25. R. I. Ovsepian and A. Pełczyński, 'On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in L^2', *Studia Math.* 54 (1975) 149–159.
26. A. Pełczyński, 'All separable Banach spaces admit for every $\varepsilon > 0$ fundamental total and bounded by $1+\varepsilon$ biorthogonal sequences', *Studia Math.* 55 (1976) 295–304.
27. R. K. Ritt, 'A condition that $\lim_{n\to\infty} n^{-1}T^n = 0$', *Proc. Amer. Math. Soc.* 4 (1953) 898–899.
28. A. M. Sinclair, 'The norm of a hermitian element in a Banach algebra', *Proc. Amer. Math. Soc.* 28 (1971) 446–450.
29. I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces* (Springer, Berlin, 1970).
30. Y. Tomilov and J. Zemánek, 'A new way of constructing examples in operator ergodic theory', *Math. Proc. Cambridge Philos. Soc.*, to appear.

N. Kalton and
S. Montgomery-Smith
Department of Mathematics
University of Missouri
Columbia
MO 65211
USA

nigel@math.missouri.edu
stephen@math.missouri.edu

K. Oleszkiewicz
Institute of Mathematics
Warsaw University
Banacha 2, 02-097
Warsaw
Poland

koles@mimuw.edu.pl

Y. Tomilov
Department of Mathematics
and Informatics
Nicholas Copernicus University
Chopin Str. 12/18
87-100 Torun
Poland

tomilov@mat.uni.torun.pl

J. London Math. Soc. (2) 70 (2004) 479–498
DOI: 10.1112/S0024610704005423

ON THE NORM OF ELEMENTARY OPERATORS

A. BLANCO, M. BOUMAZGOUR and T. J. RANSFORD

Abstract

The norm problem is considered for elementary operators of the form $U_{a,b}:\mathcal{A}\longrightarrow\mathcal{A}$, $x\longmapsto axb+bxa$ $(a,b\in\mathcal{A})$ in the special case when $\mathcal{A}$ is a subalgebra of the algebra of bounded operators on a Banach space. In particular, the lower estimate $\|U_{a,b}\|\geqslant\|a\|\|b\|$ is established when the Banach space is a Hilbert space and $\mathcal{A}$ is the algebra of all bounded linear operators.

1. *Introduction*

Let $\mathcal{A}$ be a real or complex Banach algebra. A linear mapping $\phi:\mathcal{A}\longrightarrow\mathcal{A}$ of the form

$$x\longmapsto\sum_{i=1}^{n}a_ixb_i, \tag{1}$$

where $a_i,b_i\in\mathcal{A}$ $(1\leqslant i\leqslant n)$ is called an elementary operator on $\mathcal{A}$. The *norm problem* for elementary operators consists in finding a formula which describes the norm of an elementary operator in terms of its coefficients. Every elementary operator is obviously bounded (take as upper bound, for instance, $\sum_{i=1}^{n}\|a_i\|\|b_i\|$), but until now no satisfactory *lower bounds* seem to be known for the norm of arbitrary elementary operators. (See [**12**] for a survey of this problem.)

A most satisfactory answer to the norm problem has been given when the algebra $\mathcal{A}$ of the previous paragraph is the Calkin algebra of a separable Hilbert space [**6**]. In this case, the norm of an elementary operator ϕ, as defined above, coincides with the Haagerup tensor norm of $\sum_{i=1}^{n}a_i\otimes b_i$ (see also [**11**] and [**1**, § 5.4]).

For more general Banach algebras the current state of the art is less promising. For instance, in the case of the algebra $\mathcal{B}(H)$ of all bounded operators on a *complex* Hilbert space H, besides the cases of generalized derivations $(x\longmapsto ax-xb)$ (see [**17**]) and two-sided multiplications $(x\longmapsto axb)$, no simple expression for the norm of an elementary operator seems to be known.

In this paper we shall be concerned with the elementary operator

$$\mathcal{U}_{A,B}:\mathcal{B}(E)\longrightarrow\mathcal{B}(E),\quad X\longmapsto AXB+BXA\qquad(A,B\in\mathcal{B}(E)),$$

where $\mathcal{B}(E)$ denotes the algebra of all bounded operators on a (real or complex) Banach space E. It is natural in this case to look for lower estimates of the form $c\|A\|\|B\|$. The problem is then to determine the largest possible constant c such

Received 29 April 2003; revised 17 December 2003.

2000 *Mathematics Subject Classification* 47B47 (primary), 46A32, 47B48, 47L10 (secondary).

The research of the first author was supported by a postdoctoral research fellowship at Université Laval. The research of the second author was partially supported by grants from the Agence Universitaire de la Francophonie (Programme de Coopération Interuniversitaire) and from the CRM (Montréal). That of the third author was partially supported by grants from NSERC (Canada), FQRNT (Québec) and the Canada Research Chair program.

that

$$\|\mathcal{U}_{A,B}\| \geqslant c\|A\|\|B\| \qquad (A, B \in \mathcal{B}(E)). \tag{2}$$

When E is a Hilbert space the problem has been considered by several authors [**7**, **9**, **10**, **15**, **16**]. It was conjectured in [**9**] that $c = 1$ in this particular case, and it was shown in [**10**] that $c \geqslant 2/3$. This estimate was further improved to $2(\sqrt{2}-1)$ in [**15**]. The value $c = 1$ has been established under the assumption that A and B are self-adjoint [**16**], and also under the assumption that $\|A + zB\| \geqslant \|A\|$ for all $z \in \mathbb{C}$ [**2**].

Here we settle this problem definitively by establishing the largest possible constant ($c = 1$) in the inequality (2) in the case when E is a Hilbert space (Theorem 3.7 below). We also establish the largest possible constant ($c = 2(\sqrt{2}-1)$) in the same inequality for certain normed algebras of operators acting on arbitrary Banach spaces. Moreover, we show that the estimate $2(\sqrt{2}-1)\|A\|\|B\|$ is sharp within the class of Banach algebras $\mathcal{B}(E)$.

We conclude this section with some notations and terminology.

Given a Banach space E, we denote by E' its (topological) dual. 'Operator' always means bounded linear operator. The following distinction will be important in what follows. For an operator $T : E \longrightarrow E$ we denote by $T' : E' \longrightarrow E'$ its adjoint, defined by $(T'f)(x) := f(Tx)$ $(x \in E, f \in E')$. If $\mathcal{H}$ is a Hilbert space and $V : \mathcal{H} \longrightarrow \mathcal{H}$ is a linear operator, we denote by $V^* : \mathcal{H} \longrightarrow \mathcal{H}$ its adjoint defined by $\langle h, V^*k\rangle := \langle Vh, k\rangle$ $(h, k \in \mathcal{H})$.

We denote by $\mathcal{B}(E)$, $\mathcal{F}(E)$ and $\mathcal{N}(E)$ respectively the algebras of bounded, bounded finite-rank and nuclear operators on a Banach space E. The nuclear and operator norms will be denoted by $\|\,.\,\|_N$ and $\|\,.\,\|$ respectively.

By an *operator algebra* on a Banach space E we mean a subalgebra of $\mathcal{B}(E)$ that contains the ideal $\mathcal{F}(E)$. (Note that no topology whatsoever is assumed.)

Let X and Y be normed spaces and let $T : X \longrightarrow Y$ be a linear operator. Let E and F be linear subspaces of X and Y respectively. If $T(E) \subseteq F$, we write $\|T : E \longrightarrow F\|$ for the norm of $T|_E^F$, that is, the restriction to E and corestriction to F of T. Here the norms on E and F need not be the same as those on X and Y. We also write $\|T : X \longrightarrow Y\|$ for $\|T\|$.

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathfrak{S}_2(\mathcal{H})$ the class of Hilbert–Schmidt operators on $\mathcal{H}$. Recall that $\mathfrak{S}_2(\mathcal{H})$ is itself a Hilbert space with the inner product $\langle S, T\rangle := \mathrm{tr}(T^*S)$ $(S, T \in \mathfrak{S}_2(\mathcal{H}))$. The Hilbert–Schmidt norm will be denoted by $s_2(\,.\,)$. Recall also that for a nuclear (respectively compact) operator on $\mathcal{H}$, say T, we have $\|T\|_N = \sum_i \sigma_i(T)$ (respectively $\|T\| = \max_i\{\sigma_i(T)\}$), where $(\sigma_i(T))$ denotes the sequence of singular values of T.

We denote by $\mathrm{diag}\{a_1, a_2\}$ the diagonal matrix

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

Throughout, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. The real and imaginary parts of a complex number z are denoted by $\Re(z)$ and $\Im(z)$ respectively.

2. *Some general results*

In this section we prove some results for a kind of elementary operator more general than the one mentioned in the introduction (see below). These results will be needed in the next section. Though we shall need them only for two-dimensional Hilbert spaces, we have chosen to state and prove them here in a more general form, since we believe they are of some interest in their own right, and consideration of the general case does not entail much additional effort.

Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of bounded linear operators on a (real or complex) Hilbert space $\mathcal{H}$, and let

$$\mathcal{U}_{A,B} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), \quad X \longmapsto \sum_{i=1}^{n}(A_iXB_i + B_iXA_i). \tag{3}$$

In Theorem 2.3 below, we provide a lower bound for the norm of this elementary operator. First we prove the following.

PROPOSITION 2.1. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{U}_{A,B}$ be as above. Then*

$$\begin{aligned}\|\mathcal{U}_{A,B} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})\| &= \|\mathcal{U}_{A,B} : \mathcal{N}(\mathcal{H}) \longrightarrow \mathcal{N}(\mathcal{H})\| \\ &= \sup_{\substack{X : \|X\|_N = 1,\\ \mathrm{rank} X = 1}} \|\mathcal{U}_{A,B}(X)\|_N. \end{aligned} \tag{4}$$

Proof. The first equality follows on noting that $\mathcal{U}_{A,B} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is the adjoint of $\mathcal{U}_{A,B} : \mathcal{N}(\mathcal{H}) \longrightarrow \mathcal{N}(\mathcal{H})$. The second equality is a consequence of the linearity of $\mathcal{U}_{A,B}$ and the well-known fact that the unit ball of $\mathcal{N}(\mathcal{H})$ is the closure of the convex hull of the rank-one operators of norm one. □

REMARK 2.2. Note that the proof works for any reflexive Banach space E for which $\mathcal{N}(E)' = \mathcal{B}(E)$ under trace duality. We shall return to this in the last section of this paper.

