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# Asymptotic Expansions in the Integral and Local Limit Theorems in Banach Spaces with Applications to ω-Statistics

**Vidmantas Bentkus,[1,2,4] Friedrich Götze,[1,3] and Ričardas Zitikis[1,2]**

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Let $\mathbf{B}$ be a real separable Banach space and let $X, X_1, X_2, ... \in \mathbf{B}$ denote a sequence of independent identically distributed random variables taking values in $\mathbf{B}$. Denote $S_n = n^{-1/2}(X_1 + \cdots X_n)$. Let $\pi: \mathbf{B} \to \mathbf{R}$ be a polynomial. We consider (truncated) Edgeworth expansions and other asymptotic expansions for the distribution function of the r.v. $\pi(S_n)$ with uniform and nonuniform bounds for the remainder terms. Expansions for the density of $\pi(S_n)$ and its higher order derivatives are derived as well. As an application of the general results we get expansions in the integral and local limit theorems for $\omega$-statistics

$$\omega_n^p(q) \triangleq n^{p/2} \int_{(0,1)} \{F_n(x) - x\}^p \, q(x) \, dx$$

and investigate smoothness properties of their distribution functions. Here $p \geq 2$ is an even number, $q: [0, 1] \to [0, \infty]$ is a measurable weight function, and $F_n$ denotes the empirical distribution function. Roughly speaking, we show that in order to get an asymptotic expansion with remainder term $O(n^{-\alpha})$, $\alpha < p/2$, for the distribution function of the $\omega$-statistic, it is sufficient that $q$ is nontrivial, i.e., $\text{mes}\{t \in (0, 1): q(t) \neq 0\} > 0$. Expansions of arbitrary length are available provided the weight function $q$ is absolutely continuous and positive on an nonempty subinterval of $(0, 1)$. Similar results hold for the density of the distribution function and its derivatives provided $q$ satisfies certain very mild smoothness condition and is bounded away from zero. The last condition is essential since the distribution function of the $\omega$-statistic has no density when $q$ is vanishing on an nonempty subinterval of $(0, 1)$.

[1] Supported by SFB 343 at Bielefeld.
[2] Institute of Mathematics and Informatics, Akademijos 4, 2600 Vilnius, Lithuania.
[3] Universität Bielefeld, Fakultät für Mathematik, Universitätsstrasse, Postfach 8640, 4800 Bielefeld, Germany.
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## 1. INTRODUCTION

We consider two different but closely related topics.

The first one is devoted to asymptotic analysis of remainder terms in the Central Limit Theorem (CLT) in Banach spaces—convergence rates and asymptotic expansions including the results for densities of distribution functions. This topic was intensively studied during the last 20 years by lots of authors; let us mention now only the review papers on this subject: Prohorov,[62] Sazonov,[68] Paulauskas and Račkauskas,[56] Bentkus *et al.*[8] The original motivation of investigations of the CLT in Hilbert or Banach spaces was based on applications to some statistical tests—such as the celebrated $\omega^2$-test of Cramér and von Mises. However, in many of the subsequent papers the possibility of such applications was left out.

The second topic is asymptotic analysis of $\omega$-statistics

$$\omega_n^p(q) \triangleq n^{p/2} \int_{(0,1)} \{F_n(x)-x\}^p \, q(x)\, dx$$

where $p \geqslant 2$ is an even number, $q: [0, 1] \to [0, \infty]$ is a measurable weight function, and $F_n(x) = n^{-1} \sum_{i=1}^{n} \mathbf{I}\{U_i \leqslant x\}$ denotes the empirical distribution function based on a sample $U_1,..., U_n$ taken from the uniform on $(0, 1)$ distribution. Let us mention only few papers related to this topic: Smirnov,[72,73] von Mises,[51] Anderson and Darling,[3] Stephens,[75] Kandelaki,[43] Chibisov,[19] Sazonov,[67] Rosenkrantz,[64] Kiefer,[44] Nikitin,[53] Orlov,[55] Csörgő,[24] Martynov,[49] Serfling,[70] Csörgő and Révész,[22] Götze,[33] Koroliuk and Borovskich,[45] Shorack and Wellner,[71] Bentkus and Zitikis,[10] Zitikis,[90] Bentkus *et al.*,[8] and Paulauskas and Račkauskas.[57] In many papers $\omega$-statistics have been studied without taking into account the relation between them and the analysis of the remainder term in the CLT in Banach spaces.

Our main purpose is to join the two topics mentioned. More exactly, formulating and proving the results in abstract Banach spaces we always have in mind applicability to statistical problems. This sometimes causes unexpected technical difficulties and turns us aside from the widely accepted canons of the theory. Furthermore, we look for conditions rewritable in terms of the weight function $q$ as well as easily verifyable using elementary analysis, that is, without any references to the Banach space theory. We formulate such results in Section 2. There are asymptotic expansions of an arbitrary length and with uniform and nonuniform estimates of the remainder terms given. We also study the smoothness properties of the distribution functions of $\omega$-statistics and obtain expansions for their densities as well as for their higher order derivatives. Results of this section include as partial cases many earlier results as well as their improvements.



In Section 3, we discuss asymptotic expansions in the CLT in Banach spaces for the distribution function (and for its density and higher order derivatives) of a polynomial defined on a general Banach space. This section include as partial cases many earlier results (see, for instance, Götze,[31–37] Chebotarev,[18] Yurinskii,[82–83] Bentkus,[6,7] Bentkus and Zalesskii,[9] Paulauskas and Račkauskas,[56] Zitikis,[87–89] Nagaev and Chebotarev,[52] Bentkus *et al.*,[8] etc.) as well as their improvements.

The remaining sections are rather technical and are mainly devoted to (relatively complicated) proofs of the main results.

Throughout we assume that all random variables are independent in the whole if the contradiction is not clear from the context.

We shall use the following notation.

| | |
|---|---|
| $\mathbf{N}$ | set of all natural numbers |
| $\mathbf{N}_0$ | set of all nonnegative integer numbers |
| $\mathbf{R}$ | set of all real numbers |
| $\mathbf{R}^k$ | the $k$-fold Cartesian product of $\mathbf{R}$ |
| $\mathbf{C}$ | set of all complex numbers |
| $\mathbf{B}$ | a real separable Banach space with the norm $\lVert\cdot\rVert$ |
| $\mathbf{S}$ | class of all infinitely differentiable and rapidly decreasing functions |
| $\mathbf{P}$ | probability |
| $\mathbf{E}$ | expectation |
| $\mathbf{E}_X$ | the conditional expectation with respect to $X$ (all other r.v. are fixed) |
| $\mathbf{I}$ | the indicator function |
| $U, U_1, U_2,... \in \mathbf{R}$ | a sequence of real i.i.d. r.v. such that $U$ is uniformly distributed on $(0, 1)$ |
| $F_n$ | the empirical distribution function based on the sample $U_1,..., U_n$ |
| $X, X_1, X_2,... \in \mathbf{B}$ | a sequence of i.i.d. r.v. taking values in $\mathbf{B}$ and such that $\mathbf{E}X = 0$ |
| $G, G_1, G_2,... \in \mathbf{B}$ | a sequence of i.i.d. Gaussian r.v. taking values in $\mathbf{B}$ and such that $\mathbf{E}G = 0$ and the covariances of $X$ and $G$ are the same |
| $S_n = \varepsilon(X_1 + \cdots + X_n)$ | |
| $\varepsilon = 1/\sqrt{n}$, | for $n \in \mathbf{N}$ |

| | |
|---|---|
| $X^{\nabla} = X\mathbf{I}\{\|X\| \leqslant \sqrt{n}\}$ | |
| $X_{\nabla} = X\mathbf{I}\{\|X\| > \sqrt{n}\}$ | |
| $\mathbf{s}(\cdot, \cdot)$ | Stirling's numbers of the first kind |
| $\triangleq$ | stands for "by definition" |
| $\equiv$ | stands for "identically equal to" |
| $\square$ | stands for "the end of the proof" |
| $\#$ | stands for "the number of elements in" |
| $[x]$ | stands for "the integer part of $x$" |

## 2. EXPANSIONS FOR ω-STATISTICS

Let $U, U_1, U_2,... \in \mathbf{R}$ be independent and uniformly in (0, 1) distributed random variables. Denote by $F_n$ the corresponding empirical distribution function. For even $p \geqslant 2$ and a measurable function $q$: $[0, 1] \to [0, \infty]$ define the generalized weighted Cramér-von Mises statistic (shortly $\omega$-statistic)

$$\omega_n^p(q) \triangleq n^{p/2} \int_{(0,1)} \{F_n(x) - x\}^p\, q(x)\, dx \tag{2.1}$$

which is well-defined if

$$\int_{(0,1)} x^p(1-x)^p\, q(x)\, dx < \infty \tag{2.2}$$

As partial cases we get the generalized Cramér-von Mises statistic

$$W_n^p \triangleq n^{p/2} \int_{(0,1)} \{F_n(x) - x\}^p\, dx$$

(the Cramér-von Mises statistic if $p = 2$), the generalized Anderson-Darling statistic

$$A_n^p \triangleq n^{p/2} \int_{(0,1)} \{F_n(x) - x\}^p\, \{x(1-x)\}^{-1}\, dx$$

(the Anderson-Darling statistic if $p = 2$.) We also consider the trimmed $\omega$-statistics

$${}_{\alpha,\beta}\omega_n^p(q) \triangleq n^{p/2} \int_{(\alpha,\beta)} \{F_n(x) - x\}^p\, q(x)\, dx$$

where $\alpha \geqslant 0$ and $\beta \leqslant 1$. The trimmed Cramér-von Mises ${}_{\alpha,\beta}W_n^2$ and Anderson-Darling ${}_{\alpha,\beta}A_n^2$ statistics were considered by Pettitt and Stephens.[59]



Denote

$$\omega^p(q) \triangleq \int_{(0,1)} \{Br(x)\}^p\, q(x)\, dx$$

where $x \mapsto Br(x)$ is the Brownian bridge.

Let $\mathbf{S}$ be the class of infinitely differentiable and rapidly decreasing functions, that is, $g \in \mathbf{S}$ provided that $\sup_{x \in \mathbf{R}} (1 + |x|^m)\, |(d/dx)^s\, g(x)| < \infty$ for all $m, s \in \mathbf{N}_0$.

The following theorem illustrates our general results.

**Theorem 2.1.** Let $\omega_n^p(q)$ be one of the following statistics: $W_n^p$, $A_n^p$ or ${}_{\alpha,\beta}W_n^p$, ${}_{\alpha,\beta}A_n^p$ with some $0 \leqslant \alpha < \beta \leqslant 1$. Then the distribution function $x \mapsto \mathbf{P}\{\omega^p(q) \leqslant x\}$ is infinitely differentiable, its density is of the class $\mathbf{S}$, and there exist functions $a_1, a_2,... \in \mathbf{S}$ such that for every (but fixed) $k \in \mathbf{N}_0$ we have

$$\mathbf{P}\{\omega_n^p(q) \leqslant x\} = \mathbf{P}\{\omega^p(q) \leqslant x\} + a_1(x)\, n^{-1} + \cdots + a_k(x)\, n^{-k} + R(x) \tag{2.3}$$

with the remainder $R$ satisfying

$$\sup_{x \in \mathbf{R}} |x|^m\, |R(x)| = O(n^{-k-1}), \qquad n \to \infty \tag{2.4}$$

for all $m \in \mathbf{N}_0$. Furthermore, let $s \in \mathbf{N}$. If $\omega_n^p(q)$ is either $W_n^p$ or $A_n^p$, then for all $n \geqslant p(s+1)$ the distribution function $x \mapsto \mathbf{P}\{\omega_n^p(q) \leqslant x\}$ is $s$-times differentiable and

$$\sup_{x \in \mathbf{R}} |x|^m\, |(d/dx)^s\, R(x)| = O(n^{-k-1}), \qquad n \to \infty \tag{2.5}$$

for all $m \in \mathbf{N}_0$.

The terms $a_1, a_2,...$ of the asymptotic expansions are functions depending on $q$ and $p$, which can be expressed using the distribution of the Brownian bridge. We shall describe the construction of them in Section 3. Let us note also that if $0 < \alpha < \beta < 1$, then the distribution function of each trimmed $\omega$-statistic has a jump (see Remark 2.2).

Now we are going to formulate general results.

Define

$$M(\gamma) \triangleq \int_{(0,1/2)} \left( \int_{(u,1/2)} q(x)\, dx \right)^{\gamma/p} du + \int_{(1/2,1)} \left( \int_{(1/2,u)} q(x)\, dx \right)^{\gamma/p} du$$

Let $ACM(a, b)$ denote the class of all absolutely continuous monotone functions on the open interval $(a, b)$. This means that $f \in ACM(a, b)$ has a derivative integrable on every closed subinterval of $(a, b)$, and $f'$ is either nonnegative or nonpositive on $(a, b)$.

By $ACM_p(0, 1)$ we denote the class of all functions on $(0, 1)$ which are piecewise of class $ACM$. More exactly, $q \in ACM_p(0, 1)$ if there exist a finite set of points, say $0 = x_0 < x_1 < \cdots < x_\tau < x_{\tau+1} = 1$ such that $q \in ACM(x_j, x_{j+1})$ for all $j = 0,..., \tau$.

Furthermore, $ACM_b(a, b)$ stands for the class of all functions $q \in ACM(a, b)$ such that $\sup_{x \in (a,b)} |q'(x)| < \infty$.

Throughout we impose the condition

$$\int_{(0,1)} x^{p/2}(1-x)^{p/2}\, q(x)\, dx < \infty \tag{2.6}$$

It is stronger than Eq. (2.2) and ensures that $\omega^p(q) \in \mathbf{R}$ almost surely (see Section 8).

In order to exclude the trivial case $\mathbf{P}\{\omega_n^p(q) = 0\} = 1$, we assume throughout that

$$\text{mes}\{t \in (0, 1): q(t) \neq 0\} > 0$$

**Theorem 2.2.** The distribution function $x \mapsto \mathbf{P}\{\omega^p(q) \leqslant x\}$ is of the class $\mathbf{C}^\infty$, its density is of the class $\mathbf{S}$. Let $k, m \in \mathbf{N}_0$. If $M(2k+2) < \infty$, then the functions $a_1, a_2,... \in \mathbf{S}$ in the formal asymptotic series in Eq. (2.3) are well defined. If

$$M(2k+4) + M(pm) < \infty$$

then the remainder $R$ in the asymptotic expansion of Eq. (2.3) satisfies the estimate of Eq. (2.4) provided $k+1 < p/2$.

Furthermore, let there exist a nonempty interval $(a, b) \subset (0, 1)$ such that

$$\inf_{x \in (a,b)} q(x) > 0, \qquad q \in ACM_b(a, b)$$

Then (without restrictions on $k \in \mathbf{N}_0$) the asymptotic expansion of Eq. (2.3) holds with a remainder $R$ satisfying Eq. (2.4).

The following theorem allows us to differentiate the asymptotic expansion of Eq. (2.3) (including the remainder term).

**Theorem 2.3.** Assume that the moment conditions of Theorem 2.2 hold and

$$\inf_{x \in (0,1)} q(x) > 0, \qquad q \in ACM_p(0, 1) \cap ACM_b(a, b) \tag{2.7}$$



for a nonempty interval $(a, b)$. Then for all $n \geqslant p(s+1)$ the distribution function $x \mapsto \mathbf{P}\{\omega_n^p(q) \leqslant x\}$ is $s$-times differentiable and the remainder $R$ satisfies Eq. (2.5).

**Remark 2.1.** There is a generalization of the estimates of the remainder terms in Theorems 2.2 and 2.3. Namely, if we replace the condition $M(2k+4) < \infty$ by $M(2k+2+2\delta) < \infty$ with some $0 < \delta < 1$, then the error bound $O(n^{-k-1})$ in Eqs. (2.4) and (2.5) should be replaced by $o(n^{-k-\delta})$. Of course, in this case the condition $k+1 < p/2$ in Theorem 2.2 should be replaced by $k + \delta < p/2$.

**Remark 2.2.** If $q$ vanishes on a nonempty interval $(a, b) \subset (0, 1)$, then the distribution function of $\omega_n^p(q)$ has a jump. Indeed,

$$\mathbf{P}\left\{\omega_n^p(q) = n^{p/2} \int_{(0,a)} x^p q(x)\, dx + n^{p/2} \int_{(b,1)} (1-x)^p\, q(x)\, dx\right\}$$
$$\geqslant \mathbf{P}\{U_i \in (a, b), \forall i = 1,..., n\} = (b-a)^n = \exp\{-cn\}$$

where $c = -\ln(b-a) \geqslant 0$. Thus, the condition $\inf_{x \in (0,1)} q(x) > 0$, which we have imposed to get convergence rates and asymptotic expansions in the local limit theorem, is a natural one. Of course, it can be weakened but not too much.

According to Theorem 2.3, the distribution function $x \mapsto \mathscr{F}_n(x) \triangleq \mathbf{P}\{\omega_n^p(q) \leqslant x\}$ is $s$-times differentiable provided $n \geqslant p(s+1)$. The next theorem supplements this result.

Let $\mathbf{D}^{k+r}$ be the class of functions $f: \mathbf{R} \to \mathbf{R}$ which are $k$-times differentiable and $|f(x) - f(y)| \leqslant c\,|x-y|^r$.

**Theorem 2.4.** Assume $q$ to satisfy Eqs. (2.2) and (2.7). Denote $N \triangleq n/p$. If $N = [N]$, then $\mathscr{F}_n \in \mathbf{D}^{N-1+\delta}$ for all $\delta \in [0, 1)$. If $N > [N]$, then $\mathscr{F}_n \in \mathbf{D}^{[N]+\delta}$ for all $\delta \in [0, N-[N])$.

The result of Theorem 2.4 is almost unimprovable. Indeed, if $q(x) \equiv 1$ and $p = 2$, then Theorem 2.4 states that $\mathscr{F}_n \in \mathbf{D}^{n/2-\delta}$ for all $\delta > 0$. On the other hand we know (see Stephens and Maag[76]) that $\mathscr{F}_n(x) = 0$ for $0 \leqslant x \leqslant 1/(12n)$, and $\mathscr{F}_n(x) = c(x-1)/(12n))^{n/2}$, $c > 0$, for $1/(12n) \leqslant x \leqslant (n+3)/(12n^2)$. Therefore $\mathscr{F}_n$ is not $n/2$ times differentiable.

Proofs of the results of this section are based on estimates of characteristic functions. They may be of independent interest and therefore we formulate them as the following two theorems.

