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# GENERALIZED BI-CIRCULAR PROJECTIONS ON MINIMAL IDEALS OF OPERATORS

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ABSTRACT. We characterize generalized bi-circular projections on $\mathcal{I}(\mathcal{H})$, a minimal norm ideal of operators in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

## 1. INTRODUCTION

The existence of classes of projections on a given Banach space, as well as their characterizations, are basic problems in Banach Space Theory; see [2], [3], [9], [13], and [16]. Recently, a class of projections, namely bi-circular projections, was proposed by Stachó and Zalar in [17]. A projection is called a bi-circular projection if $e^{i\alpha}P + e^{i\beta}(I-P)$ is an isometry for all $\alpha$, $\beta \in R$. Such projections have been studied in a variety of settings by Stachó and Zalar; see [18]. Fosner, Ilisevic, and C. K. Li, in [7], considered a generalization of this concept by requiring that $P + \lambda(I - P)$ is an isometry, for some $\lambda$ with $|\lambda| = 1$. We call these projections generalized bi-circular projections. In this paper, we completely characterize generalized bi-circular projections on minimal norm ideals of Hilbert space operators. It is an easy consequence of our characterization that these projections are bi-contractive. We start by recalling the basic definitions and results to be used throughout the paper.

**Definition 1.1.** We consider a Banach space $X$ with the norm $\|\cdot\|$. The operator $Q$ (on $X$) is said to be a generalized bi-circular projection if and only if $Q^2 = Q$ and there exists $\lambda \in C$, $\lambda \neq 1$, and $|\lambda| = 1$, for which $Q + \lambda(I - Q)$ is an isometry of $X$, denoted by $\tau$.

We observe that $\tau$ is a surjective isometry. In fact, if $\omega \in X$, there exists $z \in X$, namely $z = Q(\omega) + \frac{1}{\lambda}(\omega - Q(\omega))$, such that $\tau(z) = \omega$.

Let $\mathcal{H}$ be a complex separable Hilbert space of infinite dimension and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. A *symmetric norm ideal*, $(\mathcal{I}, \nu)$, in $\mathcal{B}(\mathcal{H})$ consists of a two-sided proper ideal $\mathcal{I}$ together with a norm $\nu$ on $\mathcal{I}$ satisfying the conditions:

(i) $\nu(A) = \|A\|$, for every rank 1 operator $A$.

(ii) $\nu(UAV) = \nu(A)$, for every $A \in \mathcal{I}$ and unitary operators $U$ and $V$ on $\mathcal{H}$.

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If the set of finite rank operators is dense in $\mathcal{I}$, then $\mathcal{I}$ is a minimal norm ideal; see [11].

The isometries of minimal norm ideals were characterized by Sourour, in [12]. For completeness of exposition we state Sourour's result. First, we recall the concept of transpose of an operator relative to a fixed orthonormal basis, $\{e_i\}$, for the Hilbert space. We denote by $T^t$ the transpose of the operator $T$.

**Definition 1.2.** Given $T \in B(\mathcal{H})$, the transpose $T^t$ is defined to be the unique operator in $B(\mathcal{H})$ such that

$$\langle T^t e_i, e_j \rangle = \langle T e_j, e_i \rangle.$$

**Theorem 1.3** (Sourour [12]). *If $\mathcal{I}$ is a minimal ideal in $B(\mathcal{H})$ different from $C_2(\mathcal{H})$, and $\mathcal{U}$ is a linear transformation on $\mathcal{I}$, then $\mathcal{U}$ is a surjective isometry of $\mathcal{I}$ if and only if there exist unitaries $U$ and $V$ on $\mathcal{H}$ such that*

$$\mathcal{U}(T) = UTV \quad \text{or} \quad \mathcal{U} = UT^tV$$

*for every $T \in \mathcal{I}$.*

In addition to the result of Sourour, the following theorem due to Fong and Sourour (cf. [15]) will also be used in our proofs.

