Arch. Math. 90 (2008), 341–352
© 2008 Birkhäuser Verlag Basel/Switzerland
0003/889X/040341-12, *published online* 2008-03-20
DOI 10.1007/s00013-007-2350-9

Archiv der Mathematik

Projections on tensor products of Banach spaces

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Abstract. We characterize norm hermitian operators on classes of tensor products of Banach spaces and derive results for the particular settings of injective and projective tensor products. We provide a characterization of bi-circular and generalized bi-circular projections on tensor products of Banach spaces supporting only dyadic surjective isometries.

Mathematics Subject Classification (2000). Primary 30D55; Secondary 30D05.

Keywords. Isometry, bi-circular projections, generalized bi-circular projections, injective and projective tensor products of Banach spaces.

1. Introduction. In this paper, we characterize the structure of norm hermitian operators on tensor products of Banach spaces in which the only surjective isometries are of dyadic type. Khalil in [9], Khalil-Salem in [8], and Jarosz in [11] provided classifications of surjective isometries for different tensor products of Banach spaces that assure the existence of spaces with such isometries. The structure of norm hermitian operators allows an easy characterization of those operators that are also hermitian projections. Such characterization can be transcribed for bi-circular projections, as established by Jamison in [10]. The last section extends previous results to the more general case of generalized bi-circular projections, introduced in [7], and provides characterizations of these projections in a variety of tensor product spaces. Characterizations of generalized bi-circular projections in various Banach spaces can be found in [3], [4] and [13].

We start by recalling the definitions of norm hermitian operators, bi-circular and generalized bi-circular projections, see [6] and [7].

Definition 1.1. We consider a complex Banach space X. A bounded operator S on X is said to be hermitian if and only if $\{e^{itS}\}_{t\in R}$ defines a one-parameter group of isometries. An operator P on X is said to be a bi-circular projection if and only if $P^2 = P$ and $P + \lambda(Id - P)$ is an isometry for every complex number λ of modulus 1. An operator P_λ (on X) is said to be a generalized bi-circular projection if and only if $P_\lambda^2 = P_\lambda$ and $P_\lambda + \lambda(I - P_\lambda)$ is an isometry of X, for some $\lambda \in C$, $\lambda \neq 1$, and $|\lambda| = 1$.

We observe that such isometries must be surjective. In fact, if $\omega \in X$, there exists $z \in X$, $z = P_\lambda(\omega) + \frac{1}{\lambda}(\omega - P_\lambda(\omega))$, such that $[P_\lambda + \lambda(I - P_\lambda)](z) = \omega$.

We consider the algebraic tensor product of two Banach spaces X_1 and X_2, denoted by $X_1 \otimes X_2$, equipped with some crossnorm α, cf. [12]. We denote the completion of $X_1 \otimes X_2$ relatively to this crossnorm by $X_1 \otimes_\alpha X_2$. The two most well-known crossnorms on $X_1 \otimes X_2$ are the so called projective crossnorm (denoted by ν) and injective crossnorm (denoted by λ). The corresponding completions relative to these norms are called projective and injective tensor products, commonly denoted by $X_1 \hat{\otimes} X_2$ and $X_1 \check{\otimes} X_2$, respectively. For completeness of exposition, we recall that the projective tensor norm is defined as follows

$$\nu(z) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : \; z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

the injective tensor norm is defined as follows

$$\lambda \left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sup \left\{ |\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)|, \; \|\varphi\| = \|\psi\| = 1 \right\}.$$

It is shown in [12, 5] that λ is the least crossnorm and μ the greatest crossnorm, i.e. for every reasonable crossnorm α on $X_1 \otimes X_2$, and $z \in X_1 \otimes X_2$, we have that

$$\lambda(z) \leq \alpha(z) \leq \mu(z).$$

Definition 1.2. We say that a bounded operator T on $X_1 \otimes_\alpha X_2$ is dyadic if and only if there exist bounded operators on the component spaces, denoted by T_1 and T_2, so that $T = T_1 \otimes T_2$.

It follows from the Hahn Banach theorem that the representation of a dyadic operator as the tensor product of two factors is essentially unique. If $T_1 \otimes T_2 = T_1' \otimes T_2'$ then there must exist a scalar a so that $T_1 = aT_1'$ and $T_2' = aT_2$. Moreover, given a dyadic isometry $T_1 \otimes T_2$, T_1 is an isometry if and only if T_2 is an isometry.

2. Norm Hermitian Operators on Tensor Products Spaces with Dyadic Isometries.

In this section we characterize the norm hermitian operators on a tensor product of two Banach Spaces $X_1 \otimes_\alpha X_2$, where α is a reasonable crossnorm.

