ON THE HOLOMORPHIC RIGIDITY OF LINEAR OPERATORS ON COMLEX BANACH SPACES

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1. Introduction

IN 1931 H. Cartan [**4**] proved the following uniqueness theorem: *Let $D \subset \mathbb{C}^n$ be a bounded domain and let $f : D \to D$ be a holomorphic mapping. Then f is the identity on D if it has a fixed point $a \in D$ at which the Jacobian $f'(a)$ is the identity matrix.* Another way of expressing this is as follows: In the space of all holomorphic mappings $D \to D$ a biholomorphic mapping f is uniquely determined by the first two terms $f(a)$, $f'(a)$ of its power series expansion about a. Cartan's proof uses an iteration argument that can immediately be extended to bounded domains in complex Banach spaces (clearly the Jacobian $f'(a)$ has to be interpreted as linear operator—the Fréchet derivative $df(a)$ of f at a). In finite as well as in infinite dimensions Cartan's uniqueness theorem has been the key for many important results.

In the present paper we study holomorphic mappings $f : B \to D$ between domains in complex Banach spaces that are rigid at $a \in B$ in the following sense: $f = g$ for every holomorphic mapping $g : B \to D$ with $f(a) = g(a)$ and $df(a) = dg(a)$. Our main interest is concentrated to the special case where B, D are the open unit balls of the complex Banach spaces E, F and where $a = 0$ is the origin. Then, if $f : B \to D$ with $f(0) = 0$ is rigid at the origin it necessarily must be of the form $f = L|B$ for a linear operator $L : E \to F$ with $\|L\| = 1$. Such linear operators L we also call rigid.

The paper is organized as follows:

In Section 2 we present the basic background for holomorphic mappings between domains U, V in complex Banach spaces as needed later for the linear case. In particular, for every $m \in \mathbb{N}$ and every holomorphic mapping $f : U \to V$ we define f to be m-rigid at the point $a \in U$ if f is uniquely determined within the space of all holomorphic mappings $U \to V$ by all derivatives of order $< m$ at a. This, in case $m = 2$, is just what occurs for biholomorphic f in Cartan's uniqueness theorem. We also introduce the more general notion of infinitesimal m-rigidity at a in case f is uniquely determined within families of holomorphic mappings $g_t : U \to V$ depending holomorphically on a one-dimensional parameter t. Using a theorem of Kakutani we show that every biholomorphic automorphism of a bounded domain is infinitesimally 1-rigid, and is even infinitesimally 0-rigid if U is the open unit ball of a complex Banach space.

In Section 3 we study linear operators $L : E \to F$ with $\|L\| = 1$ and call L rigid if the induced map between the open unit balls is 2-rigid at the origin. In the case of Hilbert spaces E, F for instance, L is rigid if and only if L is a (not necessarily surjective) isometry.

In Section 4 we introduce for every given linear operator $L : E \to F$ of norm 1 and every $m \in \mathbb{N}$ numerical invariants $\alpha_m \in [0, 1]$ that measure the non-rigidity of L in connection with homogeneous polynomials of degree m. It turns out that L is a complex extreme point of the unit ball in the Banach space of all polynomials $E \to F$ of degree $\leqslant m$ if and only if α_m vanishes. Furthermore, $(\alpha_m)_{m\in\mathbb{N}}$ is an increasing sequence, and L is rigid if and only if $\alpha_m = 0$ for all m (i.e. the limit α_∞ vanishes). Besides the invariants α_m we also introduce invariants π_m that measure certain eccentricities.

In Section 5 we determine the invariants α_m and estimate π_m for some special examples. Also, in the case of contractive projections L we relate rigidity properties of L with smoothness properties of the unit spheres.

In Section 6 we introduce various types of tangent spaces and correlate them to the rigidity problem.

In Section 7 we apply the methods to JB*-triples. These are generalizations of operator algebras where the algebra product is replaced by a certain ternary product, the Jordan triple product. Our main result—Theorem 7.14—solves completely the rigidity problem for w^*-closed inner ideals in JBW*-triples, the triple generalizations of W*-algebras.

NOTATION . *Throughout, E and F are complex Banach spaces with open unit balls $B \subset E$ and $D \subset F$. The notation $E \subset F$ means that E carries the induced norm from F, i.e. $B = D \cap F$. By $\mathcal{L}(E, F)$ we denote the Banach space of all bounded linear operators $E \to F$. Furthermore $\mathcal{L}(E) := \mathcal{L}(E, E)$ is the Banach algebra of all continuous endomorphisms and $E^* := \mathcal{L}(E, \mathbb{C})$ is the dual of E. The group of all invertible operators in $\mathcal{L}(E)$ is denoted by $\mathsf{GL}(E)$. The ℓ^p-sum of E and F will be denoted by $E \oplus_p F$, that is $E \oplus F$ with norm satisfying $\|(z, w)\| = \max(\|z\|, \|w\|)$ if $p = \infty$ and $\|(z, w)\|^p = \|z\|^p + \|w\|^p$ if $1 \leqslant p < \infty$. For complex Hilbert spaces we denote the inner product always by $(z|w)$ where the conjugate linear variable is w.*

The boundary of B (the unit sphere in E) is denoted by ∂B. The subset of all extreme boundary points of B is denoted by $\partial_e B$ and $\partial_{ec} B$ is the set of all complex extreme boundary points of B. Always $\Delta := \{t \in \mathbb{C} : |t| < 1\}$ is the open unit disc and $\mathbb{T} := \partial\Delta$ is the circle group. The set $\mathbb{N}$ of natural numbers always includes 0.

2. Rigid holomorphic mappings

For an open subset U of the Banach space E a mapping $f : U \to F$ is called *holomorphic* if for every $a \in U$ the Fréchet-derivative $df(a) \in \mathcal{L}(E, F)$ exists.

Then it is known that the derivative $df : U \to \mathcal{L}(E, F)$ again is holomorphic and the second derivative $d^2 f = d\,df$ takes values in $\mathcal{L}(E, \mathcal{L}(E, F))$, which as Banach space can be identified in a natural way with the space of all bounded bilinear mappings $E^2 \to F$. More generally, the n-th derivative $d^n f(a)$ exists as a bounded symmetric n-linear mapping $E^n \to F$ for every $n \in \mathbb{N}$ (by definition $d^0 f = f$).

For arbitrary subsets $S \subset E$ and $T \subset F$ a mapping $f : S \to T$ is called holomorphic if there exists an open subset $U \subset E$ and a holomorphic mapping $h : U \to F$ with $S \subset U$ and $f = h|S$. The space of all holomorphic mappings $S \to T$ will be denoted by $\mathsf{Hol}(S, T)$. With $\mathsf{Aut}(S) \subset \mathsf{Hol}(S, S)$ we denote the group of all biholomorphic automorphisms of S. For every point $a \in S$ denote by

$$\circ_a(f) = \sup\{n \in \mathbb{N} : \|f(z)\| = O(\|z - a\|^n) \qquad \text{as } z \to a\}$$

the vanishing order of the holomorphic mapping $f : S \to T$ at a. For any pair $f, g \in \mathsf{Hol}(S, T)$ then $\circ_a(f, g) := \circ_a(f - g)$ is the order of contact at a. In case a is an inner point of S the condition $\circ_a(f, g) \geqslant m$ is equivalent to $d^n f(a) = d^n g(a)$ for all $n < m$.

DEFINITION 2.1. A subset $A \subset E$ is called a *set of determinacy in E* if for every open connected neighbourhood $U \subset E$ of A the restriction operator $\mathsf{Hol}(U, \mathbb{C}) \to \mathsf{Hol}(A, \mathbb{C})$ is injective.

The following statement is an easy consequence of the Hahn–Banach theorem.

LEMMA 2.2. *Let $f : U \to F$ be a holomorphic mapping for a domain $U \subset E$. Let furthermore $R \subset F$ be a closed linear subspace and $A \subset U$ a set of determinacy in E. Then $f(U) \subset R$ if $f(A) \subset R$.*

Proof. Fix $\lambda \in F^*$ with $\lambda|R = 0$. Then the holomorphic function $\lambda \circ f$ vanishes on A.

LEMMA 2.3. *Let $A \subset E$ be a balanced set of determinacy in E. Then also $A \cap B$ is a set of determinacy in E.*

Proof. Let U be an open connected neighbourhood of $A \cap B$ and fix a holomorphic function $f : U \to \mathbb{C}$ vanishing on $A \cap B$. We have to show that $f = 0$. Since $A \cap B$ contains the origin we may assume without loss of generality that $U = B$. Expand f into a series $\sum f_n$, where every $f_n : E \to \mathbb{C}$ is homogeneous of degree n (compare Section 4). Every f_n vanishes on $A \cap B$ and hence also on A, i.e. $f_n = 0$ for all n.

DEFINITION 2.4. Let $S \subset E$ and $T \subset F$ be connected subsets, let $m \geqslant 0$ be an integer and let $\Delta \subset \mathbb{C}$ be the open unit disc. Then $f \in \mathsf{Hol}(S, T)$ is called *m-rigid* at $a \in S$ if the equality $g = f$ holds for every $g \in \mathsf{Hol}(S, T)$ with $\circ_a(f, g) \geqslant m$. If f is *m-rigid* at every point of S we call *f m-rigid every where*. The mapping f is called *infinitesimally m-rigid* at $a \in S$ if $g_t \equiv f$ holds for every family $(g_t)_{t\in\Delta}$ in $\mathsf{Hol}(S, T)$ satisfying the following properties: (i) $g_0 = f$, (ii) $\circ_a(f, g_t) \geqslant m$ for all $t \in \Delta$ and (iii) g_t depends holomorphically on t, i.e. the mapping $\Delta \times S \to T$ defined by $(t, s) \mapsto g_t(s)$ is holomorphic, where $\Delta \times S$ is considered as a subset of $\mathbb{C} \times E$.

Clearly, stronger versions of infinitesimal rigidity could be introduced by requiring for instance that (g_t) depends real analytically or only $\mathcal{C}^r$ on the parameter t. Our main interest however is in the case of linear operators where all these notions are equivalent to simple rigidity (at least if $m \geqslant 2$, compare 4.3).

All degrees of rigidity may occur: $f \in \mathsf{Hol}(\Delta, \overline{\Delta})$ defined by $f(z) = z^m$, $m \in \mathbb{N}$, is $m + 1$-rigid but not m-rigid at $0 \in \Delta$.

We start with some trivial statements

REMARK 2.5. Suppose that $f : S \to T$ is a holomorphic mapping and $g : T \to \tilde{T}$, $h : S \to \tilde{S}$ are biholomorphic mappings. Then for every $a \in S$, every integer $m \geqslant 0$ and $f := g \circ f \circ h^{-1} : \tilde{S} \to \tilde{T}$ the following holds

(i) f is (infinitesimally, resp.) $m + 1$-rigid at a if f is (infinitesimally, resp.) m-rigid at a.
(ii) f is (infinitesimally m-rigid at a if f is m-rigid at a.
(iii) $\tilde{f}$ is (infinitesimally, resp.) m-rigid at $h(a)$ if f is (infinitesimally, resp.) m-rigid at a.

