STUDIA MATHEMATICA 192 (1) (2009)

Surjective isometries on spaces of differentiable vector-valued functions

by

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Abstract. This paper gives a characterization of surjective isometries on spaces of continuously differentiable functions with values in a finite-dimensional real Hilbert space.

1. Introduction. We consider the space of continuously differentiable functions on the interval $[0,1]$ with values in a Banach space E. This function space, equipped with the norm $\|f\|_1 = \max_{x\in[0,1]}\{\|f(x)\|_E + \|f'(x)\|_E\}$, is a Banach space, denoted by $C^{(1)}([0,1],E)$.

Banach and Stone obtained the first characterization of the isometries between spaces of scalar-valued continuous functions (see [2, 15]). Several researchers derived extensions of the Banach–Stone theorem to a variety of different settings. For a survey of this topic we refer the reader to [7]. Cambern and Pathak [4, 5] considered isometries on spaces of scalar-valued differentiable functions and gave a representation for the surjective isometries of such spaces. In this paper, we extend their result to the vector-valued function space $C^{(1)}([0,1],E)$, for E a finite-dimensional Hilbert space. We also characterize the generalized bi-circular projections on $C^{(1)}([0,1],E)$.

The characterization of the extreme points of the dual unit ball of a closed subspace of the continuous functions a compact Hausdorff space due to Arens and Kelley [6, p. 441] plays a crucial role in our proofs. In addition, the following result by de Leeuw which gives a converse of the Arens–Kelley theorem, for a closed subspace $\mathcal{X}$ of $C(\Omega)$ (cf. [11]), is also essential to our methods. To state de Leeuw's result we need the following definition.

DEFINITION 1.1. The point $\omega \in \Omega$ is said to be a *peak point* for $h \in \mathcal{X}$ if $h(\omega) = 1$, $|h(\omega_1)| \le 1$ for every $\omega_1 \in \Omega$, and $|h(\omega_1)| = 1$ at some $\omega_1 \neq \omega$ if and only if $|g(\omega_1)| = |g(\omega)|$ for all $g \in \mathcal{X}$.

2000 *Mathematics Subject Classification*: 46B04, 46E15.
Key words and phrases: surjective isometries, generalized bi-circular projections.

DOI: 10.4064/sm192-1-4

THEOREM 1.2 (cf. [11, p. 61]). *If $\omega \in \Omega$ is a peak point for some $h \in \mathcal{X}$, then the functional $\Phi \in \mathcal{X}^*$ defined by $\Phi(g) = g(\omega)$ is an extreme point of the unit ball in $\mathcal{X}^*$.*

We construct an isometric embedding of $C^{(1)}([0,1],E)$ onto a closed subspace of the space of scalar-valued continuous functions on a compact set. This allows us to describe the form of the extreme points of $C^{(1)}([0,1],E)^*_1$. We denote by B_1 the unit ball in a Banach space B. We consider the isometry F from $C^{(1)}([0,1],E)$ onto a subspace $\mathcal{M}$ of the scalar-valued continuous functions on $\Omega = [0,1]\times E^*_1 \times E^*_1$, with E^* equipped with the weak* topology,

$$F(f) = F_f(x,\varphi,\psi) = \varphi(f(x)) + \psi(f'(x)).$$

The surjective isometry on the dual spaces $F^*(F^*_f)(g) = F^*_f(F_g)$ maps the extreme points of $\mathcal{M}^*_1$ onto the extreme points of $C^{(1)}([0,1],E)^*_1$. It follows from the Arens–Kelley lemma [6, p. 441] that

$$\operatorname{ext}(\mathcal{M}^*_1) \subseteq \{\Phi_\omega : \Phi_\omega(F_f) = \varphi(f(x)) + \psi(f'(x)),\ \forall f \in C^{(1)}([0,1],E)\}.$$

PROPOSITION 1.3. *If E is a smooth, separable and reflexive Banach space, over the reals or complex numbers. Then Φ is an extreme point of $\mathcal{M}^*_1$ if and only if there exists $(x,\varphi,\psi) \in \Omega$, with φ and ψ extreme points of E^*_1, such that*

$$\Phi(f) = \varphi(f(x)) + \psi(f'(x)).$$

Proof. If Φ is an extreme point of $\mathcal{M}^*_1$, then $\Phi = \Phi_\omega$ for some $\omega = (x,\varphi,\psi) \in \Omega$. If φ (or ψ) is not an extreme point of E^*_1, then there must exist distinct functionals φ_1 and φ_2 in E^*_1 such that $\varphi = (\varphi_1+\varphi_2)/2$. For $i = 1,2$, we set $\omega_i = (x,\varphi_i,\psi)$ and

$$\Phi_{\omega_i}(F_f) = \varphi_i(f(x)) + \psi(f'(x)).$$

We have $\Phi = (\Phi_{\omega_1} + \Phi_{\omega_2})/2$ and

$$|\Phi_{\omega_i}(F_f)| \le |\varphi_i(f(x))| + |\psi(f'(x))| \le \|f(x)\|_E + \|f'(x)\|_E \le \|f\|_1 = \|F_f\|_\infty.$$

On the other hand, there exist $\mathbf{a}_i \in E_1$ $(i = 1,2)$ so that $|\varphi_i(\mathbf{a}_i)| = 1$. Thus, if f_i is the constant function equal to $\mathbf{a}_i$, then $|\Phi_{\omega_i}(F_{f_i})| = 1$ and $\Phi_{\omega_i} \in \mathcal{M}^*_1$. Thus Φ is not an extreme point of $\mathcal{M}^*_1$, contradicting our initial assumption. Similar reasoning applies if $\psi \notin \operatorname{ext}(E^*_1)$.