**Theorem 2.5.** Let $s \in \mathbf{N}_0$ and $M(ps) < \infty$. If there exists a nonempty subinterval $(a, b) \subset (0, 1)$ such that

$$\inf_{x \in (a,b)} q(x) > 0, \qquad q \in ACM_b(a, b)$$

then for any (sufficiently large) $A \geq 0$ and any (sufficiently small) $\delta > 0$ we have

$$\sup_n (n^A \sup_{|t| \geq n^{p/2-1+\delta}} |(d/dt)^s \mathbf{E} \exp\{it\omega_n^p(q)\}|) < \infty \tag{2.8}$$

**Theorem 2.6.** Let $s \in \mathbf{N}_0$ and $M(ps) < \infty$. If

$$\inf_{x \in (0,1)} q(x) > 0, \qquad q \in ACM_p(0, 1)$$

then there exists a (positive) constant $c = c(q, p, s)$ such that

$$|(d/dt)^s \mathbf{E} \exp\{it\omega_n^p(q)\}| \leq cn^{cn} |t|^{s-n/p}, \qquad \forall\, |t| \geq 1 \tag{2.9}$$

Let us discuss the relationship between $\omega$-statistics and limit theorems in Banach spaces. Let $\mathbf{L}_p(q)$, $1 \leq p < \infty$, stand for the Banach space of real-valued functions $f: (0, 1) \to \mathbf{R}$ with the norm $\|f\| \triangleq \{\int_{(0,1)} |f(x)|^p q(x)\, dx\}^{1/p} < \infty$. Define the random process $X(t) \triangleq \mathbf{I}\{U \leq t\} - t$, $0 \leq t \leq 1$, and consider $X$ as a random variable taking values in the Banach space $\mathbf{L}_p(q)$. Then

$$\omega_n^p(q) = \|S_n\|^p, \qquad S_n = n^{-1/2}(X_1 + \cdots + X_n)$$

where $X_1, X_2,\ldots$ are independent copies of $X$. Since $p$ is an even number, the function $\pi(z) \triangleq \|z\|^p$ is a polynomial of the $p$th order on $\mathbf{L}_p(q)$ and $\mathbf{P}\{\omega_n^p(q) \leq x\} = \mathbf{P}\{\pi(S_n) \leq x\}$. Therefore, in order to get results for $\omega$-statistics we may (and do) apply general results in Banach space setting from the next section. Let us only note here that for $p > 0$ and $\gamma > 0$ we have (see Section 8)

$$\mathbf{E}\, \|X\|^\gamma < \infty \Leftrightarrow (2.2) \text{ and } M(\gamma) < \infty \tag{2.10}$$

and

$$\mathbf{P}\{X \in \mathbf{L}_p(q)\} = 1 \Leftrightarrow (2.2) \tag{2.11}$$

$$\mathbf{P}\{Br \in \mathbf{L}_p(q)\} = 1 \Leftrightarrow (2.6) \tag{2.12}$$

Let us describe results directly connected with the topic of the section. The paper by Bentkus *et al.*[8] contains a detailed review of the results concerning $\omega$-statistics up to 1989. Convergence rates for the distribution of $W_n^2$ were considered by Kandelaki,[43] Sazonov,[66,67] Rosenkrantz,[64] Kiefer,[44] Nikitin,[53] Orlov,[55] Csörgő,[23–24] Götze.[31] Csörgő[24] also conjectured the rate of convergence $O(n^{-1})$ and gave a formal expansion for the distribution function of $W_n^2$. It was Götze[31] who obtained the rate $O(n^{-1})$ as well as asymptotic expansions. In the book by Koroliuk and Borovskich[45] one can find similar results.



The differentiability properties of the distribution function of $W_n^2$ were investigated by Csörgő and Stachó,[25] Bentkus and Zitikis.[10] The local limit theorem as well as asymptotic expansions for the density of $W_n^2$ are proved in Bentkus and Zitikis.[10] In Zitikis[88–90] results were extended to the case of general statistics $\omega_n^2(q)$ with $q$ satisfying a little bit stronger conditions than those of the present paper.

Götze[33] proved the rate of convergence $O(n^{-1})$ as well as asymptotic expansions with uniform bounds of the remainder terms for $\omega_n^p(q)$ ($p$ is an even number) for a class of weight functions $q$.

Cotterill and Csörgő[20] using technique similar to that in Götze,[31] have obtained the rate $O(n^{-1})$ for the multivariate Cramér-von Mises statistic.

Properties similar to those we consider in Theorems 2.5 and 2.6 are discussed in Zitikis.[91] Related questions were studied in van Zwet,[92] Bickel *et al.*,[13] Götze and van Zwet,[38] Helmers,[41] Bhattacharya and Denker,[11] Callert *et al.*,[17] Csörgő and Horvath,[21] Darling,[27] David,[28] Durbin and Knott,[29] Filippova,[30] Helmers,[40] and Watson.[77]

## 3. EXPANSIONS FOR DISTRIBUTION FUNCTIONS IN BANACH SPACES

Let $\mathbf{B}$ (with norm $\|\cdot\|$) be a real separable Banach space. The topological dual of $\mathbf{B}$ is denoted by $\mathbf{B}'$. Let $X$ be a $\mathbf{B}$-valued r.v. with distribution $\mathscr{L}(X)$, and let $X_1, X_2,\ldots \in \mathbf{B}$ be independent copies of $X$. Throughout we assume that

$$\mathbf{E}X = 0, \qquad \mathbf{E}\, \|X\|^2 < \infty$$

and

$$\sup_n \mathbf{E}\, \|S_n\| < \infty \tag{3.1}$$

Without loss of generality we may assume that $\mathbf{P}\{X = 0\} < 1$. Also, we require the $X$ to be pre-Gaussian (see Pisier and Zinn[61]), i.e., there exists a centered Gaussian r.v. $G \in \mathbf{B}$ (we denote $\Phi \triangleq \mathscr{L}(G)$) such that the covariances of $X$ and $G$ coincide, that is,

$$\mathbf{E}G = 0, \qquad \mathbf{E}f^2(X) = \mathbf{E}f^2(G), \qquad \forall f \in \mathbf{B}'$$

**Remark 3.1.** By definition, $X$ satisfies the Central Limit Theorem in $\mathbf{B}$ (briefly $X \in CLT$) if the sums $S_n \triangleq (X_1 + \cdots + X_n)/\sqrt{n}$ converge weakly to a Gaussian r.v. $G$. In this paper we do not need this assumption. Let us note that $X \in CLT$ and $\mathbf{E}\,\|X\|^2 < \infty$ imply Eq. (3.1) (see Kruglov,[46] de Acosta and Giné,[2] and Araujo and Giné.[4]).

**Remark 3.2.** It is known (see Borisov[16]) that $\mathbf{E}\,\|X\|^2 < \infty$ and the fact that $X$ is pre-Gaussian imply $\mathbf{E}L(X, X) = \mathbf{E}L(G, G)$ for each continuous bilinear form $L$ on $\mathbf{B}$.

Let $\pi: \mathbf{B} \to \mathbf{R}$ be a polynomial function of degree $p \in \mathbf{N}_0$, that is

$$\pi(b) = \pi_0(b) + \cdots + \pi_p(b), \qquad \forall b \in \mathbf{B}$$

where $b \mapsto \pi_\nu(b) \triangleq \pi_\nu(b,..., b)$, and $\pi_\nu$ is a $\nu$-linear symmetric and continuous function. Without loss of generality we may assume $p \geqslant 2$. Indeed, the case $p = 0$ is trivial. In the case $p = 1$ one may apply the well-known one-dimensional results (see, e.g., Petrov[58]).

Throughout we impose the following condition:

$$\sup_{t \in \mathbf{R}} |t|^d\, \mathbf{E} \exp\{it\pi_p(G_1,..., G_p)\} < \infty, \qquad \forall d \geqslant 0 \tag{3.2}$$

where $G_1,..., G_p$ are independent copies of $G$.

**Remark 3.3.** It is known that Eq. (3.2) holds for each infinite-dimensional $G$ if $\mathbf{B}$ is a Hilbert space or $\mathbf{B} = l_p$, $p$ is an even integer (see Yurinskii[84]). In the case of $X$ (and $G$) corresponding to $\omega$-statistics this condition is fulfilled as well (see Section 8). A consequence of Eq. (3.2) is that the distribution function of $\pi(G)$ has density of the class $\mathbf{S}$.

By definition, the class $\mathbf{S}$ consists of functions $g: \mathbf{R} \to \mathbf{R}$ such that

$$\|g\|_m \triangleq \sup_{x \in \mathbf{R}} (1 + |x|^m)\, |g(x)| < \infty, \ \|g\|_{m,s} \triangleq \sup_{x \in \mathbf{R}} (1 + |x|^m)\, |(d/dx)^s g(x)| < \infty$$

for all $m \in \mathbf{N}_0$ and $s \in \mathbf{N}$.

We denote $\varepsilon \triangleq 1/\sqrt{n}$.

**Theorem 3.1.** Let $k \in \mathbf{N}_0$ and $\mathbf{E}\,\|X\|^{k+3} < \infty$. Then there exist functions $a_1,..., a_k \in \mathbf{S}$ such that

$$\mathbf{P}\{\pi(S_n) \leqslant x\} = \mathbf{P}\{\pi(G) \leqslant x\} + a_1(x)\varepsilon + \cdots + a_k(x)\,\varepsilon^k + R(x)$$

with the remainder $R$ satisfying

$$\sup_{x \in \mathbf{R}} |R(x)| = O(\varepsilon^{k+1}), \qquad n \to \infty \tag{3.3}$$



provided $k \leqslant p - 2$. If $k = p - 1$, then for each $\delta > 0$

$$\sup_{x \in \mathbf{R}} |R(x)| = O(\varepsilon^{k+1-\delta}), \qquad n \to \infty$$

If the Cramér type condition

$$\exists \delta > 0 \qquad \int_{n^{p/2-\delta} \leqslant |t| \leqslant n^{(k+1)/2}} |\mathbf{E} \exp\{it\pi(S_n)\}|\, dt = O(\varepsilon^{k+1})$$

is fulfilled, then Eq. (3.3) holds for all $k \in \mathbf{N}_0$.

**Remark 3.4.** The Cramér type condition imposed in Theorem 3.1 and similar conditions described later are the conditions on the characteristic function of the *whole* sum. As we show in Section 8, $\omega$-statistics satisfy them. The classical Cramér condition (in the finite dimensional case) is a condition on *individual* summands. Such type conditions sometimes are used in the infinite dimensional case as well (see, for example, Bentkus *et al.*[9]). Unfortunately, we are not able to verify such conditions for $\omega$-statistics.

Theorem 3.1 is an immediate consequence of more general and precise results given later. Before formulating them, let us describe the terms $a_\nu(x)$ of the asymptotic expansions. We shall define them via the Edgeworth-Cramér polynomials.

*The Edgeworth-Cramér polynomials.* These polynomials, denote them by $E_\nu = E_\nu(m_2,..., m_{\nu+2})$, of formal commutative variables $m_2,..., m_{\nu+2}$ (the so-called "moments" variables) are defined as coefficients in the formal power series expansion

$$\exp\left\{t^{-2} \ln\left(1 + \sum_{\nu=2}^{\infty} m_\nu t^\nu/\nu!\right) - m_2/2\right\} = \sum_{\nu=0}^{\infty} E_\nu(m_2,..., m_{\nu+2})\, t^\nu \tag{3.4}$$

For instance,

$$E_0 = 1,$$

$$E_1 = m_3/6,$$

$$E_2 = -m_2 m_2/8 + m_3 m_3/72 + m_4/24,$$

$$E_3 = -m_2 m_3/12 - m_2 m_2 m_3/48 + m_3 m_3 m_3/1296 + m_3 m_4/144 + m_5/120$$

*The random Edgeworth-Cramér differential operators.* To define these operators we need derivatives of measures (see Averbuch *et al.*,[5] and Daletskii and Fomin[26] for more information concerning differentiable measures). Suppose that $\Psi: \mathscr{A} \to \mathbf{R}$ is a set function (not necessary additive or $\sigma$-additive) defined on a class $\mathscr{A}$ of subsets $A \subset \mathbf{B}$. Assume that the class $\mathscr{A}$ is invariant under translations, $A \in \mathscr{A}$, $h \in \mathbf{B} \Rightarrow A + h \in \mathscr{A}$. Then we can define the directional derivative

$$D(h)\, \Psi(A) \triangleq \lim_{t \to 0} \{\Psi(A - th) - \Psi(A)\}/t, \qquad t \in \mathbf{R}$$

(if the limit exists). The first derivative $D(h)\,\Psi: \mathscr{A} \to \mathbf{R}$ is a set function. Hence we can define the successive derivatives $D(h_1) \cdots D(h_v)\, \Psi$ iteratively.

Write

$$E_v = \sum d_v(i_1, ..., i_j)\, m_{i_1} \cdots m_{i_j}$$

where the sum is taken over all integers $i_1, ..., i_j$ such that $2 \leqslant i_1 \leqslant \cdots \leqslant i_j$. From Eq. (3.4) we see that only a finite number of coefficients $d_v(\cdots)$ are nonzero. Define the polynomials

$$P_v(z_1, ..., z_v) \triangleq \sum d_v(i_1, ..., i_j)\, z_1^{i_1} \cdots z_j^{i_j}$$

of commutative variables $z_1, ..., z_v$. The random Edgeworth-Cramér differential operators are defined as follows

$$P_v \triangleq P_v(D(X_1), ..., D(X_v))$$

For instance, we have

$$P_2 = -\tfrac{1}{8} D^2(X_1)\, D^2(X_2) + \tfrac{1}{72} D^3(X_1)\, D^3(X_2) + \tfrac{1}{24} D^4(X_1)$$

*The terms of the asymptotic expansions.* Denote $A_{x,a} \triangleq \{b \in \mathbf{B}: \pi(b + a) \leqslant x\}$ for $x \in \mathbf{R}$ and $a \in \mathbf{B}$. Clearly, $\mathscr{A}_x \triangleq \{A_{x,a}: a \in \mathbf{B}\}$ is a class invariant under translations and $\Phi(A_{x,a}) = \mathbf{P}\{G \in A_{x,a}\} = \mathbf{P}\{\pi(G + a) \leqslant x\}$, where $\Phi \triangleq \mathscr{L}(G)$ stands for the distribution of $G$. Then we define

$$a_v(x) \triangleq \mathbf{E} P_v \Phi(A_x)$$

where $A_x \triangleq A_{x,0}$.

Let us now give an explicit formula for $a_v(x)$. The Stirling numbers of the first kind $\mathbf{s}(j, k)$ are numbers such that

$$x(x-1) \cdots (x - j + 1) = \sum_{r=1}^{j} \mathbf{s}(j, r)\, x^r, \qquad \text{for } j = 1, 2, ...$$

**Lemma 3.1.** We have $a_v(x) = \mathbf{E} Q_v \Phi(A_x)$, where the random differential operator

$$Q_v \triangleq Q_v(D(X_1), D(X_2), ...; D(G_1), D(G_2), ...)$$



is defined via the polynomial

$$\begin{aligned} &Q_s(z_1, z_2, ...; y_1, y_2, ...) \\ &\quad \triangleq \sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{1}{j!} \sum_{r=1}^{j} \mathbf{s}(j, r) \sum{}^{*} \frac{1}{\alpha(j)!} (z_1^{\alpha_1} - y_1^{\alpha_1}) \cdots (z_j^{\alpha_j} - y_j^{\alpha_j}) \\ &\quad \times \Bigg\{ \mathbf{I}\{i + 2(j - r) = s\} + j^{(s - i - 2j + 2r)/2} \\ &\quad \times \sum_{l=1}^{\infty} (-1)^l \sum{}^{**} \frac{1}{\beta(j)!}\, y_{j+1}^{\beta_1} \cdots y_{j+l}^{\beta_l} \Bigg\} \end{aligned}$$

of formal commutative variables $z_1, z_2, ...; y_1, y_2, ...$. The fourth sum $\sum^*$ is taken over all integer multiindexes $\alpha(j) \triangleq (\alpha_1, ..., \alpha_j)$ such that $|\alpha(j)| = 2j + i$ and $\alpha_1 \geqslant 3, ..., \alpha_j \geqslant 3$. (We use the notation $\alpha(j)! \triangleq \alpha_1! \ldots \alpha_j!$, $|\alpha(j)| \triangleq \alpha_1 + \cdots + \alpha_j$.) The last sum $\sum^{**}$ is taken over all integer multiindexes $\beta(l) = (\beta_1, ..., \beta_l)$ such that $|\beta(l)| = v - i - 2(j - r)$ and $\beta_1 \geqslant 2, ..., \beta_l \geqslant 2$.

It is easy to see that all sums in the definition of $Q_s(\ldots)$ are finite and that $Q_s(z_1, ...; y_1, ...)$ depends only on finite number of variables $z_1, z_2, ..., y_1, y_2, ...$.

**Remark 3.5.** Let us note that the differential operators $P_v$ and $Q_v$ (and the corresponding polynomials) are not the same. Let us describe a procedure allowing to get $P_v$ from $Q_v$. If $w_k(x) \triangleq w_k(x, ..., x)$ is a symmetric and continuous $k$-linear form of the argument $x \in \mathbf{B}$, then it is easy to show that $\mathbf{E} w_k(G) = 0$ if $k$ is odd, and $\mathbf{E} w_k(G) = k!/(2^m m!)\, w_k(G_1, G_1, ..., G_m, G_m)$ if $k = 2m$ is even. Furthermore, by Remark 3.2, we get

$$\mathbf{E} w_k(G_1, G_1, ..., G_m, G_m) = \mathbf{E} w_k(X_1, X_1, ..., X_m, X_m)$$

Therefore one may get the polynomial $P_v$ from the polynomial $Q_v$ proceeding as follows. First, write $Q_v$ as a sum of monomials. Then omit all monomials containing an odd degree of at least one of $y_1, ..., y_k$. Replace each product $y_1^{2m_1}, ..., y_k^{2m_k}$ by

$$(2m_1)! \ldots (2m_k)!/(2^{m_1 + \cdots + m_k} m_1! \ldots m_k!)\, z_{k+1}^2 \cdots z_{k+m_1+\cdots+m_k}^2$$

If we rewrite the polynomial obtained as

$$\sum e_v(i_1, ..., i_l)\, z_{j_1}^{i_1} \cdots z_{j_l}^{i_l}$$

where $2 \leqslant i_1 \leqslant \cdots \leqslant i_l$, we shall get the equality

$$P_v(z_1, ..., z_{v+2}) = \sum e_v(i_1, ..., i_l)\, z_1^{i_1} \cdots z_l^{i_l}$$

The following two lemmas give some properties of the terms of the asymptotic expansions.

**Lemma 3.2.** The density of the distribution function $x \mapsto \mathbf{P}\{\pi(G) \leqslant x\}$ is of the class **S**.

**Lemma 3.3.** Let $\mathbf{E}\,\|X\|^{\nu+2} < \infty$. Then the function $a_\nu$ is well-defined and $a_\nu \in \mathbf{S}$. Furthermore, if $\pi_l(b) \equiv 0$ for odd $l$, then $a_\nu(x) \equiv 0$ for odd $\nu$.

Let us continue with the results on asymptotic expansions.

Now we shall introduce quantities the remainder terms of the asymptotic expansions given here will be estimated through. Let

$$I_n = I_n(A) = I_n(m, \delta, A)$$

$$\triangleq \max_{0 \leqslant \mu \leqslant m} \int_{n^{p/2-\delta} \leqslant |t| \leqslant n^A} |(d/dt)^\mu\, \mathbf{E} \exp\{it\pi(S_n)\}|\, dt$$

where $\delta > 0$, $A \geqslant 0$ and $m \in \mathbf{N}_0$, and

$$J_n = J_n(m, \delta, s)$$

$$\triangleq \max_{0 \leqslant \sigma \leqslant s-1} \max_{0 \leqslant \mu \leqslant m} \int_{n^{p/2-\delta} \leqslant |t| < \infty} |t|^\sigma\, |(d/dt)^\mu\, \mathbf{E} \exp\{it\pi(S_n)\}|\, dt,$$

where $\delta > 0$ and $m \in \mathbf{N}_0$, $s \in \mathbf{N}$.