REMARK 2.6. Suppose that the holomorphic mappings $f_i : S \to T_i$ are (infinitesimally, resp.) m-rigid at $a \in S$ for $i = 1, 2$. Then also $f = (f_1, f_2) : S \to T_1 \times T_2$ is (infinitesimally, resp.) m-rigid at a.

As a consequence of Liouville's theorem every holomorphic mapping $E \to T$ is 1-rigid everywhere if $T \subset F$ is bounded. Also, every biholomorphic mapping $f : U \to V$ is 2-rigid everywhere as a consequence of Cartan's uniqueness theorem if $U \subset E$ is a bounded domain. We even have

PROPOSITION 2.7. *Suppose that $f : U \to V$ is a biholomorphic mapping where $U \subset E$ is a bounded domain. Then f is infinitesimally 1-rigid everywhere.*

Proof. We may assume that $U = V$ and that f is the identity on U. Fix $a \in U$ and consider on E the Carathéodory norm ν defined by

$$\nu(v) = \sup\{|df(a)v| : f \in \mathsf{Hol}(U, \Delta),\ f(a) = 0\}$$

for all $v \in E$. Then $V := (E, \nu)$ also is a complex Banach space. Now suppose that (g_t) is a family in $\mathsf{Hol}(U, U)$ depending holomorphically on $t \in \Delta$ with $g_t(a) \equiv a$ and $g_0 = f$. Then $t \mapsto dg_t(a)$ defines a holomorphic mapping $\Delta \to \mathcal{L}(V)$ with $\|dg_t(a)\| \leqslant 1$ for all t. But $dg_0(a) = \mathsf{id}$ is an extreme point of the unit ball in the Banach algebra $\mathcal{L}(V)$ by a result of Kakutani and hence $dg_t(a) = \mathsf{id}$ for all $t \in \Delta$—compare [**9**] p. 74 and p. 69. But then Cartan's uniqueness theorem gives $g_t = f$ for all $t \in \Delta$.

For the open unit balls of Hilbert spaces Proposition 2.7 can be generalized, compare also 3.6.

EXAMPLE 2.8. Suppose that $E \subset F$ are complex Hilbert spaces with open unit balls $B \subset D$. Then the canonical injection $f : B \hookrightarrow D$ is 2-rigid everywhere. Also, f is infinitesimally 1-rigid everywhere.

Proof. Denote by H the orthogonal complement of E in F and suppose that $\circ_a(f, g) \geqslant 2$ for $g \in \mathsf{Hol}(B, D)$ and some $a \in B$. Then there are holomorphic maps $\varphi : B \to B, h : B \to H$ with $g = \varphi + h$. From $\circ_a(f, \varphi) \geqslant 2$ we derive $\varphi = f$. But then $\lim_{z \to \partial B} h(z) = 0$ implies $h = 0$. That f is infinitesimally 1-rigid at $a = 0 \in B$ follows as in the proof of 2.7 since the canonical injection $E \hookrightarrow F$ is an extreme point of the unit ball in $\mathcal{L}(E, F)$. The statement for arbitrary $a \in B$ then is a consequence of 2.5 since there are $h \in \mathsf{Aut}(B)$ and $g \in \mathsf{Aut}(D)$ with $a = h(0)$ and $f = g \circ f \circ h^{-1}$.

For certain domains Cartan's uniqueness theorem can be strengthened. The first part of the following statement is due to Harris [**13**], compare also the more general Proposition 6.8.

PROPOSITION 2.9. *Let B be the open unit ball of the complex Banach space E and let $f : B \to B$ be a holomorphic mapping with $df(0) = \mathsf{id}$. Then also $f(0) = 0$ holds and hence f is the identity on B. Furthermore, f is infinitesimally 0-rigid everywhere on B.*

Proof. Put $c := f(0)$ and start with the special case $B = \Delta$. Then Schwarz lemma applied to the function $g(z) := (f(z)-c)/(\overline{c}f(z)-1)$ in $\mathsf{Hol}(\Delta, \Delta)$ gives $1 = f'(0) \leqslant (1-c\overline{c})$ and hence $c = 0$. In the general case choose $a \in \partial B$ and $\lambda \in E^*$ in such a way that $\|c\|a = c$ and $\|\lambda\| = 1 = \lambda(a)$ holds. Then $h \in \mathsf{Hol}(\Delta, \Delta)$ defined by $h(z) = \lambda \circ f(za)$ satisfies $h'(0) = 1$ and hence $\|c\| = h(0) = 0$. Finally, suppose that (g_t) is a family in $\mathsf{Hol}(B, B)$ depending holomorphically on $t \in \Delta$ with $g_0 = f$. Then by Cauchy's inequalities $\|dg_t(0)\| \leqslant 1$ holds and as before we derive $g_t = \mathsf{id}$ for all t. But then also $g_t(0) = 0$ holds, i.e. $g_t = f$ for all t.

EXAMPLE 2.10. Let E be a complex Hilbert space of finite dimension > 1. Let $B \subset E$ be the open unit ball and let $S := \partial B$ be the unit sphere of E. Suppose that $f : S \to S$ is a holomorphic mapping and that $a \in S$ is a given point. Then it is known that f extends to a holomorphic mapping $f : \overline{B} \to \overline{B}$. Therefore, if f is not constant, its restriction to B is a proper holomorphic map $B \to B$. But then by [**1**] $f|B$ is already an automorphism of B. But it is known that every $f \in \mathsf{Aut}(\overline{B})$ is linear fractional and is uniquely determined by the three derivatives $f(a)$, $df(a)$ and $d^2f(a)$ within $\mathsf{Aut}(\overline{B})$. From this we get: Either f is constant, and then f is 1-rigid on S everywhere, or $f \in \mathsf{Aut}(\overline{B})$, and then f is 3-rigid everywhere—but not 2-rigid in any point of S. For more general situations of this type compare [**23**].

PROPOSITION 2.11. *Suppose that the holomorphic mapping $f : U \to V$ is m-rigid at $a \in U$ and that Q is an arbitrary domain. Then the mapping $g : U \times Q \to V$ defined by $g(z, w) = f(z)$ is m-rigid at every point $(a, q) \in U \times Q$, provided the bounded holomorphic functions on U separate the points (this happens for instance if U is a bounded domain).*

Proof. Fix $q \in Q$ and assume that the holomorphic mapping $h : U \times Q \to V$ satisfies $\circ_{(a,q)}(h, g) \geqslant m$. We have to show that $g = h$. Consider

$$\Omega := \{(z, w) \in U \times Q : \exists \varphi \in \mathsf{Hol}(U, Q) \text{ with } \varphi(a) = q, \varphi(z) = w\}$$

and fix $(z, w) \in \Omega$ together with a corresponding φ. Then $\gamma \in \mathsf{Hol}(U, V)$ defined by $\gamma(t) = h(t, \varphi(t))$ satisfies $\circ_a(\gamma, g) \geqslant m$, i.e. $\gamma = f$ and hence $g(z, w) = h(z, w)$ for all $(z, w) \in \Omega$. Let $X \subset U$ be an open non-void subset that can be separated from a via bounded holomorphic functions on U (for instance $X = U \setminus \{a\}$). To every $x \in X$ there is a neighbourhood $Y \subset Q$ of q with $(x, y) \in \Omega$ for all $y \in Y$. This implies by the identity theorem for holomorphic functions that h coincides with g on $X \times Q$ and hence on all of $U \times Q$.

The condition on U in 2.11 cannot be omitted as the following counter example shows: Every holomorphic map $\mathbb{C} \to \Delta$ is constant and hence m-rigid everywhere while no constant map $\mathbb{C} \times \Delta \to \Delta$ is m-rigid at any point.

COROLLARY 2.12. *Let U be a bounded domain. Then for every domain Q the canonical projection $U \times Q \to U$ is 2-rigid everywhere.*

COROLLARY 2.13. *Suppose that U_i is a bounded domain and that the holomorphic mapping $f_i : U_i \to V_i$ is m-rigid at $a_i \in U_i$ for $i = 1, 2$. Then also $f_1 \times f_2 : U_1 \times U_2 \to V_1 \times V_2$ is m-rigid at $a = (a_1, a_2)$.*

3. Rigid linear operators

In the following let $B \subset E$ and $D \subset F$ always be the open unit balls. We are mainly interested in holomorphic maps $f : B \to D$ with $f(0) = 0$. Then the derivative $L := df(0)$ satisfies $\|L\| \leqslant 1$. Therefore, if f is 2-rigid at $0 \in B$ we must have $f = L|B$ and $\|L\| = 1$. It is clear that f never can be 1-rigid at 0 although it may be infinitesimally 1-rigid at 0. This motivates the following definition for the linear case.

DEFINITION 3.1. The linear operator $L \in \mathcal{L}(E, F)$ is called *rigid* if the induced map $B \to rD$ is 2-rigid at $0 \in B$ where $r = \|L\|$, that is, if for every holomorphic mapping $f : B \to rD$ with $f(0) = 0$ and $df(0) = L$ necessarily $f = L|B$ follows. We call L *strictly rigid* if for every holomorphic $f : B \to rD$ the conclusion $f = L|B$ already from the only assumption $df(0) = L$ follows. In case $E \subset F$ is a subspace we call E (strictly) rigid in F if the canonical injection $E \hookrightarrow F$ has this property.

Clearly, the study of rigidity of linear operators L always can be reduced to the case $\|L\| = 1$. The slightly more general situation in 3.1 avoids complicated constants sometimes. Rigidity in our sense is closely related to the vector valued Schwarz lemma in the following form: Suppose $f : B \to D$ with $f(0) = 0$ is holomorphic. Then the derivative $L = df(0)$ has norm $\leqslant 1$ and $\|f(z)\| \leqslant \|z\|$ holds for all $z \in B$. The question then is: *Under what conditions can $f = L|B$ be concluded?* Many authors studied this question under the additional assumption that L is isometric. It is clear that in case L isometric the Banach space E can without loss of generality be identified with a subspace $E \subset F$ via L. To simplify notation we will frequently do so.

We start with the simple situation $\dim E = 1$, compare also 5.1 for a more quantitative statement and also 7.2 for a generalization to higher dimensional E.

