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J. Math. Anal. Appl. 341 (2008) 1163–1169

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Projections as convex combinations of surjective isometries on $\mathcal{C}(\Omega)$

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Received 3 August 2007

Available online 17 November 2007

Submitted by J. Diestel

Abstract

We study properties of operators that are in the convex hull of a finite set of surjective isometries on the Banach space of complex valued continuous maps defined on a compact and connected topological space. We characterize those projections that are in the convex combination of two surjective isometries and we show that they are generalized bi-circular projections.

Keywords: Isometry; Convex combination of isometries

1. Introduction

Let $(X, \|\|)$ be a complex Banach space and let $P : X \to X$ be a linear projection. A basic problem in Banach space theory is to determine the structure of the projections on a given space and provide characterizations of their ranges. Various types of projections have been studied in the past, cf. [2,3,10,12,14]. Recently, a class of hermitian projections, namely bi-circular projections, has been a topic of research interest, see [13]. These projections are in fact hermitian projections as shown in [8]. A projection is called a bi-circular projection if $e^{i\alpha}P + e^{i\beta}(I - P)$ is an isometry for all $\alpha, \beta \in R$. This notion was generalized by Fosner, Ilisevic, and C.K. Li in [7], by requiring that $P + \lambda(I - P)$ is an isometry for some modulus 1 complex number λ, different from 1. In [7], a characterization of these projections was obtained in a finite dimensional setting for both real and complex vector spaces. Similar characterization was derived in [4] for such projections on the Banach spaces $C(\Omega)$ and the vector valued $C(\Omega, X)$, where X is a Banach space. The Banach space $\mathcal{C}(\Omega)$ consists of all continuous functions over the complex numbers equipped with the norm $\|f\|_\infty = \max_{x\in\Omega} |f(x)|$. Throughout this paper we consider Ω to be a compact and connected topological space.

Theorem 1.1. *(See [4].) If Ω is a compact and connected topological space, then P is a generalized bi-circular projection on $C(\Omega)$ if and only if there exist a homeomorphism $\phi : \Omega \to \Omega$, with $\phi^2 = \mathrm{Id}$, and a continuous function $u : \Omega \to C$, with $|u(\omega)| = 1$ and $u(\omega) = \overline{u(\phi(\omega))}$ (for every $\omega \in \Omega$), such that*

$$P(f)(\omega) = \frac{1}{2}\big[f(\omega) + u(\omega)f\big(\phi(\omega)\big)\big].$$

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0022-247X/$ – see front matter © 2007 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2007.10.070

The result in Theorem 1.1 is typical in that generalized bi-circular projections can be represented as the average of the identity with an isometric reflection, see [5] and also [4]. This raises the question of whether the convex combination of surjective isometries contain any projections. In this paper, we characterize those projections that are in the convex combination of two surjective isometries on $\mathcal{C}(\Omega)$.

In our study, the representation of surjective isometries, known as the Banach–Stone theorem, plays a crucial role, cf. [6].

Theorem 1.2 *(Banach–Stone). T is a surjective isometry of $\mathcal{C}(\Omega)$ if and only if there exist a continuous map $u:\Omega \to C$, with $|u(\omega)|=1$, and a homeomorphism $\phi:\Omega\to\Omega$ so that*

$$T(f)(\omega)=u(\omega)f\big(\phi(\omega)\big).$$

2. Projections in the convex combination of two isometries

In this section we show that the only projections expressed as a convex combination of two isometries are generalized bi-circular projections. We consider two distinct isometries in $C(\Omega)$, I_1 and I_2, with representations

$$I_i(f)(\omega)=u_i f\big(\phi_i(\omega)\big),$$

where ϕ_i is a homeomorphism of Ω and $u_i:\Omega\to C$ is a continuous map with $|u_i(\omega)|=1$.