Theorem 2.1. *If every surjective isometry on $X_1 \otimes_\alpha X_2$ is dyadic, then S is a norm hermitian operator on $X_1 \otimes_\alpha X_2$ if and only if either*

1. *$S = rId_{X_1 \otimes_\alpha X_2}$, for some $r \in \mathbb{R}$, or*
2. *There exist hermitian operators L and R, on X_1 and X_2 respectively, such that $S(x_1 \otimes x_2) = L(x_1) \otimes x_2 + x_1 \otimes R(x_2)$.*

Proof. If S is either of form (1) or (2) then we show that it is an hermitian operator. It is sufficient to prove that $T_t = e^{itS}$ is a one-parameter group of isometries. This follows trivially, whenever S is a multiple of the Id, since $T_t = e^{rit}\, Id$. If $S(x_1 \otimes x_2) = L(x_1) \otimes x_2 + x_1 \otimes R(x_2)$ then $T_t(x_1 \otimes x_2) = e^{itL}(x_1) \otimes e^{itR}(x_2)$. Each tensor factor is a one-parameter group of isometries, so is $\{T_t\}$.

Conversely, given S, an hermitian operator on $X_1 \otimes_\alpha X_2$, then $T_t = e^{itS}$ is a uniformly continuous one-parameter group of isometries on $X_1 \otimes_\alpha X_2$, see [6]. Each isometry is dyadic, hence $T_t = L_t \otimes R_t$, with L_t and R_t surjective isometries on X_1 and X_2 respectively. We show that these components also define uniformly continuous one-parameter groups of isometries. We assume that $L_0 = Id_{X_1}$ and $R_0 = Id_{X_2}$. Furthermore, we first assume that $\{T_t\}$ is a nontrivial family, i.e. for every $t \neq 0$, T_t is not a multiple of the Id.

Step I $\{L_t\}$ and $\{R_t\}$ are uniformly continuous families of operators.

$$\begin{aligned}
\|T_t - T_{t_0}\| = \;& \sup\{\alpha((T_t - T_{t_0})(z)) : \; \alpha(z) = 1\} \\
\geq \;& \sup\{\alpha\,[(T_t - T_{t_0})(x \otimes y)]\,, \; \|x\| = \|y\| = 1\} \\
= \;& \sup\{\alpha\,[(L_t(x) \otimes R_t(y) - L_{t_0}(x) \otimes R_{t_0}(y)]\,, \; \|x\| = \|y\| = 1\} \\
\geq \;& \sup\{\lambda\,[L_t(x) \otimes R_t(y) - L_{t_0}(x) \otimes R_{t_0}(y)]\,, \; \|x\| = \|y\| = 1\} \\
= \;& \sup\{\|\varphi[L_t(x) - L_{t_0}(x)]R_t(y) + \varphi(L_{t_0}(x))[R_t(y) - R_{t_0}(y)]\|_{X_2}, \\
& \qquad \|x\| = \|y\| = \|\varphi\| = 1\}.
\end{aligned}$$

We assume that there exists $x \in X_1$ (depending on t) of norm equal to 1 such that $\{L_t(x) - L_{t_0}(x),\, L_{t_0}(x)\}$ is linearly independent. The Hahn-Banach theorem asserts the existence of $\varphi \in X^*$ such that $\varphi(L_{t_0}(x)) = 1$, $\varphi(L_t(x) - L_{t_0}(x)) = 0$, and $\|\varphi\| = 1$. Therefore

$$\|T_t - T_{t_0}\| \geq \sup\{\|R_t(y) - R_{t_0}(y)\|_{X_2} : \; \|y\| = 1\} = \|R_t - R_{t_0}\|.$$

Now, we assume the existence of a sequence $\{t_n\}$ converging to t_0 such that for every n and $x \in X_1$, $\{L_{t_n}(x) - L_{t_0}(x),\, L_{t_0}(x)\}$ is linearly dependent. This means that $L_{t_n}(x) - L_{t_0}(x) = a_n(x) L_{t_0}(x)$, for scalars $a_n(x)$ depending on both t_n and x. We prove that each function $a_n(x)$ is, in fact, independent of x. We start by selecting two linearly independent vectors in X_1, say x_1 and x_2 (X_1 and X_2 are of dimension greater than 1). Therefore

$$\begin{aligned}
L_{t_n}(x_1 + x_2) - L_{t_0}(x_1 + x_2) &= a_n(x_1 + x_2) L_{t_0}(x_1 + x_2) \\
&= a_n(x_1) L_{t_0}(x_1) + a_n(x_2) L_{t_0}(x_2),
\end{aligned}$$

and $a_n(x_1) = a_n(x_1 + x_2) = a_n(x_2)$. On the other hand, for $x_1 = kx_2$ (k a scalar) we have that $a_n(x_1) = a_n(x_2)$, hence $L_{t_n} = (a_n + 1) L_{t_0}$, with $|a_n + 1| = 1$. For each n, we have