LEMMA 3.2. *Let $L : \mathbb{C} \to F$ be a linear operator with $a : L(1) \in \partial D$. Then L is rigid if and only if a is a complex extremal boundary point of D. Also, L is strictly rigid if and only if a is a (real) extremal boundary point of D.*

Proof. Put $f := L|B$ and assume that a is not complex extremal. Then there is a vector $v \in F$ with $v \neq 0$ and $a + \Delta v \subset \partial D$. But then $g(z) = z(a + zv)$ defines a holomorphic map $g : \Delta \to D$ with $dg(0) = L$, i.e. L is not rigid. Assume on the contrary that $a \in \partial B$ is complex extremal and that $g : \Delta \to D$ is a holomorphic mapping with $g(0) = 0$ and $dg(0) = L$. Then $h(z) := g(z)/z$ defines a holomorphic function $h : \Delta \to \overline{D}$ with $h(0) = a$, i.e. $h \equiv a$ and hence $f = g$, compare [**9**] p. 69 or [**21**]. Now assume that $a \in \partial B$ is not extremal. Then there is a vector $v \in E$ with $v \neq 0$ and $\|a \pm v\| = 1$. For every $\alpha, \beta \in \mathbb{C}$ then $2(\alpha a + \beta v) = (\alpha + \beta)(a + v) + (\alpha - \beta)(a - v)$ implies $2\|\alpha a + \beta v\| \leqslant |\alpha+\beta|+|\alpha-\beta|$. Define $g : \Delta \to F$ with $dg(0) = L$ by $2g(z) = 2za+(1+z^2)v$. Then $\|4g(z)\| \leqslant |1+z|^2+|1-z|^2 = 2(1+z\overline{z}) < 4$ for all $z \in \Delta$ shows $g(\Delta) \subset D$, i.e. L is not strictly rigid. It remains to show that $a \in \partial_e D$ implies strict rigidity of L. But this follows from [**12**] p. 27–28 and also from Théorème 3.6 in [**18**].

COROLLARY 3.3. *Suppose that E is arbitrary and that $A \subset \partial B$ is a set of determinacy in E. Then every $L \in \mathcal{L}(E, F)$ with $\|L\| = 1$ and $L(A) \subset \partial_{ec} D$ is rigid. If in addition $L(a) \in \partial_e D$ holds for some $a \in A$ then L is even strictly rigid. In particular, in case $\partial D = \partial_e D$ every linear isometry $L : E \to F$ is strictly rigid.*

In case of $\dim E = 2$ the situation is already much more complicated as the following example indicates. In particular, rigid linear operators of norm 1 need not be isometric even if they are bijective. Notice that by 2.9 every surjective linear isometry between complex Banach spaces is strictly rigid.

EXAMPLE 3.4. Let I be an arbitrary set of cardinality > 1. For $1 \leqslant p < q \leqslant \infty$ fixed consider $E := \ell^p(I)$ and $F := \ell^q(I)$. Then the canonical injection $L : E \to F$ is not rigid if $q = \infty$. In case that I is finite, the inverse operator

$L^{-1} : F \to E$ however always is rigid. In case $p > 1$ the operator L^{-1} is strictly rigid.

Proof. Write every $z \in B$ as tuple $z = (z(i))$, fix $j \in J$ and let $v = e_j$ (i.e. $v(i) = \delta_{ij}$ for all $i \in I$) in the following. Choose furthermore an integer $m \geqslant 2$ together with a constant $c > 0$ such that $(1 - t^p) \leqslant (1 - ct^m)^p$ holds for all $t \in [0,1]$. Then $g(z) = L(z) + c(z(k))^m v$ defines a holomorphic mapping $g : B \to D$ with $\circ_0(f, g) = m \geqslant 2$ for every $k \neq j$ in I if $q = \infty$. Now suppose that I is of finite cardinality n and put $S := (rL)^{-1}$ for $r := n^{(1/p-1/q)} = \|L^{-1}\|$. Then $A = \{z \in \partial D : |z(i)| = |z(j)| \text{ for all } i \in I\}$ is a set of determinacy in F. By 3.3 therefore L^{-1} is rigid since $S(A) \subset \partial_{ec}(B)$.

LEMMA 3.5. *Suppose that $E \subset F$ is a closed linear subspace and that A is a set of determinacy in E. Suppose that for every $a \in A$ there is a closed linear subspace $E_a \subset E$ containing the point a such that E_a is rigid in F. Then also E is rigid in F.*

Proof. Let $f : B \to D$ be a holomorphic mapping with $\circ_0(f, g) \geqslant 2$ for the canonical injection $g : B \hookrightarrow D$. For every $a \in A$ then $f(a) = g(a)$ since E_a is rigid in F. But then $f = g$ by 2.2.

PROPOSITION 3.6. *Let H, K be complex Hilbert spaces and let $L \in \mathcal{L}(H, K)$ have norm 1. Then L is rigid if and only if L is a (not necessarily surjective) isometry.*

Proof. Assume that L is rigid and let $L = V|L|$ be the polar decomposition, where $|L| = (L^*L)^{1/2}$ and $V \in \mathcal{L}(H, K)$ is a partial isometry with $\ker(V) = \ker(L)$. Let $\{E(d\lambda)\}$ be the spectral measure of $|L|$, so

$$|L| = \int_{[0,1]} \lambda E(d\lambda).$$

We claim that $|L| = \mathsf{id}$, that is $E([0, \alpha]) = 0$ for all $\alpha < 1$. Indeed, if $E([0, \beta]) \neq 0$ for some $\beta < 1$, choose a unit vector $a \in H$ fixed under the projection $P := E([0, \beta])$ and denote by $Q := \mathsf{id} - P$ the complementary projection. Define a holomorphic map $h : B \to B$ by $h(z) = (1 - \beta)(z|a)^2 a$ *where* $B \subset H$ is the open unit ball. Since $V^*V(a) = a$ holds and V^* is an isometry on $V(H) = \overline{L(H)}$ we have for all $z \in B$

$$\begin{aligned}
\|L(z) + V(h(z))\|^2 &= \||L|(z) + h(z)\|^2 \\
&= \|P|L|(z) + (1 - \beta)(P(z)|a)^2 a\|^2 + \|Q|L|(z)\|^2 \\
&\leqslant \|P|L|(z)\|^2 + (1 - \beta)^2\|P(z)\|^4 \\
&\quad + 2(1 - \beta)\|P|L|(z)\| \cdot \|P(z)\|^2 + \|Q|L|(z)\|^2 \\
&\leqslant (\beta^2 + (1 - \beta)^2)\|P(z)\|^2 + 2\beta(1 - \beta)\|P(z)\|^2 + \|Q(z)\|^2 \\
&= \|P(z)\|^2 + \|Q(z)\|^2 = \|z\|^2 < 1.
\end{aligned}$$

This is a contradiction to the rigidity of L since $V(h(a)) \neq 0$. Therefore $|L| = \mathsf{id}$ and $L = V$ is an isometry. The converse statement follows from 2.8.

COROLLARY 3.7. *Suppose that H, K are complex Hilbert spaces. Then a linear operator $L : H \to K$ is a surjective isometry if and only if L and L^* are rigid.*

COROLLARY 3.8. *Suppose that E is a complex Hilbert space of finite dimension and that $a \in B$ is a given point. Then a holomorphic mapping $f : B \to B$ is rigid at a if and only if $f \in \mathsf{Aut}(B)$.*

Proof. Suppose that f is rigid at a. Since $\mathsf{Aut}(B)$ acts transitively on B we may assume that $a = 0$. But then $f = L|B$ for some linear isometry of E by 3.6. Because of finite dimension L must be surjective, i.e. $f \in \mathsf{Aut}(B)$

A projection $P \in \mathcal{L}(F)$ is called *contractive* if $\|P\| \leqslant 1$ holds and P is called *bicontractive* if in addition also $\mathsf{id} - P$ is a contractive projection. Furthermore, a contractive projection P from F onto $E \subset F$ is called *neutral* if $\|P(z)\| = \|z\|$ always implies $z \in E$ for all $z \in F$. Neutrality is not invariant under ℓ^∞-sums, more precisely, suppose that P_i is a neutral projection on the complex Banach space F_i with range E_i for $i = 1, 2$. Then $P := P_1 \times P_2$ is a contractive projection on $F = F_1 \oplus_\infty F_2$ with image $E = E_1 \oplus_\infty E_2$, but in general P is not neutral. However, it is easily seen that $A := \{(x, y) \in E : \|x\| = \|y\|\}$ is a set of determinacy in E and that P is almost neutral in the following sense.

DEFINITION 3.9. We call a contractive projection P from F onto E *almost neutral* if there exists a set of determinacy A in E such that $z \in E$ for every $z \in F$ with $\|P(z)\| = \|z\|$ and $P(z) \in A$.

A nontrivial example for an almost neutral projection is obtained as follows (compare also 7.11). Let $H \subset K$ be complex Hilbert spaces with $H \neq K$ and $\dim(H) \geqslant 2$. Let furthermore $p : K \to H$ be the orthogonal projection. Then $P(z) = p \circ z$ defines a contractive projection from $\mathcal{L}(H, K)$ onto $\mathcal{L}(H)$ that is not neutral. With $A \subset \mathcal{L}(H)$ the unitary group we see that P is almost neutral.

PROPOSITION 3.10. *Suppose that there exists an almost neutral projection P from F onto E. Then E is rigid in F.*

Proof. Choose $A \subset E$ as in definition 3.9. Then we may assume that A is balanced and hence that $A \subset B$ holds by 2.3. Let $f : B \to D$ be a holomorphic mapping with $\circ_0(f, g) \geqslant 2$ for the canonical injection $g : B \hookrightarrow D$. Then $g = P \circ f$ holds by Cartan's uniqueness theorem. By Schwarz lemma we have for all $z \in B$

$$\|z\| = \|g(z)\| = \|Pf(z)\| \leqslant \|f(z)\| \leqslant \|z\|$$

and hence $\|Pf(z)\| = \|f(z)\|$, i.e. $f(z) \in E$ for all $z \in A$. This implies $f(z) \in E$ for all $z \in B$ by 2.2 and thus $f = g$.

A little more can be said in 3.10 if the projection P is *strongly neutral* in the following sense: *For every sequence* (z_n) *in* F *with* $\lim \|P(z_n)\| = \lim \|z_n\| < +\infty$ *always* $\lim \|z_n - P(z_n)\| = 0$ *holds*. Then it is not difficult to see that the canonical injection $B \hookrightarrow D$ is rigid everywhere. As an example of this situation we may take for every $1 \leqslant p < \infty$ any ℓ^p-sum $F = E \oplus_p W$ with the canonical projection $P : F \to E$ along W.

4. Quantitative study of rigidity

As before, let E, F be complex Banach spaces with open unit balls B, D. We call a continuous mapping $f : E \to F$ a *homogeneous polynomial of degree n* if there is a symmetric n-linear mapping $q : E^n \to F$ with $f(z) = q(z, z, \ldots, z)$ for all $z \in E$—or equivalently—if f is holomorphic and satisfies $f(tz) = t^n f(z)$ for all $z \in E$ and all $t \in \mathbb{C}$. The n-linear map q is uniquely determined by f and can be recovered from f with the polarization formula

$$q(z_1, z_2, \ldots, z_n) = (2^n n!)^{-1} \sum_{\varepsilon \in \{\pm 1\}^n} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n f(\varepsilon_1 z_1 + \varepsilon_2 z_2 + \cdots + \varepsilon_n z_n).$$

Also $d^n f(a) = n! q$ holds for every $a \in E$. Put

$$\begin{aligned} \mathcal{P}_n &:= \mathcal{P}_n(E, F) := \{\text{homogeneous polynomials } E \to F \text{ of degree } n\} \\ \mathcal{P}^n &:= \mathcal{P}^n(E, F) := \{\text{polynomials } E \to F \text{ of degree} \leqslant n\} = \bigoplus_{k \leqslant n} \mathcal{P}_k \end{aligned}$$

for every $n \in \mathbb{N}$ and denote by $\mathsf{H}^\infty(B, F)$ the Banach space of all bounded holomorphic functions $f : B \to F$ with norm $\|f\| = \sup_{z \in B} \|f(z)\|$. Then $\mathcal{P}_n$ as well as $\mathcal{P}^n$ can be considered as closed linear subspaces of $\mathsf{H}^\infty(B, F)$ for every $n \in \mathbb{N}$ and every $f \in \mathsf{H}^\infty(B, F)$ has a unique expansion

$$f = \sum_{n=0}^{\infty} f_n \qquad \text{with } f_n \in \mathcal{P}_n \text{ for all } n \in \mathbb{N},$$

converging uniformly on every subball $sB \subset B$, $s \in \Delta$. Every $f_n : E \to F$ is given by

$$f_n(z) = \int_{\mathbb{T}} (rt)^{-n} f(rtz)\, dt$$

with $z \in E$ and $r > 0$ satisfying $r_z \in B$ and dt the (normalized) Haar measure on $\mathbb{T}$. In particular, $f \mapsto f_n$ defines a contractive projection P_n from $\mathsf{H}^\infty(B, F)$ onto $\mathcal{P}_n$ for every $n \in \mathbb{N}$.