Proposition 2.1. *Let I_1 and I_2 be isometries on $C(\Omega)$. If $Q_\lambda=\lambda I_1+(1-\lambda)I_2$ (with $0<\lambda<1$) is a projection, then $\lambda=\frac{1}{2}$.*

Proof. The operator Q_λ is a projection if and only if

$$\begin{aligned}
&\lambda^2u_1(\omega)\cdot u_1\big(\phi_1(\omega)\big)f\big(\phi_1^2(\omega)\big)+\lambda(1-\lambda)u_1(\omega)\cdot u_2\big(\phi_1(\omega)\big)f\big(\phi_2\circ\phi_1(\omega)\big)\\
&\quad+\lambda(1-\lambda)u_2(\omega)\cdot u_1\big(\phi_2(\omega)\big)f\big(\phi_1\circ\phi_2(\omega)\big)+(1-\lambda)^2u_2(\omega)\cdot u_2\big(\phi_2(\omega)\big)f\big(\phi_2^2(\omega)\big)\\
&=\lambda u_1(\omega)f\big(\phi_1(\omega)\big)+(1-\lambda)u_2(\omega)f\big(\phi_2(\omega)\big),
\end{aligned}\tag{2.1}$$

for every $f\in C(\Omega)$. First, we observe that, a given $\omega\in\Omega$ determines a partition of Ω into the following four sets

$$\Omega_0=\big\{\omega\mid \phi_1(\omega)=\phi_2(\omega)\big\},\qquad \Omega_1=\big\{\omega\mid \omega=\phi_1(\omega)\neq\phi_2(\omega)\big\},\qquad \Omega_2=\big\{\omega\mid \omega=\phi_2(\omega)\neq\phi_1(\omega)\big\},$$

and $\Omega_3=\{\omega\mid \omega\neq\phi_1(\omega)\neq\phi_2(\omega)\neq\omega\}$.

If $\omega\in\Omega_0$, then Eq. (2.1) reduces to

$$\begin{aligned}
&\big[\lambda u_1(\omega)+(1-\lambda)u_2(\omega)\big]\big[\lambda u_1\big(\phi_1(\omega)\big)f\big(\phi_1^2(\omega)\big)+(1-\lambda)u_2\big(\phi_2(\omega)\big)f\big(\phi_2^2(\omega)\big)\big]\\
&\quad=\big[\lambda u_1(\omega)+(1-\lambda)u_2(\omega)\big]f\big(\phi_1(\omega)\big).
\end{aligned}$$

If $\lambda u_1(\omega)+(1-\lambda)u_2(\omega)=0$, then $u_1(\omega)-u_2(\omega)\neq 0$. Moreover, $u_1(\omega)\cdot\bar{u}_2(\omega)$ is a real number, since $\lambda=-\frac{u_2(\omega)}{u_1(\omega)-u_2(\omega)}$. This implies that $u_1(\omega)\cdot\bar{u}_2(\omega)=\pm1$ and $\lambda=\frac{1}{2}$. If $\lambda u_1(\omega)+(1-\lambda)u_2(\omega)\neq 0$, then $\lambda u_1(\phi_1(\omega))f(\phi_1^2(\omega))+(1-\lambda)u_2(\phi_2(\omega))f(\phi_2^2(\omega))=f(\phi_1(\omega))$. We have that $\lambda u_1(\phi_1(\omega))+(1-\lambda)u_2(\phi_2(\omega))=1$, for the constant function equal to 1. Furthermore, given a continuous function such that $f(\phi_1(\omega))=0$, we must have $\phi_1^2(\omega)=\phi_2^2(\omega)=\phi_1(\omega)$. Consequently, Eq. (2.1) is satisfied provided that $\omega=\phi_1(\omega)=\phi_2(\omega)$ and $u_1(\omega)=u_2(\omega)=1$.

If, for every ω, we have that $\omega\in\Omega_0$ then $I_1=I_2$, contradicting our assumption that the two isometries were distinct. We assume that $\omega\in\Omega_1$, or equivalently $\omega=\phi_1(\omega)\neq\phi_2(\omega)$. Equation (2.1) now reduces to

$$\begin{aligned}
&\lambda^2u_1^2(\omega)f(\omega)+(1-\lambda)^2u_2(\omega)\cdot u_2\big(\phi_2(\omega)\big)f\big(\phi_2^2(\omega)\big)+\lambda(1-\lambda)u_1(\omega)\cdot u_2(\omega)f\big(\phi_2(\omega)\big)\\
&\quad+\lambda(1-\lambda)u_2(\omega)\cdot u_1\big(\phi_2(\omega)\big)f\big(\phi_1\circ\phi_2(\omega)\big)\\
&=\lambda u_1(\omega)f(\omega)+(1-\lambda)u_2(\omega)f\big(\phi_2(\omega)\big).
\end{aligned}$$

In particular, for a continuous function f such that $f(\omega)=f(\phi_2(\omega))=0$, we have

$$\lambda u_1\big(\phi_2(\omega)\big)f\big(\phi_1\circ\phi_2(\omega)\big)+(1-\lambda)u_2\big(\phi_2(\omega)\big)f\big(\phi_2^2(\omega)\big)=0.$$

