AN APPLICATION OF INFINITE DIMENSIONAL HOLOMORPHY TO THE GEOMETRY OF BANACH SPACES

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§1. Introduction

The purpose of these notes is to present the proof of the following result of W. Kaup and H. Upmeier (see also [BKU]):

1.1 Theorem [KU]: *Two complex Banach spaces whose open unit balls are biholomorphically equivalent are linearly isometric.*

The notes are written for non-specialists in infinite dimensional holomorphy who are interested in the geometry of Banach spaces. Along the way we survey some results concerning the Lie algebra $aut(D)$ of all complete holomorphic vector fields on the open unit ball D of the complex Banach space E, and its associated real Banach Lie group $Aut(D)$ consisting of all biholomorphic automorphisms of D. These results are due mainly to W. Kaup, H. Upmeier and J. P. Vigue, and are the extension to infinite dimensions of the works of E. Cartan and H. Cartan on bounded symmetric domains and of the biholomorphic automorphisms of bounded domains in the context of C^n. Naturally, this survey contains very few proofs. Then we give the full details of the fact that *the orbit of the origin under the group $Aut(D)$ is preserved under biholomorphic maps*. This is the main point in the proof of Theorem 1.1. From this the proof of Theorem 1.1 is reduced to showing that *biholomorphic maps which fix the origin are necessarily linear,* and this follows easily from the versions of Schwarz's lemma and Cartan's linearity theorem. One corollary of the theory is that *there exists a closed complex subspace E_s of E so that $E_s \cap D = (Aut(D))(0)$*. Thus $Aut(D)$ acts transitively on the open unit ball $E_s \cap D$ of E_s, and so $E_s \cap D$ is a bounded symmetric domain. The space E_s is called the *symmetric part of E* and its open unit ball $D_s = E_s \cap D$ is *the symmetric part of D*. In the last section we review some examples of spaces whose symmetric part is known, in particular the *JB^*-triples* ($E_s = E$) and some spaces with an the *linear biholomorphic property* ($E_s = \{0\}$). We explain the role of the *contractive projection principle* in studying the symmetric parts of Banach spaces. Thullen's classification of Reinhardt domains in C^2 is discussed and the extension to infinite dimensions is described. It follows that the symmetric part of a Banach space with 1-unconditional basis is a band subspace which is naturally isometric to the c_0-sum of Hilbert spaces. It follows that rearrangement invariant Banach function spaces and unitary ideals have LBP provided they are not Hilbert spaces or C^*-algebras. Finally, we present an example of J.P. Vigue showing that *E_s need not be complemented in E*.

E. Cartan's classification of bounded symmetric domains in C^n is given in [CE]. For the works of H. Cartan on groups of biholomorphic automorphisms of domains in C^n see [CH1] and [CH2]. L. Harris (see [H1] and [H3]) extended the study of bounded symmetric domains and their connections to C^* algebras in the context of infinite dimensional spaces. He also extended Schwarz's lemma to the infinite dimensional setting, see [H2]. W. Kaup and H. Upmeier, and independently J. P. Vigue extended the works of E. Cartan, H. Cartan and L. Harris even further. The main point in their works is the connection between very general domains in complex Banach spaces and certain algebraic structure on the space, called the Jordan triple product. This triple product generalizes the binary product in C^*-algebras and Jordan algebras and is more suitable in many problems. For Kaup's works see [K1], [K2], [K3] and [K4]. The references to Upmeier's works are [U1] and [U2]. The book [U3] and the memoir [U4] survey the general perspective of the theory. Among Vigue's work we list first his thesis [V1] and the papers [V2], [V3], [V4] and [V5]. See also [IV].

The monograph [L] surveys the connection between bounded symmetric domains and Jordan triple systems in finite dimensional spaces. A detailed, elementary exposition of the theory of groups of biholomorphic automorphisms of bounded domains in Banach spaces and the connection to Jordan triple systems is found in [IS].

With the exception of Corollary 5.14, the results presented in the last section are not original. They are due besides Kaup, Upmeier and Vigue also to T.Barton, R. Braun, S. Dineen, Y. Friedman, B. Russo, L. L. Stacho and R. M. Timoney.

All these results deal with *complex* Banach spaces. H. Rosenthal (see [R1] and [R2]) studies the Lie algebra of linear skew Hermitian bounded operators on *real* Banach spaces.

The monographs [D1] and [FV] are general references to infinite dimensional holomorphy, and [LT1] and [LT2] to the geometry of Banach spaces.

Sections 2 and 4 below contains the details of the proof of Theorem 1.1. Section 3 contains the Lie-theoretic background. Section 5 can be considered as an appendix and it contains the information on the symmetric parts of certain families of Banach spaces.

We pass to notation and background material. In what follows E and F are complex Banach spaces. Let $n \geq 0$ and let

$$\tilde{f} : \underbrace{E \times E \times \cdots \times E}_{n \text{ factors}} \to F$$

be a continuous, symmetric, multilinear map. The associated *homogeneous polynomial of degree n* is the restriction of $\tilde{f}$ to the "diagonal", i.e. the map $f : E \to F$ defined by

$$f(z) = \bar{f}(\underbrace{z, z, \ldots, z}_{n \text{ terms}}), \quad z \in E.$$

For $n = 0$, $\bar{f}$ and f are interpreted as constant maps.

Let U be a domain in E (i.e. an open, connected subset). A function $f : U \to F$ is said to be *holomorphic* (or, *analytic)* if for each $z_0 \in U$ there is an open ball $B \subset U$ with center z_0 and a sequence $\{f_n\}_{n=0}^{\infty}$, where f_n is a homogeneous polynomial of degree n, so that

$$f(z) = \sum_{n=0}^{\infty} f_n(z-z_0), \quad z \in B. \tag{1.1}$$

The series (1.1) is called the *Taylor series* of f at z_0. It is known that f is holomorphic in U if and only if the *Frechet derivative* of f at z_0, denoted by $f'(z_0)$, exists for every $z_0 \in U$. One observes that if f is given by (1.1) then

$$f'_n(w)z = n\, \bar{f}_n(w, w, \ldots, w, z).$$

In particular,

$$f'_n(z)(z) = n\, f_n(z)$$

and

$$f'(z_0)z = f_1(z).$$

If $f : U \to F$ is holomorphic, one-to-one, $f(U)$ is open in F and $f^{-1} : f(U) \to U$ is holomorphic, then f is said to be *biholomorphic,* and the sets U and $f(U)$ are *biholomorphically equivalent.* It is known that f is biholomorphic if and only if $f'(z_0)$ is an isomorphism of E onto F for every $z_0 \in U$.

We remark that, unlike the finite dimensional case, it is an open problem whether a one-to-one holomorphic function from U onto an open subset $f(U)$ of F is biholomorphic.

A ball $B \subset U$ is *completely interior in* U *if* $dist(B, \partial U) > 0$. *If* $f : U \to F$ *is* holomorphic we set

$$\|f\|_B = \sup_{z \in B} \|f(z)\|$$

Let $H(U,F)$ denote the space of all holomorphic functions from U into F, and let $H_0(U,F)$ be the subspace consisting of all $f \in H(U,F)$ which are bounded on each ball B completely interior to U. $H_0(U,F)$ is endowed with the *topology of local uniform convergence* defined by the family of seminorms $\|\cdot\|_B$, where B ranges over all balls completely interior to U.

It is known that this topology is metrizable and complete on $H_0(U,F)$.

Remark: Let B_1, B_2 be two balls completely interior in U, and let $M \in (0,\infty)$. Then the topologies induced by the norms $\|\cdot\|_{B_1}$, $\|\cdot\|_{B_2}$ are equivalent on each set

$$H_M(U,F) = \{f \in H_0(U,F) \,;\, \sup_{z \in U} \|f(z)\| \le M\}.$$

Indeed, in case of concentric balls this follows from the Hadamard's three circles theorem. The general case follows from the observation that there exist open balls $C_1, \ldots, C_n$ with $C_1 = B_1$ and $C_n = B_2$, so that for some $\varepsilon > 0 : (C_j)_\varepsilon$ are completely interior in U and $(C_j)_\varepsilon \supset C_{j+1}$, $j = 1,2,\ldots,n-1$. Here A_ε denotes the ε-neighborhood of a set A.