Proposition 2.1 suggests estimating the right-hand side of (4) in order to obtain a lower bound for the norm of $\mathcal{U}_{A,B}$. Indeed, as shown by the second equality of the proposition, in order to estimate the right-hand side of (4), it is enough to consider rank-one operators. We point out that this idea underlies most of the results in this paper. Also note that, for a rank-one operator X, the operator $\mathcal{U}_{A,B}(X)$ is always of rank at most $2n$.

Recall that, for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ is a pre-Hilbert space with the inner product determined by $\langle h_1 \otimes k_1, h_2 \otimes k_2\rangle = \langle h_1, h_2\rangle\langle k_1, k_2\rangle$ $(h_1, h_2 \in \mathcal{H},\ k_1, k_2 \in \mathcal{K})$. We write $\mathcal{H}\tilde{\otimes}\mathcal{K}$ for its completion.

When $\mathcal{H}$ is a real Hilbert space, $\mathfrak{S}_2(\mathcal{H}) = \mathcal{H}\tilde{\otimes}\mathcal{H}$ (here $h \otimes k$ is identified with the rank-one operator $\xi \longmapsto \langle \xi, h\rangle k$ $(\xi \in \mathcal{H})$). In this terminology, $\mathcal{U}_{A,B} = \sum_i (A_i^* \otimes B_i + B_i^* \otimes A_i)$. We let $\Lambda : \mathcal{H}\tilde{\otimes}\mathcal{H} \longrightarrow \mathcal{H}\tilde{\otimes}\mathcal{H}$ denote the flip operator, which is defined on elementary tensors by $\Lambda(h \otimes k) := k \otimes h$ $(h, k \in \mathcal{H})$.

THEOREM 2.3. *Let $\mathcal{U}_{A,B} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be defined as in (3). Then*

$$\|\mathcal{U}_{A,B} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})\| \geqslant \|\mathcal{U}_{A,B} : \mathfrak{S}_2(\mathcal{H}) \longrightarrow \mathfrak{S}_2(\mathcal{H})\|. \tag{5}$$

Moreover, in the real case we have

$$\|\mathcal{U}_{A,B} : \mathfrak{S}_2(\mathcal{H}) \longrightarrow \mathfrak{S}_2(\mathcal{H})\| = \nu(\Lambda \circ \mathcal{U}_{A,B}),$$

where ν stands for the spectral radius.

To prove this theorem we need the following result on complex interpolation, which is a consequence of [**14**, Satz 8] and [**3**, Theorem 4.1.2].

PROPOSITION 2.4. *Let $\mathcal{H}$ be a complex Hilbert space, and let $T:\mathcal{B}(\mathcal{H})\longrightarrow\mathcal{B}(\mathcal{H})$ be a linear operator such that $T(\mathcal{N}(\mathcal{H}))\subseteq\mathcal{N}(\mathcal{H})$. Then $T(\mathfrak{S}_2(\mathcal{H}))\subset\mathfrak{S}_2(\mathcal{H})$ and*

$$\|T:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|\leqslant\|T:\mathcal{N}(\mathcal{H})\longrightarrow\mathcal{N}(\mathcal{H})\|^{1/2}\|T:\mathcal{B}(\mathcal{H})\longrightarrow\mathcal{B}(\mathcal{H})\|^{1/2}.$$

Proof of Theorem 2.3. If $\mathcal{H}$ is a complex Hilbert space, then (5) follows immediately from Propositions 2.4 and 2.1.

Let us consider the real case. It is easily seen that $(\Lambda\circ(B_i^*\otimes A_i))^*=\Lambda\circ(A_i^*\otimes B_i)$ $(1\leqslant i\leqslant n)$. Thus, $\Lambda\circ\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})$ is a self-adjoint operator, and, since Λ is an isometry, we obtain

$$\|\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|=\|\Lambda\circ\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|=\nu(\Lambda\circ\mathcal{U}_{A,B}).\quad(6)$$

This proves the second part of the proposition.

To show that (5) holds, we argue as follows. Suppose first that $\dim\mathcal{H}<\infty$. Since $\Lambda\circ\mathcal{U}_{A,B}$ is hermitian, there exist $\rho\in\mathbb{R}$ and $T\in\mathfrak{S}_2(\mathcal{H})$ $(T\neq0)$ such that $|\rho|=\nu(\Lambda\circ\mathcal{U}_{A,B})$ and $\Lambda\circ\mathcal{U}_{A,B}(T)=\rho T$. It follows that

$$\begin{aligned}|\rho|&=\left\|\Lambda\circ\mathcal{U}_{A,B}\left(\frac{1}{\|T\|_N}T\right)\right\|_N\leqslant\|\Lambda\circ\mathcal{U}_{A,B}:\mathcal{N}(\mathcal{H})\longrightarrow\mathcal{N}(\mathcal{H})\|\\&=\|\mathcal{U}_{A,B}:\mathcal{N}(\mathcal{H})\longrightarrow\mathcal{N}(\mathcal{H})\|.\end{aligned}$$

This, combined with (6) and (4), proves (5) in the case when $\dim\mathcal{H}<\infty$.

Now let $\dim\mathcal{H}=\infty$. Fix $\varepsilon>0$, and choose $X\in\mathcal{F}(\mathcal{H})$ such that $s_2(X)\leqslant1$ and

$$\|\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|<s_2(\mathcal{U}_{A,B}(X))+\varepsilon.\quad(7)$$

(Here we have used that $\mathcal{F}(\mathcal{H})$ is dense in $\mathfrak{S}_2(\mathcal{H})$.) Choose an orthogonal projection P on $\mathcal{H}$ such that $X=PXP$ and $\mathcal{U}_{A,B}(X)=P\mathcal{U}_{A,B}(X)P$. Then, by the finite-dimensional case,

$$\begin{aligned}s_2(\mathcal{U}_{A,B}(X))=s_2(\mathcal{U}_{PAP,PBP}(X))&\leqslant\sup_{\substack{Y\in\mathfrak{S}_2(\mathcal{H}):\\Y=PYP\text{ and }s_2(Y)\leqslant1}}s_2(\mathcal{U}_{PAP,PBP}(Y))\\&\leqslant\sup_{\substack{Y\in\mathcal{N}(\mathcal{H}):\\Y=PYP\text{ and }\|Y\|_N\leqslant1}}\|\mathcal{U}_{PAP,PBP}(Y)\|_N\leqslant\|\mathcal{U}_{A,B}:\mathcal{N}(\mathcal{H})\longrightarrow\mathcal{N}(\mathcal{H})\|.\end{aligned}$$

Combining this inequality with (7), we see that

$$\|\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|\leqslant\|\mathcal{U}_{A,B}:\mathcal{N}(\mathcal{H})\longrightarrow\mathcal{N}(\mathcal{H})\|+\varepsilon.$$

Letting $\varepsilon\to0$ and applying Proposition 2.1, we obtain the desired result. □

3. *The Hilbert space case*

In what follows, unless otherwise stated, H denotes a two-dimensional inner product space over $\mathbb{K}$. Whenever an orthonormal basis of H has been specified, we identify an operator on H with its corresponding matrix representation in this basis.

LEMMA 3.1. *Let $\{e_1,e_2\}$ be an orthonormal basis of H, and let*

$$T=\begin{pmatrix}t_{1,1}&t_{1,2}\\t_{2,1}&t_{2,2}\end{pmatrix}\in\mathcal{B}(H).$$

Then

$$\|T\|_N^2=\sum_{1\leqslant i,j\leqslant2}|t_{i,j}|^2+2|t_{1,1}t_{2,2}-t_{1,2}t_{2,1}|.\quad(8)$$

Proof. Let σ_1,σ_2 be the singular values of T. Then

$$\begin{aligned}\|T\|_N^2=(\sigma_1+\sigma_2)^2&=(\sigma_1^2+\sigma_2^2)+2\sigma_1\sigma_2\\&=s_2^2(T)+2\sqrt{\det(T^*T)}=s_2^2(T)+2|\det(T)|.\end{aligned}$$

The rest is clear. □

LEMMA 3.2. *Let $A,B\in\mathcal{B}(H)$, and let $\{e_1,e_2\}$ be an orthonormal basis of H with respect to which A and B have matrix representations $\operatorname{diag}\{1,\mu\}$ and $\left(\begin{smallmatrix}w&x\\y&z\end{smallmatrix}\right)$ respectively. Then we have the following.*

(i)

$$\left\|\mathcal{U}_{A,B}\left(\frac12\begin{pmatrix}1&1\\1&1\end{pmatrix}\right)\right\|_N^2\geqslant(1+|\mu|^2)\left(\left|w+\frac{x+y}{2}\right|^2+\left|z+\frac{x+y}{2}\right|^2\right).$$

(ii)

$$\left\|\mathcal{U}_{A,B}\left(\frac12\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\right)\right\|_N^2\geqslant(1+|\mu|^2)\left(\left|w+\frac{y-x}{2}\right|^2+\left|z+\frac{x-y}{2}\right|^2\right).$$

(iii)

$$\left\|\mathcal{U}_{A,B}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)\right\|_N^2=4|w|^2+(|x|+|y|)^2.$$

Proof. (i) It is easy to see that

$$\mathcal{U}_{A,B}\left(\tfrac12\begin{pmatrix}1&1\\1&1\end{pmatrix}\right)=\begin{pmatrix}\alpha&\beta-\gamma\\\gamma+\mu\alpha&\mu\beta\end{pmatrix},$$

where

$$\alpha=w+\frac{x+y}{2},\quad\beta=z+\frac{x+y}{2}\quad\text{and}\quad\gamma=\frac{z+y}{2}-\mu\left(\frac{w+x}{2}\right).$$

Then by Lemma 3.1 we have

$$\begin{aligned}&\left\|\mathcal{U}_{A,B}\left(\tfrac12\begin{pmatrix}1&1\\1&1\end{pmatrix}\right)\right\|_N^2\\&=|\alpha|^2+|\gamma+\mu\alpha|^2+|\beta-\gamma|^2+|\mu\beta|^2+2|\mu\alpha\beta-(\gamma+\mu\alpha)(\beta-\gamma)|\\&=|\alpha|^2+|\gamma|^2+2\Re(\overline{\gamma}\mu\alpha)+|\mu\alpha|^2+|\beta|^2-2\Re(\overline{\gamma}\beta)+|\gamma|^2\\&\quad+|\mu\beta|^2+2|\gamma(\beta-\mu\alpha-\gamma)|\\&=(1+|\mu|^2)(|\alpha|^2+|\beta|^2)+2(|\overline{\gamma}(\beta-\mu\alpha-\gamma)|-\Re(\overline{\gamma}(\beta-\mu\alpha-\gamma)))\\&\geqslant(1+|\mu|^2)(|\alpha|^2+|\beta|^2).\end{aligned}$$

(ii) The proof is analogous to that of (i).