**Theorem 3.2.** Let $k, m \in \mathbf{N}_0$. If

$$\mathbf{E}\,\|X\|^{k+3} + \mathbf{E}\,\|X\|^{pm} < \infty$$

then for all $A > 0$ the remainder $R$ in the asymptotic expansion of Theorem 3.1 allows the estimates:

$$\|R\|_m \leqslant c\{I_n(A) + \varepsilon^{k+1} + n^{-A}\}, \qquad \|R\|_{m,s} \leqslant c\{J_n + \varepsilon^{k+1}\}$$

for all $s \in \mathbf{N}$ and with some constants $c$ which do not depend on $n$.

The norms $\|\cdot\|_m$ and $\|\cdot\|_{m,s}$ here and in the following are taken with respect to the variable $x$. Traditionally an estimate of the norm $\|\cdot\|_m$ is called "nonuniform" and that of $\|\cdot\|_{m,s}$ "nonuniform in the local theorem."

**Remark 3.6.** To avoid confusion, let us agree that all inequalities in this paper mean that if the right-hand side is finite, then the left-hand side is well-defined, finite and does not exceed the right-hand side. For example, in the case $J_n < \infty$ the estimate $\|R\|_{m,s} \leqslant c\{J_n + \varepsilon^{k+1}\}$ together with



Lemmas 3.6 and 3.7 ensure that the distribution function $x \mapsto \mathbf{P}\{\pi(S_n) \leqslant x\}$ is $s$ times differentiable.

Together with the asymptotic series

$$A_{k,n}(x) \triangleq \mathbf{P}\{\pi(G) \leqslant x\} + \sum_{\nu=1}^{k} a_\nu(x)\, \varepsilon^\nu \tag{3.5}$$

described earlier, we shall also consider the following asymptotic series

$$A^{\nabla}_{k,n}(x) \triangleq \mathbf{P}\{\pi(G) \leqslant x\} + \sum_{\nu=1}^{k} a^{\nabla}_\nu(x)\, \varepsilon^\nu \tag{3.6}$$

and, for all $n > k$,

$$B_{k,n}(x) \triangleq \mathbf{P}\{\pi(G) \leqslant x\} + \sum_{\nu=1}^{k} b_{\nu,n}(x)\, \varepsilon^\nu \tag{3.7}$$

$$B^{\nabla}_{k,n}(x) \triangleq \mathbf{P}\{\pi(G) \leqslant x\} + \sum_{\nu=1}^{k} b^{\nabla}_{\nu,n}(x)\, \varepsilon^\nu \tag{3.8}$$

Let us describe the construction of them.

Coefficients $a^{\nabla}_\nu(x) \triangleq \mathbf{E} Q^{\nabla}_\nu \Phi(A_x)$ in series Eq. (3.5) are defined via the random differential operators

$$Q^{\nabla}_\nu = Q_\nu(D(X^{\nabla}_1), D(X^{\nabla}_2),...; D(G_1), D(G_2),...)$$

where the polynomial $Q_\nu$ is defined in Lemma 3.4.

The definitions of Eqs. (3.7) and (3.8) are a little bit longer. Define the polynomial

$$T_{\nu,j,n}(z_1, z_2,...; y_1, y_j,...) \triangleq n^{-j} \binom{n}{j} \sum\nolimits^{***} (z_1^{\alpha_1} - y_1^{\alpha_1}) \cdots (z_j^{\alpha_j} - y_j^{\alpha_j})/\alpha(j)!$$

where the sum $\sum^{***}$ is taken over all integer multiindexes $\alpha(j) = (\alpha_1,..., \alpha_j)$ such that $|\alpha(j)| = 2j + \nu$ and $\alpha_1 \geqslant 3,..., \alpha_j \geqslant 3$.

Using these polynomials, we define the random differential operators

$$T_{\nu,j,n} \triangleq T_{\nu,j,n}(D(X_1), D(X_2),...; D(G_1), D(G_2),...)$$

$$T^{\nabla}_{\nu,j,n} \triangleq T_{\nu,j,n}(D(X^{\nabla}_1), D(X^{\nabla}_2),...; D(G_1), D(G_2),...)$$

and put

$$b_{\nu,n}(x) \triangleq \sum_{j=1}^{\nu} \mathbf{E} T_{\nu,j,n} \Phi_{j,n}(A_x), \qquad b^{\nabla}_{\nu,n}(x) \triangleq \sum_{j=1}^{\nu} \mathbf{E} T^{\nabla}_{\nu,j,n} \Phi_{j,n}(A_x)$$

where $\Phi_{j,n} \triangleq \mathscr{L}(G\sqrt{1 - j/n})$.

**Remark 3.7.** According to Remark 3.5, $b_{\nu,n}(x)$ and $b^{\nabla}_{\nu,n}(x)$ do not change if in polynomials $T_{\nu,j,m}$ and $T^{\nabla}_{\nu,j,m}$ we omit odd degrees of $y_1,..., y_j$ and replace even degrees by appropriate products of squares. For instance, we may assume that

$$T_{1,1,n}(z_1;\, y_1) = z_1^3/6,$$
$$T_{2,1,n}(z_1;\, y_1) = (2z_1^4 - 3y_1^2 y_1^2)/48,$$
$$T_{2,2,n}(z_1, z_2;\, y_1, y_2) = (1 - 1/n)\, z_1^3 z_2^3/72,$$
$$T_{3,1,n}(z_1;\, y_1) = z_1^5/24,$$
$$T_{3,2,n}(z_1, z_2;\, y_1, y_2) = (1 - 1/n)\, z_1^3(2z_2^4 - 3y_1^2 y_2^2)/144,$$
$$T_{3,3,n}(z_1, z_2, z_3;\, y_1, y_2, y_3) = (1 - 1/n)(1 - 2/n)\, z_1^3 z_2^3 z_3^3/216$$

**Lemma 3.4.** We have $a^{\nabla}_{\nu} \in \mathbf{S}$ and, if $n > \nu$, $b^{\nabla}_{\nu,n} \in \mathbf{S}$. Moreover, for all $m, s \in \mathbf{N}_0$ we have

$$\sup_n \|a^{\nabla}_{\nu}\|_{m,s} / \{n^{\nu/2} \mathbf{E}\, \|X_{\nabla}\|^2 + \mathbf{E}\, \|X^{\nabla}\|^{\nu+2}\} < \infty,$$

$$\sup_{n>\nu} \|b^{\nabla}_{\nu,n}\|_{m,s} / \{n^{\nu/2} \mathbf{E}\, \|X_{\nabla}\|^2 + \mathbf{E}\, \|X^{\nabla}\|^{\nu+2}\} < \infty$$

If $\mathbf{E}\, \|X\|^{\nu+2} < \infty$, then $a_{\nu} \in \mathbf{S}$ and, if $n > \nu$, $b_{\nu,n} \in \mathbf{S}$. Also, we have

$$\sup_n \|a^{\nabla}_{\nu} - a_{\nu}\|_{m,s} / \mathbf{E}\, \|X\|^{\nu+2} < \infty,$$

$$\sup_{n>\nu} \|b^{\nabla}_{\nu,n} - b_{\nu,n}\|_{m,s} / \mathbf{E}\, \|X_{\nabla}\|^{\nu+2} < \infty$$

Furthermore, if $\pi_l(b) \equiv 0$ for odd $l$, then

$$a^{\nabla}_{\nu}(x) = a_{\nu}(x) = b^{\nabla}_{\nu,n}(x) = b_{\nu,n}(x) \equiv 0$$

for odd $\nu$.

Estimates of the closeness of asymptotic series in Eqs. (3.5)–(3.8) presents the following lemma.

**Lemma 3.5.** For all $m, s \in \mathbf{N}_0$ we have

$$\sup_{n>k} \|A^{\nabla}_{k,n} - B^{\nabla}_{k,n}\|_{m,s} / \{\mathbf{E}\, \|X_{\nabla}\|^2 + \varepsilon^{k+1} \mathbf{E}\, \|X^{\nabla}\|^{k+3}\} < \infty$$



and, if $\mathbf{E}\, \|X\|^{k+3} < \infty$, then

$$\sup_{n>k} \|A_{k,n} - B_{k,n}\|_{m,s} / \varepsilon^{k+1} < \infty$$

Let us continue with asymptotic expansions.

**Theorem 3.3.** Let $m \in \mathbf{N}_0$ and assume that

$$\exists \delta > 0 \qquad \mathbf{E}\, \|X\|^{2+\delta} + \mathbf{E}\, \|X\|^{pm} < \infty$$

Then, for all $k \in \mathbf{N}_0$ and $A > 0$,

$$\sup_n \|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} - A^{\nabla}_{k,n}\|_m / \{I_n(A) + K_n + n^{-A}\} < \infty, \tag{3.10}$$

$$\sup_n \|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} - A^{\nabla}_{k,n}\|_{m,s} / \{J_n + K_n\} < \infty \tag{3.11}$$

for all $s \in \mathbf{N}$, where

$$K_n \triangleq \mathbf{E}\, \|X_{\nabla}\|^2 + \varepsilon^{k+1} \mathbf{E}\, \|X^{\nabla}\|^{k+3} + \varepsilon^{pm-2} \mathbf{E}\, \|X\|^{pm}\, \mathbf{I}\{\|\varepsilon X\| > 1\}$$

**Theorem 3.4.** Let $m \in \mathbf{N}_0$ and assume that

$$\exists \delta > 0 \qquad \mathbf{E}\, \|X\|^{2+\delta} + \mathbf{E}\, \|X\|^{pm} < \infty$$

Then, for all $k \in \mathbf{N}_0$ and $A > 0$,

$$\sup_{n>k} \|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} - B^{\nabla}_{k,n}\|_m / \{I_n(A) + K_n + n^{-A}\} < \infty, \tag{3.12}$$

$$\sup_{n>k} \|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} - B^{\nabla}_{k,n}\|_{m,s} / \{J_n + K_n\} < \infty \tag{3.13}$$

for all $s \in \mathbf{N}$, where $K_n$ is defined in the previous theorem.

**Theorem 3.5.** Let $k, m \in \mathbf{N}_0$ and assume that

$$\mathbf{E}\, \|X\|^{k+3} + \mathbf{E}\, \|X\|^{pm} < \infty$$

Then, for all $A > 0$,

$$\sup_{n>k} \|\mathbf{P}\{\pi(S_n \leqslant \cdot\} - B_{k,n}\|_m / \{I_n(A) + \varepsilon^{k+1} + n^{-A}\} < \infty, \tag{3.14}$$

$$\sup_{n>k} \|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} - B_{k,n}\|_{m,s} / \{J_n + \varepsilon^{k+1}\} < \infty \tag{3.15}$$

for all $s \in \mathbf{N}$.

The paper by Bentkus *et al.*,[8] contains a review of the results obtained up to 1989. So let us mention only some papers related to the topic of the section: Götze,[31,33,35,37] Chebotarev,[18] Yurinskii,[83,84] Bentkus,[6,7] Bentkus and Zalesskii,[9] Zitikis,[88,89] Zalesskii,[84–86] Sazonov *et al.*,[69] Nagaev and Chebotarev,[52] Ulyanov,[78] Vinogradova,[81] Vakhania,[79] Kuelbs and Kurtz,[47] Rhee and Talagrand,[63] Bikelis,[14] Bhattacharya and Rao,[12] Statulevičius,[74] Sadikova,[65] Bogachev and Smolyanov,[15] Malliavin,[48] Giné and Zinn,[39] Hoffman-Jörgensen and Pisier,[42] Pisier,[60] Vakhania *et al.*[80]

## 4. EXPANSIONS FOR CHARACTERISTIC FUNCTIONS IN BANACH SPACES

In this section we preserve notation from Section 3 and assume the assumptions of that section hold.

Denote

$$\varphi_t(b) \triangleq \exp\{it\pi(b)\}, \qquad \forall b \in B$$

Let

$$\hat{a}_\nu^\nabla(t) \triangleq \mathbf{E}P_\nu^\nabla \varphi_t(G) = \mathbf{E}Q_\nu^\nabla \varphi_t(G), \qquad \hat{a}_\nu(t) \triangleq \mathbf{E}P_\nu \varphi_t(G) = \mathbf{E}Q_\nu \varphi_t(G),$$

$$\hat{b}_{\nu,n}^\nabla(t) \triangleq \sum_{j=1}^{\nu} \mathbf{E}T_{\nu,j,n}^\nabla \varphi_t(G\sqrt{1-j/n}), \quad \hat{b}_{\nu,n}(t) \triangleq \sum_{j=1}^{\nu} \mathbf{E}T_{\nu,j,n} \varphi_t(G\sqrt{1-j/n})$$

where the differential operators (for example, $D(X_j)$) act on the function $b \mapsto \varphi_t(b)$.

**Lemma 4.1.** The function $t \mapsto \mathbf{E}\varphi_t(G)$ is of the class **S**.

In this section the norms $\|\cdot\|_s$, $\|\cdot\|_{s,m}$ of the class **S** are always taken with respect to the variable $t \in \mathbf{R}$. Also, for $s, m \in \mathbf{N}_0$ and a set $D \subset \mathbf{R}$ we denote

$$\|g; D\|_s \triangleq \sup_{t\in D}(1+|t|^s)\,|g(t)|, \qquad \|g; D\|_{s,m} \triangleq \sup_{t\in D}(1+|t|^s)\,|(d/dt)^m\, g(t)|$$

Below we shall always use

$$D \triangleq \{|t| \leqslant n^{p/2-\sigma}\}, \qquad \sigma > 0$$

**Lemma 4.2.** The functions $t \mapsto t^{-1}\hat{a}_\nu^\nabla(t)$ and, if $n > \nu$, $t \mapsto t^{-1}\hat{b}_{\nu,n}^\nabla(t)$, are of the class **S**. Moreover, for all $s, m \in \mathbf{N}_0$ we have

$$\sup_n \|t^{-1}\hat{a}_\nu^\nabla\|_{s,m}/\{n^{\nu/2}\mathbf{E}\,\|X_\nabla\|^2 + \mathbf{E}\,\|X^\nabla\|^{\nu+2}\} < \infty, \tag{4.1}$$

$$\sup_{n>\nu} \|t^{-1}\hat{b}_{\nu,n}^\nabla\|_{s,m}/\{n^{\nu/2}\mathbf{E}\,\|X_\nabla\|^2 + \mathbf{E}\,\|X^\nabla\|^{\nu+2}\} < \infty \tag{4.2}$$



If $\mathbf{E}\,\|X\|^{\nu+2} < \infty$, then the functions $t \mapsto t^{-1}\hat{a}_\nu(t)$ and, if $n > \nu$, $t \mapsto t^{-1}\hat{b}_{\nu,n}(t)$ are of the class **S**, and

$$\sup_n \|t^{-1}\hat{a}_\nu^\nabla - t^{-1}\hat{a}_\nu\|_{s,m}/\mathbf{E}\,\|X_\nabla\|^{\nu+2} < \infty, \tag{4.3}$$

$$\sup_{n>\nu} \|t^{-1}\hat{b}_{\nu,n}^\nabla - t^{-1}\hat{b}_{\nu,n}\|_{s,m}/\mathbf{E}\,\|X_\nabla\|^{\nu+2} < \infty \tag{4.4}$$

Furthermore, if $\pi_l(b) \equiv 0$ for odd $l$, then

$$\hat{a}_\nu^\nabla(t) = \hat{a}_\nu(t) = \hat{b}_{\nu,n}^\nabla(t) = \hat{b}_{\nu,n}(t) \equiv 0 \tag{4.5}$$

for odd $\nu$.

We use the notation $\|t^{-1}u\|_{s,m}$ instead of $\|v\|_{s,m}$, where $v(t) \triangleq t^{-1}u(t)$. Here, we investigate the following four asymptotic series:

$$\hat{A}_{k,n}^\nabla(t) \triangleq \mathbf{E}\varphi_t(G) + \sum_{\nu=1}^{k} \hat{a}_\nu^\nabla(t)\,\varepsilon^\nu, \qquad \hat{A}_{k,n}(t) \triangleq \mathbf{E}\varphi_t(G) + \sum_{\nu=1}^{k} \hat{a}_\nu(t)\,\varepsilon^\nu,$$

$$\hat{B}_{k,n}^\nabla(t) \triangleq \mathbf{E}\varphi_t(G) + \sum_{\nu=1}^{k} \hat{b}_{\nu,n}^\nabla(t)\,\varepsilon^\nu, \qquad \hat{B}_{k,n}(t) \triangleq \mathbf{E}\varphi_t(G) + \sum_{\nu=1}^{k} \hat{b}_{\nu,n}(t)\,\varepsilon^\nu$$

Let us give some facts about the closeness of these asymptotic series.

**Lemma 4.3.** The asymptotic series $t \mapsto t^{-1}\hat{A}_{k,n}^\nabla(t)$ and, if $n > k$, $t \mapsto t^{-1}\hat{B}_{k,n}^\nabla(t)$ are of the class **S**. Moreover, for all $s, m \in \mathbf{N}_0$ we have

$$\sup_{n>k} \|t^{-1}\hat{A}_{k,n}^\nabla - t^{-1}\hat{B}_{k,n}^\nabla\|_{s,m}/\{\mathbf{E}\,\|X_\nabla\|^2 + \varepsilon^{k+1}\mathbf{E}\,\|X^\nabla\|^{k+3}\} < \infty \tag{4.6}$$

If $\mathbf{E}\,\|X\|^{k+3} < \infty$, then the asymptotic series $t \mapsto t^{-1}\hat{A}_{k,n}(t)$ and, if $n > k$, $t \mapsto t^{-1}\hat{B}_{k,n}(t)$ are of the class **S** and

$$\sup_{n>k} \|t^{-1}\hat{A}_{k,n} - t^{-1}\hat{B}_{k,n}\|_{s,m}/\varepsilon^{k+1} < \infty \tag{4.7}$$

**Corollary 4.1.** For all $s, m \in \mathbf{N}_0$ and $\sigma > 0$ we have

$$\sup_n \|t^{-1}\hat{A}_{k,n}^\nabla - t^{-1}\hat{B}_{k,n}^\nabla; D\|_{s,m}/\{\mathbf{E}\,\|X_\nabla\|^2 + \varepsilon^{k+1}\mathbf{E}\,\|X^\nabla\|^{k+3}\} < \infty \tag{4.8}$$

and, if $\mathbf{E}\,\|X\|^{k+3} < \infty$,

$$\sup_n \|t^{-1}\hat{B}_{k,n}^\nabla - t^{-1}\hat{B}_{k,n}; D\|_{s,m}/\{\varepsilon^{k+1}\mathbf{E}\,\|X_\nabla\|^{k+3}\} < \infty, \tag{4.9}$$

$$\sup_n \|t^{-1}\hat{A}_{k,n} - t^{-1}\hat{B}_{k,n}; D\|_{s,m}/\varepsilon^{k+1} < \infty \tag{4.10}$$

In the following theorem we give asymptotic expansions of the quantity

$$f_n^\nabla(t) \triangleq \mathbf{E}\varphi_t(S_n^\nabla)$$

**Theorem 4.1.** Asume that $\mathbf{E}\,\|X\|^{2+\delta}<\infty$ for some $\delta>0$. Then for all $\sigma>0$, $k\in\mathbf{N}_0$ and $s, m\in\mathbf{N}_0$ we have

$$\sup_n \|t^{-1}f_n^\nabla - t^{-1}\hat{A}_{k,n}^\nabla; D\|_{s,m}/\{\mathbf{E}\,\|X_\nabla\|^2+\varepsilon^{k+1}\mathbf{E}\,\|X^\nabla\|^{k+3}\}<\infty, \tag{4.11}$$

$$\sup_n \|t^{-1}f_n^\nabla - t^{-1}\hat{B}_{k,n}^\nabla; D\|_{s,m}/\{\mathbf{E}\,\|X_\nabla\|^2+\varepsilon^{k+1}\mathbf{E}\,\|X^\nabla\|^{k+3}\}<\infty \tag{4.12}$$

and, if $\mathbf{E}\,\|X\|^{k+3}<\infty$,

$$\sup_n \|t^{-1}f_n^\nabla - t^{-1}\hat{A}_{k,n}; D\|_{s,m}/\varepsilon^{k+1}<\infty, \tag{4.13}$$

$$\sup_n \|t^{-1}f_n^\nabla - t^{-1}\hat{B}_{k,n}; D\|_{s,m}/\varepsilon^{k+1}<\infty \tag{4.14}$$

Denote

$$f_n(t) \triangleq \mathbf{E}\varphi_t(S_n)$$

**Lemma 4.4.** Let $m\in\mathbf{N}_0$ and assume that

$$\exists\delta>0 \qquad \mathbf{E}\,\|X\|^{2+\delta}+\mathbf{E}\,\|X\|^{pm}<\infty$$

Then for all $\sigma>0$ and $s\in\mathbf{N}_0$ we have

$$\sup_n \|f_n^\nabla - f_n; D\|_{s,m}/\{\varepsilon^{pm-2}\mathbf{E}\,\|X\|^{pm}\,\mathbf{I}\{\|\varepsilon X\|>1\}\}<\infty$$

## 5. PROOFS OF THE RESULTS: SOME PRELIMINARIES

In this section we preserve the notation of Sections 3 and 4, and assume the assumptions of these sections hold.