§2 The proof of Theorem 1.1 in case $f(0) = 0$

Let E,F be complex Banach spaces with open unit balls $D(E)$ and $D(F)$. If $f : D(E) \to D(F)$ is biholomorphic (onto) then for each $z \in D(E)$, $f'(z)$ is a linear isomorphism of E onto F with inverse $(f^{-1})'(f(z))$. But in general $f'(z)$ need not be an isometry, as the following example shows. Let Δ be the open unit disc in the complex plane C, let $\varphi_a : \Delta \to \Delta$ be defined by $\varphi_a(z) = (a-z)(1-\bar{a}z)^{-1}$. Then $\varphi_a \in Aut(\Delta)$ and $\varphi'_a(0) = -1 + |a|^2$. So $\varphi'_a(0) : C \to C$ is not an isometry if $a \ne 0$. It is well known that if $\varphi \in Aut(\Delta)$ and $\varphi(0) = 0$ then φ is a rotation. The following lemma show that this is the case in general.

2.1 Lemma: *Let $f : D(E) \to D(E)$ be biholomorphic, onto. If $f(0) = 0$ then f is the restriction to $D(E)$ of a linear isometry of E onto F.*

Proof: Fix $z \in E$, and $x^* \in F^*$ with $\|z\| = 1 = \|x^*\|$, and consider $\varphi : \Delta \to \Delta$ defined via

$$\varphi(\lambda) = x^*(f(\lambda z)).$$

By the Schwarz lemma $|\varphi'(0)| \le 1$. This holds for all x^* and z, and so

$$\|f'(0)\| \le 1.$$

Similarly

$$\|(f^{-1})'(0)\| \le 1$$

thus $f'(0)$ is a linear isometry of E onto F. (Note that this already proves Theorem 1.1 in the case $f(0) = 0$). Let

$$\psi = (f^{-1})'(0) \circ f : D(E) \to D(E)$$

Then ψ is biholomorphic and

$$\psi(0) = 0, \quad \psi'(0) = I_E.$$

The following theorem yields $\psi = I_{E_{|D}}$ and hence $f = f'(0)$, thus completely the proof of the Lemma.

2.2 Theorem: *(H. Cartan's Linearity Theorem)*

Let D be a bounded domain in a complex Banach space E, let $\psi : D \to D$ be holomorphic and assume that for some $z_0 \in D$.

$$\psi(z_0) = z_0, \quad \psi'(z_0) = I_E.$$

Then $\psi = I_{E_{|D}}$.

We sketch the proof. Without loss of generality $z_0 = 0$. If $\psi \neq I_E$ then its Taylor series about 0 is

$$\psi(z) = z + \psi_m(z) + \cdots \text{(higher order terms)} \cdots$$

where $m \geq 2$ and $0 \neq \psi_m$ is a homogeneous polynomial of degree m. The n'th iterate $\psi^{[n]}$ has Taylor series

$$\psi^{[n]}(z) = z + n\,\psi_m(z) + \cdots \text{(higher order terms)} \cdots$$

But for z near 0

$$n\psi_m(z) = \frac{1}{2\pi}\int_0^{2\pi} \psi^{[n]}(e^{i\theta}z)e^{-im\theta}d\theta.$$

Hence

$$\| n\psi_m(z) \| \leq \sup\{ \| w \|,\ w \in D \} < \infty.$$

Letting $n \to \infty$ we see that $\psi_m(z) = 0$ in a neighborhood of 0, and so $\psi_m(z) = 0$ identically. This contradiction completes the proof.

We denote by $G(E) = Aut(D(E))$ resp. $G(F) = Aut(D(F))$ the group of all biholomorphic automorphisms of $D(E)$, respectively $D(F)$. Lemma 2.1 reduces Theorem 1.1 to the following result.

2.3 Theorem: *Let $f : D(E) \to D(F)$ be biholomorphic and onto. Then*

$$f(G(E)(0)) = G(F)(0).$$

Indeed, if this holds then there exists $\varphi \in G(F)$ so that $f(0) = \varphi(0)$. Thus $\psi = \varphi^{-1} \circ f : D(E) \to D(F)$ is biholomorphic and $\psi(0) = 0$. By Lemma 2.1 $\psi = \psi'(0)_{|D(E)}$ and $\psi'(0)$ is a linear isometry of E onto F. This completes the proof of Theorem 1.1 modulo Theorem 2.3.

The proof of Theorem 2.3 is based on a *biholomorphic characterization of the orbit* $G(E)(0)$, namely on establishing a property of $G(E)(0)$ which is preserved by biholomorphic maps. This is done in §4, while §3 is devoted to some Lie theoretic background.

§3. The Lie-theoretic background

Let D be the open unit ball of the complex Banach space E. Every holomorphic function $h : D \to E$ induces a *holomorphic vector field* X, written symbolically as $X = h(z)\frac{\partial}{\partial z}$, which is a differential operator on the space $H(D,E)$ of all holomorphic functions $h : D \to E$, defined via

$$(Xf)(z) = (h(z)\frac{\partial}{\partial z})f(z) = f'(z)(h(z)), \quad z \in D.$$

The *flow of X through the point* $z_0 \in D$ is the unique solution $\varphi = \varphi_X : J_{z_0} \to D$ of the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi(t) = h(\varphi(t)), & t \in J_{z_0} \\ \varphi(0) = z_0 \end{cases}$$

Here J_{z_0} is the maximal open interval containing $t_0 = 0$ in which the solution exists. The vector field X is said to be *complete* if $J_{z_0} = \Re$ for every $z_0 \in D$. In this case one gets a function

$$\varphi = \varphi_X : \Re \times D \to D$$

which is holomorphic in each variable, and satisfies for all $z \in D$

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,z) = h(\varphi(t,z)), & t \in \Re \\ \varphi(0,z) = z \end{cases}$$

Also, the function $\varphi_t : D \to D$, defined by $\varphi_t(z) = \varphi(t,z)$, belongs to $Aut(D)$ and

$$\varphi_{t+s} = \varphi_t \circ \varphi_s.$$

Thus $\{\varphi_t\}_{t\in\Re}$ is a one-parameter subgroup of $Aut(D)$ whose *generator* is X, that is for every $f \in H(D,E)$

$$X(f)(z) = \frac{\partial}{\partial t} f(\varphi_t(z))|_{t=0}.$$

One denotes

$$\exp(tX)(z) = \varphi_t(z) = \varphi(t,z).$$

In particular $\exp(X) = \varphi_1$ is the *exponential* of X.

3.1 Definition: The set of all complete holomorphic vector fields on D is denoted by $aut(D)$.

The *commutator* of $X,Y \in Aut(D)$ is defined by

$$([X,Y]f)(z) = Y(Xf)(z) - X(Yf)(z).$$

If $X = h(z)\frac{\partial}{\partial z}$, $Y = k(z)\frac{\partial}{\partial z}$ then

$$[X,Y] = (h'(z)k(z) - k'(z)h(z))\frac{\partial}{\partial z}.$$

3.2 Theorem: *Let B be any open ball completely interior to D. Then $aut(D)$ is a real Banach Lie-algebra with respect to the commutator product and the norm $\|\cdot\|_B$.*

Let us just indicate why $Aut(D)$ is closed under the operations of addition and commutation. One first identifies $aut(D)$ with the tangent space of $Aut(D)$ at the identity element $e = id_D$, with respect to the topology of local uniform convergence:

$$aut(D) = T_e(Aut(D)).$$

The meaning of this is that if $t \to g_t$, $|t| < \varepsilon$, is a smooth curve in $Aut(D)$, $g_0 = e$, and

$$h(z) = \frac{\partial}{\partial t} g_t(z)_{|t=0}$$

then $X = h(z)\frac{\partial}{\partial z} \in aut(D)$.