$$\begin{aligned}\|T_{t_n} - T_{t_0}\| \geq \ & \sup\{\alpha(L_{t_n}(x)\otimes R_{t_n}(y) - L_{t_0}(x)\otimes R_{t_0}(y)) : \ \|x\| = \|y\| = 1\} \\ = \ & \sup\{\alpha((a_n+1)L_{t_0}(x)\otimes R_{t_n}(y) - L_{t_0}(x)\otimes R_{t_0}(y)) : |x\| = \|y\| = 1\} \\ = \ & \sup\{\|L_{t_0}(x)\|_{X_1} \|(a_n+1)R_{t_n}(y) - R_{t_0}(y)\|_{X_2}, \ \|x\| = \|y\| = 1\} \\ = \ & \sup\{\|(a_n+1)R_{t_n}(y) - R_{t_0}(y)\|_{X_2} : \ \|y\| = 1\} \\ = \ & \sup\{|(a_n+1)\psi(R_{t_n}(y) - \psi(R_{t_0}(y))| : \ \|y\| = |\psi\| = 1\}.\end{aligned}$$

Moreover, if there exists a $y_n \in X_2$ (of norm 1) such that $\{R_{t_n}(y_n), R_{t_0}(y_n)\}$ is linearly independent, then let $\psi \in X_2^*$ such that $\psi(R_{t_n}(y_n)) = \overline{a_n+1}$ and $\psi(R_{t_0}(y_n)) = 0$. This would imply that $\sup\{|(a_n+1)\psi(R_{t_n}(y)) - \psi(R_{t_0}(y))| : \|y\| = |\psi\| = 1\} = 1$ and then $\|T_{t_n} - T_{t_0}\| \geq 1$. This leads to a contradiction, since $\{T_t\}$ is uniformly continuous. Therefore we assume that for every n and y, $\{R_{t_n}(y), R_{t_0}(y)\}$ is linearly dependent. As previously shown, there exist scalars depending on t_n so that $R_{t_n} = (b_n+1)R_{t_0}$ $(|b_n+1| = 1)$. Since we also have that $L_{t_n} = (a_n+1)L_{t_0}$, then $T_{t_n - t_m} = \frac{(a_n+1)(b_n+1)}{(a_m+1)(b_m+1)} Id$. Consequently, there must exist a sequence $\{\tau_n\}$, converging to zero, and modulus 1 complex numbers λ_n, such that $T_{\tau_n} = \lambda_n Id$, equivalently $e^{i\,\tau_n\,S} = e^{\ln(\lambda_n) Id}$. Since the operator S is hermitian, it has real spectrum ($\sigma(S)$), the spectrum of $\ln(\lambda_n)Id$ is clearly $\ln(\lambda_n)$. Theorem 6, in [16], implies that $\lambda_n = 1$ or $S - \ln(\lambda_n)Id = (2k_n\pi i)Id$, for some integers k_n. In either case T_{τ_n} is a multiple of the identity, contradicting our initial assumption.

We have shown that $\{R_t\}$ is a uniformly continuous family of surjective isometries. Now we prove that $\{L_t\}$ is also uniformly continuous. For every $\epsilon > 0$ there exists $\delta > 0$ so that given t with $|t - t_0| < \delta$, we have $\|T_t - T_{t_0}\| < \epsilon/2$ and $\|R_t - R_{t_0}\| < \epsilon/2$. Consequently, we have that

$$\sup\{\|R_t(y) - R_{t_0}(y)\|_{X_2} : \ \|y\| = 1\} = \|R_t - R_{t_0}\| < \epsilon/2$$

and

$$\begin{aligned}&\|T_t - T_{t_0}\| \\ &= \sup_{\|x\|=\|y\|=|\varphi\|=1}\{\|\varphi[L_t(x) - L_{t_0}(x)]R_t(y) + \varphi(L_{t_0}(x))[R_t(y) - R_{t_0}(y)]\|_{X_2}\} \\ &\geq \sup_{\|x\|=\|y\|=|\varphi\|=1}\{|\varphi[L_t(x) - L_{t_0}(x)]|\|R_t(y)\| - \|\varphi(L_{t_0}(x))\|\epsilon/2\} \\ &= \|L_t - L_{t_0}\| - \epsilon/2.\end{aligned}$$

Therefore $\|L_t - L_{t_0}\| < \epsilon$ and the uniform continuity of $\{L_t\}$ follows as well.

Step II $\{L_t\}$ and $\{R_t\}$ are weakly differentiable.

We observe that the function $f(t) = T_t$ is strongly differentiable, hence weakly differentiable. For every functional $\Phi \in (X_1 \otimes_\alpha X_2)^*$ we have that

$$\lim_{t \to t_0} \Phi \left(\frac{T_t(z) - T_{t_0}(z)}{t - t_0} \right) \quad \text{exists.}$$

In particular, this limit also exists for functionals of the form $\varphi \otimes \psi$. The linearity of f allows us to reduce the problem to the differentiability at zero. Hence, for $z = x \otimes y$, we have

$$\lim_{h \to 0} \varphi \otimes \psi \left(\tfrac{T_h(z) - z}{h} \right) = \lim_{h \to 0} \varphi \otimes \psi \left(\tfrac{(L_h \otimes R_h)(x \otimes y) - x \otimes y}{h} \right)$$

$$= \lim_{h \to 0} \varphi \otimes \psi \left[L_h(x) \otimes \tfrac{R_h(y) - y}{h} + \tfrac{L_h(x) - x}{h} \otimes y \right].$$