Now we show that Φ given by

$$\Phi(f) = \varphi(f(x)) + \psi(f(x)),$$

with $\omega = (x,\varphi,\psi) \in \Omega$ and $\varphi,\psi \in \operatorname{ext}(E^*_1)$, is an extreme point of $\mathcal{M}^*_1$. There exist $\mathbf{a}_1$ and $\mathbf{a}_2$ in E_1 such that $\varphi(\mathbf{a}_1) = e^{i\alpha_1}$ and $\psi(\mathbf{a}_2) = e^{i\alpha_2}$. We define $f \in C^{(1)}([0,1],E)$ by

$$f(t) = \frac{e^{-i\alpha_1}\mathbf{a}_1 + \lambda(t)e^{-i\alpha_2}\mathbf{a}_2}{2}$$

with

$$\lambda(t) = \begin{cases} -\frac{1}{2}(x^2 - t^2) + (x-1)(x-t) & \text{for } 0 \le t \le x, \\ -\frac{1}{2}(t^2 - x^2) + (x+1)(t-x) & \text{for } x \le t \le 1. \end{cases} \tag{1.1}$$

We observe that $\lambda(x) = 0$, $\lambda'(x) = 1$, and $|\lambda(t)| + |\lambda'(t)| = 1 - \frac{1}{2}(x-t)^2 < 1$ for all $t \ne x$. Therefore

$$|F_f(\omega)| = |\varphi(f(x)) + \psi(f'(x))| = 1.$$

If $\omega_1 \ne \omega$ with $\omega_1 = (x_1,\varphi_1,\psi_1)$ and $x_1 \ne x$, we have

$$\begin{aligned} |F_f(\omega_1)| &= |\varphi_1(f(x_1)) + \psi_1(f'(x_1))| \\ &= \left|\varphi_1\left(\frac{e^{-i\alpha_1}\mathbf{a}_1 + \lambda(x_1)e^{-i\alpha_2}\mathbf{a}_2}{2}\right) + \psi_1\left(\frac{\lambda'(x_1)e^{-i\alpha_2}\mathbf{a}_2}{2}\right)\right| \\ &\le \frac{1}{2} + \frac{|\lambda(x_1)| + |\lambda'(x_1)|}{2} < 1. \end{aligned}$$

If $x_1 = x$, and $\varphi_1 \ne \varphi$ or $\psi_1 \ne \psi$, then

$$\begin{aligned} |F_f(\omega_1)| &= |\varphi_1(f(x_1)) + \psi_1(f'(x_1))| \\ &= \left|\varphi_1\left(\frac{e^{-i\alpha_1}\mathbf{a}_1}{2}\right) + \psi_1\left(\frac{e^{-i\alpha_2}\mathbf{a}_2}{2}\right)\right| < 1, \end{aligned}$$

unless $|\varphi_1(e^{-i\alpha_1}\mathbf{a}_1)| = 1$ and $|\psi_1(e^{-i\alpha_2}\mathbf{a}_2)| = 1$. The conclusion now follows from Theorem 1.2. ∎

An extreme point of $\mathcal{M}^*_1$ is therefore represented by a triplet $(x,\varphi,\psi) \in \Omega$, with $x \in [0,1]$ and φ, ψ extreme points of E^*_1. Given the hypothesis on E we know that $\operatorname{ext}(E^*_1) = E^*_1$. If T is a surjective isometry of $C^{(1)}([0,1],E)$, then T^* maps extreme points to extreme points. Hence Proposition 1.3 asserts that given $\omega = (x,\varphi,\psi)$ there exists $\omega_1 = (x_1,\varphi_1,\psi_1)$ such that

$$\varphi[(Tf)(x)] + \psi[(Tf)'(x)] = \varphi_1(f(x_1)) + \psi_1(f'(x_1)) \tag{1.2}$$

for every $f \in C^{(1)}(\Omega,E)$.

This determines a transformation τ, on $\Omega = [0,1] \times E^*_1 \times E^*_1$, associated with the isometry T and given by

$$\tau(x,\varphi,\psi) = (x_1,\varphi_1,\psi_1).$$

LEMMA 1.4. *τ is a homeomorphism.*

Proof. We first observe that τ is well defined. Suppose there exist two triplets $\omega_1 = (x_1,\varphi_1,\psi_1)$ and $\omega_2 = (x_2,\varphi_2,\psi_2)$, both corresponding to $\omega = (x,\varphi,\psi)$. Then

$$\varphi_1(f(x_1)) + \psi_1(f'(x_1)) = \varphi_2(f(x_2)) + \psi_2(f'(x_2)). \tag{1.3}$$