Theorems 5.1–5.3 are the main results of the section.

**Theorem 5.1.** If $\pi_l(b)\equiv 0$ for odd $l$, then for $\theta\in\mathbf{R}$, all odd $\nu$, and $h_1,\dots, h_\nu\in\mathbf{B}$ we have

$$\mathbf{E}D(h_1)\dots D(h_\nu)\,\varphi_t(\theta G)\equiv 0$$

*Proof.* The function $b\mapsto\varphi_t(\theta b)\colon\mathbf{B}\to\mathbf{C}$ is even. Therefore its derivatives of odd orders are odd functions. So it is sufficient to show that $\mathbf{E}u(G)=0$ if a function $u$ is odd. But $\mathscr{L}(G)=\mathscr{L}(-G)$ and we have $\mathbf{E}u(G)=-\mathbf{E}u(-G)=-\mathbf{E}u(G)$ which completes the proof. □



To formulate other two theorems we need some notational conventions. For a r.v. $Z\in\mathbf{B}$ and a subset $C\subset\{1,\dots, n\}$ we denote

$$Z(C)\triangleq\varepsilon\sum_{j\in C}Z_j, \qquad Z(\varnothing)\triangleq 0$$

where $Z_1, Z_2,\dots$ are independent copies of $Z$.

By $\#C$ we denote the number of elements in the set $C$.

**Theorem 5.2.** Let $\nu\in\mathbf{N}_0$, and let $C\subset\{1,\dots, n\}$ be a set such that $\inf_n\{\#C/n\}>0$. Denote

$$u(t)\triangleq\mathbf{E}D(h_1)\dots D(h_\nu)\,\varphi_t(a+G(C))$$

Then for all $s, m\in\mathbf{N}_0$ there exists a constant $c$ (which does not depend on $n$) such that for all $a, h_1,\dots, h_\nu\in\mathbf{B}$ we have

$$\|u\|_{s,m}\leqslant c\,\|h_1\|\dots\|h_\nu\|(1+\|a\|^{pm+p\nu-\nu})$$

and, if $\nu\geqslant 1$,

$$\|t^{-1}u\|_{s,m}\leqslant c\,\|h_1\|\dots\|h_\nu\|(1+\|a\|^{pm+p\nu-\nu})$$

**Theorem 5.3.** Let $\nu\in\mathbf{N}_0$ and let $\mathbf{E}\,\|X\|^{2+\delta}<\infty$ for some $\delta>0$. Let $C_1, C_2, C_3$ be mutually disjoint subsets of the set $\{1,\dots, n\}$ such that $\inf_n\{\#C_j/n\}>0$ for at least one $j=1, 2, 3$. Denote

$$u(t)\triangleq\mathbf{E}D(h_1)\dots D(h_\nu)\,\varphi_t(a+G(C_1)+X^\nabla(C_2)+X(C_3))$$

If $C_3=\varnothing$, then for all $s, m\in\mathbf{N}$ and $\sigma>0$ there exists a constant $c$ (which does not depend on $n$) such that for all $a, h_1,\dots, h_\nu\in\mathbf{B}$

$$\|u; D\|_{s,m}\leqslant c\,\|h_1\|\dots\|h_\nu\|(1+\|a\|^{pm+p\nu-\nu}) \tag{5.1}$$

and, if $\nu\in\mathbf{N}$,

$$\|t^{-1}u; D\|_{s,m}\leqslant c\,\|h_1\|\dots\|h_\nu\|(1+\|a\|^{pm+p\nu-\nu}) \tag{5.2}$$

Furthermore, if the set $C_3$ is not empty, then bounds in Eq. (5.1)–(5.2) hold provided $\mathbf{E}\,\|X\|^{pm+p\nu-\nu}<\infty$. Recall that $D=\{t\colon |t|\leqslant n^{p/2-\sigma}\}$.

Before the proof of Theorems 5.2 and 5.3 we will do some preparatory work.

For a r.v. $\acute{Z} \in \mathbf{B}$, nonnegative integers $v \leqslant n$ and $w$ put

$$\Psi_Z(t, n, v, w) \triangleq \min_{C \subset \{1,\dots,v\}} \min_{**} \max_{l=1,\dots,w+1} \times \min_{***} (\mathbf{E} \exp\{itp!\, \pi_p(\tilde{Z}(D_1),\dots, \tilde{Z}(D_p))\})^{2^{-p}}$$

where $\min_{**}$ is taken over all partitions $C_1 \cup \cdots \cup C_{w+1} = C$ of the set $C$, and $\min_{***}$ is taken over all partitions $D_1 \cup \cdots \cup D_p = C_l$ of the set $C_l$; by $\tilde{Z}$ we denote a symmetrization of $Z$, that is, a random variable $\tilde{Z}$ such that $\mathscr{L}(\tilde{Z}) = \mathscr{L}(Z_1 - Z_2)$. It is easy to see that $\Psi_Z(t, n, v, w) = 1$ whenever $v < p(w+1)$.

**Lemma 5.1.** Let $L(b) = L_\nu(b,\dots, b)$, $b \in \mathbf{B}$, be a continuous $\nu$-linear form (which is not assumed to be symmetric). If

$$\mathbf{E}\, \|Z\|^\nu + \sup_n \mathbf{E}\, \|\varepsilon Z_1 + \cdots + \varepsilon Z_n\| \leqslant c_1 < \infty$$

then for an arbitrary subset $C \subset \{1,\dots, n\}$, $\# C = v$, and $w \geqslant \nu$ we have

$$|\mathbf{E} L(a + Z(C))\, \varphi_t(a + Z(C))| \leqslant c(1 + \|a\|^\nu)\, \Psi_Z(t, n, v, w)$$

where $c$ depends on $\mathscr{L}(Z)$ only through the constant $c_1$.

Before the proof of the lemma let us remind some known facts: Symmetrization inequality: For all $C \subset \{1,\dots, n\}$,

$$|\mathbf{E}\varphi_t(b + Z(C))|^{2^p} \leqslant \mathbf{E} \exp\{itp!\, \pi_p(\tilde{Z}(D_1),\dots, \tilde{Z}(D_p))\} \tag{5.3}$$

where $D_1 \cup \cdots \cup D_p = C$ is an arbitrary partition of $C$ (see Götze,[31] $p = 2$, Yurinskii,[84] $p > 2$, and, for instance, Bentkus *et al.*[8] for further references.)

A result of de Acosta[1]: For $q \geqslant 2$ there exists a constant $c$ (depending only on $q$) such that

$$\mathbf{E} \left\| \sum_{j=1}^n Z_j \right\|^q \leqslant c \left( \mathbf{E} \left\| \sum_{j=1}^n Z_j \right\| \right)^q + c \left( \sum_{j=1}^n \mathbf{E}\, \|Z_j\|^2 \right)^{q/2} + c \sum_{j=1}^n \mathbf{E}\, \|Z_j\|^q \tag{5.4}$$

whenever $Z_1,\dots, Z_n$ are centered independent r.v. taking values in a Banach space.

*Proof of Lemma 5.1.* We may assume $v \geqslant p(w+1)$ since otherwise the lemma is trivial. Write $Z(C) = V_0 + V_1 + \cdots + V_{w+1}$, where $V_0 \triangleq a + Z(C_0)$, $V_j \triangleq Z(C_j)$ for $j = 1,\dots, w+1$, and $C_0 \cup \cdots \cup C_{w+1} \subset C$ is a partition of $C$. Then



$$\mathbf{E} L(a + Z(C))\, \varphi_t(a + Z(C)) = \sum \mathbf{E} L(V_{j_1},\dots, V_{j_\nu})\, \varphi_t(a + Z(C))$$

where the sum is taken over all $0 \leqslant j_1,\dots, j_\nu \leqslant w+1$. Among $V_{j_1},\dots, V_{j_\nu}$ at least one of $V_1,\dots, V_{w+1}$ is absent, say $V_j$. Then

$$|\mathbf{E} L(a + Z(C))\, \varphi_t(a + Z(C))| \leqslant c \sum \mathbf{E}\, \|V_{j_1}\| \dots \|V_{j_\nu}\| \max_{1 \leqslant j \leqslant w+1} |\mathbf{E}_{V_j} \varphi_t(a + Z(C))|$$

and an application of Eqs. (5.3) and (5.4) concludes the proof. □

The following corollary is a consequence of Lemma 5.1.

**Corollary 5.1.** Let $m, \nu \in \mathbf{N}_0$, and

$$\mathbf{E}\, \|Z\|^{pm + p\nu - \nu} + \sup_n \mathbf{E}\, \|\varepsilon Z_1 + \cdots + \varepsilon Z_n\| \leqslant c_2 \tag{5.5}$$

Denote

$$u(t) \triangleq \mathbf{E} D(h_1) \dots D(h_\nu)\, \varphi_t(a + Z(C))$$

Then for all subsets $C \subset \{1,\dots, n\}$, $\# C = v$, and all $w \geqslant pm + p\nu - \nu$ we have

$$|(d/dt)^m\, u(t)| \leqslant c\, \|h_1\| \dots \|h_\nu\| (1 + \|a\|^{pm + p\nu - \nu})\, \Psi_Z(t, n, v, w)$$

and, if $\nu \geqslant 1$,

$$|(d/dt)^m \{t^{-1} u(t)\}| \leqslant c\, \|h_1\| \dots \|h_\nu\| (1 + \|a\|^{pm + p\nu - \nu})\, \Psi_Z(t, n, v, w)$$

for all $t \in \mathbf{R}$, where $c$ depends on the distribution $\mathscr{L}(Z)$ only through the constant $c_2$.

*Proof.* Write $u(t)$ as $\mathbf{E} L(a + Z(C))\, \varphi_t(a + Z(C))$ with an appropriate $L$ and apply Lemma 5.1. If $\nu \in \mathbf{N}$, then the function $u(t)$ is divisible by $t$ and an application of Lemma 5.1 completes the proof. □

Lemma 5.2 is a simplified version of Theorem 5.6, Ch. 2, from Kruglov.[46]

**Lemma 5.2.** Let $V_1, V_2,\dots \in \mathbf{B}$ be independent copies of a r.v. $V \in \mathbf{B}$. If

$$\mathbf{P}\{\|V\| \leqslant 1\} = 1, \qquad \mathbf{P}\{\|V_1 + \cdots + V_n\| \geqslant \lambda\} \leqslant 1/100 \tag{5.6}$$

then

$$\sup_n \mathbf{E} \exp\{\|V_1 + \cdots + V_n\|/(10+10\lambda)\} < \infty$$

This lemma and well-known properties of Gaussian random variables imply the following corollary.

**Corollary 5.2.** Let $V$ denote one of $X$, $X^\nabla$, $G$, $\tilde{X}$, $\tilde{X}^\nabla$, $\tilde{G}^\nabla$ truncated at a level $\Lambda \leqslant \sqrt{n}$ (for example, in case of $X$ we have $V = X\mathbf{I}\{\|X\| \leqslant \Lambda\}$). Then there exists a positive constant $c_1 > 0$ such that for all $R > 0$

$$\sup_n \mathbf{P}\{\|\varepsilon V_1 + \cdots + \varepsilon V_n\| \geqslant R\} \leqslant c \exp\{-c_1 R\}$$

Furthermore, for all $\tau > 0$

$$\sup_n \mathbf{E}\, \|\varepsilon V_1 + \cdots + \varepsilon V_n\|^\tau < \infty$$

**Lemma 5.3.** Suppose that $\inf_n\{v/n\} > 0$ and that $\sup_n w < \infty$. Then for each $s \in \mathbf{N}_0$ there exists a (sufficiently large) constant $c$ such that

$$\sup_{n \geqslant c} \|\Psi_G(\cdot, n, v, w)\|_s < \infty$$

If $\mathbf{E}\, \|X\|^{2+\delta} < \infty$ for some $\delta > 0$, then for each $\sigma > 0$ we have

$$\sup_n \|\Psi_X(\cdot, n, v, w); D\|_s < \infty, \qquad \sup_n \|\Psi_{X^\nabla}(\cdot, n, v, w); D\|_s < \infty$$

*Proof.* Let us first consider $\Psi_G$. Take a set $C = \{1,..., v\}$ and let a partition $C_1 \cup \cdots \cup C_{w+1} = C$ be such that $\inf_{n \geqslant c} \inf_{1 \leqslant j \leqslant w+1}\{\# C_j/n\} > 0$. It is possible to find such a partition because of $\inf_n\{v/n\} > 0$ and $\sup_n w < \infty$. Similarly, for all $l = 1,..., w+1$ we may choose $D_1 \cup \cdots \cup D_p = C_l$ so that

$$\inf_{n \geqslant c} \inf\{\# D_\tau/n \colon 1 \leqslant j \leqslant w+1,\ 1 \leqslant \tau \leqslant p\} > 0$$

Thus $\mathscr{L}(G(D_\tau)) = \mathscr{L}(c_{j,\tau} G)$ for a constant $c_{j,\tau}$ such that $\inf_{n \geqslant c} \inf_{j,\tau} c_{j,\tau} > 0$, and so the desirable estimate of $\Psi_G$ reduces to

$$\sup_{t \in \mathbf{R}} |t|^d\, E \exp\{it\pi_p(G_1,..., G_p)\} < \infty, \qquad \forall d \geqslant 0$$

which is assumed to hold.



The estimation of $\Psi_X$ and $\Psi_{X^\nabla}$ is more complicated. In order to unify the both cases, let $W$ denote either $X$ or $X^\nabla$. Obviously, we may assume $n$ and $|t|$ to be sufficiently large. Arguing as in the case of $\Psi_G$, we reduce the estimation of $\Psi_W$ to the estimation of

$$0 \leqslant T \triangleq \mathbf{E} \exp\{it\pi_p(\tilde{W}(D_1),..., \tilde{W}(D_p))\}$$

where a partition $D_1 \cup \cdots \cup D_p = F$ of a set $F \subset \{1,..., n\}$ is chosen so that $\mu \triangleq \# D_1 = \cdots = \# D_p$.

Let for a moment

$$f(\cdot) \triangleq t\varepsilon\pi_p(\tilde{W}(D_1),..., \tilde{W}(D_{p-1}), \cdot)$$

Then

$$T = \mathbf{E}(\mathbf{E}_W \exp\{if(\tilde{W})\})^\mu \tag{5.7}$$

since $\tilde{W}$, $\tilde{W}_1$, $\tilde{W}_2$,... are independent and identically distributed r.v. ($\mathbf{E}_W$ stands for the expectation with respect to $\tilde{W}$).

Let $\Lambda \geqslant 0$ be an arbitrary real number and denote

$$Z \triangleq \tilde{W}\mathbf{I}\{\|\tilde{W}\| \leqslant \Lambda\}$$

Since the r.v. $\tilde{W}$, $Z$ are symmetric we have

$$\mathbf{E}_W \exp\{if(\tilde{W})\} = \mathbf{E}_W \cos\{f(\tilde{W})\} \leqslant \mathbf{E}_W \cos\{f(Z)\} = \mathbf{E}_W \exp\{if(Z)\}$$

and so from Eq. (5.8) we get

$$T \leqslant \mathbf{E} \exp\{it\pi_p(\tilde{W}(D_1),..., \tilde{W}(D_{p-1}), Z(D_p))\}$$

Repeating the calculations, we arrive at

$$T \leqslant \mathbf{E} \exp\{it\pi_p(Z(D_1),..., Z(D_p))\} \tag{5.8}$$

Therefore, to complete the proof it remains to show that for all positive (since $Z$ is symmetric) $t \leqslant n^{p/2}$ we have

$$T_1 \triangleq \mathbf{E} \exp\{it\pi_p(Z(D_1),..., Z(D_p))\} \leqslant ct^{-A} \tag{5.9}$$

Choose $\mu \sim nt^{\theta - 2/p}$ and $\Lambda = t^a n^a$, where small but positive constants $\theta > 0$ and $a > 0$ are to be chosen later, and let us start with the proof of

$$T_1 \leqslant cT_2 + ct^{-A}, \qquad \forall t \leqslant n^{p/2-\sigma} \tag{5.10}$$

where

$$T_2 \triangleq \mathbf{E} \exp\{it\pi_p(Z(D_1),..., Z(D_{p-1}), G(D_p))\}$$

Denote

$$f(\cdot) \triangleq t\varepsilon\pi_p(Z(D_1),..., Z(D_{p-1}), \cdot),$$

$$\mathbf{I}_0 \triangleq \mathbf{I}_1 \cdots \mathbf{I}_{p-1}, \qquad \mathbf{I}_\tau \triangleq \mathbf{I}\{\|Z(D_\tau)\| \leqslant R\}, \qquad R \triangleq \sqrt{\mu/n}\ln^2 n$$

Then

$$T_1 = \mathbf{E}\{\mathbf{E}_Z \cos(f(Z))\}^\mu \leqslant \mathbf{E}\{\mathbf{E}_Z \mathbf{I}_0 \cos(f(Z))\}^\mu + \mathbf{E}\sum_{\tau=1}^{p-1}(1-\mathbf{I}_\tau) \tag{5.11}$$

By Corollary 5.2, there exists a positive constant $c_1$ such that

$$\sum_{\tau=1}^{p-1} \mathbf{E}(1-\mathbf{I}_\tau) \leqslant \sum_{\tau=1}^{p-1} \mathbf{P}\{\|Z(D_\tau)\| \geqslant R\} \leqslant c\exp\{-c_1 \ln^2 n\} \leqslant ct^{-A} \tag{5.12}$$

We shall show now that

$$\mathbf{I}_0 \mathbf{E}_Z \cos\{f(Z)\} \leqslant \exp\{ct^2\varepsilon^2 R^{2(p-1)}\Lambda^{-\delta}\} \exp\{-\mathbf{E}_G f^2(G)/2\} \tag{5.13}$$

If $\mathbf{I}_0 = 0$ then there is nothing to prove. Thus let $\mathbf{I}_0 = 1$ and so

$$\|f\| \leqslant ct\varepsilon \prod_{\tau=1}^{p-1} \|Z(D_\tau)\| \leqslant ct\varepsilon R^{p-1} \tag{5.14}$$

which implies $|f(Z)| \leqslant \Lambda \|f\| \leqslant ct\varepsilon R^{p-1}\Lambda$. If we choose $a$ and $\theta$ so that

$$a < \sigma/(p^2+2p), \qquad \theta < 2\sigma/(p^3-p^2) \tag{5.15}$$

we shall get $|f(Z)| \leqslant 1$ since Eq. (5.15) implies

$$ct^2\Lambda^2 n^{-p}\mu^{p-1}\ln^{4p} n \leqslant 1$$

Using $\cos x \leqslant 1 - x^2/4$ $\forall x$: $|x| \leqslant 1$, we get

$$\mathbf{I}_0 \mathbf{E}_Z \cos\{f(Z)\} \leqslant \mathbf{I}_0\{1 - \mathbf{E}_Z f^2(Z)/4\} \tag{5.16}$$

Furthermore,

$$\mathbf{E}_Z f^2(Z) = \mathbf{E}_W f^2(\tilde{W}) - \mathbf{E}_W f^2(\tilde{W})\,\mathbf{I}\{\|\tilde{W}\| \geqslant \Lambda\} \geqslant 2\mathbf{E}_W f^2(W) - c\Lambda^{-\delta}\|f\|^2$$

since $\mathbf{E}\|X\|^{2+\delta} < \infty$. If $W = X$, then

$$\mathbf{E}_W f^2(W) = \mathbf{E}_X f^2(X) = \mathbf{E}_G f^2(G)$$

If $W = X^\nabla$, then

$$\mathbf{E}_W f^2(W) = \mathbf{E}_X f^2(X) - \mathbf{E}_X f^2(X)\,\mathbf{I}\{\|X\| > \sqrt{n}\} \geqslant \mathbf{E}_G f^2(G) - c\Lambda^{-\delta}\|f\|^2$$



since $\Lambda \leqslant \sqrt{n}$. Using the bounds just obtained and Eq. (5.16), we get

$$\mathbf{I}_0 \mathbf{E}_Z \cos\{f(Z)\} \leqslant \mathbf{I}_0\{1 - \mathbf{E}_G f^2(G)/2 + c\|f\|^2 \Lambda^{-\delta}\}$$

which taken together with Eq. (5.14) completes the proof of Eq. (5.13).