If there exists y, so that $\{\frac{R_h(y)-y}{h}, y\}$ is linearly independent, let ψ be a functional on X_2 that attains the value 1 at y and 0 at $\frac{R_h(y)-y}{h}$. In this case, we have that

$$\lim_{h \to 0} \varphi \otimes \psi \left[L_h(x) \otimes \frac{R_h(y) - y}{h} + \frac{L_h(x) - x}{h} \otimes y \right] = \lim_{h \to 0} \varphi \left[\frac{L_h(x) - x}{h} \right].$$

Therefore $g(t) = L_t$ is weakly differentiable. Similarly, if we assume the existence of x such that $\{\frac{L_h(x)-x}{h}, x\}$ is linearly independent it follows that $h(t) = R_t$ is weakly differential. The weak differentiability of either $g(t)$, or $h(t)$ implies the weak differentiability of $h(t)$, or $g(t)$ respectively. It remains to consider the existence of a sequence h_n converging to zero such that for every $x \in X_1$ and $y \in X_2$, $\{\frac{L_h(x)-x}{h}, x\}$ and $\{\frac{R_h(y)-y}{h}, y\}$ are both linearly dependent. An analogue of a previous argument would imply that T_t is trivial, for some values of t, contradicting our assumption.

Step III $\{R_t\}$ and $\{L_t\}$ are one parameter groups of isometries.

The group condition $T_{t_1+t_2} = T_{t_1} \circ T_{t_2}$ implies that $L_{t_1+t_2} = \lambda(t_1, t_2)\, L_{t_1} \circ L_{t_2}$ and $R_{t_1+t_2} = \overline{\lambda}(t_1, t_2)\, R_{t_1} \circ R_{t_2}$, for some modulus 1 scalars. We prove that $\lambda(t_1, t_2) = 1$, for every t_1 and t_2. We recall that $T_0 = Id_{X_1 \otimes_\alpha X_2} = Id_{X_1} \otimes Id_{X_2}$ and without loss of generality we may assume that $L_0 = Id_{X_1}$ and $R_0 = Id_{X_2}$. We also have that $L_0 = Id_{X_1} = \lambda(t_1, -t_1) L_{t_1} \circ L_{-t_1} = \lambda(-t_1, t_1) L_{-t_1} \circ L_{t_1}$ and $L_{t_1} = \overline{\lambda}(t_1, -t_1) L_{-t_1}^{-1}$ implying that $Id_{X_1} = \lambda(-t_1, t_1) L_{-t_1} \circ L_{t_1} = \lambda(-t_1, t_1) \overline{\lambda}(t_1, -t_1) L_{-t_1} \circ L_{-t_1}^{-1}$. Therefore $\lambda(-t_1, t_1) = \lambda(t_1, -t_1)$ and $L_{-t_1} \circ L_{t_1} = L_{t_1} \circ L_{-t_1}$.

We clearly have $\lambda(0, t) = \lambda(t, 0) = 1$, for all t.

First, we observe that $\lambda(t_1, t_2) = \lambda(t_2, t_1)$ if and only if $L_{t_1} \circ L_{t_2} = L_{t_2} \circ L_{t_1}$. In order to prove this last statement we proceed as follows:

$$L_{3t} = \lambda(2t, t) L_{2t} \circ L_t = \lambda(2t, t) \lambda(t, t) L_t \circ L_t \circ L_t = \lambda(2t, t) L_t \circ L_{2t}$$

and

$$L_t \circ L_{2t} = L_{2t} \circ L_t.$$

This last statement is equivalent to $\lambda(2t, t) = \lambda(t, 2t)$. Inductively we show that $L_{mt} \circ L_{nt} = L_{nt} \circ L_{mt}$ and $\lambda(nt, mt) = \lambda(mt, nt)$, for n, m integers and t a real number. Therefore we have $L_{r_1} \circ L_{r_2} = L_{r_2} \circ L_{r_1}$ for rational values r_1 and r_2 and continuity implies that $L_{t_1} \circ L_{t_2} = L_{t_2} \circ L_{t_1}$ and $\lambda(t_1, t_2) = \lambda(t_2, t_1)$.

Furthermore, for arbitrary values of t, say t, t_1, t_2 we have that $\lambda(t + t_1, t_2)\lambda(t, t_1) = \lambda(t_1 + t_2, t)\lambda(t_1, t_2)$. The weak differentiability established in Step II implies the differentiability of λ, then we have

$$\partial_t \lambda(t + t_1, t_2)\lambda(t, t_1) + \lambda(t + t_1, t_2)\partial_t \lambda(t, t_1) = \partial_t \lambda(t, t_1 + t_2)\lambda(t_1, t_2).$$

Hence, for $t = t_2$, the equation above implies that $\partial_t \lambda(t_2, t_1) = 0$ and $\lambda(t_2, t_1) = C(t_1)$, a constant depending on t_1. For $t_2 = 0$, we have that $1 = \lambda(0, t_1) = C(t_1)$ and we have established that $\lambda = 1$ which completes the proof of Step III.