(iii) It is easily seen that

$$\mathcal{U}_{A,B}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)=\begin{pmatrix}2w&x\\y&0\end{pmatrix}.$$

Now apply Lemma 3.1. □

Note that

$$\tfrac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix},\quad \tfrac{1}{2}\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

all have nuclear norm 1.

LEMMA 3.3. *With the same hypotheses as in Lemma 3.2, if* $|\mu|\geqslant 1/\sqrt{3}$ *and* $s_2(B)=1$, *then* $\|\mathcal{U}_{A,B}\|\geqslant 1$.

Proof. Consider the operator

$$T=\tfrac{1}{2}\begin{pmatrix}1&0\\0&\sqrt{3}\,c\end{pmatrix},$$

where $c=\overline{\mu}/|\mu|$. Note that $s_2(T)=1$. Now

$$\mathcal{U}_{A,B}(T)=\begin{pmatrix}w & x\left(\frac{1+\sqrt{3}|\mu|}{2}\right)\\ y\left(\frac{1+\sqrt{3}|\mu|}{2}\right) & \sqrt{3}|\mu|z\end{pmatrix}.$$

When $|\mu|\geqslant 1/\sqrt{3}$ we find that

$$s_2^2(\mathcal{U}_{A,B}(T))=|w|^2+(|x|^2+|y|^2)\left(\frac{1+\sqrt{3}|\mu|}{2}\right)^2+3|\mu|^2|z|^2\geqslant s_2^2(B).$$

This proves that $\|\mathcal{U}_{A,B}:\mathfrak{S}_2(H)\longrightarrow\mathfrak{S}_2(H)\|\geqslant s_2(B)$. Now apply Theorem 2.3. □

LEMMA 3.4. *Let* $w,x,y,z,\mu\in\mathbb{K}$ *be such that* $x>3/4$, $x+|y|<1$, $z<1/2$, $0\leqslant w<z/\sqrt{3}$ *and* $|\mu|<1/\sqrt{3}$. *Then the following hold.*

(i) $2wxz+|\mu y|z^2<x^3$.

(ii) $|\mu|z+|\mu y|<x$.

(iii) $2wz+|\mu|xz<x^2$.

Proof. Parts (i) and (ii) are straightforward and we leave them to the reader. To see (iii), note that

$$2wz+|\mu|xz\leqslant\tfrac{1}{2\sqrt{3}}(1+x),$$

so it is enough to show that

$$\tfrac{1}{2\sqrt{3}}(1+x)<x^2\qquad(x>3/4).$$

This last is a simple exercise. □

LEMMA 3.5. *Let* $x,y\in\mathbb{K}$ *be such that* $|y|\leqslant x$. *Then*

$$|x^2-y^2|^2\leqslant(x+|y|)^2(x^2-\Re(y)^2).\tag{9}$$

Proof. Let $y=re^{i\theta}$. Then (9) is equivalent to

$$|x^2-r^2e^{2i\theta}|^2\leqslant(x+r)^2(x^2-r^2\cos^2\theta)\qquad(0\leqslant r\leqslant x).\tag{10}$$

It is clearly enough to consider the case $x=1$. For $x=1$, (10) can be written as

$$1-2r^2\cos 2\theta+r^4\leqslant(1+r)^2(1-r^2\cos^2\theta),$$

which, using the identity $\cos 2\theta=2\cos^2\theta-1$, can be easily reduced to

$$r^4+r^2-2r\leqslant(3r^2-2r^3-r^4)\cos^2\theta.$$

When $0\leqslant r\leqslant 1$, the left-hand side of the last inequality is non-positive, while the right-hand side is non-negative. □

PROPOSITION 3.6. *Let* $A,B\in\mathcal{B}(H)$ *(where* $\dim H=2$*). Then*

$$\|\mathcal{U}_{A,B}:\mathcal{B}(H)\longrightarrow\mathcal{B}(H)\|\geqslant\|A\|\|B\|.$$

Sketch of the proof: By Proposition 2.1, it is enough to prove that $\|\mathcal{U}_{A,B}:\mathcal{N}(H)\longrightarrow\mathcal{N}(H)\|\geqslant\|A\|\|B\|$.

Firstly, we show that we can restrict consideration to operators A and B whose matrix representations with respect to some fixed orthonormal basis of H are of a certain form.

Secondly, using the information provided by Lemmas 3.2 and 3.3, we show that if the desired inequality fails for some pair A, B of the form specified in the first part of the proof, then B must be in a determined 'neighbourhood' of $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$.

Lastly, we show that if B happens to be in the 'neighbourhood' of $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ from the previous part, then there are operators T_B and T with nuclear norm 1, such that either $\|\mathcal{U}_{A,B}(T_B)\|_N\geqslant 1$ or $\|\mathcal{U}_{A,B}(T)\|_N\geqslant 1$.

Proof of Proposition 3.6. We give the proof for the complex case. In the real case the proof is simpler and the simplifications are self-evident.

We can assume that $\|A\|=\|B\|=1$. Since $\|\mathcal{U}_{A,B}\|=\|\mathcal{U}_{WA,WB}\|$ ($W\in\mathcal{B}(H)$ unitary) and $\|\mathcal{U}_{A,B}\|=\|\mathcal{U}_{-A,B}\|$, we can clearly restrict consideration to the case when A is hermitian and 1 is an eigenvalue of A. In this case there exists an orthonormal basis of H, say $\{e_1,e_2\}$, with respect to which A has a diagonal representation, say $\operatorname{diag}\{1,\mu\}$, where $\mu\in\mathbb{R}$ and $|\mu|\leqslant 1$. Let $\left(\begin{smallmatrix}w&x\\y&z\end{smallmatrix}\right)$ be the matrix that represents B in the same basis. Since $\|\mathcal{U}_{A,B}\|=\|\mathcal{U}_{A^*,B^*}\|$, we can assume that $|x|\geqslant|y|$. Let U and V be unitary operators on H whose matrix representations in the basis $\{e_1,e_2\}$ are $\operatorname{diag}\{1,u\}$ and $\operatorname{diag}\{1,v\}$ respectively, and let $t\in\mathbb{C}$ be of modulus 1. Define $\widetilde{A}:=V^*AVU$ and $\widetilde{B}:=tV^*BVU$. It is readily seen that $\|\mathcal{U}_{A,B}\|=\|\mathcal{U}_{\widetilde{A},\widetilde{B}}\|$. Note also that, in the basis $\{e_1,e_2\}$, $\widetilde{A}$ and $\widetilde{B}$ are represented by $\operatorname{diag}\{1,u\mu\}$ and $\left(\begin{smallmatrix}tw&tuvx\\t\overline{v}y&tuz\end{smallmatrix}\right)$ respectively. We can choose t, u and v in such a way that tw, $tuvx$ and tuz are all real and non-negative. Thus, by replacing A by $\widetilde{A}$ and B by $\widetilde{B}$, if necessary, we can assume that A and B are such that there exists an orthonormal basis $\{e_1,e_2\}$ of H in which A has the representation $\operatorname{diag}\{1,\mu\}$, where $\mu\in\mathbb{C}$ with $|\mu|\leqslant 1$, and B has the representation $\left(\begin{smallmatrix}w&x\\y&z\end{smallmatrix}\right)$, with w, x and z real, non-negative and $x\geqslant|y|$.

It is clear from Lemma 3.2(iii) and Lemma 3.3 that we just need to consider those cases for which $|\mu| < 1/\sqrt{3}$ and $4w^2 + (x+|y|)^2 < 1$. Note that, since $s_2(B) \geqslant \|B\| = 1$, the last inequality implies that

$$w < z/\sqrt{3}. \tag{11}$$

Also, by Lemma 3.2 parts (i) and (ii),

$$\begin{aligned}
&\left\|\mathcal{U}_{A,B}\left(\tfrac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)\right\|_N^2 \\
&\geqslant (1+|\mu|^2)\left[\left(w+\frac{x+\Re(y)}{2}\right)^2+\left(z+\frac{x+\Re(y)}{2}\right)^2+\frac{\Im(y)^2}{2}\right]\\
&= s_2^2(A)\left[w^2+z^2+(w+z)(x+\Re(y))+\tfrac{1}{2}(x+\Re(y))^2+\tfrac{1}{2}\Im(y)^2\right]\\
&= s_2^2(A)\left[s_2^2(B)+(w+z)(x+\Re(y))-\tfrac{1}{2}(x-\Re(y))^2-\tfrac{1}{2}\Im(y)^2\right]\\
&= s_2^2(A)\left[s_2^2(B)+(w+z)(x+\Re(y))-\tfrac{1}{2}|x-y|^2\right],
\end{aligned}$$

and similarly

$$\left\|\mathcal{U}_{A,B}\left(\tfrac{1}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right)\right\|_N^2 \geqslant s_2^2(A)\left[s_2^2(B)+(z-w)(x-\Re(y))-\tfrac{1}{2}|x+y|^2\right].$$

Consequently, there is nothing else to do if $(w+z)(x+\Re(y)) \geqslant \frac{1}{2}|x-y|^2$ or if $(z-w)(x-\Re(y)) \geqslant \frac{1}{2}|x+y|^2$. Thus, let us assume that

$$z+w < \frac{1}{2}\frac{|x-y|^2}{x+\Re(y)} \tag{12}$$

and

$$z-w < \frac{1}{2}\frac{|x+y|^2}{x-\Re(y)}. \tag{13}$$

Multiplying together (12) and (13), we obtain

$$z^2-w^2 < \frac{1}{4}\frac{|x^2-y^2|^2}{x^2-\Re(y)^2}. \tag{14}$$

Since $4w^2+(x+|y|)^2<1$, it follows from (14) and Lemma 3.5 that

$$z^2 < \frac{1}{4}\frac{|x^2-y^2|^2}{x^2-\Re(y)^2}+\frac{1}{4}\left(1-(x+|y|)^2\right) \leqslant \frac{1}{4} \tag{15}$$

and in turn, from (11), that

$$w^2+z^2 < (1+\tfrac{1}{3})z^2 < \tfrac{1}{3}.$$

Taking into account that $s_2^2(B) \geqslant 1$, we conclude from the last inequality that

$$x^2+|y|^2 > \tfrac{2}{3}. \tag{16}$$

Inequality (16), together with $x+|y|<1$ and $x>|y|$, implies that

$$x > \tfrac{3}{4}. \tag{17}$$

Thus, B must be in the 'neighbourhood' of $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ determined by the inequalities (11), (15), (17) and $x+|y|<1$.