From Eqs. (5.11)–(5.13) we get Eq. (5.10) if $\mu t^2\varepsilon^2 R^{2(p-1)}\Lambda^{-\delta} \leqslant c$. But the last inequality holds because of Eq. (5.15). Thus Eq. (5.10) is proved.

Now let us prove Eq. (5.9). Repeating the calculations leading to Eq. (5.10), we can replace the sums $Z(D_1),..., Z(D_p)$ by $G(D_1),..., G(D_p)$, and for $t \leqslant n^{p/2-\sigma}$ we get

$$T_1 \leqslant cT_{p+1} + ct^{-A} \tag{5.17}$$

where

$$T_{p+1} \triangleq \mathbf{E}\exp\{it\pi_p(G(D_1),..., G(D_p))\} \leqslant \mathbf{E}\exp\{ict^{\theta p/2}\pi_p(G_1,..., G_p)\}$$

because of $\mathscr{L}(G(D_\tau)) = \mathscr{L}(\sqrt{\mu/n}\,G)$ and $\mu \sim nt^{\theta - p/2}$. Therefore the condition in Eq. (3.2) and $\theta > 0$ ensure $T_{p+1} \leqslant ct^{-A}$. This proves Eq. (5.9) and the lemma as well. □

*Proof of Theorems 5.2 and 5.3.* These theorems are easy consequences of Corollary 5.1 and Lemma 5.3. □

## 6. PROOFS OF THE RESULTS OF SECTION 4

*Proof of Lemma 4.1.* The result is a simple consequence of Theorem 5.2. □

**Lemma 6.1.** If $2 \leqslant \alpha \leqslant \beta$, then there exists a constant $c$ (depending only on $\alpha$, $\beta$, and $\mathscr{L}(X)$) such that

$$\mathbf{E}\|X\|^\alpha \leqslant c\{\mathbf{E}\|X\|^\beta\}^{(\alpha-2)/(\beta-2)}, \qquad \mathbf{E}\|X^\nabla\|^\alpha \leqslant c\{\mathbf{E}\|X^\nabla\|^\beta\}^{(\alpha-2)/(\beta-2)}$$

Furthermore, if $\mathbf{P}\{X=0\} < 1$, then

$$1 \leqslant c\{n^{(\alpha-2)/2}\mathbf{E}\|X_\nabla\|^2 + \mathbf{E}\|X^\nabla\|^\alpha\}$$

*Proof.* The first two inequalities follow from the Hölder inequality. The last inequality is obvious. □

*Proof of Lemma 4.2.* Let us prove Eq. (4.1). Fix natural numbers $\alpha_1 \geqslant 3,..., \alpha_j \geqslant 3$ such that $|\alpha(j)| = 2j + i$ (look at the definition of $\sum^*$ in

Lemma 3.1) and $\beta_1 \geqslant 2,..., \beta_l \geqslant 2$ such that $|\beta(l)| = s-i-2(j-r)$ (look at the definition of $\sum^{**}$ in Lemma 3.1). Denote

$$u(t) \triangleq \mathbf{E}\{D^{\alpha_1}(X_1^{\nabla}) - D^{\alpha_1}(G_1)\} \cdots \{D^{\alpha_j}(X_j^{\nabla}) - D^{\alpha_j}(G_j)\} \\ \times D^{\beta_1}(G_{j+1}) \cdots D^{\beta_l}(G_{j+l})\, \varphi_t(G)$$

By Theorem 5.2,

$$\|t^{-1}u\|_{s,m} \leqslant c \prod_{\tau=1}^{j} \{\mathbf{E}\,\|X^{\nabla}\|^{\alpha_\tau} + \mathbf{E}\,\|G\|^{\alpha_\tau}\} \prod_{\mu=1}^{l} \{\mathbf{E}\,\|G\|^{\beta_\mu}\}$$

and so, by Lemma 6.1,

$$\|t^{-1}u\|_{s,m} \leqslant c\{n^{v/2}\mathbf{E}\,\|X_{\nabla}\|^2 + \mathbf{E}\,\|X^{\nabla}\|^{v+2}\}$$

The proof of Eq. (4.2) is similar to that of Eq. (4.1); thus we will omit it.

Let us prove Eq. (4.4). Fix natural numbers $\alpha_1 \geqslant 3,..., \alpha_j \geqslant 3$ such that $|\alpha(j)| = 2j+v$ (look at the definition of $\sum^{***}$ just after the Eq. (3.9)), and denote

$$v(t) \triangleq \mathbf{E}\left(\prod_{\tau=1}^{j} \{D^{\alpha_\tau}(X_\tau) - D^{\alpha_\tau}(G_\tau)\} - \prod_{\tau=1}^{j} \{D^{\alpha_\tau}(X_\tau^{\nabla}) - D^{\alpha_\tau}(G_\tau)\}\right) \varphi_t(G\sqrt{1-j/n})$$

Applying the identity $z_1 \dots z_j - y_1 \dots y_j = \sum z_1 \dots z_{\tau-1}(z_\tau - y_\tau)\, y_{\tau+1} \dots y_j$ with $z_\tau \triangleq D^{\alpha_\tau}(X_\tau) - D^{\alpha_\tau}(G_\tau)$, $y_\tau \triangleq D^{\alpha_\tau}(X_\tau^{\nabla}) - D^{\alpha_\tau}(G_\tau)$ and noting that $D^{\alpha_\tau}(X_\tau) - D^{\alpha_\tau}(X_\tau^{\nabla}) = D^{\alpha_\tau}(X_{\tau\nabla})$, we have from Eq. (5.2) that

$$\|t^{-1}v(t)\|_{s,m} \leqslant c \sum_{k=1}^{j} \mathbf{E}\,\|X_{\nabla}\|^{\alpha_k} \prod_{r \neq k} \{\mathbf{E}\,\|X^{\nabla}\|^{\alpha_\tau} + \mathbf{E}\,\|G\|^{\alpha_\tau}\} \tag{6.1}$$

Lemma 6.1 and a simple calculation show that the right-hand side of Eq. (6.1) does not exceed $c\mathbf{E}\,\|X_{\nabla}\|^{s+2}$, which completes the proof of Eq. (4.4).

To prove Eq. (4.5), use Theorem 5.1; the details are trivial. □

*Proof of Lemma 4.3.* Denote

$$D^{\alpha(j)} \triangleq \prod_{i=1}^{j} \{D^{\alpha_i}(\varepsilon X_i^{\nabla}) - D^{\alpha_i}(\varepsilon G_i)\}$$



A simple rearrangement and the definition of the Stirling numbers of the first kind imply

$$\hat{A}_{k,n}^{\nabla}(t) = \sum_{v=1}^{k} \sum_{j=1}^{v} (j!)^{-1} \sum_{r=1}^{j} \mathbf{s}(j,r)\, \mathbf{I}(v+2(j-r) \leqslant k) \\ \times \sum^{***} \{\alpha(j)!\}^{-1}\, \mathbf{E}D^{\alpha(j)} \sum_{l=1}^{\infty} z_l$$

We have used the following notations: the sum $\sum^{***}$ is defined just after the Eq. (3.7);

$z_0 \triangleq \varphi_t(G)$,

$z_l \triangleq (-1)^l \sum^{3,l} j^{(\gamma_1 + \cdots \gamma_l)/2} \{\gamma(l)!\}^{-1} D^{\gamma_1}(G_{j+1}) \cdots D^{\gamma_l}(G_{j+l})\, \varphi_t(G), \qquad l \in \mathbf{N}$

the sum $\sum^{3,l}$ is taken over all $\gamma_1 \geqslant 2,..., \gamma_l \geqslant 2$ such that $|\gamma(l)| \leqslant \vartheta \triangleq k-v-2(j-r)$. Let us note that $z_l = 0$ for all $l \geqslant [\vartheta/2]+1$.

On the other hand, we have

$$\hat{B}_{k,n}^{\nabla}(t) = J_1(t) + J_2(t) \tag{6.2}$$

where

$$J_1(t) \triangleq \sum_{v=1}^{k} \sum_{j=1}^{v} \frac{1}{j!} \sum_{r=1}^{j} \mathbf{s}(j,r)\, \mathbf{I}(v+2(j-r) \geqslant k) \\ \times \sum^{***} \frac{1}{\alpha(j)!}\, \mathbf{E}D^{\alpha(j)}(G\sqrt{1-j/n})$$

and

$$J_2(t) \triangleq \sum_{v=1}^{k} \sum_{j=1}^{v} \frac{1}{j!} \sum_{r=1}^{j} \mathbf{s}(j,r)\, \mathbf{I}(v+2(j-r) \leqslant k) \\ \times \sum^{***} \frac{1}{\alpha(j)!}\, \mathbf{E}D^{\alpha(j)}(G\sqrt{1-j/n})$$

Theorem 5.2 and Lemma 6.1 show that

$$\|t^{-1}J_1\|_{s,m} \leqslant c\{\mathbf{E}\,\|X_{\nabla}\|^2 + \varepsilon^{k+1}\mathbf{E}\,\|X^{\nabla}\|^{k+3}\} \tag{6.3}$$

Thus our task is to show that the same bound holds for $\|t^{-1}\hat{A}_{k,n}^{\nabla} - t^{-1}J_2\|_{s,m}$.

Denote

$w_0 \triangleq \varphi_t(G\sqrt{1-j/n})$,

$w_l \triangleq (-1)^l \sum^{3,l} j^{(\gamma_1 + \cdots \gamma_l)/2} \{\gamma(l)!\}^{-1} D^{\gamma_1}(G_{j+1}) \cdots D^{\gamma_l}(G_{j+l})\, \varphi_t(G\sqrt{1-j/n})$

$l \in \mathbf{N}$. Since $w_0 - \sum_{l=0}^{\infty} z_l = \sum_{l=0}^{\infty}(w_l - z_l - w_{l+1})$ (let us note that $z_l = 0$ for all $l \geq [\vartheta/2]+1$) we get

$$\|t^{-1}\hat{A}_{k,n}^{\nabla} - t^{-1}J_2\|_{s,m} \leq \sum_{l=0}^{\infty} K_l$$

where

$$K_l \triangleq \left| \sum_{v=1}^{k} \sum_{j=1}^{v} \frac{1}{j!} \sum_{r=1}^{j} \mathbf{s}(j,r)\,\mathbf{I}(v+2(j-r) \leq k) \times \sum\nolimits^{***} \frac{1}{\alpha(j)!} \mathbf{E} D^{\alpha(j)}(w_l - z_l - w_{l+1}) \right|$$

(Note that $K_l = 0$ for all $l \geq [\vartheta/2]+1$.) Thus, if we show that for all $l \leq [\vartheta/2]+1$

$$\|t^{-1}K_l\|_{s,m} \leq c\{\mathbf{E}\,\|X_\nabla\|^2 + \varepsilon^{k+1}\mathbf{E}\,\|X^\nabla\|^{k+3}\} \tag{6.4}$$

then Eq. (4.6) will be proved.

To prove Eq. (6.4), let us proceed in the following way. In the definition of $z_l$ replace $\varphi_t(G)$ by $\varphi_t(G\sqrt{1-j/n} + G_{j+l+1}\sqrt{j/n})$. Then use the Taylor formula to estimate the quantity $w_l - z_l - w_{l+1}$ and replace the first summand in the Taylor expansion by 0. All these calculations are possible because of the mathematical expectation $\mathbf{E}$ in the definition of $K_l$ and the fact that $X$ is centered. Finally, we have

$$|(d/dt)^m\{t^{-1}K_l\}| \leq \cdots \sup_{\theta \in (0,1)} |(d/dt)^m \{t^{-1}\mathbf{E} \ldots D^{\vartheta - \gamma(l)+1}(G_{j+l+l}) \times \varphi_t(G\sqrt{1-j/n} + \theta G_{j+l+1}\sqrt{j/n})\}|$$

Now it remains to use Theorem 5.3 and Lemma 6.1 to get Eq. (6.4) proved. This completes the proof of Eq. (4.6).

The proof of Eq. (4.8) is similar to that of Eq. (4.6); thus we shall omit it.

Bound Eq. (4.7) is a elementary consequence of (4.4). □

*Proof of Corollary 4.1.* If $n > k$, then the corollary is a straightforward consequence of Lemma 4.3. In case $n \leq k$ go along the proof of Lemma 4.3 and instead of using Theorem 4.2 use trivial bounds for characteristic functions. □

*Proof of Theorem 4.1.* Let us first prove Eq. (4.13). Denote:

$\sum^{1,j}$ the sum taken over all integer multiindexes $r(j) = (r_1, \ldots, r_j)$ such that $1 \leq r_1 < \cdots < r_j \leq n$;



$\sum^{2,j}$ the sum taken over all integer multiindexes $\gamma(j) = (\gamma_1, \ldots, \gamma_j)$ such that $3 \leq \gamma_1 \leq k+2$ and $3 \leq \gamma_\tau \leq k + 2\tau - \gamma_1 - \cdots - \gamma_{\tau-1}$, for $\tau = 2, \ldots, j$.

Let us note that $\sum^{1,j} \ldots = 0$ whenever $j \geq n+1$, and $\sum^{2,j} \ldots = 0$ whenever $j \geq k+1$.

Also, define

$$D_{r(j)}^{\gamma(j)} \triangleq \prod_{\tau=1}^{j} \{D^{\gamma_\tau}(\varepsilon X_{r_\tau}^{\nabla}) - D^{\gamma_\tau}(\varepsilon G_{r_\tau})\},$$

$$u_0 \triangleq \mathbf{E}\varphi_t(G(\{1,\ldots,n\})) = \mathbf{E}\varphi_t(G),$$

$$u_j \triangleq \sum\nolimits^{1,j} \sum\nolimits^{2,j} \gamma(j)!^{-1}\,\mathbf{E} D_{r(j)}^{\gamma(j)} \varphi_t(G(\{1,\ldots,n\} \setminus \{r_1,\ldots,r_j\})), \qquad j \in \mathbf{N},$$

$$v_0 \triangleq \mathbf{E}\varphi_t(X^\nabla(\{1,\ldots,n\})),$$

$$v_j \triangleq \sum\nolimits^{1,j} \sum\nolimits^{2,j} \gamma(j)!^{-1}\,\mathbf{E} D_{r(j)}^{\gamma(j)} \varphi_t(w), \qquad j \in \mathbf{N},$$

where

$$w \triangleq G(\{1,\ldots,n\} \setminus \{r_1,\ldots,r_{j-1}\}) + X^\nabla(\{r_{j+1},\ldots,n\})$$

Since $v_{k+1} = 0$, we get

$$\mathbf{E}\varphi_t(S_n^\nabla) = v_0 = u_0 + \cdots + u_k + \sum_{j=0}^{k} (v_j - u_j - v_{j+1}) \tag{6.5}$$

Notice that

$$u_j = \binom{n}{j} \sum\nolimits^{2,j} \{\gamma(j)!\}^{-1}\,\mathbf{E} D_{r(j)}^{\gamma(j)} \varphi_t(G\sqrt{1-j/n})$$

and

$$\sum\nolimits^{2,j} = \sum_{v=j}^{k} \sum\nolimits^{***}, \qquad \sum_{j=1}^{k} \sum_{v=j}^{k} = \sum_{v=1}^{k} \sum_{j=1}^{v}$$

Therefore,

$$u_0 + \cdots + u_k = \hat{B}_{k,n}^{\nabla}(t)$$

and so, look at Eq. (6.5),

$$|(d/dt)^m\{t^{-1}\mathbf{E}\varphi_t(S_n^\nabla) - t^{-1}\hat{B}_{k,n}^{\nabla}(t)\}| \leq \max_{0 \leq j \leq k} |(d/dt)^m\{t^{-1}(v_j - u_j - v_{j+1})\}|$$

Put

$$x \triangleq \varepsilon X_{k_{j+1}}^{\nabla}, \qquad y \triangleq \varepsilon G_{k_{j+1}}$$

We have

$$v_j - u_j = \sum^{1,j+1} \sum^{2,j} \{\gamma(j)!\}^{-1} \mathbf{E} D^{\gamma(j)}_{r(j)} \{\varphi_t(w+x) - \varphi_t(w+y)\} \quad (6.6)$$

If we apply the Taylor formula in Eq. (6.6), we shall get

$$v_j - u_j = v_{j+1} + \sum^{1,j+1} \sum^{2,j} \{\gamma(j)!\}^{-1} \{J_1 + J_2 + J_3\}$$

where

$$J_1 \triangleq \mathbf{E} D^{\gamma(j)}_{r(j)} \{D(x) - D(y)\}\, \varphi_t(w),$$
$$J_2 \triangleq 2^{-1} \mathbf{E} D^{\gamma(j)}_{r(j)} \{D^2(x) - D^2(y)\}\, \varphi_t(w),$$
$$J_3 \triangleq \mathbf{E}(1-U)^\theta D^{\gamma(j)}_{r(j)} \{D^\theta(x)\, \varphi_t(w+Ux) - D^\theta(y)\, \varphi_t(w+Uy)\}$$

$\theta \triangleq k + 2j - |\gamma(j)|$. (As usually, $U$ stands for a r.v. uniformly distributed on $(0, 1)$.) Note that

$$D(x) - D(y) = -D(\varepsilon(X_{k_{j+1}})_\nabla), \qquad D^2(x) - D^2(y) = -D^2(\varepsilon(X_{k_{j+1}})_\nabla)$$

since the means (and covariances) of $X$ and $Y$ are equal. Therefore, it follows from Theorem 5.3 and Lemma 6.1 that

$$\|t^{-1}(v_j - u_j - v_{j+1})\|_{s,m} \leqslant c\{E\, \|X_\nabla\|^2 + \varepsilon^{k+1} E\, \|X^\nabla\|^{k+3}\}$$

which concludes the proof of bound Eq. (4.13).