The families $\{L_t\}$ and $\{R_t\}$ are one-parameter groups of uniformly continuous family of isometries, hence there exist hermitian operators L and R so that $L_t = e^{itL}$ and $R_t = e^{itR}$. Therefore we have that $T_t = e^{itS} = e^{itL} \otimes e^{itR}$ and the corresponding generator satisfies

$$S = \left(-i\,\frac{d}{dt}\right)_{t=0} e^{itS}(x_1 \otimes x_2) = L(x_1) \otimes x_2 + x_1 \otimes R(x_2).$$

This completes the proof of statement 2, provided that $\{T_t\}$ is a nontrivial family.

If we assume that, for some t_0, T_{t_0} is a multiple of the Id, then Theorem 6, in [16], implies that $\lambda = 1$ or $S - \ln(\lambda)Id = (2k\pi i)Id$, for some integer k. In either case S is a multiple of the identity. This completes the proof of the theorem. □

3. Norm Hermitian Operators on Projective and Injective tensor Products. Theorems by Khalil and Saleh [8, 9] state that surjective isometries on a class of projective tensor products are dyadic. A theorem by Jarosz states that surjective isometries, that are not reflections, on a class of injective tensor products are also dyadic. This isometry structure and the theorem 2.1 provide the infrastructure for the characterization of norm hermitian operators on Khalil-Saleh projective tensor products and Jarosz injective tensor products, as it will be shown in the forthcoming corollary 3.3. We start by stating Khalil, Khalil-Saleh and Jarosz characterizations.

Theorem 3.1. 1. *(Khalil in* [9]*) T is a surjective isometry on $L^p \hat{\otimes} L^p$ ($p > 1$) if and only if there exists surjective isometries T_1, T_2 on L^p such that $T = T_1 \otimes T_2$.*

2. *(Khalil and Saleh in* [8]*) If X and Y are an ideal pair of Banach spaces i.e. X and Y are reflexive Banach spaces so that X and Y^* are strictly convex and X^* has the approximation property (*[5]*), then every surjective isometry T on $X \hat{\otimes} Y^*$ is of the form $T = T_1 \otimes T_2$, for surjective isometries T_1, T_2 on X and Y^*.*

Theorem 3.2. *(Jarosz in* [11]*) If X_1 is a complex Banach space with trivial centralizer and X_2 a complex Banach space with strictly convex dual, then every isometry T from $X_1 \check{\otimes} X_2$ onto itself is of the form*

1. *$T(x_1 \otimes x_2) = T_1(x_1) \otimes T_2(x_2)$, where T_1 and T_2 are onto isometries.*
2. *There exists a Banach space Z such that $Z \check{\otimes} X_2$ is isometric to X_1 and T under this identification is of the form $T(z \otimes a \otimes b) = z \otimes b \otimes a$, for all $z \in Z$ and $a, b \in X_2$.*

Corollary 3.3. *Let $E = E_1 \otimes_\alpha E_2$ with E_i of any of the following forms:*

1. *$E_1 = E_2 = L^p$ and $\alpha = \nu$,*
2. *$E_2 = Y^*$ where (E_1, Y) is an ideal pair of Banach spaces and $\alpha = \nu$, or*
3. *E_1 a Banach space with trivial centralizer and E_2 a Banach space with strictly convex dual and $\alpha = \lambda$,*

then S is a hermitian operator on E if and only if either

1. *$S = rId$, for some $r \in \mathbb{R}$, or*
2. *There exist hermitian operators L on E_1, and R, on E_2, respectively, such that $S(x_1 \otimes x_2) = L(x_1) \otimes x_2 + x_1 \otimes R(x_2)$.*

Proof. If S is of any of the forms (1) or (2) then it is clearly hermitian as previously shown. If S is hermitian then $\{e^{itS}\}_t$ is a one-parameter group of isometries. This situation follows clearly from the Theorem 2.1, provided that for every t, e^{itS} is a dyadic isometry. On the other hand, if there exists a $t_0 (\neq 0)$ so that $e^{it_0 S}$ is not dyadic then there must exist an isometry onto $Z \check{\otimes} X_2$ such that, $e^{it_0 S}$ under this identification is of the form described in Jarosz theorem. This implies that $e^{2it_0 S} = Id$ therefore S is a multiple of the identity, see [16]. This completes the proof. □

4. Bi-circular Projections on Injective and Projective Tensor Products. The notion of bi-circular projection on a Banach space was first introduced by Stacho and Zalar in [17] and [18]. A projection P on a Banach space X is said to be bi-circular if $e^{ia}P + e^{ib}(I - P)$ is an isometry for all choices of real numbers a and b. These projections are in fact norm hermitian, as shown in [10]. The following theorem characterizes these projections in a class of tensor product spaces.