The operators $\{A_i\}_{i=1,\dots,m}$ and $\{B_i\}_{i=1,\dots,m}$ are bounded operators on the Banach space $X$ and $\Phi$ acts on $B(X)$ as follows:

$$\Phi(T) = A_1TB_1 + A_2TB_2 + \cdots + A_mTB_m.$$

**Theorem 1.4** (Fong and Sourour [15]). *If $\Phi(T) = O$, for all $T \in B(X)$, then $\{B_1, B_2, \cdots, B_m\}$ is linearly dependent. Furthermore, if $\{B_1, B_2, \cdots, B_n\}$ $(n \leq m)$ is linearly independent, and the $(c_{kj})$ denote constants for which*

$$B_j = \sum_{k=1}^{n} c_{kj} B_k, \quad n+1 \leq j \leq m,$$

*then $\Phi(T) \equiv 0$ for all $T \in B(X)$ if and only if*

$$A_k = -\sum_{j=n+1}^{m} c_{kj} A_j\,, \quad 1 \leq k \leq n.$$

*If $n = m$, then $A_1 = A_2 = \cdots = A_m = 0$.*

## 2. Generalized bi-circular projections on ideals of Hilbert space operators

In [4], the authors have shown that generalized bi-circular projections on certain Banach spaces are the average of the identity with an isometric reflection, i.e. an isometry $R$ such that $R^2 = Id$. We show in this paper that a similar characterization holds for ideals of Hilbert space operators. In anticipation of this result, we now give a simple characterization of isometric reflections on minimal norm ideals.

**Lemma 2.1.** *The operator $\tau$ on $\mathcal{I}$ is an isometric reflection if and only if either*

(1) *$\tau(T) = UT^tV$ with $U$ and $V$ unitary operators on $\mathcal{H}$ so that $V = \pm(U^t)^*$,*

*or*

(2) *$\tau(T) = UTV$, with $U$ and $V$ isometries of the form $U = \sqrt{\overline{\alpha}}P_0 - \sqrt{\overline{\alpha}}(Id - P_0)$ and $V = \sqrt{\alpha}P_1 - \sqrt{\alpha}(Id - P_1)$, where $P_0$ and $P_1$ are projections onto closed subspaces of $\mathcal{H}$ and $\alpha$ is a complex number of modulus 1.*



*Proof.* This lemma is a straightforward consequence of Fong and Sourour's theorem. If $\tau(T) = UT^tV$, then $\tau^2 = Id$ if and only if $UV^tTU^tV - T = 0$, for all $T \in \mathcal{I}$. Therefore, $U^tV = \alpha Id$, for some complex number $\alpha$ of modulus 1, and $VU^t = \overline{\alpha} Id$. This implies that $\alpha = \overline{\alpha}$ and hence $\alpha = \pm 1$. Consequently $V = \pm(U^t)^*$.

If $\tau(T) = UTV$, then $\tau^2 = Id$ if and only if $U^2TV^2 - T = 0$, for all $T \in \mathcal{I}$. This implies that $V^2 = \alpha Id$, with $|\alpha| = 1$, and $U^2 = \overline{\alpha} Id$. The second statement now follows from the spectral theorem applied to $U$ and $V$; see [14]. □

**Proposition 2.2.** *Let $(\mathcal{I}, \nu)$ be a separable minimal norm ideal different from $C_2$ (the Hilbert-Schmidt class) and let $Q$ be a generalized bi-circular projection on $\mathcal{I}$. If $Q$ is associated with a surjective isometry $\tau$ on $\mathcal{I}$ of the form $\tau(T) = UT^tV$ ($U$ and $V$ unitary operators on $\mathcal{H}$), then $Q$ is the average of the identity with an isometric reflection.*

*Proof.* If $Q$ is a generalized bi-circular projection, then

$$Q(T) = \frac{1}{1-\lambda}\left[-\lambda T + UT^tV\right]$$

and

$$\lambda T - (\lambda+1)UT^tV + UV^tTU^tV = O, \tag{2.1}$$

for every $T \in \mathcal{I}$. We first observe that for $\lambda = -1$, the equation (2.1) reduces to

$$-T + UV^tTU^tV = O, \quad \text{for all } T \in \mathcal{I}.$$

Fong and Sourour's theorem implies that $U^tV = \alpha Id$ and $Id = \alpha UV^t$, for some modulus 1 complex number $\alpha$. Therefore $\alpha^2 = 1$ and $V = \pm U^{*t}$. The projection $Q$ is the average of the identity with the isometric reflection: $R(T) = \pm UT^tU^{*t}$. Now, we consider $\lambda \neq -1$. We show that there are no unitary operators $U$ and $V$ for which the equation (2.1) holds for every $T \in \mathcal{I}$.