Bound Eq. (4.12) is a consequence of Eqs. (4.13) and (4.9). Bound Eq. (4.15) follows from Eqs. (4.13) and (4.10). Bound Eq. (4.14) is a consequence of Eqs. (4.15) and (4.11). □

*Proof of Lemma 4.4.* Put

$$V_k \triangleq \varepsilon X_1 + \cdots + \varepsilon X_{k-1} + \varepsilon X^\nabla_{k+1} + \cdots + \varepsilon X^\nabla_n$$

Then

$$f_n(t) - f^\nabla_n(t) = \sum_{k=1}^{n} \mathbf{E I}\{\|\varepsilon X\| \geqslant 1\} \{\varphi_t(V_k + \varepsilon X) - \varphi_t(V_k)\}$$

and so

$$\|f_n - f^\nabla_n; D\|_{s,m} \leqslant \sum_{k=1}^{n} \mathbf{E I}\{\|\varepsilon X\| \geqslant 1\} \{\|u_k; D\|_{s,m} + \|v_k; D\|_{s,m}\}$$

where

$$u_k(t) \triangleq \mathbf{E}_V \varphi_t(V_k + a), \qquad v_k(t) \triangleq \mathbf{E}_V \varphi_t(V_k), \qquad a \triangleq \varepsilon X$$



($\mathbf{E}_V$ stands for the expectation with respect to $V_k$). Applying Theorem 5.3, we get

$$\|f_n - f^\nabla_n; D\|_{s,m} \leqslant c\varepsilon^{-2} \mathbf{P}\{\|\varepsilon X\| \geqslant 1\} + c\varepsilon^{pm-2} \mathbf{E}\, \|X_\nabla\|^{pm}\, \mathbf{I}\{\|\varepsilon X\| > 1\}$$

and the lemma is proved. □

## 7. PROOF OF THE RESULTS OF SECTION 3

Let us remind that $A_{x,a} = \{b \in \mathbf{B}: \pi(b+a) \leqslant x\}$ for $x \in \mathbf{R}$ and $a \in \mathbf{B}$, and $\mathscr{A}_x = \{A_{x,a} : a \in \mathbf{B}\}$.

Put $\mathscr{A} \triangleq \bigcup\{\mathscr{A}_x : x \in \mathbf{R}\}$ and for all $\theta \in \mathbf{R}\setminus\{0\}$ define the set function

$$\Phi_\theta : A \mapsto \Phi(\theta^{-1} A) : \mathscr{A} \to \mathbf{R}$$

Clearly, if $A = A_{x,a}$, then

$$\Phi_\theta(A) = \mathbf{P}\{\pi(\theta G + a) \leqslant x\}$$

**Lemma 7.1.** The mesure $\Phi_\theta$ defined on $\mathscr{A}$ is infinitely many times differentiable. Furthermore, the distribution function $x \mapsto u(x, a) \triangleq \Phi_\theta(A_{x,a})$ has the density of the class $\mathbf{S}$ and for all $m \in \mathbf{N}_0$, $s \in \mathbf{N}$

$$\sup \|u(\cdot, a)\|_{m,s} / (1 + \|a\|^{pm}) < \infty$$

where sup is taken over all $a \in \mathbf{B}$. If $v \in \mathbf{N}$ and $v(x, a) \triangleq D(h_1) \ldots D(h_v)\, u(x, a)$, then for all $m, s \in \mathbf{N}_0$

$$\sup \|v(\cdot, a)\|_{m,s} / \{\|h_1\| \cdots \|h_v\| (1 + \|a\|^{pm - pv - v})\} < \infty$$

where sup is taken over all $a \in \mathbf{B}$, $h_1, \ldots, h_k \in \mathbf{B}\setminus\{0\}$.

*Proof.* The lemma is a consequence of Theorem 5.2 and well-known facts concerning infinitely differentiable and rapidly decreasing functions. □

*Proof of Lemma 3.2.* The lemma is a consequence of Lemma 7.1 and well-known properties of functions of the class $\mathbf{S}$. □

*Proof of Lemmas 3.3 and 3.4.* The lemmas follow from Lemma 4.2 and well-known properties of functions of the class $\mathbf{S}$. □

*Proof of Lemma 3.5.* The lemma is a consequence of Lemma 4.3 and well-known properties of functions of the class $\mathbf{S}$. □

*Proof of Theorem 3.3.* Let us first prove Eq. (3.10). By Lemmas 3.2 and 3.4,

$$\sup_n \{ \|A^{\nabla}_{k,n}\|_0 + \|A^{\nabla}_{k,n}\|_{m,1} \} < \infty$$

Therefore, Lemma 2.3 by Bentkus and Zalesskii[9] imply that, for every $A > 0$ and every distribution function $\Lambda$ with $m+1$ finite moments, the following smoothing inequality holds:

$$\|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} - A^{\nabla}_{k,n}\|_m \leqslant c(\Delta + n^{-A}) \tag{7.1}$$

where

$$\Delta \triangleq \|\mathbf{P}\{\pi(S_n) \leqslant \cdot\} * \Lambda_T - A^{\nabla}_{k,n} * \Lambda_T\|_m$$

and $\Lambda_T(x) \triangleq \Lambda(Tx)$, $T \triangleq n^{-A}$. Furthermore, one can choose $\Lambda$ so that the characteristic function $\lambda(t) = \int \exp\{itx\}\, \Lambda(dx)$ satisfy

$$\lambda \in C^{m+1}, \qquad \lambda(t) = 0 \qquad \text{for} \quad |t| > 1$$

By the choice of $\Lambda$ we have the functions $\mathbf{P}\{\pi(S_n) \leqslant \cdot\} * \Lambda_T$ and $A^{\nabla}_{k,n} * \Lambda_T$ to be infinitely differentiable.

Let $\lambda_T$ be the characteristic function of $\Lambda_T$ (we have $\lambda_T(x) = \lambda(Tx)$) and denote

$$W(s) \triangleq \int_{|t| \leqslant T} \exp(-ist)\{\mathbf{E}\varphi_t(S_n) - \hat{A}^{\nabla}_{k,n}(t)\}\, \lambda_T(t)\, dt,$$

$$V_-(x) \triangleq \int_{(-\infty, x)} W(s)\, ds, \qquad V_+(x) \triangleq \int_{(x, \infty)} W(s)\, ds,$$

$$\Delta_- \triangleq \sup_{x \leqslant 0} \max_{l = 0, m} |x^l V_-(x)|, \qquad \Delta_+ \triangleq \sup_{x \geqslant 0} \max_{l = 0, m} |x^l V_+(x)|$$

Then

$$\Delta \leqslant \Delta_- + \Delta_+ \tag{7.2}$$

and, by Eqs. (7.1) and (7.2), bound Eq. (3.10) follows from the following estimate:

$$\max\{\Delta_-, \Delta_+\} \leqslant c\{I_n + \mathbf{E}\, \|X_{\nabla}\|^2 + \varepsilon^{k+1}\mathbf{E}\, \|X^{\nabla}\|^{k+3} + \varepsilon^{pm-2}\mathbf{E}\, \|X^{\nabla}\|^{pm}\, \mathbf{I}\{\|\varepsilon X\| > 1\}\} \tag{7.3}$$

We shall prove Eq. (7.3) for $\Delta_-$ only since $\Delta_+$ can be estimated similarly.



We have

$$\Delta_- \leqslant \Delta_0 + \sup_{x \leqslant -1} \max_{l = 0, m} |x^l V_-(x)|$$

where

$$\Delta_0 \triangleq \int_{|t| \leqslant T} |\mathbf{E}\varphi_t(S_n) - \hat{A}^{\nabla}_{k,n}(t)|\, dt$$

Let us estimate $\Delta_0$. Denoting $T_0 \triangleq n^{p/2 - \sigma}$ we have

$$\Delta_0 \leqslant \int_{|t| \leqslant T_0} |\mathbf{E}\varphi_t(S_n) - \mathbf{E}\varphi_t(S^{\nabla}_n)|\, dt + \int_{|t| \leqslant T_0} |\mathbf{E}\varphi_t(S^{\nabla}_n) - \hat{A}^{\nabla}_{k,n}(t)|\, dt + I_n + \int_{T_0 \leqslant |t| \leqslant T} |\hat{A}^{\nabla}_{k,n}(t)|\, dt$$

Using Lemma 4.4, bound Eqs. (4.12), and Lemmas 4.1 and 4.2, we get that $\Delta_0$ does not exceed the right-hand side of Eq. (7.3). Therefore it remains to verify that for $x \leqslant -1$ and $l = 0, m$ the quantity $|x^l V_-(x)|$ does not exceed the right-hand side of Eq. (7.3).

Replacing $\exp\{-ist\}$ by $(i/s)^l\, (d/dt)^l \exp\{-ist\}$ and integrating by parts, we get

$$|x^l V_-(x)| = \left| x^l \int_{(-\infty, x)} s^{-l} \int_{|t| \leqslant T} \exp(-ist) \left(\frac{d}{dt}\right)^l \times (\{\mathbf{E}\varphi_t(S_n) - \hat{A}^{\nabla}_{k,n}(t)\}\, \lambda_T(t))\, dt\, ds \right|$$

(let us remind that $\lambda_T(t) = 0$ for $|t| \geqslant T$). Thus

$$|x^l V_-(x)| \leqslant \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5$$

where

$$\Delta_j \triangleq \left| x^l \int_{(-\infty, x)} s^{-l} Q_j(s)\, ds \right|, \qquad j = 1, ..., 5,$$

$$Q_1(s) \triangleq \int_{T_0 \leqslant |t| \leqslant T} \exp\{-ist\}(d/dt)^l\, (\{\mathbf{E}\varphi_t(S_n) - \hat{A}^{\nabla}_{k,n}(t)\}\, \lambda_T(t))\, dt,$$

$$Q_2(s) \triangleq \int_{1 \leqslant |t| \leqslant T_0} \exp\{-ist\}(d/dt)^l\, (\{\mathbf{E}\varphi_t(S_n) - \hat{A}^{\nabla}_{k,n}(t)\}\, \lambda_T(t))\, dt,$$

$$Q_3(s) \triangleq \int_{|t| \leqslant 1} \exp\{-ist\}(d/dt)^l\, (\{\mathbf{E}\varphi_t(S_n) - \hat{A}^{\nabla}_{k,n}(t)\}(\lambda_T(t) - \lambda(t)))\, dt,$$

$$Q_4(s) \triangleq \int_{|t|\leqslant 1} \exp\{-ist\}(d/dt)^l (\{\mathbf{E}\varphi_t(S_n) - \mathbf{E}\varphi_t(S_n^\nabla)\}\ \lambda(t))\ dt,$$

$$Q_5(s) \triangleq \int_{|t|\leqslant 1} \exp\{-ist\}(d/dt)^l (\{\mathbf{E}\varphi_t(S_n^\nabla) - \hat{A}_{k,n}^\nabla(t)\}\ \lambda(t))\ dt$$

We shall prove that

$$\Delta_1 \leqslant c\{I_n + \mathbf{E}\ \|X_\nabla\|^2 + \varepsilon^{k+1}\mathbf{E}\ \|X^\nabla\|^{k+3}\} \tag{7.4}$$

$$\max\{\Delta_2, \Delta_3\} \leqslant c\{\mathbf{E}\ \|X_\nabla\|^2 + \varepsilon^{k+1}\mathbf{E}\ \|X^\nabla\|^{k+3} + \varepsilon^{pm-2}\mathbf{E}\ \|X^\nabla\|^{pm}\ \mathbf{I}\{\|\varepsilon X\| > 1\}\}, \tag{7.5}$$

$$\Delta_4 \leqslant c\varepsilon^{pm-2}\mathbf{E}\ \|X^\nabla\|^{pm}\ \mathbf{I}\{\|\varepsilon X\| > 1\}, \tag{7.6}$$

$$\Delta_5 \leqslant c\{\mathbf{E}\ \|X_\nabla\|^2 + \varepsilon^{k+1}\mathbf{E}\ \|X^\nabla\|^{k+3}\} \tag{7.7}$$

which is what we need to complete the proof of Eq. (3.10).

Let us prove Eq. (7.4). Changing the order of integration and noting that

$$\left| x^l \int_{(-\infty,x)} s^{-l} \exp\{-ist\}\ ds \right| \leqslant 2/|t| \tag{7.8}$$

we get

$$\Delta_1 \leqslant cI_n + c \max_{0\leqslant\mu\leqslant m} \int_{T_0\leqslant|t|\leqslant T} |(d/dt)^\mu\ \hat{A}_{k,n}^\nabla(t)|\ dt$$

An application of Lemmas 4.1 and 4.2 completes the proof of Eq. (7.4).

Let us prove Eq. (7.5) for $\Delta_2$. Inequality Eq. (7.8) implies

$$\Delta_2 \leqslant c \max_{0\leqslant\mu\leqslant m} \int_{1\leqslant|t|\leqslant T_0} |(d/dt)^\mu\ \{\mathbf{E}\varphi_t(S_n) - \mathbf{E}\varphi_t(S_n^\nabla)\}|\ dt + c \max_{0\leqslant\mu\leqslant m} \int_{1\leqslant|t|\leqslant T_0} |(d/dt)^\mu\ \{\mathbf{E}\varphi_t(S_n^\nabla) - \hat{A}_{k,n}^\nabla(t)\}|\ dt$$

and whence, using Lemma 4.4 and bound Eq. (4.12), we get Eq. (7.5) for $\Delta_2$.

Let us show that Eq. (7.5) holds for $\Delta_3$. We have

$$\Delta_3 \leqslant c \max_{0\leqslant\mu\leqslant l} \Delta(\mu)$$



where

$$\Delta(\mu) \triangleq \left| x^l \int_{-\infty}^{x} s^{-l} \int_{|t|\leqslant 1} e^{-ist}(d/dt)^\mu \times \{\mathbf{E}\varphi_t(S_n) - \hat{A}_{k,n}^\nabla(t)\}(d/dt)^{l-\mu}\ \{\lambda_T(t) - \lambda(t)\}\ dt \right|$$

We shall show that each of $\Delta(\mu)$, $\mu = 0,..., l$, does not exceed the right-hand side of Eq. (7.6).

Let us start with $\Delta(l)$. By $|\lambda_T(t) - \lambda(t)| \leqslant |t| \int |x|\ d\Lambda(x) \leqslant c\ |t|$ and bound Eq. (7.8), we have that $\Delta(l)$, does not exceed

$$c \int_{|t|\leqslant 1} |(d/dt)^l\ \{\mathbf{E}\varphi_t(S_n) - \mathbf{E}\varphi_t(S_n^\nabla)\}|\ dt + c \int_{|t|\leqslant 1} |(d/dt)^l\ \{\mathbf{E}\varphi_t(S_n^\nabla) - \hat{A}_{k,n}^\nabla(t)\}|\ dt$$

This bound, Lemma 4.4 and bound Eq. (4.12) conclude the estimation of $\Delta(l)$.

Let us estimate $\Delta(\mu)$, $\mu = 0,..., l-1$. Replacing the quantity $\exp\{-ist\}$ by $(i/s)(d/dt) \exp\{-ist\}$, we get that $\Delta(\mu)$ does not exceed

$$\sup_{s\in\mathbf{R}} \left| \int_{|t|\leqslant 1} (d/dt) \exp\{-ist\}(d/dt)^\mu \times \{\mathbf{E}\varphi_t(S_n) - \hat{A}_{k,n}^\nabla(t)\}(d/dt)^{l-\mu}\ \{\lambda_T(t) - \lambda(t)\}\ dt \right|$$

Integrating by parts and noting that $\int |x|^{m+1}\ d\Lambda(x) < \infty$ (let us remind that $\mu + 1 \leqslant m$), we have

$$\Delta(\mu) \leqslant \sup_{|t|\leqslant 1} \max_{0\leqslant\mu\leqslant m} |(d/dt)^\mu\ \{\mathbf{E}\varphi_t(S_n) - \hat{A}_{k,n}^\nabla(t)\}|$$

and whence, by Lemma 4.4 and bound Eq. (4.12), the desirable bound for $\Delta(\mu)$ follows.

Let us prove Eq. (7.6). Integrating by parts and using the fact that $\lambda$ and its derivatives vanish outside the interval $(-1, 1)$, we obtain

$$\Delta_4 = \left| x^l \int_{(-\infty,x)} \int_{|t|\leqslant 1} \exp\{-ist\}\{\mathbf{E}\varphi_t(S_n) - \mathbf{E}\varphi_t(S_n^\nabla)\}\ \lambda(t)\ dt\ ds \right|$$

By the inversion formula,

$$\Delta_4 = 2\pi\ |x|^l\ |\mathbf{P}\{\pi(S_n) \leqslant \cdot\} * \Lambda(x) - \mathbf{P}\{\pi(S_n^\nabla) \leqslant \cdot\} * \Lambda(x)|$$

Consequently, we have

$$\Delta_4 \leqslant c\,|x|^l \int_{x\in\mathbf{R}} |\mathbf{P}\{\pi(S_n)\leqslant x-y\} - \mathbf{P}\{\pi(S_n^\nabla)\leqslant x-y\}|\; d\Lambda(y)$$

$$\leqslant c \int_{x\in\mathbf{R}} |x|^l\,(1+|x-y|^m)\; d\Lambda(y)\; \varepsilon^{pm-2}\mathbf{E}\;\|X^\nabla\|^{pm}\,\mathbf{I}\{\|\varepsilon X\|>1\} \tag{7.9}$$

where the last inequality is a consequence of the bound (see Remark 7.2 for some information about it)

$$\|\mathbf{P}\{\pi(S_n)\leqslant\cdot\} - \mathbf{P}\{\pi(S_n^\nabla)\leqslant\cdot\}\|_m \leqslant c\varepsilon^{pm-2}\mathbf{E}\;\|X^\nabla\|^{pm}\,\mathbf{I}\{\|\varepsilon X\|>1\} \tag{7.10}$$

Bound Eq. (7.9) clearly completes the proof of Eq. (7.6) because of $\int |x|^m\, d\Lambda(x) < \infty$.

Let us prove Eq. (7.7). First, replace $\exp\{-ist\}$ by $(i/s)(d/dt)\exp\{-ist\}$. Then use integration by parts (since $\Lambda$ has $m+1$ moments we are allowed to do it). An application of bound Eq. (4.12) completes the proof of Eq. (7.7) and of Eq. (3.10) as well.