Therefore $\phi_1 \circ \phi_2(\omega) = \phi_2(\omega)$ and $\phi_2^2(\omega) = \omega$. Equation (2.1) is now written as follows

$$\begin{aligned}&\lambda^2 u_1^2(\omega) f(\omega) + \lambda(1-\lambda) u_1(\omega) u_2(\omega) f\big(\phi_2(\omega)\big) + \lambda(1-\lambda) u_2(\omega)\cdot u_1\big(\phi_2(\omega)\big) f\big(\phi_2(\omega)\big)\\ &\quad + (1-\lambda)^2 u_2(\omega) u_2\big(\phi_2(\omega)\big) f(\omega)\\ &= \lambda u_1(\omega) f(\omega) + (1-\lambda) u_2(\omega) f\big(\phi_2(\omega)\big).\end{aligned}$$

We consider a continuous function so that $f(\omega) = 0$ and $f(\phi_2(\omega)) = 1$. Therefore $\lambda u_1(\omega) + \lambda u_1(\phi_2(\omega)) = 1$. This implies that $u_1(\omega) + u_1(\phi_2(\omega))$ is a real number, $u_1(\omega) = \overline{u_1(\phi_2(\omega))}$ and $\lambda = \frac{1}{u_1(\omega)+\overline{u_1(\omega)}} \geqslant \frac{1}{2}$. On the other hand, for a function f so that $f(\omega) = 1$ and $f(\phi_2(\omega)) = 0$, we have $\lambda^2 u_1^2(\omega) + (1-\lambda)^2 u_2(\omega) u_2(\phi_2(\omega)) = \lambda u_1(\omega)$. Therefore, $\lambda \leqslant 2\lambda^2 - 2\lambda + 1$ and $\lambda = \frac{1}{2}$. Similar considerations hold for $\omega \in \Omega_2$. It is left to consider $\omega \in \Omega_3$. Given a function f satisfying $f(\phi_1(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2 \circ \phi_1(\omega)) = 0$ and $f(\phi_2(\omega)) = 1$, (2.1) reduces to

$$\lambda^2 u_1(\omega) u_1\big(\phi_1(\omega)\big) f\big(\phi_1^2(\omega)\big) + \lambda(1-\lambda) u_2(\omega) u_1\big(\phi_2(\omega)\big) f\big(\phi_1 \circ \phi_2(\omega)\big) = (1-\lambda) u_2(\omega). \tag{2.2}$$

For a function f such that $f(\phi_2(\omega)) = f(\phi_1^2(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = 0$ and $f(\phi_1(\omega)) = 1$, Eq. (2.1) becomes

$$\lambda(1-\lambda) u_1(\omega) u_2\big(\phi_1(\omega)\big) f\big(\phi_2 \circ \phi_1(\omega)\big) + (1-\lambda)^2 u_2(\omega) u_2\big(\phi_2(\omega)\big) f\big(\phi_2^2(\omega)\big) = \lambda u_1(\omega). \tag{2.3}$$

Equation (2.2) implies that $\phi_1^2(\omega) = \phi_2(\omega)$ and $\lambda = \frac{-1+\sqrt{5}}{2}$. Furthermore, Eq. (2.3) implies that $\phi_1(\omega) = \phi_2^2(\omega)$, $(1-\lambda)^2 = \lambda$ and $\lambda = \frac{3-\sqrt{5}}{2}$, contradicting the value for λ previously determined. This shows that $\omega \notin \Omega_3$. Therefore, if Q_λ is a projection (i.e. $Q_\lambda \circ Q_\lambda(f)(\omega) = Q_\lambda(f)(\omega)$), then $\omega \in \Omega_0 \cup \Omega_1 \cup \Omega_2$. If $\omega \in \Omega_1 \cup \Omega_2$, then $\lambda = 1/2$. □

Remark 2.2. We observe that the path Q_λ connecting I_1 with I_2 consist of operators of norm 1 provided the homeomorphisms ϕ_1 and ϕ_2 are distinct. In fact, it is a consequence of Urysohn's Lemma the existence of a continuous function f so that $f(\phi_1(\omega)) = \bar{u}_1(\omega)$ and $f(\phi_2(\omega)) = \bar{u}_2(\omega)$, for $\omega \in \Omega$ such that $\phi_1(\omega) \neq \phi_2(\omega)$. Therefore $\|Q_\lambda(f)\|_\infty = 1$ and $\|Q_\lambda\| = 1$.