We fix $\lambda$, with modulus 1 and different from $-1$. If there exists a pair of unitary operators $(U, V)$ so that equation (2.1) holds for every $T \in \mathcal{I}$, then

$$T^t = \frac{1}{\lambda+1}U^*\left[\lambda T + UV^tTU^tV\right]V^*$$

and

$$T = \frac{1}{\lambda+1}V^{*t}\left[\lambda T^t + V^tUT^tVU^t\right]U^{*t}.$$

Therefore

$$T = \frac{1}{(\lambda+1)^2}\left[\lambda^2 V^{*t}U^*TV^*U^{*t} + 2\lambda T + UV^tTU^tV\right],$$

or equivalently

$$-(\lambda^2+1)T + \lambda^2 V^{*t}U^*TV^*U^{*t} + UV^tTU^tV = O, \quad \forall T \in \mathcal{I}. \tag{2.2}$$

Fong and Sourour's theorem implies that $\{Id,\ V^*U^{*t},\ U^tV\}$ is linearly dependent and we consider the following cases:

**I.** $U^tV = \alpha Id$; then $V^*U^{*t} = \overline{\alpha} Id$ and $V = \alpha U^{*t}$. Equation (2.1) becomes

$$(\lambda + \alpha^2)T - (\lambda+1)\alpha UT^tU^{*t} = O.$$

This also implies that $\left(\frac{\lambda+\alpha^2}{(\lambda+1)\alpha}\right)^2 = 1$ and $TU^t = \pm UT^t$, $\forall T \in \mathcal{I}$. We show that this is impossible. We first consider $T = e_i \otimes e_j$, with $i \neq j$. Therefore, we have that $TU^t(e_k) = \langle U^t(e_k), e_j \rangle e_i = \pm UT^t(e_k) = \pm\langle e_k, e_i \rangle U(e_j)$. Therefore for $k \neq i$

we have $\langle U(e_j), e_k\rangle = 0$, and then $U(e_j) = \nu_j e_i$, for some modulus 1 complex number $\nu_j$. On the other hand, if $T = e_i \otimes e_i$, we have $TU^t(e_k) = \langle U^t(e_k), e_i\rangle e_i = \pm U T^t(e_k) = \pm\langle e_k, e_i\rangle U(e_i)$. This implies that $\langle U(e_i), e_k\rangle = 0$, for every $k \neq i$. This implies that, for every $i$, $U(e_i) = \mu_i e_i$. Then, given $j \neq i$, we have that $U(e_j) = \nu_j e_i = \mu_j e_j$, which is impossible. Therefore there exist no $U$ and $V$ (unitary operators) so that $U^t V = \alpha Id$, and the equation (2.1) holds for every $T \in \mathcal{I}$.

**II.** $U^tV = \alpha Id + \beta V^*U^{*t}$ $(\beta \neq 0)$. We also have that $\alpha UV^t = (\lambda^2+1)Id$ and $-\beta UV^t = \lambda^2 V^{*t}U^*$. These equations imply that $U^{*t} = -\frac{\beta}{\lambda^2} V U^t V$, and $\left(1+\frac{\beta^2}{\lambda^2}\right) U^tV = \alpha\, Id$. We observe that $\alpha = 0$; then $\lambda^2 = -1$ and $\beta = \pm 1$. Therefore $U^tV = \pm V^*U^{*t}$, $(UV^t)^2 = Id$, and $(U^tV)^2 = \pm Id$. As previously considered, we let $T = e_i \otimes e_j$ $(i \neq j)$; then $(\lambda+1)\langle V(e_i), e_i\rangle U(e_j) = \langle U^tV(e_i), e_j\rangle UV^t(e_i)$. If there exists $i$ so that $\langle V(e_i), e_i\rangle \neq 0$, then $V^t(e_i) = \frac{(\lambda+1)\langle V(e_i), e_i\rangle}{\langle U^tV(e_i), e_j\rangle} e_j$, for $j \neq i$. Let $\alpha_j = \frac{(\lambda+1)\langle V(e_i), e_i\rangle}{\langle U^tV(e_i), e_j\rangle}$; then $|\alpha_j| \geq |\lambda+1||\langle V(e_i), e_i\rangle| \neq 0$. Then the series $\sum_k |\langle V^t(e_i), e_k\rangle|^2$ diverges which would contradict Parseval's identity. Therefore, we must have that for every $i$, $\langle V(e_i), e_i\rangle = 0$. This implies that $\langle U^tV(e_i), e_j\rangle = 0$, for $j \neq i$ and $U^tV(e_i) = \alpha_i e_i$. Since $(U^tV)^2 = \pm Id$ we have that $\alpha_i^2 = \pm 1$.