Let us now show that Eq. (3.11) holds. The proof of this bound is much simpler than that of Eq. (3.10) since in this case we are able to use the Fourier inversion formula for integrable functions. Namely, we get

$$\sup_{x\in\mathbf{R}} (1+|x|^m)\; |(d/dx)^s\,\{\mathbf{P}\{\pi(S_n)\leqslant x\} - A_{k,n}^\nabla(x)\}|$$

$$\leqslant c \max_{0\leqslant\sigma\leqslant s}\; \max_{0\leqslant\mu\leqslant m} \int_{\mathbf{R}} |t|^\sigma\, |(d/dt)^\mu\,\{\mathbf{E}\varphi_t(S_n) - \hat{A}_{k,n}^\nabla(t)\}|\; dt$$

$$\leqslant cJ_n + c \max_{0\leqslant\sigma\leqslant s}\; \max_{0\leqslant\mu\leqslant m} \int_{T_0\leqslant|t|} |t|^\sigma\, |(d/dt)^\mu\, \hat{A}_{k,n}^\nabla(t)|\; dt$$

$$+\, c \max_{0\leqslant\sigma\leqslant s}\; \max_{0\leqslant\mu\leqslant m} \int_{T_0\leqslant|t|} |t|^\sigma\, |(d/dt)^\mu\,\{\mathbf{E}\varphi_t(S_n) - \mathbf{E}\varphi_t(S_n^\nabla)\}|\; dt$$

$$+\, c \max_{0\leqslant\sigma\leqslant s}\; \max_{0\leqslant\mu\leqslant m} \int_{T_0\leqslant|t|} |t|^\sigma\, |(d/dt)^\mu\,\{\mathbf{E}\varphi_t(S_n^\nabla) - \hat{A}_{k,n}^\nabla(t)\}|\; dt$$

An application of Lemmas 4.1, 4.2, and 4.4; and bound by Eq. (4.12) in the right-hand side of the last inequality completes the proof of Eq. (3.11). □

**Remark 7.2.** A more general estimate than Eq. (7.10) has been proved by Sazonov *et al.*,[69] but under the additional assumption $X\in CLT$. One can adapt their proof to our setting. We give the following proof of Eq. (7.10) because it is considerably simplier (but less general) than that



of Sazonov *et al.*[69] Denote $V_j \triangleq \varepsilon(X_1^\nabla + \cdots + X_{j-1}^\nabla + X_{j+1} + \cdots + X_n)$. Then

$$|\mathbf{P}\{\pi(S_n)\geqslant x\} - \mathbf{P}\{\pi(S_n^\nabla)\geqslant x\}|$$

$$= |\mathbf{P}\{\pi(S_n)\,\mathrm{sgn}\,x \geqslant |x|\} - \mathbf{P}\{\pi(S_n^\nabla)\,\mathrm{sgn}\,x\geqslant |x|\}|$$

$$\leqslant \sum_{j=1}^n |\mathbf{P}\{\pi(V_j+\varepsilon X_j)\,\mathrm{sgn}\,x\geqslant|x|\} - \mathbf{P}\{\pi(V_j+\varepsilon X_j^\nabla)\,\mathrm{sgn}\,x\geqslant|x|\}|$$

This bound implies

$$|\mathbf{P}\{\pi(S_n)\leqslant x\} - \mathbf{P}\{\pi(S_n^\nabla)\leqslant x\}| \leqslant \sum_{j=1}^n (C_j + D_j)$$

where

$$C_j \triangleq \mathbf{P}\{\|\varepsilon X_j\|\geqslant 1\}\;\mathbf{P}\{|\pi(V_j)|\geqslant|x|\},$$

$$D_j \triangleq \mathbf{E}\mathbf{I}\{\|\varepsilon X_j\|\geqslant 1\}\;\mathbf{I}\{|\pi(V_j+\varepsilon X_j)|\geqslant|x|\}$$

The sum $\sum C_j$ does not exceed the right-hand side of Eq. (7.9); use bound of Eq. (5.4) to show it. The estimation of $\sum D_j$ is similar; let us note that

$$D_j \leqslant |x|^{-m}\;\mathbf{E}\mathbf{I}\{\|\varepsilon X_j\|\geqslant 1\}\;|\pi(V_j+\varepsilon X_j)|^m$$

and an application of Eq. (5.4) concludes the proof of Eq. (7.10). □

*Proof of Theorem 3.4.* The theorem follows from Theorem 3.3 and Lemma 3.3. □

*Proof of Theorem 3.5.* The theorem is a consequence of Theorem 3.4 and Lemma 3.4. □

*Proof of Theorem 3.2.* The theorem follows from Theorem 3.3 and Lemma 3.4. □

*Proof of Theorem 3.1.* The theorem follows from Theorem 3.2. □

*Proof of Lemma 3.1.* The lemma is a consequence of asymptotic expansions for characteristic functions given in Section 4 and properties of functions of the class **S**. □

## 8. PROOF OF RESULTS OF SECTION 2

In order to get asymptotic expansions for the distribution functions of $\omega$-statistics we are going to apply asymptotic expansions in the CLT in Banach spaces, namely, Theorems 3.5 and 3.1.

As it was noted in Section 2, we may write

$$\mathbf{P}\{\omega_n^p(q) \leqslant x\} = \mathbf{P}\{\|S_n\|^p \leqslant x\}$$

where $\|\cdot\|$ is the norm of the Banach space $\mathbf{B} = L_p(q)$. Therefore it remains to verify that the r.v. $X(t) \triangleq \mathbf{I}\{U \leqslant t\} - t$ and the Brownian bridge $G \triangleq Br$ as the limiting Gaussian variable satisfy the conditions imposed in Section 3. Verification of the moment conditions of Eqs. (2.10) and (2.11) is quite simple. Pisier and Zinn[61] have proved Eq. (2.12) (for supplementary information see, for instance, Norvaiša and Zitikis[54]). The Brownian bridge satisfies condition of Eq. (3.2) (see Lemma 8.4). The only serious problem which is left to solve is to verify the Cramér type conditions, that is, to estimate the characteristic function or, in other words, to estimate $I_n$ and $J_n$. This is the topic of the section.

While estimating the characteristic function, we use a representation (see Lemma 8.1) of $\omega$-statistic as a linear combination of functions of order statistics. Similar approach was used by Anderson and Darling,[3] Martynov,[49] Csörgő and Stachó,[25] Mason,[50] Bentkus and Zitikis,[10] and Zitikis,[88–91] etc.

The proof of Theorem 2.5 is mainly based on ideas used by van Zwet,[92] who investigated characteristic functions of $L$-statistics (see also Bickel *et al.*,[13] Helmers,[41] Götze and van Zwet,[38] etc.). This proof is more general than that used by Zitikis.[88–90]

The proof of Theorem 2.6 comes back to the paper by Bentkus and Zitikis[10]; for further information see Zitikis.[91] Theorem 2.4 is an immediate consequence of Theorem 2.6 and well-known facts of the Fourier analysis.

**Lemma 8.1.** For each $\mu = 1,\ldots, n$ the following representation of $\omega_n^p(q)$ holds:

$$\omega_n^p(q) = \sum_{j=1}^{\mu} Q_{j\mu}(U_{j:\mu}) + A_\mu \tag{8.1}$$

where we have denoted

$$\begin{aligned} Q_{j\mu}(u) \triangleq{}& n^{p/2} \int_u^1 \left\{\left(\frac{j}{n} + H_\mu(x) - x\right)^p \right.\\ &\left. - \left(\frac{j-1}{n} + H_\mu(x) - x\right)^p\right\} v(x)\, q(w)\, dx \\ &- n^{p/2} \int_0^u \left\{\left(\frac{j}{n} + H_\mu(x) - x\right)^p \right.\\ &\left. - \left(\frac{j-1}{n} + H_\mu(x) - x\right)^p\right\} \bar{v}(x)\, q(x)\, dx, \end{aligned}$$



$$\begin{aligned} A_\mu \triangleq{}& n^{p/2} \int_0^1 (H_\mu(x) - x)^p\, v(x)\, q(x)\, dx \\ &+ n^{p/2} \int_0^1 \left(\frac{\mu}{n} + H_\mu(x) - x\right)^p \bar{v}(x)\, q(x)\, dx \end{aligned}$$

and

$$v(x) \triangleq \sum_{l=0}^{p-1} \binom{2p-1}{l} x^l (1-x)^{2p-1-l}$$

$$\bar{v}(x) \triangleq \sum_{l=p}^{2p-1} \binom{2p-1}{l} x^l (1-x)^{2p-1-l}$$

$$H_\mu(x) \triangleq \frac{1}{n} \sum_{l=\mu+1}^{n} \mathbf{I}\{U_j \leqslant x\}$$

*Proof.* Denote $U_{0:\mu} \triangleq 0$ and $U_{\mu+1:\mu} \triangleq 1$. Then

$$\omega_n^p = n^{p/2} \sum_{j=1}^{\mu} \int_{(U_{j-1:\mu},\, U_{j:\mu})} \left(\frac{j-1}{n} + H_\mu(x) - x\right)^p q(x)\, dx$$

To complete the proof, write

$$q(x) = q(x)(x + 1 - x)^{2p-1} = q(x) \sum_{l=0}^{2p-1} \binom{2p-1}{l} x^l (1-x)^{2p-1-l}$$

and then replace the integral $\int_{(U_{j-1:\mu},\, U_{j:\mu})}$ by $\int_{(0,\, U_{j:\mu})} - \int_{(0,\, U_{j-1:\mu})}$ if $l \geqslant p$, and by $\int_{(U_{j-1:\mu},\, 1)} - \int_{(U_{j:\mu},\, 1)}$ if $l \leqslant p-1$. □

**Lemma 8.2.** For all $s \geqslant 0$

$$\mathbf{E}\{\omega_n^p(q)\}^s < \infty, \qquad \forall n \in \mathbf{N} \Leftrightarrow M(ps) < \infty$$

*Proof.* The implication $\Rightarrow$ is obvious. The proof of the reverse implication is based on the representation of Eq. (8.1) with $\mu = n$. □

*Proof of Theorem 2.5.* By Lemma 8.2, $\mathbf{E}\{\omega_n^p(q)\}^s < \infty$, and we have

$$|(d/dt)^s\, \mathbf{E} \exp\{it\omega_n^p(q)\}| = |\mathbf{E}\{\omega_n^p(q)\}^s \exp\{it\omega_n^p(q)\}|$$

Since $F_n(t) - t$ is the sum of r.v. $n^{-1}Z_1(t),\ldots, n^{-1}Z_n(t)$, where $Z_j(t) = \mathbf{I}\{U_j \leqslant t\} - t$, we may represent $\{\omega_n^p(q)\}^s$ as a sum having at most $n^{ps}$ summands so that each summand depends on at most $ps$ of random variables $Z_1(t),\ldots, Z_n(t)$. Rearranging i.i.d. r.v. $U_1,\ldots, U_n$ and using the fact that

$M(ps)<\infty$, we shall get the theorem proved if we show that there exists a constant $c$ such that

$$\mathbf{P}\{\sup_{|t|\geqslant n^{p/2-1+\delta}} |\mathbf{E}_{U_1,\ldots,U_\mu} \exp\{it\omega_n^p(q)\}| \leqslant cn^{-A}\} = 1 \tag{8.2}$$

with $\mu \triangleq n-ps$.

Let us use the representation of Lemma 8.1. Since $A_\mu$ is finite almost surely,

$$|\mathbf{E}_{U_1,\ldots,U_\mu} \exp\{it\omega_n^p(q)\}| = \left|\mathbf{E}_{U_1,\ldots,U_\mu} \exp\left\{it \sum_{j=1}^{\mu} Q_j(U_{j:\mu})\right\}\right|$$

where we have denoted $Q_j \triangleq Q_{j\mu}$ for notational simplicity. Henceforth, let us fix $U_{\mu+1},\ldots, U_n$ and let $U^*_{\mu+1}\leqslant \cdots \leqslant U^*_n$ be the ordered values of them. Without loss of generality we may assume $0<U^*_{\mu+1}<\cdots<U^*_n<1$. The points $U^*_{\mu+1}<\cdots<U^*_n$ divide the interval $(0, 1)$ into $ps+1$ subintervals. Therefore, the same points divide the interval $(a, b)\subset(0, 1)$ from the condition of the theorem into at most $ps+1$ subintervals. Among these subintervals there is at least one, say $J=(\alpha, \beta)$, such that $\beta-\alpha\geqslant(b-a)/(ps+1)$. Let us note that by the assumption of the theorem there exist numbers $R>0$ and $L$ such that

$$q(x)\geqslant R, \qquad |q'(x)|\leqslant L, \qquad \forall x\in(\alpha, \beta) \tag{8.3}$$

Without loss of generality we may assume that $R-10(b-a)L\geqslant R/2$. Indeed, we may replace $(a, b)$ by any other nonempty subinterval.

The remaining part of the proof is similar to the proof of Theorem 4.1 from van Zwet.[92] Therefore we shall only sketch it. Denote $N\triangleq[(\beta-\alpha)\mu/(3r)]$, where a number $r\in\mathbf{N}$ is to be chosen later. Without loss of generality we assume that $n$ is sufficiently large, $n\geqslant c_1$. Clearly, we may choose $c_1$ so that $N\geqslant 1$ and there exist numbers $k_1,\ldots, k_{r+1}\in\mathbf{N}$ such that

$$\alpha+(\beta-\alpha)/4\leqslant k_1/\mu, \quad k_{\nu+1}-k_\nu=N+1, \quad \nu=1,\ldots, r, \quad k_{r+1}/\mu\leqslant\beta-(\beta-\alpha)/4$$

Repeating the proof of Eq. (4.10) from van Zwet[92] we get that Eq. (8.2) follows from

$$\mathbf{P}\left\{\sup_{|t|\geqslant n^{p/2-1+\delta}} \sup_{*} \prod_{\nu=1}^{r} \times\left|\mathbf{E}\exp\left\{it\sum_{j=1}^{N} Q_{j+k_\nu}(s_\nu+(s_{\nu+1}-s_\nu)\, U_{j:N})\right\}\right| \leqslant cn^{-A}\right\}=1 \tag{8.4}$$



where $\sup_*$ is taken over all $s_1,\ldots, s_{r+1}\in(0, 1)$ such that

$$\alpha<s_1, \qquad s_{\nu+1}-s_\nu>(\beta-\alpha)/(4r), \qquad \nu=1,\ldots, r, \qquad s_{r+1}<\beta$$

and $U_{1:N}\leqslant\cdots\leqslant U_{N:N}$ are the order statistics of $U_1,\ldots, U_N$. Furthermore, choosing $r$ sufficiently large, we reduce Eq. (8.4) to (for more details see the proof of Lemma 4.2 in van Zwet[92])

$$\mathbf{P}\{\sup_{|t|\geqslant n^{p/2-1+\delta}} \sup_{*} \sup_{**} |\mathbf{E}\exp\{ith(U_{M:N})\}|\leqslant cn^{-\delta/2}\}=1 \tag{8.5}$$

where $\sup_{**}$ is taken over all $\nu=1,\ldots, r$ and $v_1,\ldots, v_{M-1}$, $w_{M+1},\ldots, w_N\in(0, 1)$ such that

$$\sum_{j=1}^{M-1} v_j+1+\sum_{j=M+1}^{N}(1-w_j)\geqslant N/4 \tag{8.6}$$

we have denoted $M\triangleq[N/2]$ and

$$h(u)\triangleq\sum_{j=1}^{N} Q_{j+k_\nu}(a_j+b_ju)$$

$$a_j\triangleq\begin{cases} s_\nu, & \text{for } j=1,\ldots, M,\\ s_\nu+(s_{\nu+1}-s_\nu), & \text{for } j=M+1,\ldots, N,\end{cases}$$

$$b_j\triangleq\begin{cases}(s_{\nu+1}-s_\nu)\, v_j, & \text{for } j=1,\ldots, M-1,\\ s_{\nu+1}-s_\nu, & \text{for } j=M,\\ (s_{\nu+1}-s_\nu)(1-w_j), & \text{for } j=M+1,\ldots, N\end{cases}$$

Let us examine properties of the function $h$. Since $a_j+b_ju\in(\alpha, \beta)$ for $u\in(0, 1)$,

$$h'(u)=\sum_{j=1}^{N} n^{p/2}b_jq(a_j+b_ju)\, f(u)$$

where

$$f(u)\triangleq(a_j+b_ju-h_\mu-(j+k_\nu-1)/n)^p-(a_j+b_ju-h_\mu-(j+k_\nu)/n)^p$$

Let us show that there exists a positive constant $c>0$ such that

$$h''(u)\geqslant cn^{2-p/2} \qquad \forall u\in(0, 1) \tag{8.7}$$

Indeed,

$$h''(u)\geqslant-\sum_{j=1}^{N} n^{p/2}b_j^2\, |q'(a_j+b_ju)|\, |f(u)|+\sum_{j=1}^{N} n^{p/2}b_jq(a_j+b_ju)\, f'(u)$$

Therefore

$$h''(u) \geqslant -A + \Gamma$$

where

$$A \triangleq \sum_{j=1}^{N} n^{p/2} b_j^2 \, |q'(a_j + b_j u)| \, A_j, \qquad A_j \triangleq |f(u)| - 10(\beta - \alpha) f'(u)/b_j,$$

$$\Gamma \triangleq \sum_{j=1}^{N} n^{p/2} b_j \{q(a_j + b_j u) - 10(\beta - \alpha) |q'(a_j + b_j u)|\} f'(u)$$

Let us verify that $A_j \leqslant 0$. Denote $x \triangleq a_j + b_j u - h_\mu - (j + k_\nu - 1/2)/n$, $d \triangleq 1/(2n)$. Then

$$A_j = |(x+d)^p - (x-d)^p| - 10p(\beta - \alpha)\{(x+d)^{p-1} - (x-d)^{p-1}\}$$
$$\leqslant 2p \sum_{l=1}^{p/2} \binom{p-1}{2l-2} |x|^{2l-2} d^{p-2l+1} \{|x|/(2l-1) - 10(\beta - \alpha)\}$$

Therefore, $A_j \leqslant 0$ because of $|x|/(2l-1) \leqslant |x| \leqslant 5(\beta - \alpha)$. Thus $h''(u) \geqslant \Gamma$ and using the bound

$$f'(u) \geqslant 2pb_j \sum_{l=0}^{p/2-1} \binom{p-1}{2l} x^{2l} d^{p-2l-1} \geqslant 2pb_j d^{p-1}$$

we get

$$h''(u) \geqslant c \sum_{j=1}^{N} n^{p/2} b_j^2 / n^{p-1}$$

This bound coupled with Eq. (8.6) completes the proof of Eq. (8.7).

