Acta Sci. Math. (*Szeged*)
75 (2009), 103–112

Generalized bi-circular projections on Lipschitz spaces

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Communicated by L. Kérchy

Abstract. This paper provides a description of generalized bi-circular projections on Banach spaces of Lipschitz functions.

1. Introduction

A projection P on a Banach space X is said to be a *bi-circular projection* if $e^{ia}P + e^{ib}(I - P)$ is an isometry, for all choices of real numbers a and b. These projections were first studied by Stacho and Zalar (in [17] and [18]) and shown to be norm hermitian by Jamison (in [9]).

Fosner, Illisevic, and Li have introduced a more general class of projections, designated generalized *bi-circular projections*, cf. [8]. A generalized bi-circular projection P only requires that $P + \lambda(I - P)$ is an isometry, for some $\lambda \in \mathbb{T} \setminus \{1\}$. In general, these projections are not norm hermitian.

The authors, in [8], provided a characterization of generalized bi-circular projections on finite dimensional Banach spaces. Similar characterizations of generalized bi-circular projections on Banach spaces of continuous functions, $C(\Omega)$ and $C(\Omega, X)$ were derived in [4], see also [6]. Typically, generalized bi-circular projections can be represented as the average of the identity with an isometric reflection. In this paper, we extend this representation to spaces of Lipschitz functions. We also describe special cases where such projections are not of this form.

Received February 20, 2008, and in revised form April 8, 2008.
AMS Subject Classification (2000): 30D55; 30D05.
Key words and phrases: generalized bi-circular projections, surjective isometries, spaces of Lipshitz functions.

For a given positive number $\alpha < 1$, we consider the following Banach spaces of Lipschitz functions, defined on a compact metric space $(H,\, d)$ with at least two points:

$$\mathrm{Lip}^{\alpha}(H,d) = \Big\{ f\colon H \to C \,:\, \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^{\alpha}} < \infty \Big\},$$

$$\mathrm{lip}^{\alpha}(H,d) = \Big\{ f \in \mathrm{Lip}^{\alpha}(H,d) \,:\, \lim_{\delta\to 0} \sup_{0<d(x,y)^{\alpha}<\delta} \frac{|f(x)-f(y)|}{d(x,y)^{\alpha}} = 0 \Big\},$$

$$\mathrm{Lip}^{\alpha}(H,d;h_0) = \Big\{ f \in \mathrm{Lip}^{\alpha}(H,d) : f(h_0) = 0 \Big\},$$

and

$$\mathrm{lip}^{\alpha}(H,\, d; h_0) = \{ f \in \mathrm{lip}^{\alpha}(H,\, d) : f(h_0) = 0 \}\,.$$

These spaces are equipped with the norm $\|f\| = \max\{\|f\|_{\alpha},\, \|f\|_{\infty}\}$, with $\|f\|_{\alpha} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^{\alpha}}$, cf. [13], [15], or [19].

2. Basic definitions and results

In this section we review the definition of generalized bi-circular projection, establish preliminary results to be used in forthcoming proofs, and recall Mayer-Wolf's characterization of surjective isometries on these spaces.

Definition 2.1. A linear and bounded projection P on a Banach space X is said to be a *generalized bi-circular projection* if and only if there exists a modulus 1 complex number λ, different from 1, so that $P + \lambda(Id - P)$ is an isometry T on X.

It is a consequence of Definition 2.1 that T must be a surjective isometry. Furthermore, if R is an isometric reflection, then $\frac{Id+R}{2}$ is a generalized bi-circular projection.

Our characterization of the generalized bi-circular projections on Lipschitz function spaces depends on the form of the surjective isometries on these spaces and their representation as composition operators derived by Mayer-Wolf.

We first recall some terminology introduced in [13]. A subset S of H is said to be 1-centered at $c \in H$ if $d(h,c) = 1$, for every $h \in H_1 \setminus \{c\}$. We denote by $S_c = \{h \in H|\, d(h,\, c) = 1\}$, the sphere in H centered at c, and by $D_c = S_c \cup \{c\}$.