Proposition 2.3. *If I_1 and I_2 are isometries on $\mathcal{C}(\Omega)$, then $Q = \frac{I_1+I_2}{2}$ is a projection if and only if every $\omega \in \Omega$ satisfies one of the following statements*:

(1) $\omega = \phi_1(\omega) = \phi_2(\omega)$ *and* $u_1(\omega) = u_2(\omega) = 1$, *or*
(2) $\phi_1(\omega) = \phi_2(\omega)$ *and* $u_1(\omega) = -u_2(\omega)$, *or*
(3) $\omega = \phi_i(\omega) \neq \phi_j(\omega)$, $\phi_j^2(\omega) = \omega$, $\phi_i \circ \phi_j(\omega) = \phi_j(\omega)$, $u_i(\omega) = u_i(\phi_j(\omega)) = 1$ *and the product* $u_j(\omega) \cdot u_j(\phi_j(\omega)) = 1$.

Proof. Q is a projection if and only if

$$\begin{aligned}&u_1(\omega) u_1\big(\phi_1(\omega)\big) f\big(\phi_1^2(\omega)\big) + u_1(\omega) u_2\big(\phi_1(\omega)\big) f\big(\phi_2 \circ \phi_1(\omega)\big) + u_2(\omega) u_1\big(\phi_2(\omega)\big) f\big(\phi_1 \circ \phi_2(\omega)\big)\\ &\quad + u_2(\omega) u_2\big(\phi_2(\omega)\big) f\big(\phi_2^2(\omega)\big)\\ &= 2\big\{u_1(\omega) f\big(\phi_1(\omega)\big) + u_2(\omega) f\big(\phi_2(\omega)\big)\big\}.\end{aligned} \tag{2.4}$$

We first consider $\omega \in \Omega$ so that $\phi_1(\omega) = \phi_2(\omega)$. This implies that $\phi_1^2(\omega) = \phi_1 \circ \phi_2(\omega)$ and $\phi_2 \circ \phi_1(\omega) = \phi_2^2(\omega)$. Equation (2.4) reduces to

$$\big(u_1(\omega) + u_2(\omega)\big)\big\{u_1\big(\phi_1(\omega)\big) f\big(\phi_1^2(\omega)\big) + u_2\big(\phi_1(\omega)\big) f\big(\phi_2^2(\omega)\big) - 2f\big(\phi_1(\omega)\big)\big\} = 0,$$

therefore $u_1(\omega) = -u_2(\omega)$ (statement (2) in the proposition) or

$$u_1\big(\phi_1(\omega)\big) f\big(\phi_1^2(\omega)\big) + u_2\big(\phi_1(\omega)\big) f\big(\phi_2^2(\omega)\big) = 2f\big(\phi_1(\omega)\big),$$

for every continuous function $f \in \mathcal{C}(\Omega)$. This implies that $\phi_1(\omega) = \omega$ and $u_1(\omega) = u_2(\omega) = 1$, as in the statement (1).

Now, we assume that $\phi_1(\omega) \neq \phi_2(\omega)$. We consider f in $\mathcal{C}(\Omega)$, with values in the interval $[0, 1]$, such that $f(\phi_1(\omega)) = 1$ and $f(\phi_2(\omega)) = 0$. Therefore Eq. (2.4) implies that there must exist at least two points in the set

$$\big\{\phi_1^2(\omega), \phi_1 \circ \phi_2(\omega), \phi_2 \circ \phi_1(\omega), \phi_2^2(\omega)\big\}$$

equal to $\phi_1(\omega)$. Since $\phi_1(\omega) \neq \phi_2(\omega)$, we have $\phi_1^2(\omega) \neq \phi_1 \circ \phi_2(\omega)$ and $\phi_2^2(\omega) \neq \phi_2 \circ \phi_1(\omega)$. Consequently, there are four cases to consider:

(A) $\phi_1(\omega) = \phi_1^2(\omega) = \phi_2 \circ \phi_1(\omega)$;
(B) $\phi_1(\omega) = \phi_1^2(\omega) = \phi_2^2(\omega)$;
(C) $\phi_1(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2 \circ \phi_1(\omega)$;
(D) $\phi_1(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2^2(\omega)$.

