{\langle a, x\rangle^\infty\colon x \in Co^\infty(A)\} = \min\{\langle a, x^j\rangle^\infty\colon x \in A\}.$$

Consider a finite nonempty subset A of $\mathbb{R}^n_+$ and let us say that $x^\star \in A$ is $\mathbb{B}$-redundant with respect to A if $x^\star \in Co^\infty(A\backslash\{x^\star\})$. We give a simple procedure to find all the redundant points of A.

The concept of redundancy is obviously linked to the idea of extreme point. Since we do not want to go here into the full details of extreme points and Krein–Millman like theorems in the context in $\mathbb{B}$-convexity, we will do for now with that notion of redundancy. Notice that if $x^\star \in A$ is redundant and $a \in \mathbb{R}^n_+$ then, by Proposition 4.2.2,

$$\langle a, x^\star\rangle^\infty \leq \max\{\langle a, x\rangle^\infty\colon x \in A\backslash\{x^\star\}\}$$

and therefore

$$\max\{\langle a, x\rangle^\infty\colon x \in Co^\infty(A)\} = \max\{\langle a, x\rangle^\infty\colon x \in A\backslash\{x^\star\}\}$$

If one wants to find the optimal value of $x \mapsto \langle a, x\rangle^\infty$ on a $\mathbb{B}$-polytope then, finding the redundant points reduces the amount of calculations.

First, we consider systems of max-equations, that is, systems of the form

$$(\text{MaxEq.}) \qquad \begin{cases} \max\{a_{1,1}x_1, \dots, a_{1,n}x_n\} = b_1 \\ \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \max\{a_{m,1}x_1, \dots, a_{m,n}x_n\} = b_m \end{cases}$$

where $a_i = (a_{i,1}, \dots, a_{i,n}) \in \mathbb{R}^n_+, i = 1, \dots, m,\ b = (b_1, \dots, b_m) \in \mathbb{R}^m_+$ and the solution $(x_1, \dots, x_n)$ is to be found in $\mathbb{R}^n_+$. Notice that if $b_i = 0$ then we have to take $x_j = 0$ for each j such that $a_{i,j} > 0$, and, as far as equation i is concerned, the other values x_l are irrelevant; equation i can therefore be removed from the system and the number of variables decreases. In other words, we can assume that $b_i > 0$ for all i. The system (MaxEq.) can also be written

$$\begin{cases} \langle a_1, x\rangle^\infty = b_1 \\ \quad \vdots \qquad\quad \vdots \\ \langle a_m, x\rangle^\infty = b_m \end{cases}$$

Its solution set, that is

$$S(a_1, \dots, a_m; b) = \left\{x \in \mathbb{R}^n_+\colon \bigvee_{j=1}^n x_j a^j = b\right\} = S^\star(a^1, \dots, a^n; b)$$

where $a^j = (a_{1,j}, \dots, a_{m,j})$, is $\mathbb{B}$-convex.

We can assume that for all j there is at least one index i such that $a_{i,j} > 0$; let $\sigma(j) = \{i : a_{i,j} > 0\}$ and

$$u_j = \min_{i \in \sigma(j)} \left\{\frac{b_i}{a_{i,j}}\right\}$$

Proposition 4.2.3 *If $S(a_1, \ldots, a_m; b) \neq \emptyset$ then $u = (u_1, \ldots, u_n) \in S(a_1, \ldots, a_m; b)$ and, for all $x \in S(a_1, \ldots, a_m; b)$, $x \leq u$.*

Proof If $(x_1, \ldots, x_n) \in S(a_1, \ldots, a_m; b)$ then $a_{i,j}x_j \leq b_i$ for all i and j; therefore $x_j \leq (b_i/a_{i,j})$ if $i \in \sigma(j)$, this shows that $x \leq u$. If $i \notin \sigma(j)$ then, trivially, $a_{i,j}x_j \leq a_{i,j}u_j \leq b_i$; if $i \in \sigma(j)$ we get the same inequality from $x_j \leq u_j$ and the definition of u_j. Since these inequalities hold for all i and j, we have, for all $i = 1, \ldots, m$,

$$b_i = \max\{a_{i,j}x_j : 1 \leq j \leq n\} \leq \max\{a_{i,j}u_j : 1 \leq j \leq n\} \leq b_i.$$