The set D_c is said to be a $(2,\, \alpha)$-isolated disc if $d^{\alpha}(S_c,\, H \setminus D_c) \geq 2$. We denote by $C = \{x \in H :\; d^{\alpha}(S_x,\, H \setminus D_x) \geq 2\}$.

Lemma 2.2. *There are only finitely many* $(2, \alpha)$*-isolated discs in* H*. Any two discs in* H *are either equal or disjoint.*

Proof. The compactness of H assures that $C = \{x \in H : \ d^\alpha(S_x, H \setminus D_x) \geq 2\}$ must be finite. Otherwise, there would exist an infinite and convergent sequence $\{x_n\}$ contained in C. This is impossible.

If x_i and x_j are two distinct points in C, then we show that $D_{x_i} \cap D_{x_j} = \emptyset$ or $D_{x_i} = D_{x_j}$. If we assume that $x_i \notin S_{x_j}$, then $d^\alpha(S_{x_j}, x_i) \geq 2$, since $x_i \in H \setminus D_{x_j}$. Moreover, given $a \in S_{x_i}$ and $b \in S_{x_j}$, we have that $d^\alpha(a, b) \geq d^\alpha(b, x_i) - d^\alpha(a, x_i) \geq 1$. This implies that $S_{x_i} \cap S_{x_j} = \emptyset$. On the other hand, $x_i \in S_{x_j}$ if and only if $x_j \in S_{x_i}$. In fact, if there exists $z \in S_{x_j} \setminus D_{x_i}$ then $d^\alpha(z, x_j) = 1$ and $d^\alpha(z, S_{x_i}) \geq 2$. Since $x_j \in S_{x_i}$, we have that $1 = d^\alpha(z, x_j) \geq 2$. This leads to an absurd and shows that $D_{x_j} \subset D_{x_i}$. Similarly we can prove that $D_{x_i} \subset D_{x_j}$. ■

The next lemma concerns the existence of extensions to Lipschitz functions on H, with preassigned values on a finite subset of H. This is a crucial tool in our characterization of generalized bi-circular projections. The proof is omitted since several results of this type are known and have been considered by several authors, see for example [15], [16] and [19].

Lemma 2.3. *Given a finite subset of* H*,* $A = \{h_0, h_1, \ldots, h_n\}$ *and* g_0 *a function defined on* A *so that* $g_0(h_0) = \cdots = g_0(h_{n-1}) = 0$ *and* $g_0(h_n) = 1$*, then there exists an extension of* g_0*,* $g \in \mathrm{Lip}^\alpha(H, d)$*, so that*

$$\lim_{\delta \to 0} \sup_{0 < d(x,y) \leq \delta} \frac{|g(x) - g(y)|}{d(x, y)^\alpha} = 0.$$

Remark 2.4. The previous lemma also implies the existence of extensions in $\mathrm{Lip}^\alpha(H, d)$ and $\mathrm{lip}^\alpha(H, d)$, as well as $\mathrm{Lip}^\alpha(H, d; h_0)$ and $\mathrm{lip}^\alpha(H, d; h_0)$.

Lemma 2.2 asserts that $C = \{x \in H : \ d^\alpha(S_x, H \setminus D_x) \geq 2\}$ is a finite set, say $C = \{c_1, c_2, \ldots, c_k\}$. Distinct values in C may define the same disc, i.e. $D_{x_i} = D_{x_j}$ with $x_i \neq x_j$. We denote by $H_1, H_2, \ldots, H_t$ the pairwise disjoint $(2, \alpha)$-isolated discs in H and by H_0 the complement in H of $H_1 \cup H_2 \cup \cdots \cup H_t$. Therefore $\{H_0, H_1, \ldots, H_t\}$ defines a partition of H, represented by

$$H = \bigsqcup_{i=0}^{t} H_i. \tag{2.1}$$

We state the Mayer-Wolf characterization theorems for surjective isometries.