Let $T_B = c\left(\begin{smallmatrix}x&z\\0&0\end{smallmatrix}\right)$, where $c = 1/\sqrt{x^2+z^2}$. It is clear from (8) that $\|T_B\|_N = 1$, and it is easily seen that

$$\mathcal{U}_{A,B}(T_B) = c\begin{pmatrix}2wx+yz & x^2+\mu wz+z^2\\ yx & \mu yz\end{pmatrix}.$$

(Note that, since $\|B\|=1$, we must have $c \geqslant 1$.)

Using (8), and taking into account (17), we find that

$$\begin{aligned}
\|\mathcal{U}_{A,B}(T_B)\|_N^2 &= c^2[4w^2x^2+4wx\Re(y)z+|y|^2z^2+|y|^2x^2+(x^2+z^2)^2\\
&\quad+2\Re(\mu)wz(x^2+z^2)+|\mu|^2w^2z^2+|\mu^2y^2|z^2\\
&\quad+2|y||\mu z(2wx+yz)-x(x^2+\mu wz+z^2)|]\\
&\geqslant c^2[4w^2x^2+4wx\Re(y)z+|y|^2(z^2+x^2)+(x^2+z^2)^2\\
&\quad+2\Re(\mu)wz(x^2+z^2)\\
&\quad+2|y||\mu wxz+\mu yz^2-x^3-xz^2|]\\
&\geqslant \tfrac{9}{4}w^2-4c^2wx|y|z+|y|^2+x^2+z^2+2c^2\Re(\mu)wz(x^2+z^2)\\
&\quad+2c^2|y|(x^3+xz^2-\Re(\mu)wxz-\Re(\mu y)z^2)\\
&\geqslant s_2^2(B)-4c^2wx|y|z+2c^2\Re(\mu)wz(x^2+z^2)\\
&\quad+2c^2|y|(x^3+xz^2-\Re(\mu)wxz-\Re(\mu y)z^2).
\end{aligned}$$

If $\Re(\mu)>0$ then

$$\begin{aligned}
\|\mathcal{U}_{A,B}(T_B)\|_N^2 &\geqslant s_2^2(B)-4c^2wx|y|z\\
&\quad+2c^2|y|(x^3+xz^2-\Re(\mu)wxz-\Re(\mu y)z^2)\\
&\geqslant s_2^2(B)+2c^2|y|[(x^3-2wxz-|\mu y|z^2)+xz(z-|\mu|w)]\\
&\geqslant s_2^2(B),
\end{aligned}$$

by Lemma 3.4(i) and (11).

If $\Re(\mu)<0$ and $w<|y|$ then

$$\begin{aligned}
\|\mathcal{U}_{A,B}(T_B)\|_N^2 &\geqslant s_2^2(B)-4c^2wx|y|z+2c^2\Re(\mu)|y|z(x^2+z^2)\\
&\quad+2c^2|y|(x^3+xz^2-\Re(\mu y)z^2)\\
&\geqslant s_2^2(B)+2c^2|y|[(x^3-2wxz-|\mu|x^2z)+z^2(x-|\mu|z-|\mu y|)]\\
&\geqslant s_2^2(B)
\end{aligned}$$

by Lemma 3.4(ii) and (iii).

For the remaining case, that is $\Re(\mu)<0$ and $w>|y|$, consider the operator $(1/\sqrt{2})\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$. Then

$$\mathcal{U}_{A,B}\left(\tfrac{1}{\sqrt{2}}\begin{pmatrix}1&0\\1&0\end{pmatrix}\right) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}2w+x & x\\ y+\mu w+z & \mu x\end{pmatrix}$$

and, using (8), we find that

$$\begin{aligned}\left\|\mathcal{U}_{A,B}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\1&0\end{pmatrix}\right)\right\|_N^2 &\geqslant \tfrac{1}{2}[(2w+x)^2+x^2+2|(2w+x)\mu x - x(y+\mu w+z)|]\\ &\geqslant 2w^2+2wx+x^2-\Re(\mu wx+\mu x^2-xy-xz)\\ &\geqslant 2w^2+2wx+x^2-\Re(\mu)wx-\Re(\mu)x^2+x\Re(y)+xz\\ &\geqslant 2w^2+x^2+xz+x(2w+\Re(y))\\ &\geqslant s_2^2(B),\end{aligned}$$

where the last inequality follows from $w>|y|$, together with (15) and (17).

We have thus shown that $\|\mathcal{U}_{A,B}:\mathcal{N}(H)\longrightarrow\mathcal{N}(H)\|\geqslant\|A\|\|B\|$. It just remains to apply Proposition 2.1. □

Finally, we remove the constraint imposed on the dimension of H.

THEOREM 3.7. *Let $\mathcal{H}$ be an arbitrary Hilbert space, and let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$\|\mathcal{U}_{A,B}:\mathcal{F}(\mathcal{H})\longrightarrow\mathcal{F}(\mathcal{H})\|\geqslant\|A\|\|B\|.$$

Proof. We reduce the problem to the case when $\dim\mathcal{H}=2$. The argument is analogous to that of [**16**] for Jordan algebras of operators. We include its proof here for the sake of completeness.

To simplify the notation, we assume that $\|A\|=\|B\|=1$. Fix $\varepsilon>0$, and choose $\xi,\eta\in\mathcal{H}$ such that $\|A\xi\|\geqslant 1-\varepsilon$ and $\|B\eta\|\geqslant 1-\varepsilon$. Since $\|\mathcal{U}_{A,B}\|=\|\mathcal{U}_{WA,WB}\|$ ($W\in\mathcal{B}(\mathcal{H})$ unitary), we can assume, by multiplying by a suitable W if necessary, that there exists a subspace of $\mathcal{H}$ of dimension 2, say H_1, such that $\xi,\eta,A\xi,B\eta\in H_1$. Let $P:\mathcal{H}\longrightarrow H_1$ be the orthogonal projection onto H_1, and let $\phi:\mathcal{B}(\mathcal{H})\longrightarrow\mathcal{B}(\mathcal{H})$ be the linear map defined by $\phi(X):=PXP$ ($X\in\mathcal{B}(\mathcal{H})$). Then

$$\begin{aligned}\|\mathcal{U}_{A,B}:\mathcal{F}(\mathcal{H})\longrightarrow\mathcal{F}(\mathcal{H})\| &= \sup_{\substack{X\in\mathcal{F}(\mathcal{H}):\\ \|X\|\leqslant 1}}\|AXB+BXA\|\\ &\geqslant \sup_{\substack{X\in\mathcal{F}(\mathcal{H}):\\ \|X\|\leqslant 1}}\|P(APXPB+BPXPA)P\|\\ &= \sup_{\substack{Y\in\operatorname{rg}\phi:\\ \|Y\|\leqslant 1}}\|\mathcal{U}_{\phi(A),\phi(B)}(Y)\| \geqslant \|\phi(A)\|\|\phi(B)\|,\end{aligned}$$

where the last inequality follows from Proposition 3.6, since the range of ϕ can be isometrically identified with $\mathcal{B}(l_2^2)$. Moreover, since $\|\phi(A)\|\geqslant\|A\xi\|$ and $\|\phi(B)\|\geqslant\|B\eta\|$, we have shown that

$$\|\mathcal{U}_{A,B}:\mathcal{F}(\mathcal{H})\longrightarrow\mathcal{F}(\mathcal{H})\|\geqslant(1-\varepsilon)^2.$$

Since this inequality holds for arbitrary ε, the desired result follows. □

As an immediate consequence of the last theorem we have the following corollary. (Recall from Section 1 that, by an operator algebra on a Banach space E we mean a subalgebra of $\mathcal{B}(E)$ that contains the ideal $\mathcal{F}(E)$.)

COROLLARY 3.8. *If $\mathfrak{A}$ is an operator algebra on $\mathcal{H}$ endowed with the 'operator norm' (in particular, $\mathcal{B}(\mathcal{H})$ itself), and $A,B\in\mathfrak{A}$, then $\|\mathcal{U}_{A,B}:\mathfrak{A}\longrightarrow\mathfrak{A}\|\geqslant\|A\|\|B\|$.*

In another direction, we have the following result. We are grateful to Martin Mathieu for pointing it out to us.

COROLLARY 3.9. *Let $\mathfrak{A}$ be an arbitrary C^*-algebra, and let $A,B\in\mathfrak{A}$. Let $L_A:\mathfrak{A}\longrightarrow\mathfrak{A}$, $X\longmapsto AX$ and let $R_B:\mathfrak{A}\longrightarrow\mathfrak{A}$, $X\longmapsto XB$. Then $\|\mathcal{U}_{A,B}:\mathfrak{A}\longrightarrow\mathfrak{A}\|\geqslant\|L_AR_B\|$.*

In particular, if $\mathfrak{A}$ is prime, then $\|\mathcal{U}_{A,B}:\mathfrak{A}\longrightarrow\mathfrak{A}\|\geqslant\|A\|\|B\|$.

Proof. Let $\mathfrak{A}$, A and B be as in the hypotheses. By [**1**, Theorem 5.3.12], we have

$$\|\mathcal{U}_{A,B}:\mathfrak{A}\longrightarrow\mathfrak{A}\|=\sup_{\pi}\|\mathcal{U}_{\pi(A),\pi(B)}:\pi(\mathfrak{A})\longrightarrow\pi(\mathfrak{A})\|,\tag{18}$$

where π runs over a faithful family of irreducible representations of $\mathfrak{A}$.

Let $\pi:\mathfrak{A}\longrightarrow\mathcal{B}(\mathcal{H}_\pi)$ be an irreducible representation of $\mathfrak{A}$. Since C^*-algebras have metric approximate identities, we have

$$\|\mathcal{U}_{\pi(A),\pi(B)}:\pi(\mathfrak{A})\longrightarrow\pi(\mathfrak{A})\|=\|\mathcal{U}_{\pi(A),\pi(B)}:\pi(\mathfrak{A})+\mathbb{C}I_\pi\longrightarrow\pi(\mathfrak{A})+\mathbb{C}I_\pi\|,\tag{19}$$

where I_π denotes the identity of $\mathcal{B}(\mathcal{H}_\pi)$. Moreover, since $\pi(\mathfrak{A})$ is irreducible, the bicommutant of $\pi(\mathfrak{A})+\mathbb{C}I_\pi$ is precisely $\mathcal{B}(\mathcal{H}_\pi)$, and so, by the double commutant theorem and the Kaplansky density theorem [**5**, Theorems 5.3.1 and 5.3.5], we have

$$\|\mathcal{U}_{\pi(A),\pi(B)}:\pi(\mathfrak{A})+\mathbb{C}I_\pi\longrightarrow\pi(\mathfrak{A})+\mathbb{C}I_\pi\|=\|\mathcal{U}_{\pi(A),\pi(B)}:\mathcal{B}(\mathcal{H}_\pi)\longrightarrow\mathcal{B}(\mathcal{H}_\pi)\|.\tag{20}$$

Combining (18), (19) and (20), and then applying Theorem 3.7, we obtain

$$\|\mathcal{U}_{A,B}:\mathfrak{A}\longrightarrow\mathfrak{A}\|\geqslant\sup_{\pi\text{ irred.}}\|\pi(A)\|\|\pi(B)\|.\tag{21}$$

It is known (see [**12**, p. 364]) that, for every C^*-algebra $\mathfrak{A}$, the right-hand side of the last inequality equals $\|L_AR_B\|$. This proves the first part.