This shows that $u \in S(a_1, \ldots, a_m, b)$. ■

Proposition 4.2.3 gives a very simple decision procedure for the nonemptyness of $S(a_1, \ldots, a_m; b)$; it also shows that $S(a_1, \ldots, a_m; b)$ is compact, indeed, it is defined by a set of continuous maps, and therefore closed, and, if it not empty, it is a subset of $\mathbb{R}^n_+$ with a largest element, and consequently bounded.

We illustrate the use of Proposition 4.2.3 on a simple numerical example. Let us consider the following system:

$$(S) \qquad \begin{cases} \max\{2x_1, 3x_2\} = 1 \\ \max\{4x_1, x_2\} = 2 \end{cases}$$

We have $a_1 = (2, 3)$, $a_2 = (4, 1)$, $b_1 = 1$ and $b_2 = 2$, from which we get $u_1 = \min\{1/2, 2/4\} = 1/2$ and $u_2 = \min\{1/3, 1/1\} = 1/3$. One can check that $(1/2, 1/3)$ is a solution of (S).

This example is depicted in Fig. 3. For a vector $v = (v_1, \ldots, v_n)$ with strictly positive coordinates, we denote by $\Phi_{-1}(v)$ the vector $(v_1^{-1}, \ldots, v_n^{-1})$; given vectors x and y, $x \wedge y$

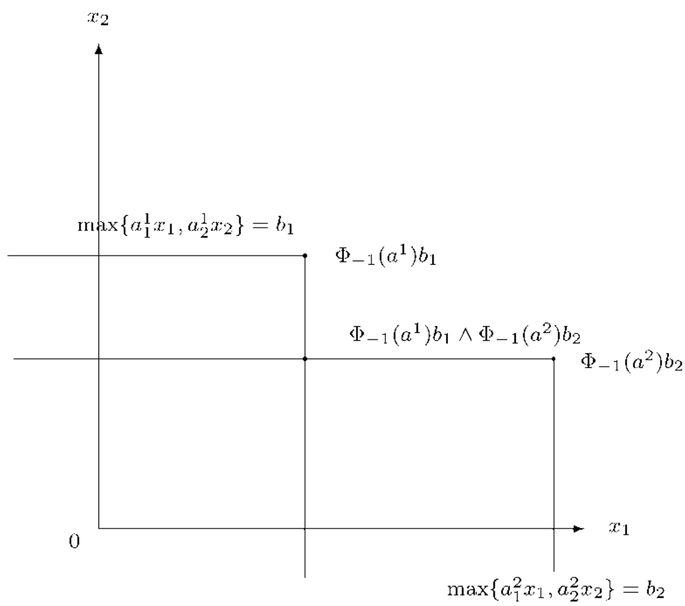

FIGURE 3 System of maximum equations.

is the infimum of x and y, that is $(x \wedge y)_i = \min\{x_i, y_i\}$. With this notation, we have $u = b_1\Phi_{-1}(a_1) \wedge b_2\Phi_{-1}(a_2)$, or, more generally, for vectors with strictly positive coordinates,

$$u = b_1\Phi_{-1}(a_1) \wedge b_2\Phi_{-1}(a_2) \wedge \cdots \wedge b_m\Phi_{-1}(a_m)$$

Next, from Proposition 4.2.3 we derive a simple procedure to find redundant points. Let $A = \{a_1, \ldots, a_m\} \subset \mathbb{R}^n_+$ and let us say that we want to decide on the redundancy of a_1. Let $a_i^+ = (a_i, 1) \in \mathbb{R}^n_+$; then a_1 is redundant if and only if $a_1 \in Co^\infty(A \setminus \{a_1\})$, that is, if and only if the system $a_1^+ = \vee_{i=2}^m x_i a_i^+$ has a solution in $\mathbb{R}^{m-1}_+$. In other words, a_1 is redundant if and only if $S^\star(a_2^+, \ldots, a_m^+; a_1^+) \neq \emptyset$, and that can easily be checked by the procedure of Proposition 4.2.3.