In the following let $\mathcal{E} := \{f \in \mathsf{H}^\infty(B, F) : f(0) = 0\}$ and denote by $\mathcal{B}$ the closed unit ball of $\mathcal{E}$. For every $t \in \overline{\Delta}$ define the commutation operator $C_t \in \mathcal{L}(\mathcal{E})$ by $f(z) \mapsto f(tz)/t$ if $t \neq 0$ and by $f \mapsto df(0) \in \mathcal{P}_1$ if $t = 0$.

LEMMA 4.1. $t \mapsto C_t$ *defines a semigroup homomorphism* $\overline{\Delta} \to \mathcal{L}(\mathcal{E})$ *with*

$\|C_t\| = 1$ for all t. For every $f = \sum_{n=1}^{\infty} f_n \in \mathcal{E}$ with $f_n \in \mathcal{P}_n$ the family (g_t) in $\mathcal{E}$ with

$$g_t := C_t(f) = \sum_{n=1}^{\infty} f_n t^{n-1}$$

depends holomorphically on $t \in \overline{\Delta}$ and $t \mapsto g_t$ defines a holomorphic curve $\Delta \to \mathcal{E}$. In case $\|f\| = \|f_1\|$ also $\|g_t\| = \|f_1\|$ holds for all t.

Proof. Fix $f \in \mathcal{B}$. Then by Schwarz lemma we have $\|f(z)\| \leqslant \|z\|$ for all $z \in B$ and hence $\|C_t\| \leqslant 1$. From $\mathcal{P} \subset \mathsf{Fix}(C_t)$ we thus get $\|C_t\| = 1$. The last statement follows from $g_1 = f$, $g_0 = f_1$ and $\|g_s\| \leqslant \|g_t\|$ if $|s| \leqslant |t|$.

Fix an operator $L \in \mathcal{L}(E, F)$ with $\|L\| = 1$ in the following. We introduce some numerical invariants that measure the size of non-rigidity of L: Let $\mathcal{A} = \mathcal{A}(L)$ be the set of all $f \in \mathcal{E}$ with $df(0) = 0$ and $\|L + f\| \leqslant 1$. Then L is rigid if and only if $\mathcal{A} = 0$. It is easily seen that $\mathcal{A}$ is closed convex in $\mathcal{E}$ and also is invariant under every operator C_t. From 4.1 we see that $L + \mathcal{A}$ is contained in the boundary $\partial\mathcal{B}$ of $\mathcal{B}$. For every $m \geqslant 2$ let

$$\mathcal{A}_m := \mathcal{A} \cap \mathcal{P}_m, \quad \alpha_m := \sup_{f \in \mathcal{A}_m} \|f\| \quad \text{and} \quad \pi_m := \sup_{f \in \mathcal{A}} \|P_m(f)\|,$$

where P_m is the projection operator $\left(\sum_n f_n\right) \mapsto f_m$ as defined above. Then clearly $\alpha_m \leqslant \pi_m \leqslant 1$ holds and every $\mathcal{A}_m$ is a balanced subset of $\mathcal{B}$, that is

$$\mathcal{A}_m = \{f \in \mathcal{P}_m : \|L + tf\| \leqslant 1 \text{ for all } t \in \overline{\Delta}\}.$$

We may use this equation to define $\mathcal{A}_m$ together with α_m also for the remaining cases $m = 0$ and $m = 1$.

LEMMA 4.2. *$\alpha_m \leqslant \alpha_{m+1}$ for all $m \in \mathbb{N}$ and in particular, the limit $\alpha_\infty := \lim \alpha_m \leqslant 1$ exists.*

Proof. Fix $f \in \mathcal{A}_m$ and let Λ be the unit ball of E^*. For every $\lambda \in \Lambda$, $z \in B$ and $t \in \overline{\Delta}$ we have

$$\|L(z) + t\lambda(z)f(z)\| \leqslant 1,$$

that is, $\lambda \cdot f$ is contained in $\mathcal{A}_{m+1}$. Then the Hahn–Banach theorem implies

$$\|f\| = \sup_{\lambda \in \Lambda} \|\lambda \cdot f\|,$$

which proves the statement.

The operators C_t may be generalized in the following way. Let μ be a regular complex Borel measure on $\overline{\Delta}$ with finite total variation and put

$$\hat{\mu}(k) := \int_{\overline{\Delta}} t^{-k}\, d\mu(t)$$

for every integer $k \leqslant 0$. Then the operator $C_\mu := \int_{\overline{\Delta}} C_t \, d\mu(t) \in \mathcal{L}(\mathcal{E})$ satisfies $\|C_\mu\| \leqslant \|\mu\|$ and for $f = \sum_{n=1}^{\infty} f_n$ as before we have

$$C_\mu(f) = \sum_{n=1}^{\infty} \hat{\mu}(1-n) f_n.$$

In particular, if μ is a probability measure on $\overline{\Delta}$, we have $\|C_\mu\| = 1 = \hat{\mu}(0)$ and

$$\|L + C_\mu(f)\| = \left\| \int_{\overline{\Delta}} C_t(L+f) \, d\mu(t) \right\| \leqslant \int_{\overline{\Delta}} \|L+f\| \, d\mu(t) \leqslant \int_{\overline{\Delta}} d\mu(t) = 1$$

for every $f \in \mathcal{A}$, i.e. C_μ maps the spaces $\mathcal{A}$ and $\mathcal{A}_m$ into themselves for every $m \in \mathbb{N}$. In the following proposition we use measures μ that are concentrated on $\mathbb{T} \subset \overline{\Delta}$.

PROPOSITION 4.3. *For every $m \geqslant 2$ and every $s \in \mathbb{C}$ with $|s| \leqslant 1/2$ the operator sP_m maps $\mathcal{A}$ into $\mathcal{A}_m$. In particular, $\alpha_m \leqslant \pi_m \leqslant 2\alpha_m$ holds and for every fixed $m \geqslant 2$ the factor 2 in this estimate is the best constant valid for all operators L uniformly.*

Proof. Consider $d\mu(t) = \mathrm{Re}(1 + 2st^{1-m})\, dt$, where dt is the Haar measure on $\mathbb{T}$. Then μ is a probability measure on $\mathbb{T}$ with $\hat{\mu}(1-n) = 0$ for all $n \geqslant 2$ except $\hat{\mu}(1-m) = s$. This implies $sP_m(f) = C_\mu(f) \in \mathcal{A}_m$ for all $f \in \mathcal{A}$. The last claim will be verified in example 5.2.

PROPOSITION 4.4. *The following conditions are equivalent for every $L \in \mathcal{L}(E, F)$ with $\|L\| = 1$.*

(i) L is a complex extreme point of the unit ball in $\mathsf{H}^\infty(B, F)$,
(ii) L is rigid,
(iii) $\alpha_m = 0$ for all m (i.e. $\alpha_\infty = 0$).

Proof. (i) $\Longrightarrow$ (ii) Suppose that L is not rigid. Then there is $f = \sum f_n \in \mathcal{B}$ with $f \neq f_1 = L$. Since $g_t := C_t(f)$ depends holomorphically on $t \in \Delta$ we get $\|L + s(g_t - g_0)\| = 1$ for all $s \in \mathbb{C}$ and $t \in \Delta$ with $|2st| \leqslant 1 - |t|$ by [**9**] p. 68. Then $g_t \neq g_0 = L$ for $t \neq 0$ implies that (i) does not hold.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i) Suppose that (i) does not hold. Then there is a non-zero holomorphic map $f : B \to D$ with $\|L + tf\| \leqslant 1$ for all $t \in \Delta$. After replacing f by the function $z \mapsto \lambda(z)^2 f(z)$ for a suitable $\lambda \in E^*$ we may assume that $f \in \mathcal{A}$. By 4.3 there is an $m \geqslant 2$ with $\alpha_m > 0$. Finally, Lemma 4.2 implies $\alpha_\infty > 0$.

The equivalence of (i) and (ii) in 4.4 is already contained in [**12**] p. 25, compare also [**6**] p. 75. In [**12**] it also has been shown that L (using our language) is strictly rigid if it is a (real) extreme point of the unit ball in $\mathsf{H}^\infty(B, F)$.

DEFINITION 4.5. For every $m \geqslant 1$ the linear operator $L \in \mathcal{L}(E, F)$ is called *m-extreme* if L is an extreme point of the unit ball in $\mathcal{P}^m(E, F)$. In case of a complex extreme point we call L *complex m-extreme*.

LEMMA 4.6. *For every $m \geqslant 2$ the conditions 'm-extreme', 'complex m-extreme' and '$\alpha_m = 0$' are equivalent. Furthermore, '$\alpha_1 = 0$' is equivalent to 'complex 1-extreme'.*

The set $\mathcal{A} \subset \mathcal{E}$ is convex and contains the origin. In general, $\mathcal{A}$ it is not circular (compare 5.2 for an example).

LEMMA 4.7. *Suppose that $\mathcal{A}$ is circular. Then the projection P_m maps $\mathcal{A}$ onto $\mathcal{A}_m$ and in particular, $\alpha_m = \pi_m$ holds for every $m \geqslant 2$.*

Proof. Fix $f \in \mathcal{A}$. Then

$$\|L(z) + t^{-m} f(tz)\| = \|L(tz) + t^{1-m} f(tz)\| \leqslant 1$$

holds for all $t \in \mathbb{T}$ and $z \in B$. This implies

$$\|L(z)+P_m f(z)\| = \left\| \int_{\mathbb{T}} (L(z)+t^{-m} f(tz))\, dt \right\| \leqslant \int_{\mathbb{T}} \|L(z)+t^{-m} f(tz)\|\, dt \leqslant 1$$

and hence $P_m(f) \in \mathcal{A}_m$.

5. Some numerical estimates

In the following, for given $L \in \mathcal{L}(E, F)$ with $\|L\| = 1$, the spaces $\mathcal{A}$, $\mathcal{A}_m$ and the numerical invariants α_m, π_m have the same meaning as in the preceding section. We want to get estimates on these invariants in the special situation where one of the spaces E, F has dimension 1. We start with the case $E = \mathbb{C}$ and a quantitative version of Lemma 3.2.