The statement (A) leads to a contradiction, since $\phi_1(\omega) = \omega$ and $\phi_1(\omega) = \phi_2(\omega)$. The statement (B) implies that $\phi_1(\omega) = \omega$ and $\phi_2^2(\omega) = \omega$. We observe that $\phi_1 \circ \phi_2(\omega) \neq \phi_1(\omega) \neq \phi_2 \circ \phi_1(\omega)$. Furthermore, given a continuous function f satisfying the conditions: $f(\phi_1(\omega)) = f(\phi_1^2(\omega)) = f(\phi_2^2(\omega)) = 1$ and $f(\phi_2(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_2 \circ \phi_1(\omega)) = 0$, Eq. (2.4) reduces to: $u_1^2(\omega) + u_2(\omega)u_2(\phi_2(\omega)) = 2u_1(\omega)$. Therefore $u_1(\omega) = 1$ and $u_2(\omega) \cdot u_2(\phi_2(\omega)) = 1$. These conditions allows us to rewrite Eq. (2.4) as follows:

$$u_2(\omega)u_1\big(\phi_2(\omega)\big)f\big(\phi_1 \circ \phi_2(\omega)\big) = u_2(\omega)f\big(\phi_2(\omega)\big).$$

Hence, we have $\phi_1 \circ \phi_2(\omega) = \phi_2(\omega)$ and $u_1(\phi_2(\omega)) = 1$, as in the third statement of the proposition, for $i = 1$. The statement (C) yields $\omega = \phi_2(\omega)$ and $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega)$. Given a continuous function f such that $f(\phi_1^2(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2(\omega)) = 0$ and $f(\phi_1 \circ \phi_2(\omega)) = f(\phi_2 \circ \phi_1(\omega)) = 1$, Eq. (2.4) becomes $u_1(\omega)u_2(\phi_1(\omega)) + u_2(\omega)u_1(\omega) = 2u_1(\omega)$. Hence $u_2(\phi_1(\omega)) = u_2(\omega) = 1$. Now, we consider a continuous function f so that $f(\phi_1^2(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2(\omega)) = 1$ and $f(\phi_1 \circ \phi_2(\omega)) = f(\phi_2 \circ \phi_1(\omega)) = 0$. Equation (2.4) now reduces to $u_1(\omega)u_1(\phi_1(\omega)) + u_2^2(\omega) = 2u_2(\omega)$ and $u_1(\phi_1(\omega)) \cdot u_1(\omega) = u_2(\omega) = 1$. Hence, we get:

$$f\big(\phi_1^2(\omega)\big) = f\big(\phi_2(\omega)\big) \quad \big(= f(\omega)\big),$$

implying that $\phi_1^2(\omega) = \omega$, as in the statement 3 of the proposition, for $i = 2$. The statement (D), as statement (A), leads to a contradiction since $\phi_2(\omega) = \omega$ and $\phi_1(\omega) = \phi_2(\omega)$. Conversely, it is straightforward to show that the average of two isometries, $I_i(f)(\omega) = u_i(\omega)f(\phi_i(\omega))$ ($i = 1$ or 2) with u_i and ϕ_i satisfying the conditions stated in the proposition, satisfies Eq. (2.4). □

Theorem 2.4. *The average of two isometries on $C(\Omega)$ is a projection if and only if it is a generalized bi-circular projection.*

Proof. We first observe that a generalized bi-circular projection Q is the average of the identity with an isometric reflection, as stated in Theorem 1.1. Conversely, we denote by Q, the average of I_1 and I_2, where $I_i(f)(\omega) = u_i(\omega)f(\phi_i(\omega))$. Proposition 2.3 allows us to define the following partition of Ω:

$$\begin{aligned} S &= \{\omega\colon \phi_1(\omega) = \phi_2(\omega)\}, \\ A_1 &= \{\omega \notin S\colon \phi_1(\omega) = \omega,\ \phi_2^2(\omega) = \omega,\ \phi_1 \circ \phi_2(\omega) = \phi_2(\omega),\ u_1(\omega) = u_1\big(\phi_2(\omega)\big) = 1, \text{ and} \\ &\qquad u_2(\omega) \cdot u_2\big(\phi_2(\omega)\big) = 1\} \end{aligned}$$

and

$$\begin{aligned} A_2 = \{&\omega \notin S\colon \phi_2(\omega) = \omega,\ \phi_1^2(\omega) = \omega,\ \phi_2 \circ \phi_1(\omega) = \phi_1(\omega),\ u_2(\omega) = u_2\big(\phi_1(\omega)\big) = 1, \text{ and} \\ &u_1(\omega) \cdot u_1\big(\phi_1(\omega)\big) = 1\}. \end{aligned}$$