We evaluate $\lambda T - (\lambda+1)UT^tV + UV^tTU^tV = O$ with $T = e_i \otimes e_j$ at $V^*(e_i)$, and obtain that

$$\lambda\langle V^*(e_i), e_j\rangle e_i - (\lambda+1)U(e_j) + \alpha_i \langle U(e_j), e_i\rangle e_i = O.$$

Therefore

$$\lambda\langle V^*(e_i), e_j\rangle \langle e_i, U(e_j)\rangle - (\lambda+1) + \alpha_i \langle U(e_j), e_i\rangle \langle e_i, U(e_j)\rangle = 0,$$

which implies that $(\lambda\overline{\alpha}_i + \alpha_i)|\langle e_i, U(e_j)\rangle|^2 = \lambda+1$, and $|\langle e_i, U(e_j)\rangle|^2 = \frac{\lambda+1}{\lambda\overline{\alpha}_i+\alpha_i)}$. Clearly $\lambda\overline{\alpha}_i + \alpha_i \neq 0$ because $\lambda \neq -1$ and $\alpha_i^2 = \pm 1$. Similarly to the previous case we have $|\langle U^*(e_i), e_k\rangle|$ must converge to zero. Thus, $\alpha = 0$ yields no solution and it is left to analyze the case where $\alpha \neq 0$ and $\beta \neq 0$. This will imply that $UV^t = \frac{\lambda^2+1}{\alpha} Id$ and hence $U^tV = \frac{\lambda^2+1}{\alpha} Id$. For $T = e_i \otimes e_j$ $(i \neq j)$, we have that $\lambda T(e_k) - (\lambda+1)UT^t(V(e_k)) + \frac{\lambda^2+1}{\alpha} UV^tT(e_k) = 0$. If $k \neq j$, then $\langle V(e_k), e_i\rangle = 0$. This implies that $\langle V^*(e_i), e_k\rangle = 0$ for all $k$. This contradiction completes the proof. □

**Proposition 2.3.** *Let $(\mathcal{I},\nu)$ be a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class) and let $Q$ be a generalized bi-circular projection on $\mathcal{I}$. If $Q$ is associated with a surjective isometry $\tau$ on $\mathcal{I}$ of the form $\tau(T) = UTV$ ($U$ and $V$ unitary operators on $\mathcal{H}$) and $\lambda \neq -1$, then $Q$ is given as*

$$Q(T) = P_FT \text{ or } Q(T) = TP_F,$$

*where $P_F$ represents a projection on $\mathcal{H}$ onto a closed subspace $F$.*

*Proof.* Since $Q$ is a generalized bi-circular projection then

$$Q(T) = \frac{1}{1-\lambda}\left[-\lambda T + UTV\right]$$

and

$$\lambda T - (\lambda+1)UTV + U^2TV^2 = O, \tag{2.3}$$

for every $T \in \mathcal{I}$.



It follows from Fong and Sourour's theorem that $\{Id, V, V^2\}$ must be linearly dependent. Therefore we have two cases to analyze: **1)** $V = \alpha Id$ and $V^2 = \alpha^2 Id$, and **2)** $V^2 = \alpha Id + \beta V$, for some complex numbers $\alpha$ and $\beta$.

**1)** If $V = \alpha Id$ and $V^2 = \alpha^2 Id$, then $|\alpha| = 1$ and $\lambda Id = \alpha(1+\lambda)U - \alpha^2U^2$. The spectral representation of $U$ is therefore of the form

$$U = \overline{\alpha}\lambda P_{ker\{\alpha U - \lambda Id\}} + \overline{\alpha} P_{ker\{\alpha U - Id\}},$$

where $P_{ker\{\alpha U - \lambda Id\}}$ represents the projection onto $ker\{\alpha U - \lambda Id\}$. Therefore $Q(T) = P_{ker\{\alpha U - Id\}}T$.