It follows from Eq. (8.7) that there exists a unique point $u_0 \in [0, 1]$ such that $|h'(u)| \geqslant |h'(u_0)|$ for all $u \in [0, 1]$. Therefore

$$|h'(u)| \geqslant |h'(u)| - |h'(u_0)| = |h'(u) - h'(u_0)| = |h''(\theta)(u - u_0)|$$

This inequality and Eq. (8.7) ensure the existence of a positive constant $c > 0$ such that

$$|h'(u)| \geqslant c\lambda n^{2-p/2}, \qquad \forall u \in [0, 1] \backslash [u_0 - \lambda, u_0 + \lambda], \qquad \forall \lambda > 0 \quad (8.8)$$

Now we shall prove Eq. (8.5) (compare with the proof of Lemma 4.1 from van Zwet.[92]) Let $b_{M:N}$ be the density of $U_{M:N}$. Then for all $\lambda > 0$ we get

$$|\mathbf{E} \exp\{ith(U_{M:N})\}|$$
$$\leqslant 2\lambda \sup_{u \in (0,1)} b_{M:N}(u) + \left| \int_{(0,1) \backslash [u_0 - \lambda, u_0 + \lambda]} \exp\{ith(u)\} \, b_{M:N}(u) \, du \right|$$



Integrating by parts, using Eq. (8.3) and the bound $\sup_{u \in (0,1)} b_{M:N}(u) \leqslant cn^{1/2}$ (see bound Eq. (4.1) in van Zwet[92]), we get

$$|\mathbf{E} \exp\{ith(U_{M:N})\}| \leqslant c\lambda n^{1/2} + cn^{p/2-3/2}/(|t| \, \lambda) \leqslant c\{n^{p/2-1}/|t|\}^{1/2}$$

choose $\lambda = n^{p/4-1} |t|^{-1/2}$ to get the last inequality above. This completes the proof of the theorem. □

*Proof of Theorem 2.6.* By Lemma 8.1 (with $\mu \triangleq n$ and $Q_j \triangleq Q_{jn}$) and Lemma 8.2, in order to prove the theorem it is sufficient to show

$$\left| \mathbf{E} Q_1^{k_1}(U_{1:n}) \cdots Q_n^{k_n}(U_{n:n}) \exp\left\{ it \sum_{j=1}^{n} Q_j(U_{1:n}) \right\} \right| \leqslant cn^{cn} |t|^{s-n/p} \quad (8.9)$$

uniformly over all $k_0, ..., k_n \in \mathbf{N}_0$ such that $k_0 + \cdots + k_n = s$.

We are going to prove Eq. (8.9) using recursion. But first let us note that without loss of generality we may assume $|t| \geqslant (2n)^p$ since otherwise the result of the theorem follows from the obvious estimate $\mathbf{E}\{\omega_n^p(q)\}^s \leqslant cn^{cn}$ (which is a consequence, for example, of Lemma 8.1).

For $z \in [0, 1]$ denote

$$\varphi_n(z) \triangleq 1, \qquad \varphi_{j-1}(z) \triangleq \int_{(z,1)} \Phi_j(x) \, dx, \qquad j = 1, ..., n$$

where

$$\Phi_j(x) \triangleq \varphi_j(x) \, Q_j^{k_j}(x) \exp\{itQ_j(x)\}$$

It is easy to see that $|\varphi_0(0)|$ is equal to the left-hand side of Eq. (8.9). Therefore Eq. (8.9) follows from

$$\|\varphi_0\|_\infty \leqslant cn^{cn} |t|^{s-n/p} \quad (8.10)$$

where $\|\varphi\|_\infty \triangleq \sup_{0 \leqslant z \leqslant 1} |\varphi(z)|$. Obviously, Eq. (8.10) follows (when $m = 0$) from

$$\|\varphi_m\|_\infty \leqslant c^{(n-m)/2} n^{p(k_{m+1} + \cdots + k_n) + 3n - 3m} |t|^{k_{m+1} + \cdots + k_n - (n-m)/p} \quad (8.11)$$

$m = 0, ..., n$. Let us show that Eq. (8.11) is a consequence of

$$\|\varphi_{j-1}\|_\infty \leqslant c \sum_{m=j}^{n} \|\varphi_m\|_\infty \, n^{p(k_j + \cdots + k_m) + 3} |t|^{k_j + \cdots + k_m - (m-j+1)/p} \quad (8.12)$$

for all $j = n, ..., 1$. Indeed, if $m = n$, then Eq. (8.11) reduces to the bound $\|\varphi_n\|_\infty \leqslant 1$ which is fulfilled by the definition of $\varphi_n$. Assume now that

Eq. (8.11) is fulfilled for $m=j,\dots,n$. Then in order to prove Eq. (8.11) with $m=j-1$ use Eq. (8.12).

So it remains to prove Eq. (8.12) for all $j=1,\dots,n$.

Let $x_0,\dots,x_{\tau+1}\in[0,1]$ be taken from the definition of the class $ACM_p(0,1)$. Denote $\lambda\triangleq|t|^{-1/p}$ and let $A(\lambda)\triangleq\bigcup_{a\in A}\{x\in[0,1]: |x-a|<\lambda\}$ be the $\lambda$-neighborhood of the set

$$A\triangleq\{x_0,\dots,x_{\tau+1}\}\cup\{(k+l-1)/(2n): 1\leqslant k\leqslant l\leqslant n\}$$

We have

$$|\varphi_{j-1}(z)|\leqslant\Delta_1+\Delta_2 \tag{8.13}$$

where

$$\Delta_1\triangleq\int_{(z,1)\cap A(\lambda)}\Phi_j(x)\,dx,\qquad \Delta_2\triangleq\int_{(z,1)\setminus A(\lambda)}\Phi_j(x)\,dx$$

Let us show that

$$\|\Delta_1\|_\infty\leqslant c\,\|\varphi_j\|_\infty\,n^{pk_j+2}\,|t|^{k_j-1/p} \tag{8.14}$$

If $s=0$, then Eq. (8.14) follows immediately from $\operatorname{mes}\{A(\lambda)\}\leqslant 2\lambda(n^2+\tau+1)$. Therefore let $s\geqslant 1$. Using the Hölder inequality we get

$$|\Delta_1|\leqslant\|\varphi_j\|_\infty\left\{\int_{(z,1)}|Q_j(x)|^s\,dx\right\}\{\operatorname{mes}\{A(\lambda)\}\}^{1-k_j/s}$$

and Eq. (8.14) follows easily from the definition of $Q_j$, the fact that $M(ps)<\infty$ and $\operatorname{mes}\{A(\lambda)\}\leqslant 2\lambda(n^2+\tau+1)$.

In view of Eq. (8.14) it remains to show that $|\Delta_2|$ does not exceed the right-hand side of Eq. (8.12). Clearly, there are at most $cn^2$ ($c$ is a positive constant) disjoint subintervals $I_1,\dots,I_w$, $w\leqslant cn^2$ such that $(z,1)\setminus A(\lambda)=I_1\cup\cdots\cup I_w$. Therefore, in order to get the desirable estimate of $\Delta_2$, it is sufficient to show that for all $I\in I_1,\dots,I_w$ the quantity

$$\Delta_3\triangleq\left|\int_I\Phi_j(x)\,dx\right|$$

does not exceed the right-hand side of Eq. (8.12) divided by $n^2$.

For $1\leqslant j\leqslant l\leqslant n$, denote

$$\Psi(j,l)\triangleq\int_I\prod_{m=j}^{l}Q_j^{k_j}(x)\prod_{m=j}^{l-1}\{Q_j'(x)+\cdots+Q_m'(x)\}^{-1}\times\exp\left\{it\sum_{m=j}^{l}Q_m(x)\right\}\varphi_l(x)\,dx$$

(Notice that $\Delta_3=|\Psi(j,j)|$.)



Let us show that

$$\left|\sum_{m=j}^{l}Q_m'(x)\right|\geqslant n^{p/2-1}\lambda^{p-1}R,\qquad\forall x\in I \tag{8.15}$$

where $R\triangleq\inf_{0\leqslant x\leqslant 1}q(x)$. (In particular, bound Eq. (8.15) shows that the quantity $\Psi(j,l)$ is well-defined).

We have

$$\sum_{m=j}^{l}Q_m'(x)=n^{p/2}f(x)\,q(x),\qquad f(x)\triangleq(x-(j-1)/n)^p-(x-l/n)^p \tag{8.16}$$

It is easy to verify that $f$ is strictly monotone on $(0,1)$ and that the point $z_0\triangleq(l+j-1)/(2n)$ is unique satisfying $f(z_0)=0$. Terefore

$$\inf_{x\in I}|f(x)|\geqslant|f(z_0-\lambda)|\geqslant\lambda^{p-1}/n \tag{8.17}$$

Now Eqs. (8.16) and (8.17) and $q(x)\geqslant R$ imply Eq. (8.15).

Let us estimate $|\Psi(j,l)|$ from the previous example. Replacing $\exp\{it\sum_{m=j}^{l}Q_m(x)\}$ in the definition of $\Psi(j,l)$ by

$$\left\{it\sum_{m=j}^{l}Q_m(x)\right\}^{-1}(d/dx)\exp\left\{it\sum_{m=j}^{l}Q_m(x)\right\}$$

and then integrating by parts, we get

$$|\Psi(j,l)|\leqslant c(\Delta_4+\Delta_5)\,\|\varphi\|_\infty/|t|+|\Psi(j,l+1)|/|t| \tag{8.18}$$

where $\Psi(j,n+1)\triangleq 0$,

$$\Delta_4\triangleq\sup_{x\in I}\prod_{m=j}^{l}|f_m(x)/g_m(x)|,\qquad \Delta_5\triangleq\int_{x\in I}\left|(d/dx)\prod_{m=j}^{l}f_m(x)/g_m(x)\right|,$$
$$f_m(x)\triangleq Q_m^{k_m}(x),\qquad g_m(x)\triangleq Q_j'(x)+\cdots+Q_m'(x)$$

We shall show that

$$\max\{\Delta_4,\Delta_5\}\leqslant\Delta_6\triangleq cn^{p(k_j+\cdots+k_l)/2}/\lambda^{p(k_j+\cdots+k_l)+(p-1)(l-j+1)} \tag{8.19}$$

Bounds of Eqs. (8.18) and (8.19), and the equality $|\Psi(j,j)|=\Delta_3$ taken together complete the estimation of $\Delta_3$ and the proof of the theorem as well.

So let us prove Eq. (8.19). The bound $\Delta_4 \leqslant \Delta_6$ is a consequence of Eq. (8.15) and the following estimate

$$\sup_{z\in I} |Q_j(z)| \leqslant cn^{p/2} \left\{\int_{(\lambda,1)} (1-x)^p\, q(x)\, dx + \int_{(0,1-\lambda)} \lambda^p q(x)\, dx\right\} \leqslant cn^{p/2}\lambda^{-p} \tag{8.20}$$

The proof of $\Delta_5 \leqslant \Delta_6$ is more complicated. We have

$$\Delta_5 \leqslant \sum_{v=j}^{l} (a_v + b_v) \tag{8.21}$$

where

$$a_v \triangleq \int_{x\in I} \left| f'_v \prod_{j\leqslant m\leqslant l, m\neq v}^{l} f_m \Big/ \prod_{m=j}^{l} g_m \right|,$$

$$b_v \triangleq \int_{x\in I} \left| g'_v \prod_{j\leqslant m\leqslant l, m\neq v}^{l} f_m \Big/ \prod_{m=j}^{l} g_m \right|$$

(if $k_v = 0$, then $a_v = 0$).

Let us estimate $a_v$. Since $a_v = 0$ if $k_v = 0$, assume $k_v \geqslant 1$. We have

$$f'_v = k_v Q_v^{k_v-1} Q'_v = k_v Q_v^{k_v-1}\{g_v - g_{v-1}\}, \qquad g_{j-1} \triangleq 0$$

Therefore

$$a_v \leqslant k_v \int_I \left| Q_v^{k_v-1} \prod_{j\leqslant m\leqslant l, m\neq v}^{l} (f_m/g_m) \right| + k_v \int_I \left| Q_v^{k_v-1} g_{v-1} \prod_{j\leqslant m\leqslant l, m\neq v}^{l} f_m \Big/ \prod_{m=j}^{l} g_m \right|$$

Bounds of Eqs. (8.15) and (8.20) imply

$$a_v \leqslant \Delta_6/n \tag{8.22}$$

and

$$b_v \leqslant c(n^{p/2}\lambda^{-p})^{k_j + \cdots + k_l}\, n^{(j-l)(p/2-1)}\lambda^{(j-l)(p-1)} \int_I |g'_v/g_v| \tag{8.23}$$

We have

$$g_v(x) = n^{p/2} f(x)\, q(x), \qquad f(x) = (x-(j-1)/n)^p - (x - v/n)^p$$



Therefore

$$\int_I |g'_v/g_v| \leqslant n^{-p/2} \int_I |f'/(f^2 g)| + n^{-p/2} \int_I |g'/(g^2 f)| \leqslant n^{-p/2} R^{-1} \int_I |f'/f^2| + n^{1-p/2} \int_I |g'/g^2|$$

because of $q \geqslant R$ and $f \geqslant \lambda^{p-1}/n$ (see Eq. (8.17)). Furthermore, $\int_I |g'/g^2| \leqslant 2/R$ since $q$ is monotone on $I$. The function $f$ is monotone on $I$ as well and $f \geqslant \lambda^{p-1}/n$. Therefore $\int_I |f'/f^2| \leqslant 2(\lambda^{p-1}/n)^{-1}$. These remarks show that $\int_I |g'_v/g_v| \leqslant cn^{1-p/2}\lambda^{1-p}$ and therefore Eq. (8.23) imply $b_v \leqslant \Delta_6/n$. This bound, together with Eqs. (8.21) and (8.22), proves Eq. (8.19) for $\Delta_5$. This completes the proof of the theorem. □

We conclude the section with some remarks concerning condition of Eq. (3.2). Let $b \mapsto \pi_p(b,..., b)$, $b \in \mathbf{B}$, be a continuous symmetric $p$-linear form. Let $G, G_1,..., G_p \in \mathbf{B}$ be centered i.i.d. Gaussian r.v. Denote

$$\varphi_G(t) \triangleq \mathbf{E} \exp\{it\pi_p(G_1,..., G_p)\}$$

As usually we define the covariance operator $C\colon \mathbf{B}' \to \mathbf{B}$ of a r.v. $G \in \mathbf{B}$ via

$$(Cf, g) \triangleq \mathbf{E} f(G)\, g(G), \qquad (f, x) \triangleq f(x), \qquad f, g \in \mathbf{B}', \qquad x \in \mathbf{B}$$

The following lemma allows us while estimating $\varphi_G(t)$ to replace $G_1,..., G_p$ by (almost) arbitrary other Gaussian r.v.

**Lemma 8.3.** Let $e_j \in \mathbf{B}'$, $j = 1, 2,...$, and

$$0 < c^2 \triangleq \sum_{j=1}^{\infty} \mathbf{E} e_j^2(G) = \sum_{j=1}^{\infty} (Ce_j, e_j) < \infty$$

where $C$ is the covariance operator of $G$. Then

$$\varphi_G(t) \leqslant \varphi_Z(c^{-p}t) \triangleq \mathbf{E} \exp\{ic^{-p}t\pi_p(Z_1,..., Z_p)\}$$

where $Z, Z_1,..., Z_p$ are i.i.d. Gaussian r.v. such that $\mathscr{L}(Z) = \mathscr{L}(\sum_{j=1}^{\infty} \eta_j Ce_j)$ and $\eta_1, \eta_2,... \in \mathbf{R}$ are i.i.d. standard normal r.v.

*Proof.* The idea of the proof is taken from Yurinskii.[84] Denote $f(b) \triangleq \pi_p(b, G_2,..., G_p)$, $b \in \mathbf{B}$. Then

$$\varphi_G(t) = \mathbf{E} \exp\{-t^2(Cf, f)/2\}$$

If $D$ denotes the covariance of $Z$, then

$$(Df, f) = \mathbf{E} f^2(Z) = \sum_{j=1}^{\infty} f^2(Ce_j) = \sum_{j=1}^{\infty} (Ce_j, f)^2 \leqslant c^2(Cf, f)$$

since $(C\cdot, \cdot)$ is a (semi)inner product and therefore $(Cg, h)^2 \leqslant (Cg, g)(Ch, h)$. Thus

$$\varphi_G(t) \leqslant \mathbf{E} \exp\{-t^2(Df, f)/(2c^2)\} = \mathbf{E} \exp\{it\pi_p(Z_1, G_2, \ldots, G_p)/c\}$$

Repeating this procedure we complete the proof of the lemma. □

As an example of application we have the following

**Corollary 8.1.** If a Gaussian r.v. $G \in \mathbf{R}^m$ is not concentrated in a proper subspace of the $m$-dimensional space $\mathbf{R}^m$ and

$$\pi_p(x, \ldots, x) = x_1^p + \cdots + x_m^p, \qquad x = (x_1, \ldots, x_m) \in \mathbf{R}^m$$

then $\varphi_G(t) \leqslant c\, |t|^{-m/p}$.

*Proof.* According to Lemma 8.3, we may replace $G_i$ by $Z_i$ with independent identically distributed coordinates. Now simple calculations complete the proof. □

**Lemma 8.4.** The Brownian bridge $Br$, as the Gaussian r.v. $G$, and the multilinear form

$$\pi_p(b, \ldots, b) \triangleq \int_{(0,1)} b^p(t)\, q(t)\, dt$$

satisfy condition of Eq. (3.2) provided $p \in \mathbf{N}$ and the weight function is not identically equal to zero, that is $\operatorname{mes}\{t \in (0, 1): q(t) \neq 0\} > 0$.

*Proof.* Clearly, for each $m \in \mathbf{N}$ there exist functions $f_1, \ldots, f_m \in C^\infty(0, 1)$ with disjoint compact (in the open interval $(0, 1)$) supports such that

$$a_j \triangleq \int_{(0,1)} f_j^p(t)\, q(t)\, dt \neq 0, \qquad \forall j = 1, \ldots, m$$

We are going to apply Lemma 8.3. Let us choose $e_j \triangleq -f_j''$ for $j = 1, \ldots, m$, and $e_j \triangleq 0$ for $j > m$. It is easy to verify that

$$Ce_j = f_j \quad \text{for} \quad j = 1, \ldots, m, \quad \text{and} \quad c^2 = \sum_{j=1}^m \int_{(0,1)} f_j'^2(t)\, dt < \infty$$

Therefore

$$\varphi_G(t) \leqslant \varphi_Z(c^{-p}t) = \mathbf{E} \exp\left\{itc^{-p} \sum_{j=1}^m a_j \eta_{1j} \cdots \eta_{pj}\right\}$$

where $\eta_{sj} \in \mathbf{R}$, $s = 1, \ldots, p$, $j = 1, \ldots, m$ are i.i.d. standard normal r.v. Therefore $\varphi_G(t) \leqslant c\, |t|^{-m/p}$ and Eq. (3.2) follows since $m \in \mathbf{N}$ is arbitrary. □



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# Harnack Inequalities and Difference Estimates for Random Walks with Infinite Range

**Gregory F. Lawler**[1] **and Thomas W. Polaski**[2]

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Difference estimates and Harnack inequalities for mean zero, finite variance random walks with infinite range are considered. An example is given to show that such estimates and inequalities do not hold for all mean zero, finite variance random walks. Conditions are then given under which such results can be proved.

**KEY WORDS:** Random walk; harmonic functions; difference equations; Harnack inequality.

## 1. INTRODUCTION

Let $S(n)$ be a random walk as in Ref. 9 taking values in $Z^d$ with transition probability $p(x, y) = p(y - x)$. Associated with the random walk is the corresponding Laplacian $\Delta = \Delta_p$ acting on functions on $Z^d$ by

$$\Delta f(x) = \sum_{y \in Z^d} p(x, y)(f(y) - f(x))$$

A function $f$ is called *harmonic (with respect to p)* on the set $A \subset Z^d$ if $\Delta f(x) = 0$ for all $x \in A$. It is well known that if $A$ is finite and $F: A^c \to Z^d$, then the unique function $f$ which is harmonic on $A$ and equals $F$ on $A^c$ is given for $x \in A$ by

$$f(x) = E^x[F(S(\tau))]$$

[1] Department of Mathematics, Duke University, Box 90320, Durham, North Carolina 27708-0320. Research supported by the National Science Foundation.
[2] Department of Mathematics, Winthrop University, Rock Hill, South Carolina 29733.