We construct a continuous function $u : \Omega \to C$ of modulus 1 and a homeomorphism ϕ such that

$$u_1(\omega)f\big(\phi_1(\omega)\big) + u_2(\omega)f\big(\phi_2(\omega)\big) = f(\omega) + u(\omega)f\big(\phi(\omega)\big), \tag{2.5}$$

for every $f \in \mathcal{C}(\Omega, C)$ and $\omega \in \Omega$. Let

$$\phi(\omega) = \begin{cases} \phi_1(\omega) & \text{if } \omega \in A_2, \\ \phi_2(\omega) & \text{if } \omega \in A_1, \\ \phi_1(\omega) = \phi_2(\omega) & \text{if } \omega \in S. \end{cases}$$

We show that ϕ is a homeomorphism. We first show that ϕ is continuous. In fact, given a net $\{\omega_\alpha\}$, in Ω, converging to ω_*, we have that the net $\{\phi(\omega_\alpha)\}$ converges to $\phi(\omega_*)$. This is a straightforward consequence of the continuity of both ϕ_1 and ϕ_2. We also have that ϕ is a bijection. We consider ω_1 and ω_2 such that $\phi(\omega_1) = \phi(\omega_2)$. Without loss of generality we assume that $\omega_1 \in A_1$ and $\omega_2 \in A_2$. Therefore, $\phi_2(\omega_1) = \phi_1(\omega_2)$ and $\omega_1 = \phi_2 \circ \phi_2(\omega_1) = \phi_2 \circ \phi_1(\omega_2) = \phi_1(\omega_2)$. Consequently, $\omega_1 = \phi_1(\omega_1) = \phi_1^2(\omega_2) = \omega_2$. Furthermore, ϕ is surjective. Given $\omega \in A_1$, we have that $\phi_2(\omega)$ is also in A_1 and $\phi(\phi_2(\omega)) = \omega$. Similarly for $\omega \in A_2$. The surjectivity of ϕ now follows since $\phi_i(A_i) \cap S = \emptyset$. The continuity of ϕ^{-1} follows from the continuity of ϕ and the compactness of Ω. The function u is determined so that Eq. (2.5) holds true for all continuous functions, in particular for $f \equiv 1$. Hence $u = u_1 + u_2 - 1$. We observe that u is modulus 1 since for $\omega \in A_1$ (or A_2) we have that $u(\omega) = u_2(\omega)$, (or $u_1(\omega)$, respectively). If $\omega \in S$ then $u(\omega) = \pm 1$. Furthermore, we have that $u(\omega) \cdot u(\phi(\omega)) = 1$. $\square$

Generalized bi-circular projection are the average of the identity operator and an involution isometry, i.e. an isometry L so that $L \circ L = \mathrm{Id}$. This motivates the following definition of n-circular projection.

Definition 2.5. L is said to be an n-isometry on a Banach space if and only if L is an isometry such that $L^n = \mathrm{Id}$. Further, Q is a generalized n-circular projection if and only if there exists an n-isometry L of X such that

$$Q = \frac{1}{n}\left[\mathrm{Id} + L + L^2 + \cdots + L^{n-1}\right]$$

where n is the smallest positive integer for which $L^n = \mathrm{Id}$.

Remark 2.6. 1. The point spectrum of an n-isometry (L) consists of the nth roots of 1. We denote these roots by λ_0 $(= 1)$, $\lambda_1, \ldots, \lambda_n$. A theorem from Taylor, cf [11], implies that $L = Q_0 + \lambda_1 Q_1 + \cdots + \lambda_n Q_n$, where Q_k is the projection onto the kernel of $L - \lambda_k \mathrm{Id}$. In particular, this implies that the projection Q_0 is a generalized n-circular projection.

2. It follows from the Banach–Stone Theorem (see [1]) that L is an n-isometry on $\mathcal{C}(\Omega)$ if and only if there exist a homeomorphism ϕ such that $\phi^n = \mathrm{Id}$ and a continuous function $u : \Omega \to C$ such that $|u(\omega)| = 1$, $u(\omega)u(\phi(\omega)) \cdots u(\phi^{n-1}(\omega)) = 1$ and $L^k(f)(\omega) = u(\omega) \cdot u(\phi(\omega)) \cdots u(\phi^{k-1}(\omega)) f(\phi^k(\omega))$.