If $x_1 \ne x_2$, we select a function $f \in C^{(1)}([0,1],E)$ constant equal to $\mathbf{a}$, an arbitrary vector in E_1, on a neighborhood of x_1, say O_{x_1}, and equal to zero on a neighborhood of x_2, say $\mathcal{O}_{x_2}$, with $\mathcal{O}_{x_1} \cap \mathcal{O}_{x_2} = \emptyset$. Equation (1.3) implies that $\varphi_1(\mathbf{a}) = 0$, so $\varphi = 0$. This contradicts $\varphi \in E_1^*$ and shows that $x_1 = x_2$. If f is now chosen to be constant equal to $\mathbf{a}$, an arbitrary vector in E_1, then (1.3) reduces to $\varphi_1(\mathbf{a}) = \varphi_2(\mathbf{a})$, thus $\varphi_1 = \varphi_2$. If f is given by $f(x) = (x - x_1)\mathbf{a}$ then (1.3) implies that $\psi_1 = \psi_2$. Therefore τ is well defined. Similar arguments and the invertibility of T imply that τ is a bijection. The continuity of τ follows from the weak* continuity of T^*. ∎

2. Properties of the homeomorphism τ. In this section we explore further properties of the homeomorphism τ to be used in our characterization of surjective isometries on $C^{(1)}([0,1], E)$, with E a real and finite-dimensional Hilbert space.

For a fixed $x \in [0,1]$ we define the map $\tau_x : E_1^* \times E_1^* \to [0,1]$ by $\tau_x(\varphi, \psi) = \pi_1 \tau(x, \varphi, \psi)$, with π_1 representing the projection on the first component.

The next lemma holds for a finite-dimensional Banach range space, the proof does not require an inner product structure.

LEMMA 2.1. *If E a finite-dimensional Banach space, then τ_x is a constant function.*

Proof. If τ_x is not constant, then its image is a non-degenerate subinterval of $[0,1]$. We select a basis for E^*, say $\{\varphi_1, \ldots, \varphi_k\}$, consisting of functionals of norm 1. We select an element

$$y \in \tau_x(E_1^* \times E_1^*) \setminus \{\tau_x(\varphi_i, \varphi_i), \tau_x(\varphi_i, -\varphi_i)\}_{i=1,\ldots,k}.$$

Then we set $\tau(x, \varphi_i, \varphi_i) = (x_i, \eta_i, \xi_i)$, $\tau(x, \varphi_i, -\varphi_i) = (y_i, \alpha_i, \beta_i)$, and $\tau(x, \varphi_0, \psi_0) = (y, \eta, \xi)$. We select $g \in C^{(1)}([0,1], E)$ such that, for all $i = 1, \ldots, k$, $g(x_i) = g(y_i) = g'(x_i) = g'(y_i) = 0_E$, $g(y) = u$ and $g'(y) = v$, where u and v are such that $\eta(u) = 1$ and $\xi(v) = 1$. Therefore we have

$$\varphi_i((Tg)(x)) + \varphi_i((Tg)'(x)) = \eta_i(g(x_i)) + \xi_i(g'(x_i)) = 0$$

and

$$\varphi_i((Tg)(x)) - \varphi_i((Tg)'(x)) = \alpha_i(g(y_i)) + \beta_i(g'(y_i)) = 0.$$

These equations imply that $\varphi_i((Tg)(x)) = 0$ and $\varphi_i((Tg)'(x)) = 0$ for all i. Hence $Tg(x) = (Tg)'(x) = 0_E$, implying that $2 = \eta(g(y)) + \xi(g'(y)) = 0$. This contradiction establishes the claim. ∎

For fixed $x \in [0,1]$ and $\varphi \in E_1^*$, we define the map $\tau_{(x,\varphi)} : E_1^* \to E_1^*$ by

$$\tau_{(x,\varphi)}(\psi) = \varphi_1 \quad \text{provided that} \quad \tau(x, \varphi, \psi) = (x_1, \varphi_1, \psi_1).$$

LEMMA 2.2. *If E is a finite-dimensional real Hilbert space then, for any fixed $x \in [0,1]$ and $\varphi \in E_1^*$, $\tau_{(x,\varphi)}$ is constant.*

Proof. The Riesz Representation Theorem allows us to set notation as follows: $\varphi, \psi \in E_1^*$ are completely determined by the inner product with a single vector u, v respectively. Hence we define $\tau : [0,1] \times E_1 \times E_1 \to [0,1] \times E_1 \times E_1$ by $\tau(x, u, v) = (x_1, u_1, v_1)$, and for every $f \in C^{(1)}([0,1], E)$,

$$\langle (Tf)(x), u \rangle + \langle (Tf)'(x), v \rangle = \langle f(x_1), u_1 \rangle + \langle f'(x_1), v_1 \rangle. \tag{2.1}$$

For fixed x and u, we let $F_{(x,u)} : E_1 \to E_1$ be given by $F_{(x,u)}(v) = \pi_2(\tau(x, u, v))$, where π_2 is the projection on the second component. We prove the lemma by showing that $F_{(x,u)}$ is constant. For simplicity we denote $F_{(x,u)}$ by just F, unless the dependence on x, u has to be emphasized.