Next, if $X = h(z)\frac{\partial}{\partial z}$, $Y = k(z)\frac{\partial}{\partial z}$ are in $aut(D)$, then it is elementary to prove that

$$\frac{\partial}{\partial t} \varphi_X(t,\varphi_Y(t,z))_{|t=0} = h(z) + k(z).$$

Hence $X + Y \in aut(D)$. To show that $[X,Y] \in aut(D)$ we consider

$$g_t(z) = \varphi_X(t,\varphi_Y(t,\varphi_X(-t,\varphi_Y(-t,z)))).$$

Clearly $g_0(z) = z$, $\frac{\partial}{\partial t} g_t(z)_{|t=0} = 0$. Moreover,

$$\lim_{t\to 0} \frac{g_t(z)-z}{t^2} = h'(z)k(z) - k'(z)h(z),$$

showing that $[X,Y] \in aut(D)$.

The Lie algebra $aut(D)$ is *purely real* in the sense that

$$aut(D) \cap i\ aut(D) = \{0\}.$$

Indeed, let $X = h(z)\frac{\partial}{\partial z} \in (aut(D) \cap i\ aut(D))$. Since $[X,iX] = 0$ the one-parameter groups $\{\exp(tX)\}_{t\in\Re}$ and $\{\exp s(iX)\}_{s\in\Re}$ commute. Thus for fixed $z \in D$

$$F(t+is) = \exp(tX)(\exp s(iX)(z))$$

is a bounded entire function in $t + is$. By Liouville's theorem, F is a constant. Thus $0 = F'(0) = \frac{\partial}{\partial t} F_{|t=s=0} = h(z)$. Since $z \in D$ is arbitrary, $X = 0$ as asserted.

The following is classical.

3.3 Theorem (H.Cartan's Uniqueness Theorem): *Fix $a \in D$.*

(i) Suppose that $\varphi,\psi \in Aut(D)$ satisfy

$$\varphi(a) = \psi(a) \quad ,\varphi'(a) = \psi'(a)$$

then $\varphi = \psi$

(ii) Suppose $h\frac{\partial}{\partial z}$, $k\frac{\partial}{\partial z} \in aut(D)$ satisfy

$$h(a) = k(a),\ \ h'(a) = k'(a)$$

then $h = k$.

Proof: (i) Let $\sigma = \varphi \circ \psi^{-1}$. Then $\sigma(a) = a$ and $\sigma'(a) = I_E$. Hence $\sigma = I_{E_{|D}}$ by Theorem 2.2.

(ii) Without loss of generality assume $k = 0$. Let $X = h\frac{\partial}{\partial z}$, let $\varphi = \varphi_X : \Re \times D \to D$ be the flow of X, and let $\varphi_t(z) = \varphi(t,z)$. Clearly, $\varphi_t(a) = a$ for every $t \in \Re$ because this function solves the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} \varphi_t(a) = h(\varphi_t(a)) \\ \varphi_0(a) = a \end{cases}$$

Next, we claim that $\frac{\partial}{\partial z} \varphi_t(z)_{|z=a} = I_E$ for every $t \in \Re$. Having this we get from the first part that $\varphi_t(z) = z$ for all t and z, and so $h(z) = 0$ identically. To prove the claim we consider $\frac{\partial^2}{\partial t \partial z} \varphi_t(z)$ at $z = a$. First

$$\frac{\partial^2\varphi_t(z)}{\partial t\partial z} = \frac{\partial}{\partial z}(\frac{\partial}{\partial t}\varphi_t(z))$$
$$= \frac{\partial}{\partial z} h(\varphi_t(z))$$
$$= h'(\varphi_t(z)) \frac{\partial}{\partial z} \varphi_t(z)$$

and since $h'(a) = 0$, we get

$$\frac{\partial^2}{\partial t\partial z} \varphi_t(z)_{|z=a} = 0.$$

Hence

$$\frac{\partial}{\partial t}(\frac{\partial}{\partial z} \varphi_t(z)_{|z=a}) = 0.$$

So $\frac{\partial}{\partial z}\varphi_t(z)_{|z=a}$ is constant with respect to t, hence

$$\frac{\partial}{\partial z}\varphi_t(z)_{|z=a} = \frac{\partial}{\partial z} \varphi_0(z)_{|z=a}$$
$$= \frac{\partial}{\partial z}z_{|z=a} = I_E.$$

This establishes the claim and completes the proof.

□

The following fundamental result is due to J.P. Vigue. It is the topological version of Cartan's Uniqueness Theorem. Here $L(E)$ denotes the space of all bounded linear operators on E.

3.4 Theorem [V1]: *Fix $a \in D$. Then the map $T : aut(D) \to E \oplus L(E)$ defined via*

$$T(h\frac{\partial}{\partial z}) = (h(a), h'(a))$$

is an isomorphism into. In particular, the image of T is closed in $E \oplus L(E)$. Thus the norms $\|h\|_B$ and $\|h(a)\|_E + \|h'(a)\|_{L(E)}$ are equivalent on $aut(D)$.

The following simple result is the key for the analysis of the structure of $aut(D)$ and is the basis for the connection between the holomorphic structure of D and the structure of E.

3.5 Lemma: *Let $h\frac{\partial}{\partial z} \in aut(D)$.*

(i) h is quadratic in the sense that its Taylor series at 0 is $h = h_0 + h_1 + h_2$ with $\{h_n\}$ homogeneous polynomial of degree n;

(ii) $h_1\frac{\partial}{\partial z}$ and $(h_0+h_2)\frac{\partial}{\partial z}$ belong to $aut(D)$

(iii) $h_0 = 0$ if and only if $h_2 = 0$.

Proof: D is circular about 0 so the rotations

$$\varphi_t(z) = e^{it}z, \quad t \in \Re, \ z \in D$$

form a continuous one-parameter subgroup of G. The infinitesimal generator is the vector field $u\frac{\partial}{\partial z} \in aut(D)$ with $u(z) = iz$. Let $T : aut(D) \to aut(D)$ be defined via

$$Th = [u, h]$$

where we drop the "$\frac{\partial}{\partial z}$" notation for convenience. Clearly, T is continuous with respect to local uniform convergence. Moreover, if h_n is a n-homogeneous polynomial then

$$(Th_n)(z) = u'(z)h_n(z) - h'_n(z)u(z) = i(1-n)h_n(z).$$

Let $h(z) = \sum_{n=0}^{\infty} h_n(z)$, then

$$(Th)(z) = \sum_{n=0}^{\infty} i(1-n)h_n(z).$$

If $p(x)$ is a polynomial with real coefficients then $p(T) : aut(D) \to aut(D)$ and

$$p(T)h = p(T)(\sum_{n=0}^{\infty} h_n) = \sum_{n=0}^{\infty} p(i(1-n))\, h_n.$$

Taking $p(x) = x(1+x^2)$ we get that

$$p(T)h = i \sum_{n=0}^{\infty} n(n-1)(n-2)h_n.$$

The vector field $p(T)h$ belongs to $aut(D)$, it vanishes at 0 and has vanishing derivative there. By Theorem 3.3 (ii), $p(T)h = 0$. so $h_n = 0$ for $n \geq 3$. Next, take $p(x) = 1+x^2$. Then the same argument yields

$$p(T)h = \sum_{n=0}^{2} n(n-2)h_n = h_1 \in aut(D).$$

Also, $h_0 + h_2 = h - h_1 \in aut(D)$. Finally, if $h_0 = 0$ then $h_2 = h - h_1 \in aut(D)$ and $h_2(0) = 0$, $h'_2(0) = 0$. Hence $h_2 = 0$ by Theorem 3.3(ii). And if $h_2 = 0$ then $h_0 \in aut(D)$. But the integral curves of a constant vector field are line segments, so h_0 is incomplete unless it is zero.

□

3.6 Theorem [KU],[BKU]: *Let*

$$E_s = \{h(0);\ h\frac{\partial}{\partial z} \in aut(D)\}.$$

Then E_s is a closed, complex linear subspace of E.

Proof: The fact that E_s is a closed, real linear subspace of E follows easily from Theorem 3.4. To prove that it is complex linear we apply the technique of the proof of Lemma 3.5. Let $h = h_0 + h_1 + h_2 \in aut(D)$, then

$$Th = [u,h] = i(h_0 - h_1) \in aut(D).$$

So $h_0 \in E_s$ implies $ih(0) \in E_s$ and so E_s is complex linear.