Theorem 2.5. (cf. [13]) (1) *T is a surjective isometry on* $\mathrm{Lip}^\alpha(H,d;h_0)$ *or on* $\mathrm{lip}^\alpha(H,d;h_0)$ *if and only if there exist a modulus* 1 *complex number θ and a distance preserving ϕ on H so that*

$$T(f)(x)=\theta[f(\phi(x))-f(\phi(h_0))].$$

(2) *T is a surjective isometry on* $\mathrm{Lip}^\alpha(H,d)$ *or on* $\mathrm{lip}^\alpha(H,d)$ *if and only if there exist a modulus* 1 *complex valued continuous function θ, on H, constant whenever restricted to subsets of α-diameter less than* 2, *and a bijection ϕ on H that sends H_0 onto H_0 and sends each H_i* (1*-centered disc at c_i*) *onto $H_{\tau(i)}$* (1*-centered disk at $c_{\tau(i)}$, with τ the permutation of $\{1,\ldots,t\}$ induced by ϕ*) *so that*

$$T(f)(\xi)=\begin{cases}\theta(\xi)[f(\phi(c_i))-f(\phi(\xi))], & \text{if } \xi\in H_i\cap S_{c_i},\\ \theta(c_i)\,f(\phi(c_i)), & \text{if } \xi=c_i,\end{cases}$$

and

$$T(f)(\xi)=\theta(\xi)\,f(\phi(\xi)),\ \ \text{for } \xi\in H_0.$$

3. Generalized bi-circular projections on Banach spaces of Lipschitz functions

In this section we show that generalized bi-circular projections on Banach spaces of Lipschitz functions are typically given as the average of the identity with an isometric reflection. In addition, we also describe pathological examples of generalized bi-circular projections that are not representable in this form.

Theorem 3.1. *If H is a connected and compact metric space, then a projection on* $\mathrm{Lip}^\alpha(H,\,d)$, $\mathrm{Lip}^\alpha(H,\,d;h_0)$, $\mathrm{lip}^\alpha(H,\,d)$, *or* $\mathrm{lip}^\alpha(H,\,d;h_0)$ *is a generalized bi-circular projection if and only if it is the average of the identity operator with an isometric reflection.*

Proof. If P is the average of the identity with an isometric reflection then P is clearly a generalized bi-circular projection.

If P is a generalized bi-circular projection, then there exists a modulus 1 complex number λ so that $P+\lambda(Id-P)$ is a surjective isometry, denoted by T. This isometry satisfies the quadratic equation

$$\lambda Id-(1+\lambda)T+T^2=0. \tag{3.1}$$

Theorem 2.5 asserts that T has the representation

$$T(f)(x) = \theta[f(\phi(x)) - f(\phi(h_0))],$$

for $f \in \mathrm{Lip}^\alpha(H, d; h_0)$ or $\mathrm{lip}^\alpha(H, d; h_0)$, or

$$T(f)(x) = \theta\, f(\phi(x)),$$

for $f \in \mathrm{Lip}^\alpha(H, d)$ or $\mathrm{lip}^\alpha(H, d)$.

Hence, (3.1) has the form

$$\lambda f(h) - (1+\lambda)\theta[f(\phi(h)) - f(\phi(h_0))] + \theta^2[f(\phi^2(h)) - f(\phi^2(h_0))] = 0$$

or

$$\lambda f(h) - (1+\lambda)\theta f(\phi(h)) + \theta^2 f(\phi^2(h)) = 0,$$

respectively.