We choose $f(t) = \mathbf{a}$, a unit vector. Then

$$\langle (Tf)(x), u \rangle + \langle (Tf)'(x), \pm v \rangle = \langle \mathbf{a}, F(\pm v) \rangle.$$

This implies that

$$\langle (Tf)(x), u \rangle = \left\langle \mathbf{a}, \frac{F(v) + F(-v)}{2} \right\rangle$$

for every $v \in E_1$. The function $G : E_1 \to E$ defined by $G(v) = F(v) + F(-v)$ is therefore constant, denoted by w. As a consequence, for every v_0 and v_1 in E_1, we have

$$\langle F(v_0), F(-v_0) \rangle = \langle F(v_1), F(-v_1) \rangle, \qquad \langle F(v_0), F(v_1) \rangle = \langle F(-v_0), F(-v_1) \rangle.$$

Therefore

$$\|F(v_0) - F(-v_0)\|^2 = 2 - 2\langle F(v_0), F(-v_0) \rangle = 2 - 2\langle F(v_1), F(-v_1) \rangle$$

and

$$\|F(v_0) - F(-v_0)\|^2 = \|F(v_1) - F(-v_1)\|^2.$$

Moreover, the function $H : E_1 \to \mathbb{R}$ given by $H(v) = \|F(v) - F(-v)\|$ is also constant. This implies that

$$\langle F(v_0) - F(-v_0), F(v_0) + F(-v_0) \rangle = 0.$$

If $v \in E_1$ is such that $\{F(v), F(-v)\}$ is linearly independent, we set Π_v to be the two-dimensional space spanned by $F(v)$ and $F(-v)$. Clearly $w \in \Pi_v$. In the plane Π_v, we represent $F(v)$ by $(w/\|w\|)e^{i\alpha}$ and $F(-v)$ by $(w/\|w\|)e^{-i\alpha}$. This is the polar representation for $F(v)$ and $F(-v)$ in Π_v, with w identified with the positive direction of the x-axis. Without loss of generality, we choose $\alpha \in (0, \pi)$. This, in particular, implies that $w = F(v) + F(-v) = (2\cos(\alpha)/\|w\|)w$ and $2\cos(\alpha) = \|w\|$. The value of α is then uniquely determined, so $\{F(v), F(-v)\}$ are the only two values in the range of F belonging to the plane Π_v. The line $\overline{Ow}$ divides the line segment $\overline{F(v)F(-v)}$ into two equal parts. Since G is a constant function we have

$$F(E_1) \subseteq \left(\frac{w}{2} + \{w\}^\perp\right) \cap \mathcal{S}\left(\frac{w}{2}, \frac{\|F(v) - F(-v)\|}{2}\right)$$

with $\mathcal{S}(x, \delta)$ representing the set of points in E at distance δ from $x \in E$, and $\{w\}^{\perp}$ representing the space orthogonal to the span of w. We also notice that $F(v_0) \neq \pm F(-v_0)$ for every v_0.

These considerations imply that F maps the $n-1$-sphere $\mathrm{ext}(E_1)$ to a set homeomorphic to a subset of an $n-2$-sphere, and F sends antipodal points to antipodal points. We now show that such a map cannot exist.

First, for $n = 2$ this would mean that F would map S^1 onto two points, which is impossible since S^1 is connected and F is continuous. The general case is a consequence of the Borsuk–Ulam Theorem (see [13, p. 266]).

Therefore $\{F(v), F(-v)\}$ is linearly dependent, and as a consequence, we consider the following two possibilities:

(i) $F(v) = F(-v)$ for every v,
(ii) $F(-v) = -F(v)$ for every v.

In case (i), we have $F(v) = w/2$ for every v, so F is constant.

In case (ii), given two different vectors v_0 and v_1 in E_1 we have

$$\langle (Tf)(x), u\rangle + \left\langle (Tf)'(x), \frac{v_0+v_1}{\|v_0+v_1\|}\right\rangle = \left\langle a, F\left(\frac{v_0+v_1}{\|v_0+v_1\|}\right)\right\rangle,$$
$$\langle (Tf)(x), u\rangle - \left\langle (Tf)'(x), \frac{v_0+v_1}{\|v_0+v_1\|}\right\rangle = \left\langle a, F\left(-\frac{v_0+v_1}{\|v_0+v_1\|}\right)\right\rangle.$$

Hence $\langle (Tf)(x), u\rangle = 0$ and

$$\left\langle (Tf)'(x), \frac{v_0+v_1}{\|v_0+v_1\|}\right\rangle = \left\langle a, F\left(\frac{v_0+v_1}{\|v_0+v_1\|}\right)\right\rangle.$$

This implies that

$$\text{(2.2)}\quad \frac{F(v_0)+F(v_1)}{\|v_0+v_1\|} = F\left(\frac{v_0+v_1}{\|v_0+v_1\|}\right), \qquad \|F(v_0)+F(v_1)\| = \|v_0+v_1\|.$$

Equation (2.2) implies that

$$\langle F(v_0), F(v_1)\rangle = \langle v_0, v_1\rangle,$$

or F is norm preserving. We define a map $\Theta : [0,1]\times E_1 \to C(E_1, E_1)$ by $\Theta(x,u)(v) = F_{(x,u)}(v)$. It follows from Lemma 1.4 that Θ is continuous. Furthermore, we have shown that, for each $(x,u) \in [0,1]\times E_1$, $\Theta(x,u)$ is either constant or an isometry in E_1.