□

3.7 Definition: Let $a \in E_s$. We denote by $h_a(z) = a - q_a(z)$ the unique element of $aut(D)$ satisfying $h_a(0) = a$ and $h'_a(0) = 0$. Here q_a is a homogeneous polynomial of degree 2. We denote the associated symmetric bilinear map by $q_a(z,w)$. The *partial triple product* on E is the map

$$\{\ \} : E \times E_s \times E \to E$$

defined via

$$\{z\ a\ w\} = q_a(z,w).$$

Remark: The map $a \to q_a$ is clearly real-linear. The proof of Theorem 3.6 yields $q_{ia} = -iq_a$. Hence $a \to q_a$ is a conjugate-linear map. Thus the partial triple product $\{z\ a\ w\}$ is bilinear and symmetric in z and w and conjugate-linear in a. See [K1], [K3] and [V2] for further investigation.

The *exponential map* $\exp : aut(D) \to Aut(D)$ is not injective. But it becomes so when restricted to an appropriate neighborhood of 0, in which it is given by a convergent power series (and so the inverse map is given by the inverse power series), see [IS, chapter 6]. Precisely

3.8 Lemma: *Let B be an open ball in D and let $\delta > 0$ be so that $B_\delta = \{z \in D\ ; dist(z,B) < \delta\}$ is completely interior in D. Let*

$$\mathcal{M} = \{h\frac{\partial}{\partial z} \in aut(D)\ ;\ \|h\|_{B_\delta} < \frac{\delta}{2e}\}$$

(i) for any $X = h\frac{\partial}{\partial z} \in \mathcal{M}$ the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} X^n(id_D)$$

converges to $\exp X$ in the norm $\|\cdot\|_B$

(ii) let $\mathcal{N} = \exp(\mathcal{M})$. Then the map $\exp : \mathcal{M} \to \mathcal{N}$ is one-to- one (hence a bijection) and is real analytic when $\mathcal{N}$ is taken with the norm $\|\cdot\|_B$.

3.9 Definition: *The analytic topology on $Aut(D)$* is defined in terms of the following system of open neighborhoods of the identity

$$\{\exp(\frac{1}{n}\mathcal{M})\ ;\ n = 1,2,\cdots\}$$

where $\mathcal{M}$ is the set defined in Lemma 3.8.

Remark: The analytic topology is finer than (and in general different from) the topology of local uniform convergence.

3.10 Definition: A Hausdorff topological space M is an *analytic Banach manifold* over the field K (of either the real or complex scalars) if

(i) for every $x \in M$ there exists an open neighborhood U in M and a homeomorphism h_U of U onto some open subset W_U in a Banach space over K,

(ii) If U_1, U_2 are open subsets of M which arise in this way and $U_1 \cap U_2 \neq \phi$, then the map $h_{U_2} \circ h_{U_1}^{-1} : W_1 \to W_2$ is bianalytic.
(If $K = \Re$ than "analytic" means "real analytic").

3.11 Definition: A *submanifold* N of an analytic Banach manifold M over the field K is a subset having the following property:

For every $x \in N$ there exists an open neighborhood U of x in M and an open neighborhood W of 0 in some Banach space X over K, a K-linear closed subspace Y of X and a bianalytic map h of U onto W such that

$$h(U \cap N) = W \cap Y.$$

We remark that the neighborhood U of x is taken in M rather than N. Thus the topology of N is the relative topology induced from M (in some books such submanifolds are called "imbedded" submanifolds).

3.11 Definition: A *Banach Lie-group* over K is a topological group G which is also a Banach manifold over K, such that the algebraic operations (product, inverse) are analytic (from $G \times G$ into G and from G into G respectively).

The following fundamental result is due to H. Upmeier [U1], [U2] and J.P. Vigue [V1] independently. See also [IS, chapter 6].

3.12 Theorem: *Let D be the open unit ball of the complex Banach space E. Then $G = Aut(D)$, endowed with the analytic topology, is a real Banach Lie- group.*

Let G^0 denote the connected component of the identity in $G = Aut(D)$. Clearly, G^0 is a closed, normal subgroup of G. Let $E_s = aut(D)(0)$ be the subspace of E considered in Theorem 3.6.

3.13 Theorem: (a) *The set* $D_s = E_s \cap D$ *is invariant under every member of* G.

(b) $G^0(0)$ *is an open neighborhood of* 0 *in* D_s.

The proof is based on differentiation of the evaluation map

$$G^0 \ni \psi \to \psi(0) \in D_s$$

and the application of Theorem 3.4 and the implicit function theorem.

§4. Biholomorphic characterization of the orbit of the origin under the group of biholomorphic automorphisms.

Let E be a complex Banach space, and let D be its open unit ball. We let $G = Aut(D)$ be the group of all biholomorphic automorphisms of D endowed with the analytic topology, and let G^0 be the connected component of the identity in G. $E_s = aut(D)(0)$ and $D_s = E_s \cap D$.

4.1 Definition:

$$\Omega = \Omega(E) = \{z \in D\ ; G^0(z) \text{ is a closed complex submanifold of } D\}.$$

4.2 Main Lemma: $G(0) = G^0(0) = \Omega = D_s$.

This lemma provides a biholomorphic characterization of the orbit of the origin $G(0)$ and thus proves Theorem 2.3, hence Theorem 1.1.

Indeed, assuming Lemma 4.2, let $f : D(E) \to D(F)$ be biholomorphic. Denote $G(E) = Aut(D(E))$, and $G(F) = Aut(D(F))$. Then the map $\varphi \to f \circ \varphi \circ f^{-1}$ is an isomorphism of $G(E)$ onto $G(F)$ as real Banach Lie groups. In particular

$$f \circ G^0(E) \circ f^{-1} = G^0(F).$$

If $z \in G(E)(0) = \Omega(E)$, then $G^0(E)(z)$ is a closed, complex submanifold of $D(E)$. Hence $f(G^{(0)}(E)(z))$ is a closed, complex submanifold of $D(F)$. But

$$f(G^0(E)(z)) = (f \circ G^0(E) \circ f^{-1})(f(z))$$
$$= G^0(F)(f(z)).$$

So $f(z) \in \Omega(F) = G(F)(0)$. It follows that $f(G(E)(0) \subseteq G(F)(0)$, and applying the same argument with f^{-1} we get $f(G(E)(0)) = G(F)(0)$ as desired.

Proof of the Main Lemma 4.2.

Step 1: $\Omega \subseteq G^0(0)$.

It is enough to show that

$$z \in \Omega \setminus \{0\} \quad \text{implies} \quad 0 \in G^0(z)$$

because $0 \in G^0(z)$ if and only if $z \in G^0(0)$. Let $[z]$ be the one-dimensional complex subspace generated by z. $G^0(z)$ is closed in D, so $[z] \cap G^0(z)$ is closed in $[z] \cap D$. $[z] \cap G^0(z)$ contains $e^{i\theta}z$ for all $\theta \in [0,2\pi)$, since G^0 contains the rotation subgroup.

Claim: $[z] \cap G^0(z)$ is open in $[z] \cap D$.

Having this we clearly get $[z] \cap G^0(z) = [z] \cap D$, since $[z] \cap D$ is connected. In particular, $0 \in G^0(z)$ as desired.

To prove the claim apply the fact that $G^0(z)$ is a complex submanifold. Let U be an open ball contained in D with center z, W an open neighborhood of 0 in E, Y a closed complex linear subspace of E and let $h : U \to W$ be a biholomorphic map satisfying $h(U \cap G^0(z)) = W \cap Y$. The open subset $[z] \cap U$ of $[z]$ contains $\Gamma_\varepsilon = \{e^{i\theta}z;\ |\theta| < \varepsilon\}$ for some $\varepsilon > 0$, and the holomorphic function

$$h_1 = h_{|[z] \cap U}$$

maps Γ_ε into the subspace Y. Let $\pi : E \to E/Y$ be the canonical quotient map, then $h_2 = \pi \circ h_1$ is holomorphic in $[z] \cap U$ and $h_{2|\Gamma_\varepsilon} \equiv 0$. It follows by ordinary function theory that $h_2 \equiv 0$, and so

$$h_1([z] \cap U) = h([z] \cap U) \subseteq W \cap Y.$$

Applying h^{-1} we get

$$[z] \cap U \subseteq G^0(z) \cap U.$$

so $[z] \cap U \subseteq [z] \cap G^0(z)$, and hence z is an interior point of $[z] \cap G^0(z)$. If

$w \in [z] \cap G^0(z)$, $w \neq 0$, then w is an interior point of $[w] \cap G^0(w) = [z] \cap G^0(z)$. Thus $[z] \cap G^0(z)$ is an open subset of $[z] \cap D$. This proves the claim and thus completes the proof of step 1.