Moreover, if $\phi = Id$, then (3.1) reduces to

$$\lambda f(x) - (1+\lambda)\theta f(x) + \theta^2 f(x) = 0.$$

Therefore $\theta = 1$ or $\theta = \lambda$, which leads to $P = Id$ or $P = 0$. If $\phi \neq Id$, then there exists h so that $\phi(h) \neq h$. Without loss of generality, we may assume that $h \neq h_0$, $\phi(h) \neq h_0$ and $\phi(h_0) \neq h$, whenever the spaces under consideration are $\mathrm{Lip}^\alpha(H, d; h_0)$ or $\mathrm{lip}^\alpha(H, d; h_0)$. This is possible since H has no isolated points and ϕ is continuous. Lemma 2.3 asserts the existence of f so that $f(h) = f(\phi^2(h_0)) = f(\phi^2(h)) = f(h_0) = f(\phi(h_0)) = 0$ and $f(\phi(h)) = 1$. Therefore $(1+\lambda)\theta = 0$ and $\lambda = -1$. Hence equation (3.1) reduces to $T^2 = Id$ and Lemma 2.3 implies that $\phi^2 = Id$ and $\theta = \pm 1$.

■

The next proposition abridges the form of isometries that can be associated with generalized bi-circular projections.

Proposition 3.2. *Let H be a compact metric space. If T is a surjective isometry associated with a generalized bi-circular projection P (i.e. $P + \lambda(Id - P) = T$), then T is of the form:*

(1) $T(f)(x) = \theta\,[f(\phi(x)) - f(\phi(h_0))]$, *for* $f \in \mathrm{Lip}^\alpha(H, d; h_0)$ *or* $\mathrm{lip}^\alpha(H, d; h_0)$, *with ϕ a distance preserving bijection of H, and θ a modulus 1 complex number, or*

(2) $T(f)(x) = \theta(x) f(\phi(x))$, *for* $f \in \mathrm{Lip}^\alpha(H, d)$ *or* $\mathrm{lip}^\alpha(H, d)$, *with ϕ a bijection on H and an isometry over sets of α-diameter less than 2, and θ a modulus 1 complex valued function defined on H, which is constant over subsets of α-diameter less than 2.*

Proof. We consider H partitioned as defined in 2.1

$$H = \bigsqcup_{i=0}^{t} H_i,$$

with $t > 0$. We show that an isometry T on $\mathrm{Lip}^\alpha(H,d)$ or $\mathrm{lip}^\alpha(H,d)$, cannot be associated with a generalized bi-circular projection. Theorem 2.5 asserts that T is given as follows:

$$T(f)(\xi) = \begin{cases} \theta(\xi)[f(\phi(c_i)) - f(\phi(\xi))], & \text{if } \xi \in H_i \bigcap S_{c_i}, \\ \theta(c_i)(f(\phi(c_i))), & \text{if } \xi = c_i. \end{cases}$$

We consider a nontrivial disc, D_{c_1}. Let ξ be a point in D_{c_1} different from c_1. The isometry T must satisfy the equation $\lambda Id - (1+\lambda)T + T^2 = 0$. This implies that for every f and $\xi \in H$, we must have

$$\lambda f(\xi) - (1+\lambda)\theta[f(\phi(c_1)) - f(\phi(\xi))] + \theta^2[f(\phi^2(c_1)) - f(\phi^2(\xi)] = 0.$$

If we set f the constant function equal to 1, the equation above reduces to $\lambda = 0$. This leads to a contradiction, which completes the proof.

∎

The next two propositions provide a characterization of generalized bi-circular projections on the four spaces of Lipschitz functions. We first consider the spaces $\mathrm{Lip}^\alpha(H,d;h_0)$ and $\mathrm{lip}^\alpha(H,d;h_0)$.

Proposition 3.3. *If H is a compact metric space with at least four points, then a projection on*

$$\mathrm{Lip}^\alpha(H,d;h_0),\ \textit{or}\ \mathrm{lip}^\alpha(H,d;h_0)$$

is a generalized bi-circular projection if and only if it is the average of the identity operator with an isometric reflection.