The continuity of Θ and the connectedness of the domain $[0,1]\times E_1$ implies that the range of Θ consists only of constant functions on $C(E_1,E_1)$ or only of norm preserving functions on E_1 that map antipodal points onto antipodal points. This last assertion follows from the fact that the distance between one such norm preserving map on E_1 and a constant function is greater than or equal to $\sqrt{2}$. In fact, let $F_{x_0,u_0} = \Theta(x_0,u_0)$ be a constant function, everywhere equal to $\mathbf{a}$, and $F_{x_1,u_1} = \Theta(x_1,u_1)$ be norm preserving on E_1 with $F_{x_1,u_1}(-v) = -F_{x_1,u_1}(v)$ for all $v \in E_1$. Then we have

$$\|F_{x_0,u_0} - F_{x_1,u_1}\|_\infty = \max_{v\in E_1}\{\|F_{x_0,u_0}(v) - F_{x_1,u_1}(v)\|_E\}.$$

Furthermore,

$$\|F_{x_0,u_0}(v) - F_{x_1,u_1}(v)\|_E = \|\mathbf{a} - F_{x_1,u_1}(v)\|_E,$$
$$\|F_{x_0,u_0}(-v) - F_{x_1,u_1}(-v)\|_E = \|\mathbf{a} + F_{x_1,u_1}(v)\|_E,$$

implying that

$$4 = \|\mathbf{a} - F_{x_1,u_1}(v)\|_E^2 + \|\mathbf{a} + F_{x_1,u_1}(v)\|_E^2 \le 2\max\{\|\mathbf{a} \pm F_{x_1,u_1}(v)\|_E^2\}.$$

Consequently,

$$\|F_{x_0,u_0} - F_{x_1,u_1}\|_\infty \ge \sqrt{2}.$$

As mentioned before, this implies that the range of Θ contains only constant functions or only norm preserving maps. Now we show that the assumption that the range of Θ contains only norm preserving maps that send antipodal points to antipodal points leads to a contradiction.

In fact, if the range of Θ contains only such maps, then for a fixed constant function f on $[0,1]$ equal to $\mathbf{a} \in E_1$, we have

$$\langle (Tf)(x), u\rangle + \langle (Tf)'(x), v\rangle = \langle a, F_{(x,u)}(v)\rangle$$

and

$$\langle (Tf)(x), u\rangle - \langle (Tf)'(x), v\rangle = \langle a, F_{(x,u)}(-v)\rangle.$$

Therefore $\langle (Tf)(x), u\rangle = 0$ for all u and x, and so Tf is zero. This completes the proof. ∎

REMARK 2.3. We mention that we can also prove, following a similar strategy, that for a fixed $x \in [0,1]$ and $\psi \in E_1^*$, the map $\tau_{(x,\psi)} : E_1^* \to E_1^*$ such that

$$\tau_{(x,\psi)}(\varphi) = \psi_1$$

is constant. This result is stated in Lemma 3.2 of the next section.

3. Surjective isometries of $C^{(1)}([0,1],E)$. In this section we establish that surjective isometries on $C^{(1)}([0,1],E)$ are composition operators. First, we prove preliminary results about surjective isometries on these spaces. The space E is a finite-dimensional Hilbert space. The Riesz Representation Theorem allows us to associate a unique unit vector to each functional in E_1^*. Then we represent $\tau : [0,1]\times E_1\times E_1 \to [0,1]\times E_1\times E_1$ given by $\tau(x,u,v) = \tau(x,u_1,v_1)$ with u, v, u_1, v_1 corresponding to φ, ψ, φ_1, ψ_1 respectively.

LEMMA 3.1. *If E is a finite-dimensional real Hilbert space and T is a surjective isometry on $C^{(1)}([0,1],E)$ then T maps constant functions to constant functions.*

Proof. We assume that there exists a constant function $f \in C^{(1)}([0,1],E)$ with $f(t) = \mathbf{a}$, a vector in E, such that Tf is not constant. This means there exists $x_0 \in [0,1]$ such that $(Tf)'(x_0) \neq 0_E$. We choose a vector v_0 in E_1 orthogonal to $(Tf)'(x_0)$, i.e. $\langle (Tf)'(x_0), v_0\rangle = 0$. We set $\tau(x_0,u,v) = (x_1,u_1,v_1)$. Then

$$\langle (Tf)(x_0), u\rangle + \langle (Tf)'(x_0), v\rangle = \langle \mathbf{a}, u_1\rangle. \tag{3.1}$$

Lemma 2.2 implies that

$$\langle (Tf)(x_0), u\rangle + \langle (Tf)'(x_0), v_0\rangle = \langle \mathbf{a}, u_1\rangle. \tag{3.2}$$

Therefore $\langle (Tf)(x_0), u\rangle = \langle \mathbf{a}, u_1\rangle$ and $\langle (Tf)'(x_0), v\rangle = 0$ for every v. This contradicts our initial assumption that $(Tf)'(x_0) \neq 0_E$, and completes the proof. ∎

For a fixed $x \in [0,1]$ and $v \in E_1$, we define $\tau_{(x,v)} : E_1 \to E_1$ by

$$\tau_{(x,v)}(u) = v_1 \quad \text{provided that} \quad \tau(x,u,v) = (x_1,u_1,v_1).$$

LEMMA 3.2. *If E is a finite-dimensional real Hilbert space then, for fixed $x \in [0,1]$ and $v \in E_1$, $\tau_{(x,v)}$ is constant.*