Step 2: $G^0(0) = D_s$.

By Theorem 3.13 $G^0(0)$ is an open subset of D_s. Also, D_s is invariant under all members of G^0. Therefore G^0 is identified naturally with a subgroup of $Aut(D_s)$. To prove that $G^0(0)$ is all of D_s, it is enough to show that $G^0(0)$ is closed in D_s, and then apply the connectivity of D_s. To this end we need

4.3 Lemma: *Let Z be a complex Banach space with unit ball U. Then there exists a $Aut(U)$-invariant metric ρ on U, such that the norm topology and the topology induced by ρ coincide on U.*

Assuming this we continue the proof with $Z = E_s$, $U = D_s$. Since $G^0(0)$ is open in D_s, there exists $\varepsilon > 0$ so that the ρ-ball

$$B_\rho(0,\varepsilon) = \{x \in D_s \; ; \; \rho(0,x) < \varepsilon\}$$

is contained in $G^0(0)$. But ρ is G^0-invariant, and so

$$B_\rho(z,\varepsilon) = \{x \in D_s \; ; \; \rho(z,x) < \varepsilon\} \subseteq G^0(0)$$

for every $z \in G^0(0)$. This clearly implies that $G^0(0)$ is closed in D_s, hence $D_s = G^0(0)$.

Proof of Lemma 4.3: Let ρ_0 be the Poincare metric in Δ, namely

$$\rho_0(a,b) = \frac{1}{2}\log\left[\frac{1+\left|\dfrac{a-b}{1-\bar{a}b}\right|}{1-\left|\dfrac{a-b}{1-\bar{a}b}\right|}\right] \; ; \qquad a,b \in \Delta$$

ρ_0 is the unique (up to a constant multiple) $Aut(\Delta)$- invariant metric in Δ. We take ρ to be the *Caratheordory metric* on U, that is

$$\rho(z,w) = \sup\{\rho_0(f(z),f(w)); f \in H(U,\Delta)\}.$$

It is clear that ρ is $Aut(U)$-invariant, symmetric and satisfies the triangle inequality. The fact that ρ is locally equivalent to the distance given by the norm (and hence ρ is also a metric, not just a semi metric) is due to the fact that ρ_0 is locally equivalent to the Euclidean metric on Δ. Indeed, $\rho_0(a,b) \geq |a-b|$, and so

$$p(z,w) \geq \sup_{z^* \in Z^*, \|z^*\|=1} \rho_0(z^*(z),z^*(w))$$

$$\geq \sup_{z^* \in Z^*, \|z^*\|=1} |z^*(z)-z^*(w)| = \|z-w\|.$$

Conversely, let $z \in U$, and let $0 < \delta$ be so that $B = \{w \in Z \; ; \; \|z-w\| < \delta\} \subseteq U$. Let $w \in B \setminus \{z\}$, then the intersection of B with the hyperplane through z and w is identified with $\delta\Delta$, and so

$$\rho(z,w) \leq \rho_0\left(\frac{\|z-w\|}{\delta}, 0\right) \leq C_\delta \|z-w\|.$$

□

Step 3: $G(0) \subseteq \Omega$

We begin with the remark that the normality of G^0 in G implies that Ω is G-invariant. This was proved in the discussion following the statement of Lemma 4.2. By Step 2 and Theorem 3.13, $G^0(0) = D_s$ is a closed complex submanifold of D. Hence $0 \in \Omega$. Since Ω is G-invariant we get $G(0) \subseteq \Omega$, completing the proof.

Step 4: Conclusion of the proof of Lemma 4.2.

We have,

$G(0) \subseteq \Omega$	(Step3)
$\subseteq G^0(0)$	(Step1)
$= D_s$	(Step2)
$\subseteq G(0)$	($G^0 \subseteq G$).

Therefore equality holds everywhere and the proof is complete.

□

4.4 Corollary : *Let $G_0 = \{\varphi \in G \; ; \; \varphi(0) = 0\}$ be the isotropy subgroup at 0. Then*

$$G = G^0 G_0 = G_0 G^0.$$

Proof: Let $\psi \in G$. By Lemma 4.2 there exists $\psi_1 \in G^0$ so that $\psi(0) = \psi_1(0)$. Therefore $\psi_2 = \psi_1^{-1}\psi \in G_0$ and so $\psi = \psi_1\psi_2$. This shows that $G = G^0 G_0$. The normality of G^0 in G yields now that $G = G_0 G^0$.

§5. The symmetric part of a Banach space

Let E be a complex Banach space with an open unit ball D. Let $G = Aut(D)$ be the group of all biholomorphic automorphisms of D and let G^0 the connected component of the identity in G. $\underline{g} = aut(D)$ denotes the Lie algebra of all complete, holomorphic vector fields on D.

5.1 Definition: The *symmetric part of* D is

$$D_s = G(0) = G^0(0).$$

The *symmetric part of* E is the closed, complex subspace of E

$$E_s = \underline{g}(0) = \{h(0)\,;\, h\frac{\partial}{\partial z} \in \underline{g}\}.$$

Notice that according to Lemma 4.2

$$D_s = E_s \cap D.$$

So D_s is the open unit ball of E_s. D_s is invariant under both G and G^0, thus they are identified with subgroups of $Aut(D_s)$. D_s is *homogeneous,* i.e. $Aut(D_s)$ acts on it *transitively,* because G and G^0 act transitively. Moreover D_s is symmetric in the sense that for each $z \in D_s$ there exists a $s_z \in Aut(D_s)$ so that

$$s_z(z) = z,\ s_z^2 = \text{identity, and } s'_z(z) = -I_E.$$

s_z is called the *symmetry at* z; according to Theorem 3.3 (Cartan's Uniqueness Theorem) s_z is the unique member of $Aut(D_s)$ with these properties. Indeed $s_0 = -id_{D_s}$ is the symmetry at 0. If $\varphi \in Aut(D_s)$ and $z = \varphi(0)$, then $s_z = \varphi \circ s_0 \circ \varphi^{-1}$.

The following fundamental result is due to L. L. Stacho [S2].

5.2 Theorem: *(contractive projection principle) Let* E *be a complex Banach space and let* $P : E \to E$ *a contractive (i.e. norm 1) projection. Let* $h\frac{\partial}{\partial z}$ *be a complete holomorphic vector field on* D *and define*

$$\tilde{h} = Ph_{|P(E)\cap D} : P(E) \cap D \to P(E).$$

Then $\tilde{h}\frac{\partial}{\partial z}$ *is a complete, holomorphic vector field on* $P(E) \cap D$.

Theorem 5.2 is a corollary of a very general result proved in [S2]. In our case a more direct proof can be given based on the following characterization of complete, holomorphic vector fields:

Let $X = h\frac{\partial}{\partial z}$ *be a holomorphic vector field on* D. *Then* X *is complete if and only if* h *extends holomorphically to a neighborhood of* $\bar{D}$ *and* X *is tangent to* ∂D *in the sense that whenever* $x \in E$, $x^* \in E^*$ *satisfy* $\|x\| = 1 = \|x^*\|$ *and* $x^*(x) = 1$, *then* $x^*(h(x)) = 0$. [See also [U4, Lecture 4].