Proof. If P is given as the average of the identity operator with an isometric reflection then it is a generalized bi-circular projection. We are then reduced to show that if P is a generalized bi-circular projection then $\lambda = -1$, thus P is given as the average of the identity operator with an isometric reflection. Proposition 3.2 describes the form of surjective isometries on $\mathrm{Lip}^\alpha(H,d;h_0)$, and $\mathrm{lip}^\alpha(H,d;h_0)$. A generalized bi-circular projection is associated with an isometry of the form

$$T(f)(x) = \theta[f(\phi(x)) - f(\phi(h_0))].$$

Therefore we must have

$$\theta^2[f(\phi^2(x)) - f(\phi^2(h_0))] - (1+\lambda)\theta[f(\phi(x)) - f(\phi(h_0))] + \lambda f(x) = 0. \tag{3.2}$$

If $\phi(h_0) = h_0$ and there exists $x \in H$ so that $x \neq h_0$, then (3.2) reduces to

$$\theta^2 f(\phi^2(x)) - (1+\lambda)\theta f(\phi(x)) + \lambda f(x) = 0.$$

If, in addition $\phi(x) \neq x$, then Lemma 2.3 assures the existence of f so that $f(\phi(x)) = 1$ and $f(x) = f(\phi^2(x)) = 0$. Therefore $\lambda = -1$ and P is given as the average of the Id with an isometric reflection. If, for every $x \neq h_0$, $\phi(x) = x$ then $\theta = 1$ or $\lambda = \theta$. Therefore P is the identity or the zero projection, respectively.

Now, we assume that $h_0 \neq \phi(h_0)$ and consider the following two cases:

[I.] There exists $x \in H$ so that $x \notin \{h_0, \phi(h_0), \phi^{-1}(h_0)\}$ and $\phi(x) \neq x$. Under these assumptions, Lemma 2.3 assures the existence of f so that $f(\phi(x)) = 1$ and $f(\phi^2(x)) = f(\phi^2(h_0)) = f(\phi(h_0)) = f(x) = 0$, leading to $\lambda = -1$.

[II.] For every $x \in H$ and $x \notin \{h_0, \phi(h_0), \phi^{-1}(h_0)\}$, $\phi(x) = x$. Under these assumptions, Lemma 2.3 assures the existence of f so that $f(\phi(h_0)) = 1$ and $f(\phi^2(h_0)) = f(x) = 0$ and (3.2) reduces to $(1+\lambda)\theta = 0$. This implies $\lambda = -1$ and completes the proof.

■

Remark 3.4. We observe that the previous proof only requires the existence of a point in H that is not in the set $\{h_0,\, \phi(h_0),\, \phi^{-1}(h_0)\}$, whenever $h_0 \neq \phi(h_0)$. Moreover, it only requires the existence of $x \neq h_0$ if $\phi(h_0) = h_0$. If H has at least four points these two assertions are necessarily true. We now study the remaining case, namely $H = \{h_0, \phi(h_0),\, \phi^2(h_0)\}$ and $\phi^3(h_0) = h_0$.

Proposition 3.5. *Let $H = \{h_0, \phi(h_0),\, \phi^2(h_0)\}$, so that $h_0 \neq \phi(h_0) \neq \phi^{-1}(h_0)$ and $\phi^3(h_0) = h_0$. If P is a generalized bi-circular projection on $\mathrm{Lip}^\alpha(H, d; h_0)$, or $\mathrm{lip}^\alpha(H, d; h_0)$, then*

$$P(f)(\xi) = \frac{1}{1-\lambda}\left[-\lambda f(\xi) + \overline{\lambda} f(\phi(\xi)) - \overline{\lambda} f(\phi(h_0))\right],$$

with $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Proof. Equation (3.2) evaluated at $x = \phi(h_0)$ yields:

$$[\theta^2 + (1+\lambda)\theta] f(\phi^2(h_0)) = [\lambda + (1+\lambda)\theta] f(\phi(h_0)). \tag{3.3}$$

Lemma 2.3 assures the existence of f so that $f(\phi(h_0)) = 1$ and $f(\phi^2(h_0)) = f(h_0) = 0$. Therefore $\theta = \overline{\lambda} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. The operator

$$P(f)(\xi) = \frac{1}{1-\lambda}\left[-\lambda f(\xi) + \overline{\lambda} f(\phi(\xi)) - \overline{\lambda} f(\phi(h_0))\right]$$

is a projection. Since $\lambda^2 = \overline{\lambda}$, we have

$$P^2(f)(\xi) = \frac{1}{(1-\lambda)^2}\{\overline{\lambda} f(\xi) - 2f(\phi(\xi)) + \lambda f(\phi^2(\xi)) + 2f(x) - \lambda f(\phi(x))\}.$$