Examples.

(1) Consider $\Omega = \{z \in C\colon |z| = 1\}$, $\phi(z) = e^{\frac{2\pi}{n} i} z$, $u(z) = 1$. Therefore $Q(f)(z) = \frac{1}{2}[f(z) + f(\phi(z)) + f(\phi^2(z)) + \cdots + f(\phi^{n-1}(z))]$ is a n-circular projection.

(2) It follows from Theorem 1.1 that generalized bi-circular projections on $\mathcal{C}([0, 1])$ are 2-circular projections, however there are no n-circular projections for $n \geqslant 3$. This follows from the fact that there are no homeomorphism of the interval with period n, for $n \geqslant 3$.

Remark 2.7. Similar techniques to those applied in the previous results allow us to show that a projection Q in the convex combination of three surjective isometries is given by

$$Q(f)(\omega) = \frac{f(\omega) + f(\phi(\omega)) + f(\phi^2(\omega))}{3},$$

where ϕ is a homeomorphism of Ω such that $\phi^3 = \mathrm{Id}$. This suggests that for higher values of n the same result might also hold.

3. Operators in the convex combination of isometries

In this section we study operators in the linear convex combination of finitely many isometries on $\mathcal{C}(\Omega)$. We start with a definition to distinguish isometries.

Definition 3.1. We say that two isometries $I_1(f)(\omega) = u_1(\omega) f(\phi_1(\omega))$ and $I_2(f)(\omega) = u_2(\omega) f(\phi_1(\omega))$ are essentially distinct if and only if ϕ_1 and ϕ_2 are distinct homeomorphisms.

Proposition 3.2. *If I_1 and I_2 are essentially distinct isometries, then $\|I_1 - I_2\| = 2$. If I_1 and I_2 are not essentially distinct isometries, then $\|I_1 - I_2\| = \|u_1 - u_2\|_\infty$.*

Proof. If I_1 and I_2 are essentially distinct, then there exists ω such that $\phi_1(\omega) \neq \phi_2(\omega)$ and hence an Urysohn's function f, with modulus 1 values, such that $f(\phi_1(\omega)) = \bar{u}_1(\omega)$ and $f(\phi_2(\omega)) = -\bar{u}_2(\omega)$. The first statement in the proposition follows from

$$\|I_1 - I_2\| \geqslant \big\|I_1(f) - I_2(f)\big\|_\infty \geqslant \big|I_1(f)(\omega) - I_2(f)(\omega)\big| = 2.$$

If I_1 and I_2 are not essentially distinct, then

$$\|I_1 - I_2\| = \sup_{\|f\|_\infty = 1} \big\|\big[u_1(\omega) - u_2(\omega)\big] f\big(\phi_1(\omega)\big)\big\|_\infty \leqslant \|u_1 - u_2\|_\infty \sup_{\|f\|_\infty = 1} \big\|f\big(\phi_1(\omega)\big)\big\|_\infty = \|u_1 - u_2\|_\infty.$$

On the other hand, we have that $\|I_1 - I_2\| \geqslant \|u_1 - u_2\|_\infty$, which concludes the proof. □

Definition 3.3. We say that two isometries I_1 and I_2 are isometrically connected if there exists a continuous map $\mathcal{I}$ from the closed interval $[0, 1]$ into the set of all surjective isometries $\mathcal{SI}$, $\mathcal{I} : [0, 1] \to \mathcal{SI}$, such that $\mathcal{I}(0) = I_1$ and $\mathcal{I}(1) = I_2$. Each isometry I determines a unique isometric component.

The following corollary is a consequence of the previous proposition.

Corollary 3.4.

(1) *The set of all surjective isometries is closed.*
(2) *Two isometries at a distance less than 2 are isometrically connected.*
(3) *Two essentially distinct isometries belong to distinct isometric components.*

Proof. (1) Given a Cauchy sequence of isometries $\{I_n\}$, after a certain order any two isometries are at a distance less than 2 and hence, by Proposition 3.2, all the corresponding homeomorphisms are equal. Therefore, a sequence of isometries is Cauchy if and only if the associated sequence of modulus 1 maps $\{u_n\}$ is a Cauchy sequence. Such sequence is convergent and hence the original sequence of isometries also converges. This completes the proof of the first statement.