Theorem 4.1. *If every surjective isometry on $X_1 \otimes_\alpha X_2$ is dyadic, then S is a hermitian projection on $X_1 \hat{\otimes} X_2$ if and only if $S = Id_{X_1} \otimes R$ or $L \otimes Id_{X_2}$ where L and R are hermitian projections on X_1 and X_2, respectively.*

Proof. We begin by assuming that S is a hermitian projection then by the previous theorem

$$S(x_1 \otimes x_2) = L(x_1) \otimes x_2 + x_1 \otimes R(x_2),$$

where L and R are hermitian operators on X and Y respectively. Since S is a projection then

$$[L^2 - L](x_1)\otimes x_2 + 2L(x_1)\otimes R(x_2) + x_1\otimes[R^2 - R](x_2) = 0. \tag{4.1}$$

Hahn-Banach Theorem leads to a contradiction if there exists $x_i \in X_i$ $(i = 1, 2)$ so that either $\{x_1, L(x_1), [L^2 - L](x_1)\}$ or $\{x_2, R(x_2), [R^2 - R](x_2)\}$ is linearly independent. Therefore for every $x_i \in X_i$, $\{x_1, L(x_1), [L^2 - L](x_1)\}$ and $\{x_2, R(x_2), [R^2 - R](x_2)\}$ are linearly dependent. If there exists $x_1 \neq 0$ so that $L(x_1) = ax_1$ then equation 4.1 reduces to

$$x_1\otimes\left[(a^2 - a)x_2 + 2aR(x_2) + (R^2 - R)(x_2)\right] = 0,$$

and $R^2 + (2a-1)R + (a^2 - a)\, Id \equiv 0$. A theorem due to Taylor (see [15]) applied to R implies that $R = -a\, P_1 + (1-a)P_2 = P_2 - aId$, where P_1 and P_2 are projections such that $P_1 \circ P_2 = P_1 \circ P_2 = 0$ and a is a real number. These projections are also hermitian projections. The equation 4.1 reduces to

$$\left[(L^2 - L)(x_1) - 2aL(x_1) + (a^2 + a)x_1\right]\otimes x_2 + [2L(x_1) - 2ax_1]\otimes P_2(x_2) \equiv 0.$$

Therefore, for x_2 in the range of P_2, the last equation implies that $L^2 + (1 - 2\,a)L + (a^2 - a)Id = 0$ and $L = aQ_1 + (a-1)Q_2$, with Q_1 and Q_2 two orthogonal projections. Since $S(x_1\otimes x_2) = -Q_2(x_1)\otimes x_2 + x_1\otimes P_2(x_2)$, equation 4.1 now implies that $Q_2(x_1)\otimes P_2(x_2) = 0$ and then either Q_2 or P_2 is the zero projection. Clearly if $S = Id_{X_1}\otimes R$ or $L\otimes Id_{X_2}$, with L and R hermitian projections on X_1 and X_2 respectively, S is an hermitian projection. This completes the proof. □

Remark 4.2. 1. S is a hermitian projection on $L^p\hat{\otimes}L^p$ $(p > 1)$ or on a projective tensor product $X\hat{\otimes}Y^*$ where (X, Y) is an ideal pair of Banach spaces, if and only if $S = Id_X\otimes S_Y$ or $S_X\otimes Id_Y$ where S_X and S_Y are hermitian projections on X and Y respectively.

2. S is a hermitian projection on $X_1\check{\otimes}X_2$, with X_1 a Banach space with trivial centralizer and X_2 a Banach space with strictly convex dual, if and only if $S = Id_{X_1}\otimes S_{X_2}$ or $S_{X_1}\otimes Id_{X_2}$ where S_{X_2} and S_{X_1} are hermitian projections on X_2 and X_1 respectively.

5. Generalized Bi-Circular Projections as Dyadic Operators. A generalization of bi-circular projections was recently introduced by Fosner, Illisevic, and Li in [7]. It concerns with projections P_λ on X so that $P_\lambda + \lambda(I - P_\lambda)$ is an isometry of X, for some modulus 1 complex number $\lambda \neq 1$.

In the next theorem $\mathcal{B}(X_1\otimes_\alpha X_2)$ represents the bounded operators on $X_1\otimes_\alpha X_2$.

Theorem 5.1. *If $\lambda \in \mathbb{T}$ with $\lambda \neq 1$, then $P_\lambda \in \mathcal{B}(X_1\otimes_\alpha X_2)$ is a projection associated with a dyadic isometry (i.e. $P_\lambda + \lambda(I - P_\lambda)$ is dyadic) if and only if P_λ is dyadic.*

Proof. Given a generalized dyadic projection P_λ, the isometry $P_\lambda + \lambda(Id - P_\lambda)$ is clearly dyadic.