**2)** If $V^2 = \alpha Id + \beta V$, for some complex numbers $\alpha$ and $\beta$, then $\lambda Id = -\alpha U^2$ and $(\lambda+1)U = \beta U^2$. This implies that $|\alpha| = 1$ and that $\alpha = -\frac{\lambda\beta^2}{(\lambda+1)^2}$. Therefore the spectral theorem applied to $V$ implies the following representation:

$$V = \frac{\beta\lambda}{\lambda+1} P_{Ker(V - \frac{\beta\lambda}{\lambda+1}Id)} + \frac{\beta}{\lambda+1} P_{Ker(V - \frac{\beta}{\lambda+1}Id)}.$$

We also notice that $P_{V - \frac{\beta}{\lambda+1}Id} + P_{V - \frac{\beta\lambda}{\lambda+1}Id} = Id$. Therefore, if we denote $Ker(V - \frac{\beta}{\lambda+1}Id)$ by $F$, we have that

$$Q(T) = \frac{1}{1-\lambda}\left[-\lambda T + UTV\right] = TP_F.$$

This completes the proof of the proposition. □

**Corollary 2.4.** *If $Q(T) = P_FT$ or $Q(T) = TP_F$, where $P_F$ represents a projection on $\mathcal{H}$, then $Q$ is the average of the identity with an isometric reflection.*

*Proof.* We consider $Q(T) = P_FT$ and show that $Q$ is the average of the identity with an isometric reflection. If $\lambda$ is a modulus 1 complex number, let $U_\lambda$ be defined on $\mathcal{H}$ by $U_\lambda(v) = P_F(v) + \lambda(Id - P_F)(v)$. It is easy to see that $U_\lambda$ is a surjective isometry. In fact, given $v \in \mathcal{H}$ we have that

$$\|U_\lambda(v)\|^2 = \|P_F(v)\|^2 + \|(Id - P_F)(v)\|^2 = \|v\|^2.$$

The surjectivity follows, since $U_\lambda(P_F(v) + \lambda^{-1}(Id - P_F)(v)) = v$. On the other hand, we have that $\tau(T) = 2P_FT - T$ is an isometry of $\mathcal{I}$ since it can be written as $\tau(T) = U_{-1}TId$. It follows that $Q(T) = \frac{1}{2}(Id + \tau)(T)$. If $Q(T) = TP_F$ the proof follows similarly. □

**Proposition 2.5.** *Let $(\mathcal{I},\nu)$ be a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class) and let $Q$ be a generalized bi-circular projection on $\mathcal{I}$. If $Q$ is associated with a surjective isometry $\tau$ on $\mathcal{I}$ of the form $\tau(T) = UTV$ ($U$ and $V$ unitary operators on $\mathcal{H}$) and $\lambda = -1$, then $Q$ is the average of the identity with an isometric reflection on $\mathcal{H}$.*

*Proof.* The operator $Q(T) = \frac{1}{2}\left[T + UTV\right]$ is a projection if and only if $T = U^2TV^2$ for every operator $T \in \mathcal{I}$. Therefore we have that $V^2 = \alpha Id$ and $U^2 = \overline{\alpha} Id$. □

**Theorem 2.6.** *If $(\mathcal{I},\nu)$ is a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class), then $Q$ is a generalized bi-circular projection on $\mathcal{I}$ if and only if $Q$ is the average of the identity with an isometric reflection.*

*Proof.* If a projection $Q$ is the average of the identity with an isometric reflection, denoted by $R$, then $R = Q - (Id - Q)$ and $Q$ is a generalized bi-circular projection with $\lambda = -1$. Conversely, if $Q$ is a generalized bi-circular projection the statement in this theorem follows from Proposition 2.3, Corollary 2.4, and Proposition 2.5. □

The following corollary is an immediate consequence of the previous results.

**Corollary 2.7.** *Every generalized bi-circular projection on $\mathcal{I}$ is a bi-contractive projection, i.e.* $\|P\| \leq 1$ *and* $\|Id - P\| \leq 1$.