(2) If two isometries I_1 and I_2 are at a distance less than 2, then $\|I_1 - I_2\| = \|u_1 - u_2\|_\infty < 2$, where u_1 and u_2 are the associated modulus 1 factors and ϕ the corresponding homeomorphism. We define the following path of isometries

$$\mathcal{I}(\lambda) = \frac{\lambda u_1 + (1-\lambda)u_2}{|\lambda u_1 + (1-\lambda)u_2|} f\big(\phi(\omega)\big),$$

where we observe that $|\lambda u_1 + (1-\lambda)u_2| \neq 0$ since u_1 and u_2 are modulus 1, and $\|u_1 - u_2\|_\infty < 2$.

(3) If I_1 and I_2 are essentially distinct, then the associated homeomorphisms ϕ_1 and ϕ_2 are distinct. Let $\mathcal{I}$ be a continuous path on $[0, 1]$ with values on the set of surjective isometries such that $\mathcal{I}(0) = I_1$ and $\mathcal{I}(1) = I_2$. We denote by ϕ_λ the homeomorphism associated with the isometry $\mathcal{I}(\lambda)$, for $\lambda \in [0, 1]$. A continuity argument implies that ϕ_λ is locally constant, and it follows that $\phi_1 = \phi_2$, contradicting our initial assumption. □

Remark 3.5. Two isometries, not essentially distinct and with associated multiplicative factors at a distance equal to 2, are isometrically connected if and if the multiplicative factors are homotopic in $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. As for example, if $I_1(f)(\omega) = f(\phi(\omega))$ and $I_2(f)(\omega) = -f(\phi(\omega))$, then $\mathcal{I}(\lambda)(f)(\omega) = e^{\pi\lambda i} f(\phi(\omega))$. On the other hand, if $\Omega = S^1$, $I_1(f)(\omega) = f(\omega)$ and $I_2(f)(\omega) = \omega f(\omega)$, there is no path of isometries connecting I_1 with I_2. Otherwise S^1 would be contractible which is impossible, see [9].

Definition 3.6. The n-simplex determined by n surjective isometries is defined to be $\Delta(I_1, I_2, \ldots, I_n) = \{\sum_{k=1}^n \lambda_k I_k\colon 0 \leqslant \lambda_k \leqslant 1,\ \text{and}\ \sum_k \lambda_k = 1\}$.

We also establish when a convex combination of surjective isometries is an operator of norm 1. The convex combination of finitely many surjective isometries consists of operators with norm $\leqslant 1$.

Proposition 3.7. *If T is a convex combination of n surjective isometries, with associated homeomorphisms $\{\phi_i\}_{i=1,\ldots,n}$ and multiplicative factors $\{u_i\}_{i=1,\ldots,n}$, then $\|T\| < 1$ if and only if for every $\omega \in \Omega$ there exist i and j $(i \neq j)$ with $\phi_i(\omega) = \phi_j(\omega)$ and $u_i(\omega) \neq u_j(\omega)$.*

Proof. Let $T(f)(\omega) = \sum_{i=1}^n \lambda_i u_i(\omega) f(\phi_i(\omega))$ with $\lambda_i \geqslant 0$ and $\sum_{i=1}^n \lambda_i = 1$. We assume that for every $\omega \in \Omega$ there exist i and j $(i \neq j)$ such that $\phi_i(\omega) = \phi_j(\omega)$ and $u_i(\omega) \neq u_j(\omega)$. Therefore for a given ω we have that

$$\left|T(f)(\omega)\right| \leqslant \left|\lambda_i u_i(\omega) + \lambda_j u_j(\omega)\right| + \sum_{k \neq i,j} \lambda_k < 1,$$

for every f with $\|f\|_\infty = 1$. Since Ω is compact we have that $\|T\| < 1$. Conversely, we assume that there exists $\omega \in \Omega$ so that either

(a) $\phi_i(\omega) \neq \phi_j(\omega)$, for all $i \neq j$, or
(b) there exist $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ with $\phi_{i_p}(\omega) = \phi_{i_q}(\omega)$ (then $u_{i_p}(\omega) = u_{i_q}(\omega)$, for $p, q \in \{1, \ldots, k\}$).

Case (a). We select a Urysohn's function f with $\|f\|_\infty = 1$ such that $f(\phi_i(\omega)) = \bar{u}_i(\omega)$. Therefore $T(f)(\omega) = 1$ and $\|T\| = 1$.

Case (b). A similar argument used in Case (a) leads to a contradiction and completes the proof. □

Acknowledgment

The author is most grateful to Professor James Jamison for many insightful discussions throughout the preparation of this manuscript.

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