We prove the converse, if T denotes a dyadic isometry associated with P_λ, then P_λ is also dyadic. Since P_λ is a projection then T must satisfy the algebraic equation $T^2 - (\lambda+1)T + \lambda I = 0$. Furthermore $T = T_1 \otimes T_2$, hence we have that

(5.1)
$$T_1^2(x) \otimes T_2^2(y) - (\lambda+1)T_1(x) \otimes T_2(y) + \lambda x \otimes y = 0, \text{ for all } x \in X_1, \text{ and } y \in X_2,$$

with $x \otimes y$ interpreted as an operator from the dual X_1^* into X_2. For every $\varphi \in X_1^*$ the equation (5.1) yields

$$\varphi(T_1^2(x))T_2^2(y) - (\lambda+1)\varphi(T_1(x))T_2(y) + \lambda\varphi(x)y = 0. \tag{5.2}$$

We first assume that $\lambda = -1$, then (5.2) reduces to $\varphi(T_1^2(x))T_2^2(y) = \varphi(x)y$. For each $x \in X_1$ we have that $T_1^2(x) = a_x x$ (for some scalar a_x) and $T_2^2 = \overline{a}_x Id$. The linearity of T_1 implies that a_x is independent of x. Hence $T_1^2 = aId$ and $T_2^2 = \overline{a}Id$. Therefore $P_\lambda = \frac{Id+T}{2} = Id$, which is clearly dyadic. Now, we assume $\lambda \neq -1$. If, in addition, there exists $x_1 \in X_1$ so that $\{x_1, T_1(x_1), T_1^2(x_1)\}$ is linearly independent then the Hahn-Banach theorem assures the existence of a functional in X_1^* such that $\varphi(T_1(x_1)) = 1$ and $\varphi(T_1^2(x_1)) = \varphi(x_1) = 0$. This leads to a contradiction. Hence, for all $x \in X_1$, the set $\{x, T_1(x), T_1^2(x)\}$ must be linearly dependent. If there exists $x \in X_1$ such that $\{x, T_1(x)\}$ is linearly independent then $T_1^2(x) = a\,x + bT_1(x)$, for some scalars a and b. Then equation (5.2) reduces to

$$[a\varphi(x) + b\varphi(T_1(x))]T_2^2(y) - (\lambda+1)\varphi(T_1(x))T_2(y) + \lambda\varphi(x)y = 0.$$

We select a functional φ such that $\varphi(x) = 1$ and $\varphi(T_1(x)) = 0$. Hence $aT_2^2(y) + \lambda y = 0$, for all $y \in X_2$. This implies that $T_2^2 = cId$, for some c of modulus 1. We also select a functional ψ such that $\psi(x) = 0$ and $\psi(T_1(x)) = 1$. Then $b\,c\,y - (\lambda+1)T_2(y) = 0$ and $T_2 = d\,Id$ for some scalar d of modulus 1. The equation (5.2) becomes $\phi(d^2T_1^2(x) - (\lambda+1)\,dT_1(x) + \lambda x) = 0$, for all $\phi \in X_1^*$. Therefore $d^2T_1^2 - (\lambda+1)dT_1 + \lambda Id = 0$. The projection P_λ is given as follows

$$P_\lambda(x \otimes y) = \frac{1}{1-\lambda}\,[-\lambda x \otimes y + T_1(x) \otimes T_2(y)] = \frac{1}{1-\lambda}\,[-\lambda x + d\,T_1(x)] \otimes y.$$

We set $S_1(x) = \frac{-\lambda x + d\,T_1(x)}{1-\lambda}$, hence $P_\lambda = S_1 \otimes Id$. The remaining case assumes that for every $x \in X_1$, $\{x, T_1(x)\}$ is linearly dependent. For each x, there exists a modulus 1 scalar e_x such that $T_1(x) = e_x\,x$. The linearity of T_1 assures that e_x is independent of x and then $T_1 = eId$. The equation (5.2) becomes

$$e^2\phi(x)\,T_2^2(y) - (\lambda+1)\,e\phi(x)\,T_2(y) + \lambda\phi(x)y = 0,$$

for every $\phi \in X_1^*$ and $y \in X_2$. Therefore $e^2T_2^2 - e(\lambda+1)T_2 + \lambda Id = 0$ and $P_\lambda = Id \otimes S_2$ with $S_2(y) = \frac{-\lambda y + eT_2(y)}{1-\lambda}$. □

It is a consequence of the previous proof the following corollary.

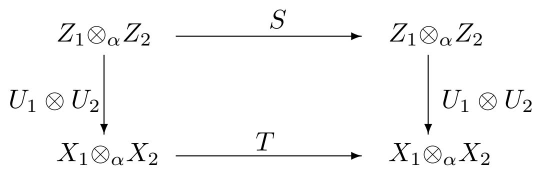

Figure 1. S and T are tensor conjugate if and only if the diagram commutes.

Corollary 5.2. *If* $\lambda \neq -1$, $P_\lambda \in \mathcal{B}(X_1 \otimes_\alpha X_2)$ *is a generalized bi-circular projection associated with a dyadic isometry if and only if* P_λ *is either of the form* $S_1 \otimes Id$ *or* $Id \otimes S_2$, *with* S_i *a generalized bi-circular projection on* X_i.