Proof. We follow the steps in the proof of Lemma 2.2 with the following modification. We consider functions of the form $f(t) = (t - x_1)\mathbf{a}$ with $\mathbf{a}$ representing some unit vector in E, and set $F(u) = v_1$ with u and v_1 associated with the functions φ and ψ_1, respectively. A similar strategy to that followed in Lemma 2.2 allows us to conclude that either F is constant or $(Tf)'$ is zero. If $(Tf)'$ is zero, then Tf is constant. Lemma 3.1 and the surjectivity of T imply that f must be constant. This contradiction completes the proof. ∎

LEMMA 3.3. *If E is a finite-dimensional real Hilbert space, x and x_1 are such that $\tau(x,u,v) = (x_1,u_1,v_1)$, and $f \in C^{(1)}([0,1],E)$, then $f(x_1) = 0_E$ implies that $(Tf)(x) = 0_E$.*

Proof. Equation (2.1) reduces to

$$\langle (Tf)(x), u\rangle + \langle (Tf)'(x), v\rangle = \langle f'(x_1), v_1\rangle.$$

Now considering $u_0 \in E_1$, Lemmas 2.1 and 3.2 imply that

$$\langle (Tf)(x), u_0\rangle + \langle (Tf)'(x), v\rangle = \langle f'(x_1), v_1\rangle.$$

Therefore $\langle (Tf)(x), u - u_0\rangle = 0$. Since u_0 is chosen arbitrarily in E_1 we conclude that $T(f)(x) = 0_E$. ∎

LEMMA 3.4. *If E is a finite-dimensional real Hilbert space and T is a surjective isometry on $C^{(1)}([0,1],E)$, then there exists a surjective isometry U on E and a homeomorphism σ on the interval $[0,1]$ such that*

$$T(f)(t) = U(f(\sigma(t)))$$

for every $f \in C^{(1)}([0,1],E)$.

Proof. We define $U(v) = T(\tilde{v})(0)$ with $\tilde{v}$ representing the constant function equal to v. Since T is a surjective isometry, U is also a surjective isometry on E. Given $f \in C^{(1)}([0,1],E)$ and $x_1 \in [0,1]$ we denote by f_1 the function given by $f_1(t) = f(t) - f(x_1)$. Lemma 3.3 implies that $T(f_1)(x) = 0_E$. Therefore

$$T(f)(x) = U(f(x_1)).$$

We set $\sigma(x) = x_1$; Lemmas 1.4 and 2.1 imply that σ is a homeomorphism. ∎

THEOREM 3.5. *If E is a finite-dimensional real Hilbert space, then T is a surjective isometry on $C^{(1)}([0,1],E)$ if and only if there exists a surjective isometry on E such that for every f,*

$$T(f)(x) = U(f(\sigma(x)))$$

with $\sigma = \mathrm{Id}$ or $\sigma = 1 - \mathrm{Id}$.

Proof. It is clear that a composition operator of the form described in the theorem is a surjective isometry on $C^{(1)}([0,1],E)$. Conversely, if T is a surjective isometry then Lemma 3.4 asserts the existence of a surjective isometry U on E and a homeomorphism σ on the interval $[0,1]$ such that

$$T(f)(t) = U(f(\sigma(t)))$$

for every $f \in C^{(1)}([0,1],E)$. In particular, if $f(x) = x\mathbf{a}$ with $\mathbf{a}$ an arbitrary vector in E, then $T(f)(x) = \sigma(x)U(\mathbf{a})$. This implies that σ is continuously differentiable. Similar considerations applied to T^{-1} imply that σ^{-1} is also continuously differentiable. Therefore σ is a diffeomorphism of $[0,1]$. Since $\|T(f)\|_1 = \max_x\{\|Tf(x)\|_E + \|(Tf)'(x)\|_E\}$ and $Tf(x) = Uf(\sigma(x))$ with U an isometry on E, we have

$$\begin{aligned}\|Tf\|_1 &= \max\{\|Uf(\sigma(x))\|_E + \|Uf'(\sigma(x))\|_E|\sigma'(x)|\} \\ &= \max\{\|f(\sigma(x))\|_E + \|f'(\sigma(x))\|_E|\sigma'(x)|\}\end{aligned} \tag{3.3}$$

and

$$\|f\|_1 = \max\{\|f(x)\|_E + \|f'(x)\|_E\} = \|f(x_0)\|_E + \|f'(x_0)\|_E$$

for some $x_0 \in [0,1]$. Therefore $|\sigma'(\sigma^{-1}(x_0))| \leq 1$. On the other hand, $T^{-1}(f)(x) = U^{-1}f(\sigma^{-1}(x))$ and

$$\begin{aligned} rl\|T^{-1}f\|_1 &= \max\{\|U^{-1}f(\sigma^{-1}(x))\|_E + \|U^{-1}f'(\sigma^{-1}(x))\|_E|(\sigma^{-1})'(x)|\} \\ &= \max\{\|f(\sigma^{-1}(x))\|_E + \|f'(\sigma^{-1}(x))\|_E|(\sigma^{-1})'(x)|\}.\end{aligned}$$