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## References

[1] Behrends, E., *M-Structure and the Banach-Stone Theorem*, Lecture Notes in Mathematics **736** (1979), Springer-Verlag. MR547509 (81b:46002)
[2] Berkson, E., *Hermitian projections and orthogonality in Banach spaces*, Proc. London Math. Soc. **24:3** (1972), 101–118. MR0295123 (45:4191)
[3] Bernau, S. J. and Lacey, H. E., *Bicontractive projections and reordering of $L_p$–spaces*, Pac. Journal of Math. **69:2** (1977), 291–302. MR0487416 (58:7054)
[4] Botelho, F. and Jamison, J. E., *Generalized bi-circular projections on $C(\Omega, X)$*, Rocky Mountain Journal (to appear).
[5] Friedman, Y. and Russo, B., *Contractive projections on $C_0(K)$*, Transactions of the AMS **273:1** (1982), 57–73. MR664029 (83i:46062)
[6] Fleming, R. and Jamison, J., *Isometries on Banach Spaces* (2003), Chapman & Hall.
[7] Fosner, M., Ilisevic, D. and Li, C., *G-invariant norms and bicircular projections*, Linear Algebra and Applications **420** (2007), 596–608. MR2278235
[8] Groot, J. and Wille, R., *Rigid continua and topological group-pictures*, Arch. Math. **IX** (1958), 441–446. MR0101514 (21:324)
[9] Jamison, J. E., *Bicircular projections on some Banach spaces*, Linear Algebra and Applications **420** (2007), 29–33. MR2277626
[10] Lindenstrauss, J. and Tzafriri, L., *Classical Banach Spaces I* (1977), Springer Verlag. MR0500056 (58:17766)
[11] Shatten, R., *Norm Ideals of Completely Continuous Operators* (1970), Springer-Verlag, Berlin. MR0257800 (41:2449)
[12] Sourour, A. R., *Isometries of norm ideals of compact operators*, J. Funct. Anal. **43** (1981), 69–77. MR639798 (84e:47061)
[13] Spain, P. G., *Ultrahermitian projections on Banach spaces*, preprint (2006).
[14] Taylor, A. E., *Introduction to Functional Analysis* (1957), John Wiley & Sons Inc. MR0098966 (20:5411)
[15] Fong, C. K. and Sourour, A. R., *On the operator identity $\sum A_k X B_k = 0$*, Can. J. Math., **XXXI** (1979), 845–857. MR540912 (80h:47020)
[16] Randrianantoanina, B., *Norm-One projections in Banach spaces*, Taiwanese Journal of Mathematics. **5:1** (2001), 35–95. MR1816130 (2003d:46001)
[17] Stachó, L. L. and Zalar, B., *Bicircular projections on some matrix and operator spaces*, Linear Algebra and Applications **384** (2004), 9–20. MR2055340 (2005a:46146)
[18] Stachó, L. L. and Zalar, B., *Bicircular projections and characterization of Hilbert spaces*, Proc. Amer. Math. Soc. **132** (2004), 3019–3025. MR2063123 (2005b:46152)

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# RATIONAL FUNCTIONS WITH LINEAR RELATIONS

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Abstract. We find all polynomials $f, g, h$ over a field $K$ such that $g$ and $h$ are linear and $f(g(x)) = h(f(x))$. We also solve the same problem for rational functions $f, g, h$, in case the field $K$ is algebraically closed.

## 1. Introduction

Around 1920, Fatou, Julia and Ritt made profound investigations of functional equations. In particular, they wrote at length on commuting rational functions: that is, $f, g \in \mathbb{C}(x)$ with $f(g(x)) = g(f(x))$. Fatou and Julia [7, 8] found all solutions when the Julia set of $f$ or $g$ is not the Riemann sphere. This includes the case of polynomials of degree at least 2, where up to conjugacy by a linear polynomial, either $f = x^n$ and $g = x^m$ are power polynomials, or $f = T_n$ and $g = T_m$ are Chebychev polynomials, or $f$ and $g$ have a common iterate. Using different methods which did not require the Julia set hypothesis, Ritt [12] determined precisely when two polynomials have a common iterate, and moreover [14] he found all commuting rational functions. Years later, Eremenko [6] proved Ritt's results using methods of modern iteration theory.

Julia showed that commuting rational functions have the same Julia set. Conversely, much subsequent work has shown that rational functions with the same Julia set are related to commuting rational functions (cf. [9] and the references therein). In particular, for polynomials this relationship involves composition with a rotational symmetry of the Julia set.

Several authors have considered analogous questions over fields $K$ of positive characteristic, but there are few satisfactory results. There are new types of examples, for instance any two additive polynomials $\sum_i a_i x^{p^i}$ over the prime field $\mathbb{F}_p$ commute.

In fact, challenges already arise in finding the commuting polynomials $f, g \in K[x]$ in the special case $\deg(g) = 1$. Wells [15] and Mullen [10] solved this problem over finite fields $K$, so long as $\deg(f) < \#K$. Park [11] proved similar results. Eigenthaler and Nöbauer [5] solved the problem in various special cases, for instance if $\deg(f) = \text{char}(K)$. In this paper we solve the problem in general, and more generally we find all $f, g, h \in K[x]$ with $\deg(g) = \deg(h) = 1$ such that $f \circ g = h \circ f$:

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