5.3 Corollary: *Let* P *be a contractive projection in* E. *Then*

$$P(E_s) \subseteq (P(E))_s.$$

Proof: Let $h\frac{\partial}{\partial z} \in aut(D)$, and let $\tilde{h} = Ph_{|P(E)\cap D}$. Then by Theorem 5.2 $\tilde{h}\frac{\partial}{\partial z} \in aut(P(E)\cap D)$ and

$$\tilde{h}(0) = Ph(0) \in (P(E))_s.$$

Since $h(0)$ is an arbitrary point in E_s, we get $P(E_s) \subseteq (P(E))_s$. □

We turn to some examples of Banach spaces whose symmetric parts are known.

*JB**- **TRIPLE SYSTEMS**

Our first example, in fact a family of examples, deals with the case where $E_s = E$ and $D_s = D$, that is E and D are symmetric. In this case the partial triple product introduced in §3 via $\{xyz\} = q_y(x,z)$ becomes a triple product, namely it is defined now for all $x,y,z \in E$. One denotes for $x,y \in E$ $(x \,\square\, y)z = \{xyz\}$. In his fundamental work [K1] (see also [V2]) W. Kaup establishes the following properties of the map $\square : E \times E \to L(E)$.

(i) $\square$ *is continuous and sesquilinear;*

(ii) The Jordan Triple Identity: for every $x,y,u,v \in E$,

$$[x \,\square\, y, u \,\square\, v] = \{xyu\} \,\square\, v - u \,\square\, \{vxy\}.$$

(iii) Positivity: for every $x \in E$, $x \,\square\, x \in L(E)$ is a positive operator in the sense that $i(x \,\square\, x) \in aut(D)$ and the spectrum of $x \,\square\, x$ is non-negative;

(iv) The C^**-condition:* For every $x \in E$

$$\|x \,\square\, x\| = \|x\|^2.$$

A Banach space E together with a map $[\,]: E \times E \to L(E)$ satisfying (i) - (iv) is called a *JB^*-triple.* Kaup proved also that *if E is a JB^*-triple then D is symmetric,* i.e. $D_s = D$ and $E_s = E$. Kaup's work [K3] goes even further, he shows that *every bounded symmetric domain in a complex Banach space E is biholomorphically equivalent to a bounded, symmetric domain which is convex and circular about the origin,* i.e. the open unit ball of an equivalent norm on E (This extends earlier results of Vigue [V1], who did not prove convexity). This establishes the equivalence of the categories of JB^*-triple systems and bounded symmetric domains with base point.

JB^*-triples generalize both C^*-algebras, JB^*-algebras (which are the complexifications of Jordan Banach algebras) and Hilbert spaces. In the case of C^*-algebra the triple product is defined by

$$\{xyz\} = (xy^*z + zy^*x)/2.$$

In the case of JB^*-algebra with Jordan product $x \circ y$, the triple product is

$$\{xyz\} = x \circ (y^* \circ z) + z \circ (y^* \circ x) - (x \circ z) \circ y^*.$$

In the case of Hilbert space with inner product (x,y) the triple product is given by

$$\{xyz\} = ((x,y)z + (z,y)x)/2.$$

Let us mention some important example of JB^*-subtriples of $L(H)$, the bounded operators on a Hilbert space H: the symmetric operators ($A^T = A$), the anti symmetric operators ($A^T = -A$). Here A^T denotes the transpose of A with respect to some orthonormal basis. Another subtriple, the spin factor, is very important in quantum mechanics but its precise description is a bit more involved.

The action of G on D in case E is a JB^*-triple is by *generalized Mobius transformations.* In case E is a C^*-algebra with a unit 1 these Mobious transformations were studied by Potapov, see [IS, chapter 8]. For every $a, z \in D$ define

$$M_a(z) = (1-aa^*)^{-1/2}(a-z)(1-a^*z)^{-1}(1-a^*a)^{1/2}.$$

One shows that $M_a \in G$ and that M_a is the symmetry which interchanges 0 and a (hence M_a fixes the midpoint along the geodesic from 0 to a). In the commutative case, i.e. $E = C(K)$ with K a compact Hausdorff space, M_a takes the familiar simple form

$$M_a(z) = (a-z)/(1-\bar{a}z); \qquad a, z \in D.$$

In case E is a Hilbert space the Mobius transformations are given in terms of the orthogonal projection P_a on Ca:

$$M_a(z) = (a - P_a z + (1-\|a\|^2)^{1/2}(I-P_a)z)/(1-(z,a)).$$

A recent result of Y. Friedman and B. Russo [FR3], generalizing the Gelfand- Naimark theorem for C^*-algebras, shows that the above example are not too general: Every JB^*-triple is isometrically and triple- isomorphic to a subtriple of

$$\Big(\sum_\alpha \oplus L(H_\alpha)\Big)_\infty \oplus_\infty C(K, C^6)$$

where $\{H_\alpha\}$ are Hilbert spaces, and $C(K,C^6)$ is the space of all continuous functions on a compact Hausdorff space K with values in C^6, the exceptional JB^*- triple of all 3×3 Hermitian matrices over the Cayley numbers.

Unlike C^*-algebras, the category of JB^*-triple systems is closed under the action of contractive projections. This fundamental result is due to Y. Friedman and B. Russo [FR!], [FR2], to W. Kaup [K4] and to L. L. Stacho [S2] independently. It extends the earlier works [AF] and [ES].

5.4 Theorem: *Let E be a JB^*-triple and let $P: E \to E$ be a contractive projection. Then $P(E)$ is also a JB^*-triple when endowed with the triple product*

$$[xyz] = P(\{xyz\}); \qquad x, y, z \in P(E).$$

We present Kaup's proof.

Proof: Since $E_s = E$, Corollary 5.3 gives

$$P(E) = P(E_s) \subseteq (P(E))_s \subseteq P(E).$$

Hence $P(E) = (P(E))_s$. By Kaup's theorem mentioned above $P(E)$ is a JB^*-triple with triple product $[xyz]$. Let $a \in P(E)$ and consider

$$h_a(z) = a - \{zaz\}, \qquad z \in D$$

sp $h_a \dfrac{\partial}{\partial z}$ is the unique element of $aut(D)$ satisfying $h_a(0) = a$ and $h'_a(0) = 0$. Similarly, if

$$k_a(z) = a - [zaz]; \qquad z \in P(E) \cap D$$

$k_a \dfrac{\partial}{\partial z}$ is the unique element of $aut(P(E) \cap D)$ for which $k_a(0) = a$ and $k'_a(0) = 0$. Let

$$\bar{h}_a = Ph_{a\,|P(E)\cap D}.$$

Then by Theorem 5.2, $\bar{h}_a \dfrac{\partial}{\partial z} \in aut(P(E) \cap D)$ and clearly $\bar{h}_a(0) = Pa = a$ and $\bar{h}'_a(0) = 0$. It follows that $\bar{h}_a = k_a$. Thus

$$P\{zaz\} = [zaz]; \qquad a,z \in P(E) \cap D.$$

Polarizing, we get

$$P\{xaz\} = [xaz]; \qquad x,a,z \in P(E) \cap D.$$

□

Let us mention one more result concerning JB^*-triples. It is obvious that *the ℓ_∞-sum of a family of JB^*-triple is also a JB^*-triple,* where the triple product is defined coordinatewise. (Equivalently, the unit ball is a product of symmetric balls, hence symmetric). It follows that *the ultra product of a family of JB^*-triples is naturally a JB^*-triple.* Next, it is well-known that for every Banach space E the second dual E^{**} is naturally isometric to a 1-complemented subspace of an ultra power of E. Combining these facts with Theorem 5.4, S. Dineen proved

5.5 Theorem [D2]: *The second dual E^{**} of a JB^*-triple E is in a natural way a JB^*-triple, and E is a subtriple of E^{**} under the canonical embedding.*

Modifying the construction of Dineen, T.barton and R. M. Timoney [BT] were able to improve Theorem 5.5 by showing that *the triple product on E^{**} must be separately w^*-continuous.* Dineen [D3] generalized [D2] by showing that if $\{E_j\}$ are Banach spaces with open unit balls $\{D_j\}$, and if $h_j \frac{\partial}{\partial z} \in aut(D_j)$ are uniformly bounded then $(h_j \frac{\partial}{\partial z})$ give rise to a complete holomorphic vector field on the unit ball of the ultra product $\prod_j E_j/U$. From this he concluded Theorem 5.5 and also that *every biholomorphic automorphism of the unit ball of a complex Banach space E can be extended to a biholomorphic automorphism of the unit ball of E^{**}.*