Therefore $|(\sigma^{-1})'(\sigma(x_0))| = 1/|\sigma'(x_0)| \leq 1$ and so $|\sigma'(x_0)| \geq 1$. To conclude that $|\sigma'(x)| = 1$ for all x, we need to show that for every $x \in [0,1]$ there exists f such that $\|f\|_1 = \|f(x)\|_E + \|f'(x)\|_E$ and $\|f'(x)\|_E \neq 0$. We consider

$f_x(t) = \lambda_x(t)\mathbf{a}$ with $\mathbf{a}$ a unit vector in E and λ_x given as in (1.1)

$$\lambda_x(t) = \begin{cases} -\frac{1}{2}(x^2 - t^2) + (x-1)(x-t) & \text{for } 0 \le t \le x, \\ -\frac{1}{2}(t^2 - x^2) + (x+1)(t-x) & \text{for } x \le t \le 1. \end{cases} \tag{3.4}$$

Hence $|\sigma'| = 1$ and so $\sigma = \mathrm{Id}$ or $= 1 - \mathrm{Id}$. ∎

REMARK 3.6. If the range space E is an infinite-dimensional separable Hilbert space then there are nonsurjective isometries. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis and U be the operator defined by $U(e_n) = e_{2n}$. The isometry $T : C^{(1)}([0,1],E) \to C^{(1)}([0,1],E)$ given by $T(f)(x) = U(f(x))$ is not surjective. It is not clear, whenever E is finite-dimensional, if there are isometries on $C^{(1)}([0,1],E)$ which are not surjective.

Theorem 3.5 was stated for range spaces that are finite-dimensional Hilbert spaces over the reals, and we now extend our characterization to finite-dimensional Hilbert spaces over the complex numbers.

COROLLARY 3.7. *If E is a finite-dimensional complex Hilbert space, then T is a surjective isometry on $C^{(1)}([0,1],E)$ if and only if there exists a surjective isometry U on E such that, for every f,*

$$T(f)(x) = U(f(\sigma(x)))$$

with $\sigma = \mathrm{Id}$ or $\sigma(x) = 1 - \mathrm{Id}$.

Proof. The space E is equipped with an inner product over $\mathbb{C}$, denoted by $\langle\cdot,\cdot\rangle$. This inner product induces a norm on E, denoted by $\|\cdot\|$, and the norm $\|\cdot\|_1$ is defined on the space $C^{(1)}([0,1],E)$. We define another inner product $(\cdot,\cdot)$ on E by

$$(u,v) = \mathrm{Re}\,\langle u,v\rangle.$$

The space E with multiplication only by real scalars and equipped with this real inner product $(\cdot,\cdot)$, is a Hilbert space, denoted by $\widetilde{E}$. The induced norm is denoted by $|\!|\!|\cdot|\!|\!|$ and

$$|\!|\!|f|\!|\!|_1 = \sup_{x\in[0,1]} \{|\!|\!|f(x)|\!|\!| + |\!|\!|f'(x)|\!|\!|\}$$

is the corresponding norm on $C^{(1)}([0,1],\widetilde{E})$. The identity map id : $(E, |\!|\!|\cdot|\!|\!|) \to (E, \|\cdot\|)$ is real linear. Furthermore, given $u \in E$ we have

$$|\!|\!|u|\!|\!|^2 = (u,u) = \mathrm{Re}\,\langle u,u\rangle = \|u\|^2.$$

Consequently, $(\widetilde{E}, |\!|\!|\cdot|\!|\!|)$ and $(E, \|\cdot\|)$ are linearly isometric as real Banach spaces. If T is a surjective isometry on $C^{(1)}([0,1],E)$, then $\widetilde{T}$, on $C^{(1)}([0,1],\widetilde{E})$, given by $\widetilde{T}(f) = T(f)$ is also a surjective isometry. In fact,

$$|\!|\!|\widetilde{T}f|\!|\!|_1 = \sup_{t\in[0,1]} \{|\!|\!|\widetilde{T}f(t)|\!|\!| + |\!|\!|(\widetilde{T}f)'(t)|\!|\!|\} = \sup_{t\in[0,1]} \{\|f(t)\| + \|f'(t)\|\} = \|f\|_1.$$

Theorem 3.5 now asserts that there exists a real isometry U on $\widetilde{E}$ and $\sigma = \mathrm{Id}$ or $1 - \mathrm{Id}$ so that $\widetilde{T}(f)(t) = U(f(\sigma(t)))$. Then it follows that $T(f)(t) = U(f(\sigma(t)))$. It also follows that U is a complex linear isometry by considering constant functions. This concludes the proof. ∎

4. Generalized bi-circular projections on $C^{(1)}([0,1],E)$. In this section we give a characterization of all generalized bi-circular projections on $C^{(1)}([0,1],E)$ with E a finite-dimensional complex Hilbert space. We starting by reviewing the definition of generalized bi-circular projection.

DEFINITION 4.1 (cf. [8]). A bounded linear projection P on $C^{(1)}([0,1],E)$ is said to be a *generalized bi-circular projection* if and only if there exists a modulus 1 complex number λ, different from 1, so that $P + \lambda(\mathrm{Id} - P)$ is an isometry T on $C^{(1)}([0,1],E)$.