Example (See Fig. 4) Let $A = \{(1,4), (4,1), (2,2), (3,3)\}$. $(1,4)$ is redundant if and only if the following system has a solution.

$$(S^1) \qquad \begin{cases} \max\{4\rho_2, 2\rho_3, 3\rho_4\} = 1 \\ \max\{1\rho_2, 2\rho_3, 3\rho_4\} = 4 \\ \max\{\rho_2, \rho_3, \rho_4\} = 1 \end{cases}$$

We find $u_1 = \min\{1/4, 4/1, 1\}$, $u_2 = \min\{1/2, 4/2, 1\}$, $u_3 = \min\{1/3, 4/3, 1\}$, and therefore $u = (1/4, 1/2, 1)$. Since $\max\{u_1, u_2, u_3\} \neq 1$ we conclude that (S^1) has no solution, and consequently, that $(1,4)$ is not redundant. Similarly, one can check that $(4,1)$ and $(2,2)$ are not redundant and that $(3,3)$ is redundant.

The procedure described in Proposition 4.2.3 can also be used to ascribe to points of a $\mathbb{B}$-polytope $Co^\infty(A)$ a canonical set of coordinates. Let $A = \{a_1, \ldots, a_k\} \in \mathbb{R}^n_+$; if $b \in Co^\infty(A)$ then there is a set of scalars $(t_1, \ldots, t_k) \in [0,1]^k$ such that $\max\{t_1, \ldots, t_k\} = 1$ and $b = \vee_{j=1}^k t_j a_j$, in other words $(t_1, \ldots, t_k) \in S^\star(a_1^+, \ldots, a_k^+; b^+\}$;

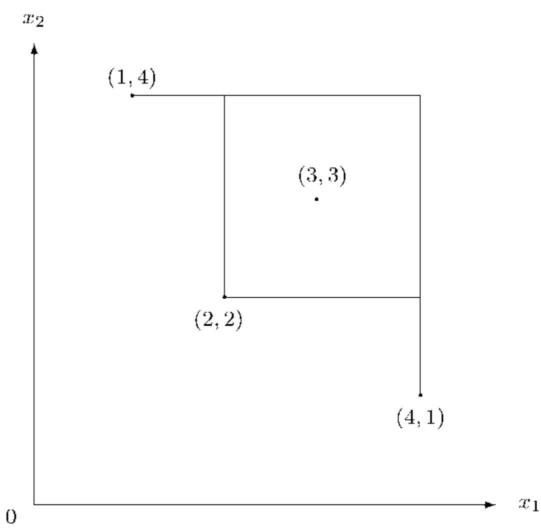

FIGURE 4 Finding the redundant points.

from Proposition 4.2.3 we know that $S^{\star}(a_1^+, \ldots, a_k^+; b^+\}$ has a maximal element $u = (u_1, \ldots, u_k)$. We then have $b = \vee_{j=1}^{k} u_j a_j$ and $\max\{u_1, \ldots, u_k\} = 1$.

5 CONCLUSION

We have given the basic definitions and structural properties pertaining to $\mathbb{B}$-convexity, mainly in $\mathbb{R}_+^n$. In the last section we have seen that we can expect a nice duality theory and that the computational side of $\mathbb{B}$-convexity can be reasonably carried out. Hahn–Banach like theorems and the structure of $\mathbb{B}$-polytopes and Krein–Millman like theorems, will be the subject matter of a forthcoming paper. A deeper topological study of $\mathbb{B}$-convex sets will give us fixed point theorems, and their applications. The characterization of $\mathbb{B}$-convex subsets of $\mathbb{R}^n$ remains to be done.

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