RIENHARDT DOMAINS AND SPACES WITH 1-UNCONDITIONAL BASIS

Let E be a complex Banach space of sequences $z = (z(n))$ so that the *standard unit vectors* $\{e_k\}_{k=1}^{\infty}$ (defined via $e_k(n) = \delta_{n,k}$) form a *normalized, 1-unconditional basis for E. See [LT1], [LT2] for these and* related notations. A domain U in E is called a *Reinhardt domain* (with respect to the $\{e_k\}_{k=1}^{\infty}$) if it contains the origin and if

$$(z(n)) \in U \quad \text{if and only if} \quad (e^{i\theta_n} z(n)) \in U$$

for all choices of $\theta_n \in R, n = 1,2, \cdots$ with $\theta_n = 0$ for all but finitely many indices n. The Reinhardt domain U is *normalized* if $e_k \in \partial U$ and $\lambda e_k \in U$ imply $|\lambda| < 1$. Clearly every Reinhardt domain is linearly equivalent to a normalized one. Finally, a convex, normalized Reinhardt domain in E is simply the open unit ball of E in an equivalent norm in which $\{e_k\}_{k=1}^{\infty}$ is still a 1-unconditional basis.

The study of Reinhardt domains depends on the following two-dimensional result of P. Thullen. Let us define for $r > 0$

$$U(r) = \{(z,w) \in C^2;\ |z|^2 + |w|^{1/r} < 1\}$$

and let

$$U(0) = \{(z,w) \in C^2;\ |z| < 1, |w| < 1\}.$$

5.6 Theorem [T]. *Let U be a bounded, normalized Reinhardt domain in C^2.*

(i) Either $Aut(U)$ consists of linear maps, or $U = U(r)$ for a unique $r \in [0,\infty)$, up to a permutation of the coordinates;

(ii) $U(0)$ and $U(1/2)$ are symmetric, being the open unit balls of the 2-dimensional ℓ_∞ and ℓ_2 spaces respectively;

(iii) For $0 < r, r \neq 1/2$, $Aut(U(r))$ consists of all maps

$$\Phi(z,w) = (\varphi(z), \lambda(\varphi'(z))^r w)$$

with $\varphi \in Aut(\Delta)$ and $|\lambda| = 1$. In particular the symmetric part of $U(r)$ is

$$U(r)_s = Aut(U(r))(0) = \Delta \times \{0\}.$$

This classical theorem shows in particular that the Riemann mapping theorem is not valid in dimension n for $n > 1$. For instance

5.7 Corollary: *The bidisc $U(0) = \Delta \times \Delta$ and the unit ball $U(1/2)$ in C^2 are not biholomorphically equivalent.*

5.8 Corollary: *for $p \neq 2,\infty$ all biholomorphic automorphisms of*

$$\{(z,w) \in C^2;\ |z|^p + |w|^p < 1\}$$

are linear. In particular this domain is not biholomorphically equivalent to $U(0)$ or $U(1/2)$.

Theorem 5.6 was extended by T. Sunada [SU] to C^n and then by L. L. Stacho [S2], J. P. Vigue [V5] and T. Barton [B1], [B2] to the infinite dimensional setting. See also [BDT] for this and for the study of Reinhardt decompositions of Banach spaces. To describe these results we adopt the notation

$$E_A = \overline{span}\ \{e_k\}_{k \in A},\ U_A = U \cap E_A$$

for every $A \subseteq N$.

5.9 Theorem [BDT]: *Let U be a bounded normalized Reinhardt domain in E.*

(i) There exists a subset I of the positive integer N with complement $J = N \setminus J$ and a partition $\mathcal{P}$ of I so that $E_s = E_I$ and so that the sequence $\{e_i\}_{i \in I}$ is isometrically equivalent to the unit vector basis of $(\sum_{p \in \mathcal{P}} \oplus \ell_2(p))_{c_0}$;

(ii) $(Aut(U))(0) = U_I$;

(iii) There exist non-negative numbers $\{r_{p,j}\ ; p \in \mathcal{P}, j \in J\}$ satisfying

$$\sup_{j \in J} \sum_{p \in \mathcal{P}} r_{p,j} < \infty \qquad (*)$$

so that $(z(n)) \in U$ if and only if

(a) $(z_p)_{p \in \mathcal{P}} \in c_0(\mathcal{P})$ and $\max_{p \in \mathcal{P}} z_p < 1$, where $z_p = (\sum_{i \in \mathcal{P}} |z(i)|^2)^{1/2}, p \in \mathcal{P}$

(b) $\sum_{j \in J} (\prod_{p \in \mathcal{P}} (1 - z_p^2)^{-r_{p,j}}) z(j) e_j \in U_J$.

Moreover, if U is convex then U_J is convex and

$$\sup_{j \in J} \sum_{p \in \mathcal{P}} r_{p,j} \leq 1. \qquad (**)$$

5.10 Remarks: (i) The first part of Theorem 5.9 concerning general Reinhardt domains admits a converse. Given $I, J, \mathcal{P}$ and $\{r_{p,j}\}$ satisfying (*) there exists a Reinhardt domain U consisting of all $(z(n))$ so that the conditions (a) and (b) are satisfied. For the convexity part of Theorem 5.9 - it is not true that in general the convexity of U_J and (**) imply the convexity of U. See [B1] and [BDT] for these matters.

(ii) If $0 < r < \infty$, $r \neq 1/2$, then $(z, w) \in U(r)$ if and only if

$$|z| < 1 \quad \text{and} \quad \frac{|w|}{(1 - |z|^2)^r} < 1.$$

So, in the notation of Theorem 5.9, $I = \{1\}$, $\mathcal{P} = \{I\}$, $J = \{2\}$, $r_{I,2} = r$ and $U_J = \{0\} \times \Delta$.

(iii) The subspaces $E_p, p \in \mathcal{P}$ of E are known as *Hilbert components* of E. See [KW].

5.11 Corollary: *E is symmetric (in the sense that $E = E_s$) if and only if E is the c_0-sum of a sequence of Hilberts spaces: In particular, if E is a symmetric sequence space (i.e. the unit vector basis $\{e_k\}_{k=1}^{\infty}$ form a 1-symmetric basis of E) then either $E_s = \{0\}$ or $E_s = E$. In the last case, either $E = \ell_2$ or $E = c_0$.*

SPACES WITH THE LINEAR BIHOLOMORPHIC PROPERTY

A complex Banach space E with open unit disc D is said to have the *linear biholomorphic property (LBP*, for short) if $G = Aut(D)$ consists of linear operators. Equivalently, if $\underline{g} = aut(D)$ consists of linear operators. *LBP* is the extreme opposite to symmetricity since E has *LBP* if and only if its symmetric part is trivial, namely $E_s = \{0\}$.

The main tool in studying *LBP* in Banach spaces is the following.

5.12 Proposition [S2]: *Suppose $\mathcal{P}$ is a family of contractive projections in E, satisfying*

(i) $P(E)$ has LBP for every $P \in \mathcal{P}$

(ii) $\bigcap_{P \in \mathcal{P}} \ker P = \{0\}$.

Then E has LBP.

Proof: For every $P \in \mathcal{P}$ we have by (i) and Corollary 5.3

$$P(E_s) \subseteq (P(E))_s = \{0\}.$$

Thus $E_s \subseteq \ker P$ for every $P \in \mathcal{P}$. Therefore $E_s = \{0\}$ by (ii). □

5.13 Corollary ([S1],[S2],[BKU]): *Let E be a rearrangement-invariant Banach function space on either N,[0,1] or $[0,\infty)$. If E is not a Hilbert space or a C^*- algebra then E has LBP.*

For the study of rearrangement-invariant spaces see [LT2]. If the measure space is N then E is a symmetric sequence space and the assertion follows from Corollary 5.11. For the other two cases the assertion follows from Proposition 5.1 by taking $\mathcal{P}$ to be the family of all conditional expectations with respect to finite σ-algebras, generated by finite families of disjoint measurable sets with the same measure.