If $\mathfrak{A}$ is prime, then $\|L_AR_B\|=\|A\|\|B\|$ [**8**, Proposition 2.3], and the last assertion follows. □

4. *Jordan algebras of self-adjoint operators*

It was shown in [**16**] that, for Jordan algebras of self-adjoint operators, the greatest possible value of c in the inequality (2) is 1 (see also [**7**] for a different proof). In this section we give a new short proof of this result.

We denote by $\mathcal{B}_h$ (respectively $\mathcal{N}_h$) the linear space of bounded (respectively nuclear) hermitian operators on l_2. Also, we write $J:l_2\longrightarrow l_2'$ for the conjugate-linear isometry defined by $(Jh)(k):=\langle k,h\rangle$ ($h,k\in l_2$).

Our next lemma may be thought of as a self-adjoint version of Proposition 2.1 above.

LEMMA 4.1. *Let $A,B\in\mathcal{B}_h$. Then*

$$\begin{aligned}\|\mathcal{U}_{A,B}:\mathcal{B}_h\longrightarrow\mathcal{B}_h\| &= \|\mathcal{U}_{A,B}:\mathcal{N}_h\longrightarrow\mathcal{N}_h\|\\ &= \sup_{\xi\in l_2:\|\xi\|=1}\|\mathcal{U}_{A,B}(J\xi\otimes\xi)\|_N.\end{aligned}\tag{22}$$

Proof. Let $\mathfrak{A}$ be a self-adjoint subspace of $\mathcal{B}(l_2)$, and let α be a norm on $\mathfrak{A}$ satisfying $\alpha(T) = \alpha(T^*)$ $(T \in \mathfrak{A})$. We set $\mathfrak{A}_h = \{T \in \mathfrak{A} : T = T^*\}$.

Recall that a linear functional $\rho : \mathfrak{A} \longrightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) is said to be *hermitian* if $\rho(T^*) = \overline{\rho(T)}$ $(T \in \mathfrak{A})$. It is easily seen that, if ρ is hermitian, then $\|\rho\| = \|\rho|_{\mathfrak{A}_h}\|$. Indeed, let $\varepsilon > 0$, let $T \in \mathfrak{A}$ with $\alpha(T) = 1$ be such that $\|\rho\| - \varepsilon \leqslant |\rho(T)|$, and let a be a complex number such that $|\rho(T)| = a\rho(T)$. Define $L = aT$. Then

$$\|\rho\| - \varepsilon \leqslant |\rho(T)| = \rho(L) = \frac{1}{2}(\rho(L) + \rho(L^*)) = \rho\Big(\frac{L + L^*}{2}\Big)$$

and $\alpha((L + L^*)/2) \leqslant \alpha(L) = \alpha(T)$.

It is easy to verify that the linear functionals

$$\rho_T : \mathcal{N}(l_2) \longrightarrow \mathbb{K},\ X \longmapsto \operatorname{tr}(\mathcal{U}_{A,B}(T)X),$$

where $T \in \mathcal{B}_h$, and

$$\rho_X : \mathcal{B}(l_2) \longrightarrow \mathbb{K},\ T \longmapsto \operatorname{tr}(\mathcal{U}_{A,B}(X)T),$$

where $X \in \mathcal{N}_h$, are both hermitian. Moreover, it is well known that $\|T\|_N = \|T^*\|_N$ $(T \in \mathcal{N}(l_2))$, and that $\|T\| = \|T^*\|$ $(T \in \mathcal{B}(l_2))$ (see for instance [**20**, III.G.5]). Thus, from the previous paragraph, we have $\|\rho_T\| = \|\rho_T|_{\mathcal{N}_h}\|$ and $\|\rho_X\| = \|\rho_X|_{\mathcal{B}_h}\|$. It follows that

$$\begin{aligned} \|\mathcal{U}_{A,B} : \mathcal{N}_h \longrightarrow \mathcal{N}_h\| &= \sup_{X \in \mathcal{N}_h : \|X\|_N \leqslant 1} \|\mathcal{U}_{A,B}(X)\|_N \\ &= \sup_{\substack{X \in \mathcal{N}_h : \\ \|X\|_N \leqslant 1}} \ \sup_{\substack{T \in \mathcal{B}(l_2) : \\ \|T\| \leqslant 1}} |\rho_X(T)| = \sup_{\substack{X \in \mathcal{N}_h : \\ \|X\|_N \leqslant 1}} \ \sup_{\substack{T \in \mathcal{B}_h : \\ \|T\| \leqslant 1}} |\rho_X(T)| \\ &= \sup_{\substack{T \in \mathcal{B}_h : \\ \|T\| \leqslant 1}} \ \sup_{\substack{X \in \mathcal{N}_h : \\ \|X\|_N \leqslant 1}} |\rho_T(X)| = \sup_{\substack{T \in \mathcal{B}_h : \\ \|T\| \leqslant 1}} \ \sup_{\substack{X \in \mathcal{N}(l_2) : \\ \|X\|_N \leqslant 1}} |\rho_T(X)| \\ &= \sup_{T \in \mathcal{B}_h : \|T\| \leqslant 1} \|\mathcal{U}_{A,B}(T)\| = \|\mathcal{U}_{A,B} : \mathcal{B}_h \longrightarrow \mathcal{B}_h\|. \end{aligned}$$

This proves the first equality.

To prove the second one, recall that, for $X \in \mathcal{N}_h$, we have the Schmidt decomposition $X = \sum_{i=1}^{\infty} \sigma_i(X) J x_i \otimes x_i$, where (x_i) is an orthonormal system on l_2, and $(\sigma_i(X))$ is the sequence of singular values of X. Moreover, $\|X\|_N = \sum_i \sigma_i(X)$. Thus, if $X \in \mathcal{N}_h$ is such that $\|X\|_N \leqslant 1$ and $\|\mathcal{U}_{A,B}(X)\| \geqslant \|\mathcal{U}_{A,B} : \mathcal{N}_h \longrightarrow \mathcal{N}_h\| - \varepsilon$, then for some i (possibly depending on ε) we must have $\|\mathcal{U}_{A,B}(Jx_i \otimes x_i)\| \geqslant \|\mathcal{U}_{A,B} : \mathcal{N}_h \longrightarrow \mathcal{N}_h\| - \varepsilon$. As this holds for every ε,

$$\|\mathcal{U}_{A,B} : \mathcal{N}_h \longrightarrow \mathcal{N}_h\| \leqslant \sup_{\xi \in l_2 : \|\xi\| = 1} \|\mathcal{U}_{A,B}(J\xi \otimes \xi)\|_N.$$

The opposite inequality is obvious. □

REMARK 4.2. The same result holds for n-tuples $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ of elements of $\mathcal{B}_h$ and the elementary operator $\mathcal{U}_{A,B} : \mathcal{B}_h \longrightarrow \mathcal{B}_h$ defined by $\mathcal{U}_{A,B}(X) := \sum_i (A_i X B_i + B_i X A_i)$ $(X \in \mathcal{B}_h)$. The proof is verbatim that of the previous lemma.

THEOREM 4.3. *Let $A, B \in \mathcal{B}_h$. Then*

$$\|\mathcal{U}_{A,B} : \mathcal{B}_h \longrightarrow \mathcal{B}_h\| \geqslant \|A\|\|B\|.$$

Proof. We can assume that $\|A\| = \|B\| = 1$. As shown in [**16**], it is enough to prove the result for $A, B \in \mathcal{B}_h(H)$, where H is a Hilbert space of dimension 2. Since A is hermitian, there exists an orthonormal basis of H, say $\{e_1, e_2\}$, in which A has matrix representation $\operatorname{diag}\{1, \mu\}$. By multiplying e_2 by a scalar of modulus 1, if necessary, we can also assume that the matrix that represents B in the basis $\{e_1, e_2\}$, say $\left(\begin{smallmatrix} w & z \\ x & z \end{smallmatrix}\right)$, is real. Moreover, by multiplying e_2 by -1 if necessary, we can further assume that $\operatorname{sign}(x(w + z)) = 1$. To see this, recall that $w = \langle Be_1, e_1\rangle$, $x = \langle Be_2, e_1\rangle$ and $z = \langle Be_2, e_2\rangle$, so only the sign of x changes if we multiply e_2 by -1. Now Lemma 3.2(i), applied with the basis $\{e_1, e_2\}$, gives

$$\begin{aligned} \Big\|\mathcal{U}_{A,B}\Big(\tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\Big)\Big\|_N^2 &\geqslant s_2^2(A)[(w + x)^2 + (z + x)^2] \\ &= s_2^2(A)[s_2^2(B) + 2x(w + z)]. \end{aligned}$$

It is readily seen that

$$\|\mathcal{U}_{A,B} : \mathcal{N}_h \longrightarrow \mathcal{N}_h\| \geqslant s_2(A)s_2(B).$$

The desired result now follows from this and Lemma 4.1. □

5. *Operator algebras on arbitrary Banach spaces*

Throughout this section, unless otherwise stated, E denotes an arbitrary Banach space. As is customary, we identify (algebraically) $\mathcal{F}(E)$ with $E' \otimes E$. In particular, $\lambda \otimes x \in E' \otimes E$ is identified with the rank-one operator $e \longmapsto \lambda(e)x$ $(e \in E)$.

In this section we consider operator algebras on E which are endowed with a norm α that satisfies the following.

(A.1) $\alpha(T) \geqslant \|T\|$ $(T \in \mathfrak{A})$.

(A.2) $\alpha(\lambda \otimes x) = \|\lambda\|\|x\|$ $(\lambda \otimes x \in \mathcal{F}(E))$.

We shall need the following theorem. In the case of a Hilbert space, it can be deduced from results of Magajna and Turnšek [7, Propositions 3.1 and 3.2]. In fact, their proofs extend without difficulty to the case of dual Banach spaces, and, with a little more effort, to arbitrary Banach spaces. However, rather than describing all the details, we prefer to give a simple, direct proof.