The isometry T must satisfy the following operator quadratic equation:

$$T^2 - (1+\lambda)T + \lambda\,\mathrm{Id} = 0.$$

Since T is a surjective isometry, Theorem 3.5 implies the existence of a surjective isometry U on E such that

$$U^2 f(x) - (1+\lambda)U(f(\sigma(x)) + \lambda f(x) = 0.$$

Therefore if $\lambda = -1$ then $U^2 = \mathrm{Id}$ and P is the average of the identity with an isometric reflection $R(f)(x) = U(f(\sigma(x)))$. If $\lambda \neq -1$, then $\sigma(x) = x$ for every $x \in [0,1]$ and $U^2 - (1+\lambda)U + \lambda\,\mathrm{Id} = 0$. Hence

$$P(f) = \frac{U - \lambda\,\mathrm{Id}}{1-\lambda} f(x).$$

We summarize the previous considerations in the following proposition.

PROPOSITION 4.2. *Let E be a finite-dimensional complex Hilbert space. Then P is a generalized bi-circular projection on $C^{(1)}([0,1],E)$ if and only if there exists a generalized bi-circular projection P_E on E so that $Pf(x) = P_E(f(x))$.*

REMARK 4.3. We wish to thank the referee for several helpful suggestions that resulted in a substantial improvement of this paper. The referee also suggested that the proof of our main result could be shortened by using results by Jarosz and Pathak in [9].

References

[1] R. Arens and J. Kelley, *Characterization of spaces of continuous functions over compact Hausdorff spaces*, Trans. Amer. Math. Soc. 62 (1947), 499–508.

[2] S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.

[3] F. Botelho and J. E. Jamison, *Generalized bi-circular projections on* $C(\Omega, X)$, Rocky Mountain J. Math. (2008), to appear.
[4] M. Cambern, *Isometries of certain Banach algebras*, Studia Math. 25 (1965), 217–225.
[5] M. Cambern and V. D. Pathak, *Isometries of spaces of differentiable functions*, Math. Japon. 26 (1981), 253–260.
[6] N. Dunford and J. T. Schwartz, *Linear Operators Part I: General Theory*, Interscience, New York, 1958.
[7] R. Fleming and J. Jamison, *Isometries on Banach Spaces*: Vol. I, *Function Spaces*, Vol. II, *Vector-Valued, Function Spaces*, Chapman & Hall, 2003, 2007.
[8] M. Fosner, D. Ilišević and C. K. Li, *G-invariant norms and bicircular projections*, Linear Algebra Appl. 420 (2007), 596–608.
[9] K. Jarosz and V. Pathak, *Isometries between function spaces*, Trans. Amer. Math. Soc. 305 (1988), 193–206.
[10] K. S. Lau, *A representation theorem for isometries of* $C(X, E)$, Pacific J. Math. 60 (1975), 229–223.
[11] K. de Leeuw, *Banach spaces of Lipschitz functions*, Studia Math. 21 (1961), 55–66.
[12] N. V. Rao and A. K. Roy, *Linear isometries on some function spaces*, Pacific J. Math. 38 (1971), 177–192.
[13] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.
[14] L. L. Stacho and B. Zalar, *Bicircular projections and characterization of Hilbert spaces*, Proc. Amer. Math. Soc. 132 (2004), 3019–3025.
[15] M. Stone, *A general theory of spectra, II*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 83–87.

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Received June 2, 2008
Revised version October 14, 2008 (6366)

STUDIA MATHEMATICA 192 (1) (2009)

Weighted variable L^p integral inequalities for the maximal operator on non-increasing functions

by

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Abstract. Let B_p be the Ariño–Muckenhoupt weight class which controls the weighted L^p-norm inequalities for the Hardy operator on non-increasing functions. We replace the constant p by a function $p(x)$ and examine the associated $L^{p(x)}$-norm inequalities of the Hardy operator.

1. Introduction. The weights $w : \mathbb{R}_+ \to \mathbb{R}_+$ for which the Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

on non-negative non-increasing functions f (we write simply $f\downarrow$) is bounded:

$$\int_0^\infty Hf(x)^p w(x)\,dx \le c_* \int_0^\infty f(x)^p w(x)\,dx, \quad 1 \le p < \infty, \tag{1}$$

have been characterized by Ariño and Muckenhoupt [1] by the condition

$$w \in B_p : \quad \int_r^\infty \left(\frac{r}{x}\right)^p w(x)\,dx \le c\int_0^r w(x)\,dx. \tag{2}$$

A different proof of (1)⇔(2) was given by me in [7] where it is also apparent that in the implication (2)⇒(1) the constant c_* can be taken to be $(c+1)^p$. For (1)⇒(2) one uses the test function $f = \chi_{[0,r]}$ and (2) follows with $c = c_*$. We also note that for $f\downarrow$, $Hf(x)$ equals $Mf(x)$, the Hardy–Littlewood maximal function.

In the past few years a great deal of attention has been paid to the problem of the boundedness of M on variable L^p-spaces. If $p : \mathbb{R}^n \to [1,\infty)$ and $w : \mathbb{R}^n \to \mathbb{R}_+$, let $L^{p(x)}(w)$ be the collection of all functions $f : \mathbb{R}^n \to \mathbb{R}$

2000 *Mathematics Subject Classification*: Primary 42B25.
Key words and phrases: weights, Hardy operator, variable L^p.

DOI: 10.4064/sm192-1-5