5.14 Corollary: *[A] Let E be a symmetric sequence space different from ℓ_2 and c_0 and let S_E be the associated unitary ideal of operators on ℓ_2. Then S_E has LBP.*

S_E is defined as the Banach space of all operators T on ℓ_2 whose sequence of singular numbers $(s_n(T))$ belongs to E, normed by

$$\| T \|_{S_E} = \| (s_n(T)) \|_E .$$

See [GK]. Corollary 5.14 follows from Proposition 5.12 and Corollary 5.13 by taking $\mathcal{P}$ to be the family of all generalized diagonal projections P of the form

$$P(T) = \sum_{n \in I} (Th_n, f_n)(\cdot, h_n) f_n$$

where $\{h_n\}_{n \in I}$, $\{f_n\}_{n \in I}$ are (finite or infinite) orthonormal sequences. The point is that the sequence, $\{(\cdot, h_n) f_n\}_{n \in I}$ is isometrically equivalent to the sequence $\{e_n\}_{n \in I}$ in E, and so $P(S_E)$ has *LBP* if $|I| \geq 2$.

Let $H_1, \ldots, H_n$ be Hilbert spaces with $\dim (H_j) \geq 2$. Let $H_1 \hat{\otimes} H_2 \hat{\otimes} \cdots \hat{\otimes} H_n$ be the injective tensor product, i.e. the space of all bounded n-linear maps $f : H_1 \times H_2 \times \cdots \times H_n \to C$, normed by $\| f \| = \sup \{ | f(z_1, \ldots, z_n) | ; z_j \in H_j, \| z_j \| \leq 1 \}$. If $n = 2$, $H_1 \hat{\otimes} H_2$ is identified with the space of all compact operators from H_1 into H_2 which is a JB^*-subtriple of $L(H_1 \oplus H_2)$. For $n > 2$ the situation changes drastically

5.15 Proposition [S2]: *For $n > 2$ the space $H_1 \hat{\otimes} H_2 \hat{\otimes} \cdots \hat{\otimes} H_n$ has LBP.*

In an another direction we have

5.16 Proposition [BKU]: *for $1 \leq p < \infty, p \neq 2$, the Hardy spaces H^p have LBP.*

UNIFORM ALGEBRAS

A *uniform algebra* A is a closed subalgebra of $C(K)$, K a compact Hausdroff space, which contains the unit 1 and separates the points of K. A is endowed with the supremum norm.

5.17 Proposition [BKU]: *Let A be a uniform algebra. Then the symmetric part of A is its maximal C^*-subalgebra, i.e.*

$$A_s = \{ f \in A ; \bar{f} \in A \}.$$

From this it is elementary to see

5.18 Corollary [BKU]: *Let A be either the disc algebra or H^∞ over the unit disc Δ. Then $A_s = C1$.*

E_s NEED NOT BE COMPLEMENTED IN E

A natural question concerning the symmetric part E_s of a Banach space E is whether E_s must be complemented in E. The following unpublished example of J. P. Vigue answers this negatively.

5.19 Proposition: *There exists an equivalent 1-symmetric norm $|||\cdot|||$ on ℓ_∞ so that if $E = (\ell_\infty, |||\cdot|||)$ then $E_s = c_0$. In particular E_s is not isomorphic to a complemented subspace of E.*

Indeed, fix $0 < \varepsilon < 1$ and let D_0 and D_∞ be the open unit balls of c_0 respectively ℓ_∞. Define

$$D = (1-\varepsilon) D_0 + \varepsilon D_\infty .$$

Then D is an open, convex subset of D_∞, $\varepsilon D_\infty \subset D$ and D is invariant under permutations and changes of complex signs of the coordinates. Thus D is the open unit ball of ℓ_∞ in an equivalent, 1-symmetric norm $|||\cdot|||$. Set $E = (\ell_\infty, |||\cdot|||)$. Let $G = Aut(D)$, and let G^0 be the connected component of the identity in G. We claim that

$$G^0(0) = D_0 .$$

Clearly, having this the proposition follows from Lemma 4.2 and the fact that $D_0 = D \cap D_0$. To prove the claim, observe first that

$$D = \{ z = (z_n) \in D_\infty ; \limsup_{n \to \infty} |z_n| < \varepsilon \}.$$

Next, consider the maps of the form

$$\varphi(z_1, z_2, \ldots, z_n, \cdots) = (\varphi_1(z_1), \varphi_2(z_2), \ldots, \varphi_n(z_n), \ldots) \qquad (*)$$

where $z_j \in \Delta$ and $\varphi_j \in Aut(\Delta)$. It is elementary to verify that such φ is a member of $Aut(D)$ if and only if $\varphi(0) = (\varphi_j(0)) \in D_0$. Let $\tilde{G}$ denotes the set of these maps (*) with $\varphi(0) \in D_0$. Then $\tilde{G}$ is a connected subgroup of $Aut(D)$. The main point is that $\tilde{G}$ is open in $Aut(D)$, see [V3, Th. 1.8]. But an open subgroup of a topological group is always closed. Thus $\tilde{G}$ is open, closed and connected. Hence $\tilde{G} = G^0$. Thus the symmetric part of D is

$$G^0(0) = \tilde{G}(0) = D_0 .$$

□

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A DENSITY CONDITION FOR ANALYTICITY OF THE RESTRICTION ALGEBRA

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1. INTRODUCTION

This paper has to be considered as an appendix to [1] (see especially remark (2) at the end). More precisely, we will use the method developed in [1] to obtain a significant improvement of the density criterion for analyticity of the restriction algebra proved by Katznelson and Malliavin in [4]. A rather detailed discussion of their results can also be found in the book of Graham and McGehee [3] (see Open Problem section). [3] is also one standard reference work for background material. Next I recall some definitions and facts. Let G be a compact Abelian group, $\Gamma = \hat{G}$ the dual group of G. For $\Lambda \subset \Gamma$, let

$$A(\Lambda) = \{\hat{f}\,|_{\Lambda}; f \in L^1(G)\}$$

be the restriction algebra (of Fourier transforms of L^1 - functions on G).

The set $\Lambda \subset \Gamma$ is dissociated if any ($\pm$ 1,0) relation on the characters of Λ is trivial, thus

$$\Sigma_\Lambda \varepsilon_\gamma \gamma = 0,\ \varepsilon_\gamma = 1,-1,0 \Rightarrow \varepsilon_\gamma = 0 \quad \text{for } \gamma \neq 0.$$

The set $\Lambda \subset \Gamma$ is called a Sidon set provided there is a constant $C > 0$ such that

$$C\, \| \Sigma_\Lambda a_\gamma \gamma \|_{C(G)} \geq \Sigma |a_\gamma| \tag{1}$$

holds for all scalar sequences $(a_\gamma)_{\gamma\in\Lambda}$. The smallest constant C satisfying inequality (1) is called the Sidon constant $S(\Lambda)$ of Λ. Here, $C(G)$ refers to the space of continuous functions on G endowed with the uniform norm and characters γ are considered as functions on the group.

Dissociated sets are special cases of Sidon sets. The interpolating measures are then given by the standard Riesz products.

The algebra $A(\Lambda)$ is analytic provided only analytic functions operate on $A(\Lambda)$. Recall that $F:[-1,1]\to C$ operates on $A(\Lambda)$ provided

$$\varphi\in A(\Lambda), \varphi(\gamma)\in\,]-1,1[\, \Rightarrow F\circ\varphi\in A(\Lambda)$$

It is known that for $A(\Lambda)$ to be analytic, it is necessary and sufficient that for some $c>0$

$$N(\Lambda,t) \equiv \sup_{\varphi\ real,\, \|\varphi\|_A \leq 1} \| e^{it\varphi} \|_A \geq e^{ct}. \tag{2}$$

for all $t>0$ large enough. Here of course $\|\ \|_A$ refers to the $A(\Lambda)$ - norm (= quotient norm) , i.e.

$$\|\varphi\|_A = \inf\{\|f\|_{L^1(G)}; \hat{f}(\gamma) = \varphi(\gamma) \quad \text{for } \gamma\in\Lambda\}.$$