THEOREM 5.1. *Let $A, B \in \mathcal{B}(E)$, and let α and β be norms on $\mathcal{F}(E)$ that satisfy conditions* (A.1) *and* (A.2) *above. Then*

$$\|\mathcal{U}_{A,B} : (\mathcal{F}(E), \alpha) \longrightarrow (\mathcal{F}(E), \beta)\| \geqslant 2(\sqrt{2} - 1)\|A\|\|B\|.$$

Proof. First note that, if $x, y \in E$ and $\lambda, \tau \in E'$ are all vectors of norm 1, then

$$\begin{aligned} \|\mathcal{U}_{A,B} : (\mathcal{F}(E), \alpha) \longrightarrow (\mathcal{F}(E), \beta)\| &\geqslant \beta(\mathcal{U}_{A,B}(\tau \otimes y)) \geqslant \|\mathcal{U}_{A,B}(\tau \otimes y)\| \\ &= \|A'\tau \otimes By + B'\tau \otimes Ay\| \\ &\geqslant |\tau(Ax)\lambda(By) + \tau(Bx)\lambda(Ay)|. \end{aligned} \tag{23}$$

Without loss of generality, we can assume that $\|A\| = \|B\| = 1$. Fix $\varepsilon > 0$, and choose $x, y \in E$ and $x', y' \in E'$, all of norm 1, such that $x'(Ax) > 1 - \varepsilon$ and $y'(By) > 1 - \varepsilon$. It follows from (23), with $\lambda = x'$ and $\tau = y'$, that

$$\|\mathcal{U}_{A,B}\| \geqslant (1 - \varepsilon)^2 - |y'(Ay)||x'(Bx)|. \tag{24}$$

By (23) again, this time with $\lambda = \tau = x'$ and y replaced by x,

$$\|\mathcal{U}_{A,B}\| \geqslant 2(1-\varepsilon)|x'(Bx)|. \tag{25}$$

Similarly,

$$\|\mathcal{U}_{A,B}\| \geqslant 2(1-\varepsilon)|y'(Ay)|. \tag{26}$$

Set $c = (1-\varepsilon)^{-2}\|\mathcal{U}_{A,B}\|$. Then (24), (25) and (26) combined give $c \geqslant 1 - c^2/4$, whence $c \geqslant 2(\sqrt{2}-1)$. On letting $\varepsilon \longrightarrow 0$, we obtain the desired result. □

COROLLARY 5.2. *Let $\mathfrak{A}$ be an operator algebra on E, endowed with a norm α that satisfies* (A.1) *and* (A.2), *and let $A, B \in \mathfrak{A}$. Then*

$$\|\mathcal{U}_{A,B} : \mathfrak{A} \longrightarrow \mathfrak{A}\| \geqslant 2(\sqrt{2}-1)\|A\|\|B\|.$$

Proof. This is immediate from Theorem 5.1. □

Note that if $\mathfrak{A}$ is an ideal of $\mathcal{B}(E)$ and α has in addition the ideal property (that is $\alpha(VT) \leqslant \|V\|\alpha(T)$ and $\alpha(TW) \leqslant \alpha(T)\|W\|$ whenever $V, W \in \mathcal{B}(E)$ and $T \in \mathfrak{A}$) then we clearly have the upper bound $2\|A\|\|B\|$ for the norm of $\mathcal{U}_{A,B}$.

It turns out that for certain Banach spaces E, the constant $2(\sqrt{2}-1)$ of Corollary 5.2 is best possible. Indeed, we have the following.

PROPOSITION 5.3. *Let*

$$A := \begin{pmatrix} 1 & 0 \\ \sqrt{2}-1 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1-\sqrt{2} & 0 \\ 1 & 0 \end{pmatrix}$$

be operators acting on l_∞^2, where the matrix representations are with respect to the unit vector basis. Then

$$\|\mathcal{U}_{A,B} : \mathcal{B}(l_\infty^2) \longrightarrow \mathcal{B}(l_\infty^2)\| = 2(\sqrt{2}-1)\|A\|\|B\| \tag{27}$$

and

$$\|\mathcal{U}_{A',B'} : \mathcal{B}(l_1^2) \longrightarrow \mathcal{B}(l_1^2)\| = 2(\sqrt{2}-1)\|A'\|\|B'\|. \tag{28}$$

Proof. Exactly the same argument as in the proof of Proposition 2.1 (see Remark 2.2) shows that

$$\begin{aligned}\|\mathcal{U}_{A,B} : \mathcal{B}(l_\infty^2) \longrightarrow \mathcal{B}(l_\infty^2)\| &= \|\mathcal{U}_{A,B} : \mathcal{N}(l_\infty^2) \longrightarrow \mathcal{N}(l_\infty^2)\| \\ &= \sup_{\substack{\lambda\otimes x \in \mathcal{N}(l_\infty^2):\\ \|\lambda\|=\|x\|=1}} \|\mathcal{U}_{A,B}(\lambda \otimes x)\| \\ &= \sup_{\substack{\lambda\otimes x \in \mathcal{N}(l_\infty^2):\\ \|\lambda\|=\|x\|=1}} \|A'\lambda \otimes Bx + B'\lambda \otimes Ax\|_N.\end{aligned}$$

Recall that, if T is a linear operator from l_∞^2 into a Banach space X, and $\{e_1, e_2\}$ is the unit vector basis of l_∞^2, then $\|T\|_N = \|Te_1\|_X + \|Te_2\|_X$, where $\|\,.\,\|_X$ stands for the norm on X (see [**19**, Proposition 8.7]). With the help of this formula, and taking into account our definitions of A and B, we find that

$$\begin{aligned}&\sup_{\substack{\lambda\otimes x \in \mathcal{N}(l_\infty^2):\\ \|\lambda\|=\|x\|=1}} \|A'\lambda \otimes Bx + B'\lambda \otimes Ax\|_N \\ &\quad= \sup_{\substack{\lambda\otimes x \in \mathcal{N}(l_\infty^2):\\ \|\lambda\|=\|x\|=1}} \|\lambda(Ae_1)Bx + \lambda(Be_1)Ax\|_{l_\infty^2} \\ &\quad= \sup_{\substack{\lambda,\xi \in l_1^2:\\ \|\lambda\|=\|\xi\|=1}} \sup_{\substack{x\in l_\infty^2:\\ \|x\|=1}} |\lambda(Ae_1)\xi(x_1Be_1) + \lambda(Be_1)\xi(x_1Ae_1)| \\ &\quad= \sup_{\substack{\lambda,\xi \in l_1^2:\\ \|\lambda\|=\|\xi\|=1}} |\lambda(Ae_1)\xi(Be_1) + \lambda(Be_1)\xi(Ae_1)| \\ &\quad= \max_{\lambda,\xi\in\{e_1^*,e_2^*\}} |\lambda(Ae_1)\xi(Be_1) + \lambda(Be_1)\xi(Ae_1)| \\ &\quad= \max\{|2(\sqrt{2}-1)|, |1 + (\sqrt{2}-1)(1-\sqrt{2})|\} \\ &\quad= 2(\sqrt{2}-1).\end{aligned}$$

(Here x_1 stands for the first coordinate of x in the basis $\{e_1, e_2\}$.) Since $\|A\| = \|B\| = 1$, this establishes (27).

To prove (28) we argue as follows. As with l_∞^2, we have

$$\|\mathcal{U}_{C,D} : \mathcal{B}(l_1^2) \longrightarrow \mathcal{B}(l_1^2)\| = \|\mathcal{U}_{C,D} : \mathcal{N}(l_1^2) \longrightarrow \mathcal{N}(l_1^2)\| \qquad (C, D \in \mathcal{B}(l_1^2)).$$

Then, for A and B as above, and taking into account the previous part, we obtain

$$\begin{aligned}\|\mathcal{U}_{A',B'} : \mathcal{B}(l_1^2) \longrightarrow \mathcal{B}(l_1^2)\| &= \|\mathcal{U}_{A',B'} : \mathcal{N}(l_1^2) \longrightarrow \mathcal{N}(l_1^2)\| \\ &= \sup_{x\otimes\lambda \in \mathcal{N}(l_1^2)} \|Ax \otimes B'\lambda + Bx \otimes A'\lambda\|_N \\ &= \sup_{\lambda\otimes x \in \mathcal{N}(l_\infty^2)} \|B'\lambda \otimes Ax + A'\lambda \otimes Bx\|_N \\ &= \|\mathcal{U}_{A,B} : \mathcal{N}(l_\infty^2) \longrightarrow \mathcal{N}(l_\infty^2)\| \\ &= \|\mathcal{U}_{A,B} : \mathcal{B}(l_\infty^2) \longrightarrow \mathcal{B}(l_\infty^2)\| = 2(\sqrt{2}-1).\end{aligned}$$

Note that $\|A'\| = \|B'\| = 1$. The rest is clear. □

COROLLARY 5.4. *Let E be either $l_1(\Gamma)$ or $l_\infty(\Gamma)$, where Γ is an arbitrary set with at least two elements. Then*

$$\min_{\substack{A,B\in\mathcal{B}(E):\\ \|A\|=\|B\|=1}} \|\mathcal{U}_{A,B} : \mathcal{B}(E) \longrightarrow \mathcal{B}(E)\| = 2(\sqrt{2}-1). \tag{29}$$

Proof. We give the proof for $E = l_1(\Gamma)$, the proof when $E = l_\infty(\Gamma)$ being completely analogous.

Let $E = l_1(\Gamma)$. Plainly, E contains a 1-complemented subspace, F say, which is isometrically isomorphic to l_1^2. Let $P : E \longrightarrow F$ (respectively $\imath : F \longrightarrow E$) be a norm-1 projection onto F (respectively the natural inclusion map). By Proposition 5.3, there exist $A, B \in \mathcal{B}(F)$ such that

$$\|\mathcal{U}_{A,B} : \mathcal{B}(F) \longrightarrow \mathcal{B}(F)\| = 2(\sqrt{2}-1)\|A\|\|B\|. \tag{30}$$

Define $\widetilde{A}:=\imath\circ A\circ P$ and $\widetilde{B}:=\imath\circ B\circ P$. Then $\|\widetilde{A}\|=\|A\|$ and $\|\widetilde{B}\|=\|B\|$. Moreover, it is easily verified that

$$\|\mathcal{U}_{\widetilde{A},\widetilde{B}}:\mathcal{B}(E)\longrightarrow\mathcal{B}(E)\|=\|\mathcal{U}_{A,B}:\mathcal{B}(F)\longrightarrow\mathcal{B}(F)\|.$$

This last, combined with (30), shows that the left-hand side of (29) is no greater than $2(\sqrt{2}-1)$. The opposite inequality follows from Corollary 5.2. □

REMARK 5.5. Clearly, essentially the same proof applies to any Banach space E with the property that, for every $\varepsilon>0$, there is a $(1+\varepsilon)$-complemented subspace of E, say F_ε, whose Banach–Mazur distance to l_1^2 (respectively l_∞^2) is no greater than $1+\varepsilon$. Examples include the $C(K)$ spaces (with K a compact Hausdorff space), the Tsirelson space, and l_p-sums of sequences of finite-dimensional Banach spaces of the form $(l_1^{n_k})$ (respectively $(l_\infty^{n_k})$), where (n_k) is a sequence of positive integers and $1<p<\infty$.