LEMMA 5.1. *Let $L : \mathbb{C} \to F$ be a linear operator with $\|L\| = 1$. Then*

$$\alpha_m = \alpha_0 = \sup\{\|v\| : v \in F, \|a + tv\| \leqslant 1\} \qquad \text{for all } t \in \Delta$$

for all $m \in \mathbb{N}$ where $a := L(1) \in \partial D$.

Proof. Suppose that $f \in \mathcal{A}_m$. Then $f(z) = z^m v$ for some $v \in F$ with $\|a + tv\| \leqslant 1$ for all $t \in \mathbb{T}$. This shows $v \in \mathcal{A}_0$ and thus $\alpha_m \leqslant \alpha_0$.

EXAMPLE 5.2. Let K be a locally compact Hausdorff space and $F := \mathcal{C}_0(K)$. Fix a function $a \in F$ with $\|a\| = 1$ and put $r := \|1 - |a|\|$. Let $L : \mathbb{C} \to F$ be defined by $L(z) = za$. Then by 3.2 the operator L is rigid if and only if $r = 0$ holds. For $v \in F$ the condition $\|a + tv\| \leqslant 1$ for all $t \in \Delta$ is equivalent to $|a| + |v| \leqslant 1$ which implies $\alpha_m = 1 - r$ for all $m \in \mathbb{N}$. We claim that in general the invariants π_m differ from α_m, i.e. the set $\mathcal{A}$ is not circular. To see this, define for every $c \in D$ the holomorphic mappings $g_c : \Delta \to D$ by

$$g_c(z) := \frac{c+z}{1+z\overline{c}} = c + (1 - c\overline{c}) \sum_{m=1}^{\infty} (-\overline{c})^{m-1} z^m,$$

where every $z \in \mathbb{C}$ is identified with the constant function $\equiv z$ on K. Then for all $0 < s, t < 1$ and $z \in \Delta$ we have

$$\|g_{sa}(z) - g_{ta}(z)\| = \left\| \frac{(s-t)(a - z^2\overline{a})}{(1+sz\overline{a})(1+tz\overline{a})} \right\| \leqslant 2(1-|z|)^{-2}|s-t|.$$

This implies that the local uniform limit

$$g := \lim_{s \nearrow 1} g_{sa} \in \mathsf{Hol}(\Delta, \overline{D})$$

exists. Now consider the F-valued function f on Δ defined by

$$f(z) := zg(z) - L(z) = (1 - a\overline{a}) \sum_{m=2}^{\infty} (-\overline{a})^{m-2} z^m.$$

Then f is contained in $\mathcal{A}$ and hence

$$\pi_m \geqslant r^{m-2}(1-r^2) \qquad \text{for all } m \geqslant 2.$$

In particular, $\pi_2 > \alpha_2$ holds if $0 < r < 1$. Consequently, $\mathcal{A}$ is not circular in this case. Also, because of $\lim_{r \to 1} r^{m-2}(1-r^2)/(1-r) = 2$ the example shows that for every fixed $m \geqslant 2$ the estimate $\pi_m \leqslant 2\alpha_m$ in Proposition 4.3 cannot be improved with a universal constant < 2.

Next we consider the case $F = \mathbb{C}$, that is, when L is a linear form on E. For every pair of vectors $a, v \in E$ with $\|a + tv\| \geqslant \|a\|$ for all $t \in \mathbb{C}$ put

$$\delta_a(v) := \limsup_{t \to 0} \frac{\log(\|a+tv\| - \|a\|)}{\log|t|} \in [1, +\infty],$$

where t runs in $\mathbb{C}^*$. Then $\delta_a(sv) = \delta_a(v)$ holds for all $s \in \mathbb{C}^*$ and also $\delta_a(0) = +\infty$.

Now let $L : E \to \mathbb{C}$ be a linear form with $\|L\| = 1$ in the following. Assume that there exists a unit vector $a \in E$ with $L(a) = 1$. Then every $z \in E$ can be uniquely written as $z = (u, v)$ with $u = L(z)$ and $v = z - ua \in V := \ker(L)$. We would like to relate rigidity properties of L to smoothness properties of the unit sphere of E at the point $a = (1, 0)$.

PROPOSITION 5.3. *For every integer $m \geqslant 2$ and $\delta := \inf_{v \in V} \delta_a(v)$ we have*

(i) L is m-extreme (i.e. $\alpha_m = 0$) if $m < \delta$,

(ii) L is not m-extreme if $m > \delta$ and V has the following property: To every $w \in V$ there is a non-zero linear form $\lambda \in V^*$ such that $(u, v) \mapsto (u, \lambda(v)w)$ defines a linear operator of norm 1 on E. This condition is always satisfied if E has dimension 2.

Proof. (i) Suppose that $h \in \mathcal{A}$ is homogeneous of degree m. For every $(u, v) \in B$ and every $t \in \mathbb{T}$ we have $|u + th(u, v)| \leqslant \|(u, v)\|$ and hence

$$|u| + |h(u, v)| \leqslant \|(u, v)\|.$$

Fix $0 < u < 1$ in the following and put $w := v/u$. Dividing by u then gives

$$|u^{m-1}h(1, w)| \leqslant \|(1, w)\| - 1$$

for all $w \in V$ near $0 \in V$. Fix an arbitrary vector $v \in V$ and put $p(t) := u^{m-1}h(1, tv)$ for all $t \in \mathbb{C}$. Then p is a polynomial of degree $\leqslant m$ in t. Choose $\beta > m$ together with a sequence (t_n) in $\Delta \setminus \{0\}$ satisfying

$$\frac{\log(\|(1, t_n v)\| - 1)}{\log |t_n|} \geqslant \beta \qquad \text{for all } n$$

and $\lim t_n = 0$. This implies

$$\|(1, t_n v)\| - 1 \leqslant |t_n|^{\beta} \quad \text{and hence} \quad |p(t_n)| \leqslant |t_n|^{\beta}$$

for all n big enough. Since p has degree $< \beta$ this implies $p = 0$ and hence $h(1, v) = 0$ for all $v \in V$. But this means $h = 0$.

(ii) Because of $m > \delta$ there exists $r > 0$ and a non-zero vector $w \in V$ with $|t|^m \leqslant \|(1, tw)\| - 1$ for all $t \in r\Delta$. Choose $\lambda \in V^*$ as above. We may assume that $|\lambda(v)| < 1$ holds for all $(u, v) \in B$. There exists $c > 0$ such that

$$2c \leqslant 1 \qquad \text{and} \qquad 1 + c|t|^m \leqslant \|(1, tw)\| \quad \text{if } 0 \leqslant |t| \leqslant 2.$$

Consider the homogeneous polynomial $h(u, v) = c\lambda(v)^m$ of degree m on E. We claim that $h \in \mathcal{A}_m$, i.e. $|u + h(u, v)| \leqslant 1$ for all $(u, v) \in B$. Indeed, in case $|\lambda(v)| < 2|u|$ we have

$$\begin{aligned}|u + c\lambda(v)^m| &\leqslant |u|(1 + c|u^{m-1}\lambda(v/u)|^m) \leqslant |u|(1 + c|\lambda(v/u)|^m)\\ &\leqslant |u| \cdot \|(1, \lambda(v/u)w)\| = \|(v, \lambda(v)w)\| < 1.\end{aligned}$$

In case $|\lambda(v)| \geqslant 2|u|$ we have $|u + c\lambda(v)^m| \leqslant |\lambda(v)/2| + c < 1$, which proves the claim.

PROPOSITION 5.4. *Let V be a complex Banach space and put $E := \mathbb{C} \oplus_p V$ for fixed $1 \leqslant p \leqslant +\infty$. Let $L : E \to \mathbb{C}$ be the canonical projection and $a = (1, 0)$. Then for every $v \in V \subset E$ with $v \neq 0$ we have $\delta_a(v) = p$. In particular, L is m-extreme if $m < p$. Since V satisfies the condition in 5.3.ii, the operator L is not m-extreme if $m > p$. Actually, we have $\alpha_m \geqslant 1/p$ if $m = p$ and $\alpha_m = 1$ if $m > p$. In case $V = \mathbb{C}$ and $p = 2$ the equality $\alpha_2 = 1/2$ holds.*

Proof. Fix a linear form $\lambda \in V^*$ with $\|\lambda\| = 1$ and define the homogeneous polynomial $h : E \to \mathbb{C}$ by $h(u, v) := c\lambda(v)^m$ for $c > 0$ to be determined. Then $\|h\| = c$ is clear. By elementary calculus we see: $h \in A_m$ if $m > p$ and $c = 1$ or if $m = p$ and $c = 1/p$. In case $V = \mathbb{C}$ and $p = 2$ one shows that $\mathcal{A}_2 = \{(u, v) \mapsto cv^2 : |c| \leqslant 1/2\}$.

6. Tangent spaces

We start with an example that motivates the following definitions.

EXAMPLE 6.1. Let E be a complex Hilbert space with open unit ball B and let $F := E \oplus E$ with norm

$$\|(z, w)\| = \sup_{t\in\mathbb{R}} \|((\cos t)z + (\sin t)w)\|.$$

Denote by P the projection on F defined by $P(z, w) = (z, 0)$ and identify E with $P(F)$ in the obvious way. The projection P is bicontractive but not almost neutral. Indeed, $\mathrm{Re}(z|w) = 0$ and $\|w\| \leqslant \|z\|$ implies $\|(z, w)\| = \|z\|$. Our methods so far do not guarantee that $E \subset F$ is rigid. To get this, suppose that $f : E \to E$ is a homogeneous polynomial of degree $m \geqslant 2$ satisfying

$$\|(z, f(z))\| \leqslant 1 \qquad \text{for all } z \in B.$$

This is easily seen to be equivalent to $\mathrm{Re}(f(z)|z) = 0$ for all unit vectors $z \in E$ and hence for all $z \in E$ since f is homogeneous. Geometrically this means that every vector $f(z)$ is tangent at z to the sphere with radius $\|z\|$ about the origin. The same holds for if in place of f, i.e.

$$(f(z)|z) = 0 \qquad \text{for all } z \in E.$$

But then polarization gives $(f(z)|w) = 0$ for all $z, w \in E$, i.e. $f = 0$ and therefore E is rigid in F by Proposition 4.4. It can be shown that F is isometrically isomorphic to the complex Banach space of all $\mathbb{R}$-linear operators $X \to E$, where X is a real Hilbert space of real dimension 2.