Proof. It was shown in the previous that if P_λ is associated with a dyadic isometry then

$$P_\lambda(x \otimes y) = S_1(x) \otimes y \text{ or } P_\lambda(x \otimes y) = x \otimes S_2(y)$$

with $S_1(x) = \frac{-\lambda x + d\,T_1(x)}{1-\lambda}$ and $S_2(y) = \frac{-\lambda y + e\,T_2(y)}{1-\lambda}$. The operators S_1 and S_2 are generalized bi-circular projections on the component spaces since $d^2 T_1^2 - (\lambda + 1) d\,T_1 + \lambda Id = 0$ and $e^2 T_2^2 - e\,(\lambda + 1) T_2 + \lambda Id = 0$. It is trivial to check the converse. □

Definition 5.3. Given the Banach spaces X_1, X_2, Z_1 and Z_2, we consider the tensor products $X_1 \otimes_\alpha X_2$ and $Z_1 \otimes_\alpha Z_2$, representing the completions of $X_1 \otimes X_2$ and $Z_1 \otimes Z_2$ relative to the crossnorm α. A bounded operator S on $Z_1 \otimes_\alpha Z_2$ is said to be **tensor conjugate** to a bounded operator T, on $X_1 \otimes_\alpha X_2$, if and only if there exists a dyadic isometry $U_1 \otimes U_2$ with isometric factors $U_i : Z_i \to X_i$ such that (see Figure 1)

$$T = (U_1 \otimes U_2) \circ S \circ (U_1^{-1} \otimes U_2^{-1}). \tag{5.3}$$

Remark 5.4. Isometric properties are preserved under tensor conjugacy. The equation 5.3 also implies that $T^k = (U_1 \otimes U_2) \circ S^k \circ (U_1^{-1} \otimes U_2^{-1})$, for every positive integer k. In particular, we conclude that, for $X_1 = X_2$, the operator $S(a \otimes b) = b \otimes a$ is not tensor conjugate to a dyadic one.

If X_1 has trivial centralizer (see [1] for the definition) and X_2 has strictly convex dual it was shown in [11] that there exists a Banach space Z so that the injective tensor product $Z \check{\otimes} X_2$ is isometric to X_1, we denote such isometry by U. If T is a nondyadic surjective isometry on $X_1 \check{\otimes} X_2$ then $S = U \otimes Id_{X_2} \circ T \circ U^{-1} \otimes Id_{X_2}$, acting on the basis element $z \otimes a \otimes b$, yields $z \otimes b \otimes a$. If we assume that a given generalized bi-circular projection P_λ, on $X_1 \check{\otimes} X_2$, is associated with such an isometry T then $P_* = U^{-1} \otimes Id_{X_2} \circ P_\lambda \circ U \otimes Id_{X_2}$ is a projection ($P_*^2 = P_*$). Therefore we have that $(\lambda + 1)(S - Id) = 0$, and hence $\lambda = -1$ or $S = Id$. In either case P_λ is the average of the Id with an isometric reflection ($S^2 = Id$).

Corollary 5.5. *If X_1 is a complex Banach space with trivial centralizer and X_2 a complex Banach space with strictly convex dual then every generalized bi-circular projection P on $X_1 \check{\otimes} X_2$ is of the form*

$$P_1 \otimes Id_{X_2}, \; Id_{X_1} \otimes P_2, \; or \; \frac{Id_{X_1 \otimes X_2} + R}{2},$$

where P_i are generalized bi-circular projections on X_i and R is an isometric reflection on $X_1 \check{\otimes} X_2$.

Proof. If the isometry associated with P is dyadic then corollary 5.2 applies. Otherwise Jarosz's theorem asserts the existence of a Banach space Z such that X_1 is isometrically isomorphic to $Z \otimes X_2$ where the isometry associated with P, denoted by R, is tensor conjugate to a reflection, hence $R^2 = Id$. Since $R^2 - (1+\lambda)R + \lambda Id = 0$ then $\lambda = -1$ and $P = \frac{Id_{X_1 \check{\otimes} X_2} + R}{2}$. This completes the proof. □

Corollary 5.6. 1. *Every generalized bi-circular projection, P_λ with $\lambda \neq -1$, on $L^p \hat{\otimes} L^p$ is of the form*

$$P_1 \otimes Id \; or \; Id \otimes P_2,$$

where P_i is a generalized bi-circular projection on L^p.

2. *If X and Y define an ideal pair of Banach spaces, every generalized bi-circular projection, P_λ with $\lambda \neq -1$, on $X \hat{\otimes} Y^*$ is of the form*

$$P_1 \otimes Id_{Y^*} \; or \; Id_X \otimes P_2,$$

where P_1 is a generalized bi-circular projection on X and P_2 is a generalized bi-circular projection on Y^* .

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Received: 26 February 2007

Revised: 30 May 2007