We end the section with a result that underlines the relationship between the topology considered in an operator algebra and the norm of the elementary operators under study.

PROPOSITION 5.6. *Let*

$$A:=\begin{pmatrix}1 & 0\\ 0 & \sqrt{2}-1\end{pmatrix}\quad\text{and}\quad B:=\begin{pmatrix}1-\sqrt{2} & 0\\ 0 & 1\end{pmatrix}$$

be operators acting on the Hilbert space l_2^2. Then

$$\|\mathcal{U}_{A,B}:\mathfrak{S}_2(l_2^2)\longrightarrow\mathfrak{S}_2(l_2^2)\|=2(\sqrt{2}-1).$$

Proof. It is easily seen that $\mathcal{U}_{A,B}:\mathfrak{S}_2(l_2^2)\longrightarrow\mathfrak{S}_2(l_2^2)$ is represented in the orthonormal basis $\{e_i\otimes e_j\}_{1\leqslant i,j\leqslant 2}$ of $\mathfrak{S}_2(l_2^2)$ by a diagonal matrix whose entries in the main diagonal are all of modulus $2(\sqrt{2}-1)$. The rest is clear. □

6. *Some questions and remarks*

Let E be a Banach space and let $\mathfrak{A}$ be a subalgebra of $\mathcal{B}(E)$. In the previous sections we have considered the problem of finding lower estimates for the norm of elementary operators of the form: $U_{A,B}:\mathfrak{A}\longrightarrow\mathfrak{A}$, $X\longmapsto AXB+BXA$ $(A,B\in\mathfrak{A})$. The following questions have arisen in a natural way while investigating this problem and we believe they deserve further consideration.

There is an obvious lack of symmetry in the statement of Theorem 2.3. This is due to the fact that we have not found in the literature an analogue of Proposition 2.4 for real Hilbert spaces, and also because we have not succeeded in extending the argument given in the proof of Theorem 2.3 for real Hilbert spaces to the complex case. It seems natural, however, to expect the norm of $\mathcal{U}_{A,B}$ to be the same as the norm of $\sum_i(A_i^*\otimes B_i+B_i^*\otimes A_i)$, also in the complex case. (Note that, in the complex case, $\mathfrak{S}_2(\mathcal{H})$ is no longer the same as $\mathcal{H}\tilde{\otimes}\mathcal{H}$.) This would provide a way to estimate $\|\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|$. Thus, we are led to ask the following question.

QUESTION 6.1. Let $\mathcal{H}$ be a complex Hilbert space. Is it true that

$$\|\mathcal{U}_{A,B}:\mathfrak{S}_2(\mathcal{H})\longrightarrow\mathfrak{S}_2(\mathcal{H})\|=\Big\|\sum_i(A_i^*\otimes B_i+B_i^*\otimes A_i):\mathcal{H}\tilde{\otimes}\mathcal{H}\longrightarrow\mathcal{H}\tilde{\otimes}\mathcal{H}\Big\|$$

$(A_i,B_i\in\mathcal{B}(\mathcal{H}),\ 1\leqslant i\leqslant n)$?

It is expected that, for Banach spaces E with 'few' 1-complemented subspaces, a formula like (4) could play an essential role in studying the norm problem for elementary operators on the algebra $\mathcal{B}(E)$. For then, an argument like the one given in the proof of Theorem 3.7 might be no longer applicable. Thus, we formally raise the following question.

QUESTION 6.2. Does the equality

$$\|\mathcal{U}_{A,B}:\mathcal{B}(E)\longrightarrow\mathcal{B}(E)\|=\|\mathcal{U}_{A,B}:\mathcal{N}(E)\longrightarrow\mathcal{N}(E)\|\tag{31}$$

hold for every Banach space E?

It seems likely that the last question has an affirmative answer, but we have been unable to prove this result in its full generality.

In a brief remark after Proposition 2.1, we pointed out that the result of this proposition could be extended without any additional effort to the case in which E is reflexive and $\mathcal{N}(E)'=\mathcal{B}(E)$. Here we establish (31) for Banach spaces E with the *metric approximation property* (see the definition below).

We first recall that a Banach space E is said to have the *approximation property* if the identity operator on E can be uniformly approximated on compact subsets of E by continuous finite-rank operators. If, in addition, the finite-rank operators can always be chosen with norm no greater than some constant λ, then E is said to have the *bounded approximation property*. When $\lambda=1$, E is said to have the *metric approximation property*. If E has the approximation property then $\mathcal{N}(E)=E'\hat{\otimes}E$ holds isometrically. Recall also that for any pair of Banach spaces, X and Y, there is a linear isometry $\phi:\mathcal{B}(X,Y')\longrightarrow(X\hat{\otimes}Y)'$ which is defined by $\phi(T)(\sum_i x_i\otimes y_i):=\sum_i(Tx_i)(y_i)$ $(\sum_i x_i\otimes y_i\in X\hat{\otimes}Y)$. Thus, in particular, $(E'\hat{\otimes}E)'$ can be isometrically identified with $\mathcal{B}(E')$ and $\mathcal{B}(E,E'')$. (All this can be found, for instance, in [**4**, Chapter VIII; **13**, § 1.7.10 and § 1.7.11].)

THEOREM 6.3. *Let E be a Banach space with the metric approximation property, in particular, any reflexive Banach space with the approximation property (see* [**4**, Corollary VIII.4.2]*). Then* (31) *holds.*

Proof. Recall that the unit ball of $E'\hat{\otimes}E$ is the closure of the convex hull of the elementary tensors of norm one. It follows from this fact and the duality $(E'\hat{\otimes}E)'=\mathcal{B}(E')$ (see above), that

$$\begin{aligned}\|\mathcal{U}_{A,B}:\mathcal{B}(E)\longrightarrow\mathcal{B}(E)\| &= \sup_{\substack{T\in\mathcal{B}(E):\\ \|T\|\leqslant 1}}\|\mathcal{U}_{A',B'}(T')\|\\ &= \sup_{\substack{T\in\mathcal{B}(E):\\ \|T\|\leqslant 1}}\ \sup_{\substack{\lambda\otimes x\in E'\hat{\otimes}E:\\ \|\lambda\otimes x\|\leqslant 1}}|(\mathcal{U}_{A',B'}(T')\lambda)(x)|\\ &= \sup_{\substack{\lambda\otimes x\in E'\hat{\otimes}E:\\ \|\lambda\otimes x\|\leqslant 1}}\ \sup_{\substack{T\in\mathcal{B}(E):\\ \|T\|\leqslant 1}}|(A'\lambda)(TBx)+(B'\lambda)(TAx)|.\end{aligned}$$

On the other hand, taking into account the same fact about the unit ball of $E'\hat{\otimes}E$ and the duality $(E'\hat{\otimes}E)' = B(E, E'')$ (see above), we obtain

$$\|\mathcal{U}_{A,B} : E'\hat{\otimes}E \longrightarrow E'\hat{\otimes}E\| = \sup_{\substack{\lambda\otimes x \in E'\hat{\otimes}E:\\ \|\lambda\otimes x\|\leqslant 1}} \|A'\lambda\otimes Bx + B'\lambda\otimes Ax\|_\wedge$$
$$= \sup_{\substack{\lambda\otimes x \in E'\hat{\otimes}E:\\ \|\lambda\otimes x\|\leqslant 1}} \sup_{\substack{\mathcal{T}\in\mathcal{B}(E,E''):\\ \|\mathcal{T}\|\leqslant 1}} |(\mathcal{T}Bx)(A'\lambda) + (\mathcal{T}Ax)(B'\lambda)\|.$$

To finish the proof, we just need to show that, for any set of vectors $e_1, e_2 \in E$ and $f_1, f_2 \in E'$, we have

$$\sup_{\substack{\mathcal{T}\in\mathcal{B}(E,E''):\\ \|\mathcal{T}\|\leqslant 1}} |(\mathcal{T}e_1)(f_1) + (\mathcal{T}e_2)(f_2)| \leqslant \sup_{\substack{\mathcal{T}\in\mathcal{B}(E):\\ \|\mathcal{T}\|\leqslant 1}} |f_1(\mathcal{T}e_1) + f_2(\mathcal{T}e_2)|. \quad (32)$$

Let M denote the left-hand side of the last inequality. Fix $\varepsilon > 0$, and choose $\mathcal{T} \in \mathcal{B}(E, E'')$ such that $\|\mathcal{T}\| \leqslant 1$ and $M - \varepsilon \leqslant |(\mathcal{T}e_1)(f_1) + (\mathcal{T}e_2)(f_2)|$. Since E has the metric approximation property, there is $P \in \mathcal{F}(E)$ such that $Pe_i = e_i$ $(i = 1, 2)$ and $\|P\| \leqslant 1 + \varepsilon$. Let $e_i'' = \mathcal{T}e_i$ $(i = 1, 2)$. By the principle of local reflexivity [**20**, II.E.14], there exists a linear operator $W : rg\,\mathcal{T}P \longrightarrow E$ with $\|W\| \leqslant 1+\varepsilon$ such that $e_i''(f_i) = f_i(We_i'')$ $(i = 1, 2)$. Define $T := (1+\varepsilon)^{-2} W \circ \mathcal{T}|_{rgP}^{rg\mathcal{T}P} \circ P$. Then $T \in \mathcal{B}(E)$, $\|T\| \leqslant 1$ and

$$M - \varepsilon \leqslant |(\mathcal{T}e_1)(f_1) + (\mathcal{T}e_2)(f_2)| \leqslant (1+\varepsilon)^2 |f_1(Te_1) + f_2(Te_2)|.$$

Since ε is arbitrary, this proves (32). □

A closer look at the proof of Proposition 3.6 reveals that we actually have the better estimate:

$$\|\mathcal{U}_{A,B}\| \geqslant \max\{\|A\|s_2(B), s_2(A)\|B\|\} \qquad (A, B \in \mathcal{B}(l_2^2)).$$

We conjecture that the following is true.