Let F be a an arbitrary complex Banach space with open unit ball D. For every $a \in F$ denote by S_a the set of all $\lambda \in F^*$ with $\lambda(a) = \|\lambda\| \cdot \|a\|$ and $\|\lambda\| = \|a\|$. Then S_a is a non-void convex subset of F^* with $S_{ta} = \bar{t}S_a$ for all $t \in \mathbb{C}$ and hence also $S_a(v) := \{\lambda(v) : \lambda \in S_a\}$ is convex in $\mathbb{C}$ for every $v \in F$. Put

$$T_a^{\mathbb{R}} := \{v \in F : S_a(v) \subset i\mathbb{R}\} \quad \text{and} \quad T_a := \{v \in F : S_a(v) = \{0\}\}.$$

Then the $\mathbb{R}$-linear subspace $T_a^{\mathbb{R}}$ for $a \neq 0$ may be considered as the real tangent space at a to the sphere $\{v \in F : \|v\| = \|a\|\}^a$ and $T_a = T_a^{\mathbb{R}} \cap iT_a^{\mathbb{R}}$ is called the *complex tangent space at* a. We call $a \in F$ a *smooth point* if S_a consists of a single functional or equivalently if $F = \mathbb{C}_a + T_a$. For every smooth $a \in F$ denote by s_a the unique functional in S_a. For instance, if F is a complex Hilbert space, then every $a \in F$ is smooth and $s_a(v) = (v|a)$ holds for all $a, v \in F$.

Our definition of tangent space implies in particular $T_0 = F$ for the origin. This simplifies later notations and also means that for T_a only the case $a \neq 0$ counts. The following characterization of tangent spaces in terms of differentiability conditions seems to be known.

REMARK 6.2. For every $0 \neq a \in F$ the vector $v \in F$ is in $T_a^{\mathbb{R}}$ (in T_a respectively) if and only if

$$\lim_{t \to 0} \frac{\|a + tv\| - \|a\|}{t} = 0$$

holds where t runs in $\mathbb{R}$ (in $\mathbb{C}$ respectively).

Simple examples show that $\{(a, v) \in F^2 : v \in T_a\}$ is not closed in F^2 in general. Therefore, denote by C_a the set of all $v \in F$ such that there exist sequences (a_n), (v_n) in F with $a = \lim a_n$, $v = \lim v_n$ and $v_n \in T_{a_n}$ for all n. Then C_a is a closed complex cone in F with $T_a \subset C_a$. For every subset $A \subset F$ we put

$$T_A := \bigcap_{a \in A} T_a \quad \text{and} \quad C_A := \bigcap_{a \in A} C_a.$$

To indicate the dependence on F we also write $T_a(F)$, $T_A(F)$ and $C_A(F)$ instead of T_a, T_A and C_A. For arbitrary complex Banach spaces $E \subset F \subset R$ the identity $T_E(F) = F \cap T_E(R)$ is clear by the Hahn–Banach theorem. For every contractive projection P from F onto E and every $a \in E$ we have $P(T_a(F)) = T_a(E)$. Also, for every $a \in F$ and every skew-hermitian operator $\delta \in \mathcal{L}(F)$, i.e. $\|\exp(t\delta)\| = 1$ for all $t \in \mathbb{R}$, the vector $\delta(a)$ belongs to the tangent space $T_a^{\mathbb{R}}$.

LEMMA 6.3. *For every closed linear subspace $E \subset F$ the space T_E is a closed linear subspace of F with $E \cap T_E = 0$. Furthermore, $R := E + T_E$ is closed in F and the projection $R \to E$ along T_E is contractive.*

Proof. For every $a \in E$ and $v \in T_a$ we have $\|a + v\| \geqslant \|a\|$. In particular, $a = 0$ if $a + v = 0$, i.e. $E \cap T_E = 0$. The projection $P : R \to E$ along T_E is contractive. This implies for every Cauchy sequence (z_n) in R that also (Pz_n) and $(z_n - Pz_n)$ are Cauchy sequences in E and T_E, respectively. Therefore (z_n) converges in R and thus R is closed in F.

We call the linear subspace $E \subset F$ *smooth in F* if $F = E + T_E$.

EXAMPLE 6.4. Let K be a locally compact Hausdorff space and $F := C_0(K)$. For every unit vector $a \in F$ put

$$\Sigma_a := \{s \in K : |a(s)| = 1\}.$$

Then S_a is the space of all linear forms

$$f \longmapsto \int_{\Sigma_a} f(s)\overline{a(s)}\, d\mu(s)$$

where $\mu \geqslant 0$ is a regular Borel measure on Σ_a with $\mu(\Sigma_a) = 1$. In particular, if $E \subset F$ is a closed linear subspace and

$$\Omega := \{s \in K : |a(s)| = 1 \qquad \text{for some unit vector } a \in E\}$$

then $T_E(F) = \{f \in F : f|\Omega = 0\}$. Therefore, $T_E(F) = 0$ if and only if Ω is dense in K. Also, $a \in F$ is smooth if and only if $T_a = C_a$ holds. The subspace E is smooth in F if and only if the restriction operator $E \to C_0(\overline{\Omega})$ is surjective and then in particular E has to separate the points of $\overline{\Omega}$.

DEFINITION 6.5. The complex Banach space E is called a JB*-*triple* if the group $\mathsf{Aut}(B)$ acts transitively on the unit ball $B \subset E$. More generally, for an arbitrary complex Banach space F with open unit ball D a closed linear subspace $E \subset F$ is called a JB*-subtriple of F if the group $\{g \in \mathsf{Aut}(D) : g(B) = B\}$ acts transitively on B. Then clearly E is a JB*-triple by itself.

The name JB*-triple comes from the fact that for every JB*-subtriple $E \subset F$ there is a natural triple product mapping $\{\} : F \times E \times F \to F$ such that $\{zaw\}$ is symmetric bilinear in $(z, w) \in F^2$, antilinear in $a \in E$ and such that for every $a \in E$ the polynomial $a - \{zaz\} \in \mathcal{P}^2(F, F)$ is a complete holomorphic vector field on D tangent to the subspace $E \subset F$, compare [**3**] and [**15**].

EXAMPLE 6.6. Let F be a C*-algebra or more generally a JB*-algebra with unit e, compare [**11**]. Then F is also a JB*-triple. Furthermore, the self-adjoint part $J := \{z \in F : z^* = z\}$ is a JB-algebra and $T_e^{\mathbb{R}}(F) = iJ = \{z \in F : z^* = -z\}$ (compare 7.8). In particular, $T_E(F) = 0$ holds for every closed linear subspace $E \subset F$ with $e \in E$.

DEFINITION 6.7. We say that the pair $E \subset F$ of complex Banach spaces satisfies

(i) Property P if $f(E) \subset T_E(F)$ holds for every holomorphic mapping $f : E \to F$ satisfying $f(z) \in T_z(F)$ for all $z \in E$.
(ii) Property Q if there exists a complex Banach space $R \supset F$ such that E is a JB*-subtriple of R.

The assumptions in 6.7 may be weakened in several ways: From the power series expansion of holomorphic functions it is clear that in 6.7.i instead of arbitrary holomorphic mappings f only homogeneous polynomials f have to be checked. Also, the condition $f(z) \in T_z$ only has to be assumed for all $z \in A$ where A is some set of determinacy in E. Clearly, E is a JB*-triple if Property Q holds.

PROPOSITION 6.8. *Suppose that $E \subset F$ are complex Banach spaces and denote by $L : E \hookrightarrow F$ the canonical injection. Suppose furthermore that $f : B \to D$ is a holomorphic mapping with $df(0) = L$. Then $f(0) \in T_E(F)$ and $g(z) \in T_z(F)$ holds for all $z \in B$ and $g := f - L$. In particular, if $T_E(F) = 0$*

and $E \subset F$ satisfies Property P*, we have $f = L|B$ and hence E is strictly rigid in F.*

Proof. For every $a \in \partial B$ and $\lambda \in S_a$ consider the function $h \in \mathsf{Hol}(\Delta, \Delta)$ defined by $h(t) = \lambda \circ f(ta)$. Then $h'(0) = 1$ implies $h(t) = t$ and hence $\lambda(g(t_a)) = 0$ for all t. This means $g(z) \in T_z$ for all $z \in B$ and in particular $f(0) = g(0) \in T_E(F)$. Now suppose that $T_E(F) = 0$ holds and that E is not rigid in F. Then we may assume that $g \neq 0$ is a homogeneous polynomial of degree $m \geqslant 2$ by Proposition 4.3. Clearly, $g(z) \in T_z$ holds for all $z \in E$ by homogeneity and hence Property P cannot be satisfied.

Obviously every linear subspace $E \subset F$ of dimension 1 satisfies Property P. Further examples are obtained in the following way.

LEMMA 6.9. *Suppose that P is a contractive projection from F onto E. Let $f \in \mathsf{Hol}(E, F)$ satisfy $f(z) \in T_z(F)$ for all $z \in E$. Then $f(E) \subset \ker P$.*

Proof. $g := P \circ f$ satisfies $g(z) \in T_z(E)$ for all $z \in E$ which implies $g = 0$ by the special case $E = F$ of the following proposition.

PROPOSITION 6.10. *The pair $E \subset F$ satisfies Property* P *if E is smooth in F.*

Proof. Because of 6.3 and 6.9 we only have to consider the special case $E = F$. Fix a holomorphic map $f : E \to E$ with $f(a) \in T_a$ for all $a \in \partial B$. Then f is a complete holomorphic vector field on the unit ball $B \subset E$ by [**19**], compare also [**20**], p. 28. But if has the same property and hence is also complete on B, i.e. $f = 0$.

COROLLARY 6.11. *Let E, W be arbitrary complex Banach spaces and $F = E \oplus_p W$ the ℓ^p-sum for $1 \leqslant p \leqslant \infty$. Then the pair $E \subset F$ satisfies Property* P.

Proof. Let $f \in \mathsf{Hol}(E, F)$ satisfy $f(z) \in T_z(F)$ for all $z \in E$ and denote by $P : F \to E$ the canonical projection along W. In case $p = 1$ we have $T_z(F) = T_z(E)$ for all $z \in E$ and the statement follows by 6.10. In case $p > 1$ the subspace E is smooth in F.

For every measure space (X, μ) and every p with $2 \leqslant p < \infty$ proposition 6.10 applied to $E = F = \mathrm{L}^p(X, \mu)$ gives $f = 0$ for every holomorphic function $f : E \to E$ satisfying

$$\int_X f(z)\overline{z}|z|^{p-2}\, d\mu = 0$$

for all $z \in E$. In case $p = 2$ this is trivial (compare the reasoning in 6.1) and for $p > 2$ a direct proof also can be obtained by taking real derivatives with respect to z and then considering their complex linear as well as their complex antilinear parts. Notice that every $a \in E$ is smooth and that for $a \neq 0$ the corresponding supporting functional s_a is given by

$$s_a(v) = \|a\|^{2-p} \int_X v\overline{a}|a|^{p-2}\, d\mu.$$

All these considerations remain valid for $1 < p < 2$ if for every $a \in E$ the function $\overline{a}|a|^{p-2}$ on X is interpreted in an appropriate way.

Property P does not always hold.