Moreover, $P^2(f)(\xi) = P(f)(\xi)$ can be tested at each point. ∎

Remark 3.6. We observe that the projections derived in the previous proposition cannot be written as the average of the identity with an isometric reflection. In fact, if P was the average $\frac{Id+R}{2}$, then

$$R(f)(x) = \frac{1}{1-\lambda}\{-(1+\lambda) f(x) + 2\overline{\lambda} f(\phi(x)) - 2\overline{\lambda} f(\phi(h_0))\}.$$

It is easy to show that R does not preserve norm of f, if $f(h_0) = 0$ and $f(\phi(h_0)) = f(\phi^2(h_0)) = 1$.

Proposition 3.7. *If H is a compact metric space, then a projection on $\mathrm{Lip}^\alpha(H,d)$, or $\mathrm{lip}^\alpha(H,d)$ is a generalized bi-circular projection if and only if it is the average of the identity operator with an isometric reflection.*

Proof. As in Proposition 3.3, we are reduced to show that if P is a generalized bi-circular projection then $\lambda = -1$. Hence P is given as the average of the identity operator with an isometric reflection. Proposition 3.2 asserts that if T is an isometry associated with a generalized bi-circular projection then it must be of the form

$$T(f)(x) = \theta(x)(f(\phi(x)).$$

Therefore (3.1) reduces to

$$\theta(x)\theta(\phi(x))\, f(\phi^2(x)) - (1+\lambda)\theta(x) f(\phi(x)) + \lambda f(x) = 0.$$

Therefore, if $\lambda = -1$ then $\phi^2(x) = x$, for every x. If $\lambda \neq -1$ and there exists x_0 so that $\phi(x_0) \neq x_0$ then there exists $f \in \mathrm{lip}^\alpha(H,d)$ so that $f(\phi(x_0)) = 1$ and $f(x_0) = f(\phi^2(x_0)) = 0$. This leads to a contradiction. Consequently, whenever $\lambda \neq -1$, $\phi(x) = x$ for every x. Under such conditions, (3.1) reduces to $\theta(x)^2 - (1+\lambda)\theta(x) + \lambda = 0$, and $\theta(x) = 1$ or $\theta(x) = \lambda$. We set $H_1 = \{x \in H : \theta(x) = 1\}$ and $H_2 = \{x \in H : \theta(x) = \lambda\}$. This is possible whenever $d^\alpha(H_1, H_2) \geq 2$. Then

$$P(f)(x) = \begin{cases} f(x), & \text{for } x \in H_1, \\ 0, & \text{for } x \in H_2. \end{cases}$$

Therefore $P = \frac{Id+R}{2}$, with R an isometric reflection given by:

$$R(f)(x) = \begin{cases} f(x), & \text{for } x \in H_1, \\ -f(x), & \text{for } x \in H_2. \end{cases}$$

■

Propositions 3.3 and 3.7 allow us to state in the next theorem the complete characterization of generalized bi-circular projections in spaces of Lipschitz functions.

Theorem 3.8. *If H is a compact metric space with at least four points, then a projection on $\mathrm{Lip}^\alpha(H,d)$, $\mathrm{Lip}^\alpha(H,d;h_0)$, $\mathrm{lip}^\alpha(H,d)$, or $\mathrm{lip}^\alpha(H,d;h_0)$ is a generalized bi-circular projection if and only it is given by the average of the identity operator with an isometric reflection.*

Remark 3.9. It follows from Theorems 3.1 and 3.8 that every generalized bi-circular projection is bi-contractive but it remains open the question if every bi-contractive projection of these spaces of Lipschitz functions must be a generalized bi-circular projections.

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