CONJECTURE 6.4.

$$\|\mathcal{U}_{A,B}\| \geqslant s_2(A)s_2(B) \qquad (A, B \in \mathcal{B}(l_2^2)). \quad (33)$$

It should be pointed out that (33) cannot be generalized to higher dimensions (consider, for instance, a pair of orthogonal projections P and Q such that $P+Q = I$ and $PQ = QP = 0$). However, the results of Section 3 suggest that, at least for Hilbert–Schmidt operators A and B, a lower estimate for $\|\mathcal{U}_{A,B}\|$ of the form $\sqrt{(\sigma_1^2(A) + \sigma_2^2(A))(\sigma_1^2(B) + \sigma_2^2(B))}$ might be possible.

QUESTION 6.5. Do there exist Banach spaces for which the inequality (2) holds with a constant c strictly larger than 1?

We believe not. This belief is supported by the results of the previous section, and the following proposition.

PROPOSITION 6.6. *Let E be an infinite-dimensional Banach space. Then the largest possible value of the constant c in the inequality* (2) *is never greater than* 1.

Proof. Let $\lambda, \varphi \in E'$ and $x \in E$ be all vectors of norm 1. Then

$$\|\mathcal{U}_{\lambda\otimes x, \varphi\otimes x} : \mathcal{B}(E) \longrightarrow \mathcal{B}(E)\|$$
$$= \sup_{\substack{T\in\mathcal{B}(E):\\ \|T\|=1}} \sup_{\substack{z\in E:\\ \|z\|=1}} \|((\lambda\otimes x)T(\varphi\otimes x) + (\varphi\otimes x)T(\lambda\otimes x))(z)\|$$
$$= \sup_{\substack{T\in\mathcal{B}(E):\\ \|T\|=1}} \sup_{\substack{z\in E:\\ \|z\|=1}} |\lambda(Tx)\varphi(z) + \varphi(Tx)\lambda(z)|$$
$$= \sup_{\substack{y,z\in E:\\ \|y\|=\|z\|=1}} |\lambda(y)\varphi(z) + \varphi(y)\lambda(z)|,$$

from which it follows that

$$\inf_{\substack{A,B\in\mathcal{B}(E):\\ \|A\|=\|B\|=1}} \|\mathcal{U}_{A,B} : \mathcal{B}(E) \longrightarrow \mathcal{B}(E)\| \leqslant \inf_{\substack{\lambda,\varphi\in E':\\ \|\lambda\|=\|\varphi\|=1}} \sup_{\substack{y,z\in E:\\ \|y\|=\|z\|=1}} |\lambda(y)\varphi(z) + \varphi(y)\lambda(z)|. \quad (34)$$

If $H = l_2^2$ and $\{e_1, e_2\}$ is the unit vector basis then

$$\inf_{\substack{\lambda,\varphi\in H':\\ \|\lambda\|=\|\varphi\|=1}} \sup_{\substack{y,z\in H:\\ \|y\|=\|z\|=1}} |\lambda(y)\varphi(z) + \varphi(y)\lambda(z)|$$
$$\leqslant \sup_{\substack{y,z\in H:\\ \|y\|=\|z\|=1}} |\langle y, e_1\rangle\langle z, e_2\rangle + \langle y, e_2\rangle\langle z, e_1\rangle|$$
$$\leqslant \tfrac{1}{2} \sup_{\substack{y,z\in H:\\ \|y\|=\|z\|=1}} \{|\langle y, e_1\rangle|^2 + |\langle z, e_2\rangle|^2 + |\langle y, e_2\rangle|^2 + |\langle z, e_1\rangle|^2\}$$
$$= \tfrac{1}{2} \sup_{\substack{y,z\in H:\\ \|y\|=\|z\|=1}} \{\|y\|^2 + \|z\|^2\} = 1.$$

By the version of Figiel, Lindenstrauss and Milman of the theorem of Dvoretsky on spherical sections of convex bodies (see for instance [**19**, Theorem 7.1]), for every $\varepsilon > 0$ there exists a subspace $F_\varepsilon \subset E$ whose Banach–Mazur distance to l_2^2 is not greater than $1 + \varepsilon$. Combining this fact with the last result it is readily seen that the right-hand side of (34) (and hence c) cannot be greater than 1. □

Assume it is true that the largest possible value of the constant c in the inequality (2) is never greater than 1 (otherwise restrict consideration to infinite-dimensional Banach spaces). The results of the previous section suggest that it may well be the case that the (largest) value of the constant, $c = 1$, in the inequality (2) occurs only in the case when E is a Hilbert space. This motivates us to ask our last question.

QUESTION 6.7. Let E be a Banach space for which (2) holds with $c = 1$. Is E necessarily a Hilbert space?

Finally, we point out that the norm problem itself, which consists in finding a formula for $\|\mathcal{U}_{A,B}\|$ in terms of A and B, remains unsolved, even in the case when E is a Hilbert space.

REMARK 6.8. After this paper had been submitted we learned of a different proof of Theorem 3.7 by Timoney [18]. In the same paper he also proves Conjecture 6.4.

References

1. P. ARA and M. MATHIEU, *Local multipliers of C^*-algebras*, Monographs in Mathematics (Springer, London, 2003).
2. M. BARRAA and M. BOUMAZGOUR, 'A lower bound of the norm of the operator $X \to AXB + BXA$', *Extracta Math.* 16 (2001) 223–227.
3. J. BERGH and J. LÖFSTRÖM, *Interpolation spaces*, Grundlehren der Mathematischen Wissenschaften 223 (Springer, 1976).
4. J. DIESTEL and J. J. UHL, *Vector measures*, Mathematical Surveys and Monographs 15 (American Mathematical Society, Providence, RI, 1977).
5. R. V. KADISON and J. R. RINGROSE, *Fundamentals of the theory of operator algebras – Vol. I* (Academic Press, New York, 1983).
6. B. MAGAJNA, 'The Haagerup norm on the tensor product of operator modules', *J. Funct. Anal.* 129 (1995) 325–348.
7. B. MAGAJNA and A. TURNŠEK, 'On the norm of symmetrised two-sided multiplications', *Bull. Austral. Math. Soc.* 67 (2003) 27–38.
8. M. MATHIEU, 'Elementary operators on prime C^*-algebras I', *Math. Ann.* 284 (1989) 223–244.
9. M. MATHIEU, 'Properties of the product of two derivations of a C^*-algebra', *Canad. Math. Bull.* 32 (1989) 490–497.
10. M. MATHIEU, 'More properties of the product of two derivations of a C^*-algebra', *Bull. Austral. Math. Soc.* 42 (1990) 115–120.
11. M. MATHIEU, 'Elementary operators on Calkin algebras', *Irish Math. Soc. Bull.* 46 (2001) 33–42.
12. M. MATHIEU, 'The norm problem for elementary operators', *Recent progress in functional analysis*, North-Holland Mathematical Studies 189 (ed. K. D. Bierstedt *et al.*, North-Holland, Amsterdam, 2001).
13. T. W. PALMER, *Banach algebras and the general theory of *-algebras – Vol. I* (Cambridge University Press, 1994).
14. A. PIETSCH and H. TRIEBEL, 'Interpolationstheorie für Banachideale von beschränkten linearen Operatoren', *Studia Math.* 31 (1968) 95–109.
15. L. L. STACHO and B. ZALAR, 'On the norm of Jordan elementary operators in standard operator algebras', *Publ. Math. Debrecen.* 49 (1996) 127–134.
16. L. L. STACHO and B. ZALAR, 'Uniform primeness of Jordan algebra of symmetric elementary operators', *Proc. Amer. Math. Soc.* 126 (1998) 2241–2247.
17. J. STAMPFLI, 'The norm of a derivation', *Pacific J. Math.* 33 (1970) 737–747.
18. R. M. TIMONEY, 'Norms and CB norms of Jordan elementary operators', *Bull. Sci. Math.* 127 (2003) 597–609.
19. N. TOMCZAK-JAEGERMANN, *Banach–Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics 38 (Longman, 1989).
20. P. WOJTASZCZYK, *Banach spaces for analysts* (Cambridge University Press, 1991).

A. Blanco, M. Boumazgour and T. J. Ransford
Département de Mathématiques et de Statistique
Université Laval
Québec
Canada G1K 7P4

J. London Math. Soc. (2) 70 (2004) 499–511
DOI: 10.1112/S0024610704005605

WEIGHTED COMPOSITION OPERATORS ON THE BERGMAN SPACE

ŽELJKO ČUČKOVIĆ AND RUHAN ZHAO

ABSTRACT

Bounded, compact and Schatten class weighted composition operators on the Bergman space are characterized by the use of generalized Berezin transforms. An estimate of the essential norms of weighted composition operators on the Bergman space is given. Most of the results remain true for Hardy spaces and weighted Bergman spaces.

1. *Introduction*

Let D be the open unit disk in the complex plane $\mathbb{C}$. Let u be an analytic function on the unit disk D and φ be an analytic self-map of D. The weighted composition operator uC_φ is defined as follows: for an analytic function f on D, $(uC_\varphi)f(z)=u(z)f(\varphi(z))$. These operators can be considered as a combination of a multiplication operator and a composition operator. They appear naturally. It is known that the isometries between the Hardy spaces H^p, $1\leqslant p<\infty$, $p\neq 2$, are weighted composition operators (see [6; 10, p. 148]). A similar result holds for the isometries on the Bergman space L^p_a; see [11].

Recently, several authors have studied weighted composition operators on different spaces of analytic functions. The boundedness and compactness of weighted composition operators on the Hardy spaces were characterized in [3], using a Carleson measure condition. In [19, 20], the boundedness and compactness of these operators on the Bloch spaces and α-Bloch spaces were characterized. In [15, 17], essential norms of weighted composition operators between α-Bloch spaces were obtained and estimated for different cases. In this paper we will characterize bounded, compact and Schatten class weighted composition operators on the Bergman space L^2_a, but our boundedness and compactness results can be appropriately generalized to Hardy spaces and weighted Bergman spaces. We will also give estimates of essential norms of these operators on Hardy and weighted Bergman spaces. Recall that an analytic function $f\in L^p_a$ if and only if

$$\|f\|_p^p=\int_D |f(z)|^p\,dA(z)<\infty,$$

where

$$dA(z)=\frac{1}{\pi}\,dx\,dy$$

is the normalized Lebesgue measure on the unit disk. For $p=2$ we simply denote the corresponding norm as $\|\cdot\|$.

Received 9 June 2003; revised 2 December 2003.
2000 *Mathematics Subject Classification* 47B38 (primary).