EXAMPLE 6.12. Let E, W be complex Banach spaces and let $A \in \mathcal{L}(E)$ with $\|A\| = 1$ be an operator such that $E_1 := \{z \in E : \|Az\| = \|z\|\}$ is a linear subspace with $0 \neq E_1 \neq E$. Let F be the Banach space $E \oplus W$ with norm given by

$$\|(z, w)\| = \max(\|z\| \cdot \|Az\| + \|w\|)$$

for all $z \in E$ and $w \in W$. Then for all $z \in E \subset F$ we have

$$T_z(F) = \begin{cases} T_z(E) & z \in E_1 \setminus \{0\} \\ T_z(E) \oplus W & \text{otherwise.} \end{cases}$$

Therefore, if $0 \neq \lambda \in E^*$ satisfies $\lambda(E_1) = 0$ and $0 \neq v \in W$ is a given vector, then $f(z) := \lambda(z)v \in T_z(F)$ defines a holomorphic map $f : E \to F$ with $f(E) \not\subset T_E(F) = \{0\}$, but clearly $f(E) \subset C_E(F)$ holds.

PROPOSITION 6.13. *Suppose that $E \subset F$ satisfies Property Q and that $f : B \to F$ is a holomorphic mapping with*

$$\lim_{z \to a} \lambda \circ f(z) = 0$$

for every $a \in \partial B$, $\lambda \in S_a$ and z running over the open unit ball B of E. Then $f(B) \subset T_E(F)$.

Proof. Because of $T_E(F) = F \cap T_E(R)$ for every JB*-triple $R \supset F$ we may assume without loss of generality that F is a JB*-triple containing E as a subtriple. Fix $a \in \partial B$ and $\lambda \in S_a$. Then we also have $\lim_{z \to a} \lambda \circ f(tz) = 0$ for all $t \in \mathbb{T}$. Therefore, if we put $g(s) := \lambda \circ f(sa)$ for $s \in \Delta$, the holomorphic function $g : \Delta \to \mathbb{C}$ satisfies $\lim_{|s| \to 1} g(s) = 0$, i.e. $g \equiv 0$ and hence $f(sa) \in T_a$ for all $s \in \Delta$. This shows $f(0) \in T_E := T_E(F)$. Fix an arbitrary point $c \in B$. Then there exists a complete holomorphic vector field $X^\alpha := (\alpha - \{z\alpha z\})$ on the open unit ball D of F with $g(0) = c$, $g(B) = B$, $dg(0) = \exp(L)$ and $L(E) \subset E$ for $g := \exp(X^\alpha) \in \mathsf{Aut}(D)$ and a certain hermitian operator $L \in \mathcal{L}(F)$, compare [**17**] Proposition 2.6. For every real t the isometry $\exp(itL) \in \mathsf{GL}(F)$ leaves the subspaces E and T_E invariant. This implies that also L and consequently also $dg(0)$ leaves T_E invariant. Define the holomorphic mapping $\tilde{f} : B \to F$ by $f(g(w)) := dg(w)\tilde{f}(w)$ for all $w \in B$. Because of $f(c) = dg(0)\tilde{f}(0)$ we only have to show that also $\tilde{f}$ satisfies the assumptions of the proposition since then $\tilde{f}(0) \in T_E$ by the above reasoning. For this fix $b \in \partial B$ and $\mu \in S_b$. By [**16**] g extends to a biholomorphic mapping $g : U \to V$ for suitable open neighbourhoods

U, V of $\overline{D}$ in F. Consider $a := g(b) \in \partial B$ and $\lambda := \mu \circ dg(b)^{-1} \in F^*$. Then $S := \{g(z) : z \in U, \mu(z) = 1\}$ is a complex-analytic hypersurface of V with $S \cap D = \emptyset$. Therefore also the corresponding tangent hyperplane $\{z \in F : \lambda(z) = \lambda(a)\}$ at $a \in S$ does not intersect D, i.e. $\lambda \in \mathbb{C}S_a$ and thus

$$\lim_{w \to b} \mu \circ \tilde{f}(w) = \lim_{w \to b} \mu \circ dg(w)^{-1} \circ f(g(w)) = \lim_{z \to a} \lambda \circ f(z) = 0.$$

COROLLARY 6.14. *Property* Q *implies Property* P.

THEOREM 6.15. *Suppose that $E \subset F$ are complex Banach spaces satisfying Property* P *and $T_E(F) = 0$. For B, the open unit ball of E, let $\mathcal{F}$ be the space of all holomorphic mappings $f : \overline{B} \to F$ with $f(a) \in T_a^{\mathbb{R}}(F)$ for every $a \in \partial B$. Then $\mathcal{F} \subset \mathsf{Hol}(\overline{B}, F)$ is an $\mathbb{R}$-linear subspace with $\mathcal{F} \cap i\mathcal{F} = 0$ and every $f \in \mathcal{F}$ is a polynomial of degree at most 2. Every f is uniquely determined in $\mathcal{F}$ by $f(0)$ and $df(0)$.*

Proof. Fix $f \in \mathcal{F}$ and expand it on $\overline{B}$ into the uniformly convergent series $f = \sum f_n$ with $f_n \in \mathcal{P}_n(E, F)$ for every $n \in \mathbb{N}$. Fix $a \in \partial B$, $\lambda \in S_a$ and define $c_n := \lambda \circ f_n(a) \in \mathbb{C}$ for all n. Then we also have $\mathrm{Re}(\overline{t}\lambda \circ f(ta)) = 0$ for every $t \in \mathbb{T}$ and hence

$$\sum_{n=0}^{\infty}(t^{n-1}c_n + t^{1-n}\overline{c}_n) = 2\,\mathrm{Re}(\overline{t}\lambda \circ f(ta)) = 0$$

for all $t \in \mathbb{T}$. Since the coefficients of a Fourier series are uniquely determined we get

$$(*) \qquad c_0 + \overline{c}_2 = c_1 + \overline{c}_1 = 0 \qquad \text{and} \qquad c_n = 0 \quad \text{for all } n \geqslant 3.$$

On the other hand, for every $f \in \mathsf{Hol}(\overline{B}, F)$, condition $(*)$ for every $a \in \partial B$, $\lambda \in S_a$ and $c_k = \lambda \circ f_k(a)$ is also sufficient for f to be in $\mathcal{F}$, i.e. $f_n = 0$ for all $n \geqslant 3$ as a consequence of Property P and hence $f = f_0 + f_1 + f_2$ is a polynomial of degree at most 2. In particular $\mathcal{F} \cap i\mathcal{F} = 0$ follows. In case $f_0 = 0$ the quadratic function f_2 is in $\mathcal{F} \cap i\mathcal{F}$, i.e. every $f \in \mathcal{F}$ is uniquely determined by f_0 and f_1.

Suppose that $E \subset F$ is a JB*-subtriple. For every $a \in E$ the polynomial $h : F \to F$ defined by $h(z) = a - \{zaz\}$ is a complete holomorphic vector field on D and therefore the restriction $f = h|\overline{B}$ is in the space $\mathcal{F}$. The proof of Proposition 6.15 therefore gives the decomposition $\mathcal{F} = \mathcal{K} \oplus \mathcal{P}$ where

$$\mathcal{K} = \{f \in \mathcal{F} : f(0) = 0\} = \mathcal{F} \cap \mathcal{L}(E, F) \text{ and}$$
$$\mathcal{P} = \{f \in \mathcal{F} : df(0) = 0\} = \{z \mapsto a - \{zaz\} : a \in E\}.$$

7. The JB*-triple case

Property P together with $T_E(F) = 0$ is sufficient but by no means necessary for the rigidity of $E \subset F$. For instance, if F is a complex Hilbert space and $E \neq F$ is an arbitrary closed linear subspace, then E is rigid in F as a consequence of 3.3 or of 3.10. On the other hand, $E \subset F$ satisfies Property P as a consequence of 6.14 and $T_E(F) \neq 0$ is the orthogonal complement of E in the Hilbert space F. Therefore, the tangent spaces $T_a(F)$ and $T_E(F)$ still seem to big for some rigidity questions.

For every $a \in F$ denote by $\Theta_a = \Theta_a(F) \subset F$ the smallest closed linear subspace containing every $v \in F$ with $\|a + tv\| = \|a\|$ for all $t \in \mathbb{C}$ with $|t| \leqslant \|a\|$. Then Θ_a is a linear subspace of T_a with $\Theta_0 = F$ and $\Theta_{sa} = \Theta_a$ for all $s \in \mathbb{C}^*$. Clearly, $a \in \partial D$ is a complex extreme boundary point of D if and only $\Theta_a = 0$. In case $E \subset F$ also $\Theta_a(E) = \Theta_a(F) \cap E$ holds for all $a \in E$. The following result is well known, compare also Théorème 3.1 in [**18**].

LEMMA 7.1. *Let U be a domain in a complex Banach space and suppose that $f : U \to F$ is a holomorphic mapping with $f(U) \subset \overline{D}$. Then $f(U)$ is contained in the affine subspace $(a + \Theta_a)$ for every $a \in f(U) \cap \partial D$.*

Proof. We may assume that $U = \Delta$ and $a = f(0) \in \partial D$. Fix an arbitrary $c \in \Delta \setminus \{0\}$ and consider

$$v := \frac{1 - |c|}{2|c|}(f(c) - a) \in F.$$

Then [**9**], p. 68, implies $a + \Delta v \subset \overline{D}$ and hence $a + \Delta v \subset \partial D$, i.e. $(f(c) - a) \in \Theta_a$.

PROPOSITION 7.2. *Let $E \subset F$ be arbitrary complex Banach spaces and suppose that the balanced set*

$$A := \{a \in E : \Theta_a(E) = \Theta_a(E) = \Theta_a(F) \text{ or } a = 0\}$$

is a set of determinacy in E. Then E is rigid in F.

Proof. Let $f : B \to D$ be a holomorphic mapping with $f(0) = 0$ and $df(0) : E \hookrightarrow F$ the canonical injection. Fix an arbitrary unit vector $a \in A$ and consider $h(t) = f(ta)/t \in \overline{D}$ for all $t \in \Delta$. Then $h(t) \in (a + \Theta_a(F)) \subset E$ for all $t \neq 0$ by 7.1, i.e. $f(z) \in E$ for all $z \in A \cap B$. Since A is balanced in E we derive $f(B) \subset B$ by 2.3 and 2.2. But then Cartan's uniqueness theorem implies that f is linear.

Proposition 7.2 for the special case of finite dimensions and A dense in E essentially already occurs in [**22**], compare Théorème 5.2. The proof is different from ours and does not extend to infinite dimensions. In the following we want to get rigidity also in cases where the set A in Proposition 7.2 is not a set of determinacy (even where $A = \{0\}$, compare the discussion at the end of this section).

As before in the case of the tangent spaces we put

$$\Theta_E := \Theta_E(F) := \bigcap_{a \in E} \Theta_a(F)$$

for every closed linear subspace $E \subset F$. Then $\Theta_E(F) \subset T_E(F)$ is a closed linear subspace and the following analogue of Proposition 6.13 holds. The proof is similar to the one of 6.13.

PROPOSITION 7.3. *Suppose that $E \subset F$ satisfy Property* Q. *Let $f : B \to F$ be a holomorphic mapping with*

$$\lim_{z \to a} \lambda \circ f(z) = 0$$

for every $a \in \partial B$ and every $\lambda \in F^*$ with $\lambda(\Theta_a) = 0$. Then $f(B) \subset \Theta_E(F)$.

COROLLARY 7.4. *Let $f : E \to F$ be a holomorphic mapping with $f(a) \in \Theta_a$ for all $a \in E$. Then $f(E) \subset \Theta_E(F)$ if $E \subset F$ satisfy Property* Q.

For the rest of the section let F be a JB*-triple with triple product $\{abc\}$. By the symmetry in the outer variables the triple product is uniquely determined by all triple products of the form $\{aba\}$. For every $a, b \in F$ denote by $a\Box b \in \mathcal{L}(F)$ the operator $z \mapsto \{abz\}$. Then $\Box$ can be understood as an operator-valued positive-definite hermitian product on F, compare [**15**] for details. In particular, we write $a \perp b$ if $a\Box b = 0$ or—equivalently—if $b\Box a = 0$. For every subset $A \subset F$ call $A^\perp := \{z \in F : z \perp A\}$ the *annihilator* of A in F.

Examples of JB*-triples are for instance all Hilbert spaces with triple product given by $\{zaz\} = (z|a)z$ or more generally all spaces $\mathcal{L}(H, K)$ with triple product $\{zaz\} = za^*z$ where H, K are arbitrary complex Hilbert spaces and * is the usual adjoint of operators. The class of subtriples of all $\mathcal{L}(H, K)$ includes in particular the class of all C*-algebras.

For every JB*-triple F and every $a \in F$ the smallest closed subtriple of F containing a is isometrically isomorphic to a space $\mathcal{C}_0(K)$ with $K \subset (0, \infty) \subset \mathbb{R}$ and $K \cup \{0\}$ compact. For the study of rigidity and tangent spaces in JB*-triples therefore the following example is helpful.

EXAMPLE 7.5. Let $F = \mathcal{C}_0(K)$, the linear subspace $E \subset F$ and $\Omega \subset K$ be as in Example 6.4. Then F is a JB*-triple and it is seen easily that $\Theta_E(F)$ is the closure of $\{f \in F : \overline{\Omega} \cap \operatorname{support}(f) = \emptyset\}$ in F, i.e. $\Theta_E(F) = T_E(F)$ by Stone–Weierstraß.

PROPOSITION 7.6. *For every closed linear subspace $E \subset F$ and every unit vector $a \in F$ we have*

(i) $(u - \{aua\} + \{vwa\} - \{wva\}) \in T_a^{\mathbb{R}}(F)$ *for all $u, v, w \in F$,*
(ii) $E^\perp \subset \Theta_E(F)$.

Proof. (i) The polynomial $f(z) = u - \{zuz\} + \{vwz\} - \{wvz\}$ is a complete holomorphic vector field on the open unit ball D of F, compare [**15**]. Therefore the solution $g : \mathbb{R} \to F$ of the initial value problem $g'(t) = f(g(t))$, $g(0) = a$

satisfies $g(\mathbb{R}) \subset \partial D$ and hence $\mathrm{Re}(\lambda \circ g)$ has a critical value in 0 for every $\lambda \in S_a$, i.e. $g'(0) = f(a) \in T_a^{\mathbb{R}}$.

(ii) Follows from the fact that $v \perp w$ implies $\|v + w\| = \max(\|v\|, \|w\|)$.

COROLLARY 7.7. $\mathsf{Ker}(E) := \{z \in F : Q(E)z = 0\} \subset T_E(F)$ *for every linear subspace* $E \subset F$.

The element $e \in F$ is called a *tripotent* if $\{eee\} = e$ holds. Every tripotent e induces a direct sum decomposition $F_1 \oplus F_{1/2} \oplus F_0$ (the *Peirce decomposition* with respect to e), where $F_k = F_k(e)$ is the k-eigenspace of the operator $e\Box e$ in F. Every Peirce space F_k is a JB*-subtriple and the canonical projection $F \to F_k$ is contractive. For $k = 1/2$ the Peirce projection is even bicontractive.

A special rôle is played by the conjugate linear operators $Q(a)$ on F defined by $z \mapsto \{aza\}$. These satisfy the fundamental formula $Q(Q(a)b) = Q(a)Q(b)Q(b)$. For every tripotent $e \in F$ the operator $Q(e)$ splits F into real subspaces $F = F^1 \oplus F^{-1} \oplus F^0$ where F^k is the k-eigenspace of $Q(e)$. Then $F_1 = F^1 \oplus F^{-1}$, $F^{-1} = iF^1$ and $F^0 = F_{1/2} \oplus F_0$.

LEMMA 7.8. $T_e^{\mathbb{R}}(F) = F^{-1} \oplus F^0$, $T_e(F) = F^0$ *and* $\Theta_e(F) = F_0$ *for every tripotent* $e \in F$.

Proof. Proposition 7.6 implies $F_0 \subset \Theta_e$, $F^{-1} \oplus F_{1/2} = \{\{eve\} - \{vee\} : v \in F\} \subset T_e$ and hence also $F^{-1} \oplus F^0 \subset T_e^{\mathbb{R}}$. Now consider a vector $v \in T_e^{\mathbb{R}} \cap F^1$ and denote by $V \subset F$ the closed (complex) subtriple generated by v and e. Then V coincides in the JB*-algebra $F_1(e)$ with the closed complex subalgebra generated by the unit e and the self-adjoint element v. In particular, V is a unital associative JB*-algebra and hence isometrically isomorphic to $\mathcal{C}(K)$ for some compact subset $K \subset \mathbb{R}$ in such a way that $e(s) = 1$ and $v(s) = s \geqslant 0$ for all $s \in K$. But then $\int_K v(s)\,d\mu(s) = 0$ for every Borel measure $\mu \geqslant 0$ implies $K = \{0\}$ and hence $T_e^{\mathbb{R}} = F^{-1} \oplus F^0$ as well as $\Theta_e \subset T_e = F^0$. The proof will be finished if we show that $w \in F_0$ for all $w \in \Theta_e$. For this we may assume that $w \in F_{1/2}$ and $e + \Delta w \subset \partial D$ holds. Consider the complete holomorphic vector field $f(z) = \{zez\} - e$ on D and denote by $t \mapsto g_t(z)$ for every $z \in \overline{D}$ the solution of $\partial g_t(z)/\partial t = f(g_t(z))$ to the initial value $g_0(z) = z$. Then $g_t \in \mathsf{Hol}(\overline{D}, \overline{D})$ for all $t \in \mathbb{R}$. Furthermore, $f(e) = 0$ and $df(e) = 2e\Box e$ imply $g_t(e) = e$ and $dg_t(e) = \exp(te\Box e)$ for all t, compare [**17**] p. 210. For every t define $h_t \in \mathsf{Hol}(\Delta, F)$ by $h_t(s) = g_t(e + sw) \in \partial D$. Then $\{h_t : t \in \mathbb{R}\}$ is a bounded family of holomorphic mappings. Therefore also the set of all derivatives $\{h_t'(0) = e^t w : t \in \mathbb{R}\}$ must be bounded in F, i.e. $w = 0$.

LEMMA 7.9. *For every tripotent* $e \in F$ *and every closed linear subspace* $E \subset F_1(e)$ *we have (i)* $F^0(e) \subset T_E(F)$, *(ii)* $F_0(e) \subset \Theta_E(F)$.

Proof. For every unit vector $a \in E$ and every $u \in F^0(e)$ we have $u = u - \{aua\} \in F^0(e)$ by 7.6.i, proving (i). The second statement follows from $E \perp F_0$.

LEMMA 7.10. *Suppose that $E \subset F$ is a JB*-subtriple with the following property: To every $v \neq 0$ in F there exists a tripotent $e \in E$ with $\{eve\} \neq 0$ (i.e. the Peirce-1-component of v with respect to the tripotent e does not vanish). Then $T_E(F) = 0$ and hence E is rigid in F.*

Proof. Suppose that $v \neq 0$ for a vector $v \in T_E(F)$. Choose a tripotent $e \in E$ with $\{eve\} \neq 0$. Then $v \in F^0(e)$ by 7.8, a contradiction.

As an application of 7.10 we see for instance that for $F = \mathcal{L}(H, K)$ the subspace $\mathcal{K}(H, K)$ of all compact operators is a rigid subspace of F for every pair of complex Hilbert spaces H, K.

PROPOSITION 7.11. *Let e be a tripotent in the JB*-triple F and denote by $P = Q(e)^2$ the Peirce projection from F onto $E := F_1(e)$. Then the following conditions are equivalent.*

(i) $F_0(e) = 0$,
(ii) *P is almost neutral,*
(iii) *E is rigid in F.*

Proof. (i) $\Longrightarrow$ (ii) E is a JB*-algebra with unit e in the product $a \circ b = \{aeb\}$ and the involution $a^* = \{eae\}$. The selfadjoint part $V := F^1(e)$ is a JB-algebra and $\Omega := \exp(V)$ is an open convex cone in V. The generalized unit circle $A := \exp(iV)$ is a set of determinacy in E. This follows from the fact that the real analytic mapping $\varphi : V \to E$ defined by $v \mapsto \exp(iv)$ has real differential $d\varphi(0) : V \to E$ given by $v \mapsto iv$. Every $a \in A$ is a tripotent with $F_{1/2}(a) = W := F_{1/2}(e)$. Indeed, $a = \exp(2iv)$ holds for some $v \in V$ and $\lambda := \exp(i(v\Box e + e\Box v)) \in \mathsf{GL}(F)$ is a triple automorphism with $\lambda(c) = a$ and $\lambda(W) = W$. Therefore it is enough to show that $\|e + w\| = 1$ for $w \in W$ implies $w = 0$. Suppose on the contrary that $w \neq 0$ holds. Then $c := \{eww\} \in \overline{\Omega}$ and $c \neq 0$, compare [**16**] p. 183. But this is not possible—the closed real subalgebra of V generated by e and c can be realized as $\mathcal{C}(K, \mathbb{R})$ in such a way that e is the function $\equiv 1$ on the compact space K and $c \geqslant 0$.

(ii) $\Longrightarrow$ (iii) Follows from Proposition 3.10.

(iii) $\Longrightarrow$ (i) Is trivial because of $e^\perp = F_0(e)$.

For every tripotent $e \in F$ the Peirce spaces $F_1(e)$ and $F_0(e)$ are inner ideals of F—by definition, a closed linear subspace $J \subset F$ is called an *inner ideal* if $\{JFJ\} \subset J$ holds. Every inner ideal $J \subset F$ is a subtriple of F and with every tripotent $e \in J$ also the whole Peirce space $F_1(e)$ is contained in J. By [**7**] the inner ideals of F can be uniquely characterized in the class of all closed subtriples $E \subset F$ by the unique norm preserving extension property of linear functionals: *To every $\lambda \in E^*$ there exists a unique $\sigma \in F^*$ with $\|\lambda\| = \|\sigma\|$ and $\sigma|E = \lambda$.*The following proposition is a characterization of inner ideals within the bigger class of all closed linear subspaces of F in terms of holomorphic automorphisms of the open unit ball D of F.

PROPOSITION 7.12. *Let E with open unit ball B be a closed linear subspace of the JB*-triple F with open unit ball D. Then E is an inner ideal of F if and only if $g(B)$ is convex in F for every $g \in \mathsf{Aut}(D)$.*

Proof. Fix an arbitrary $c \in D$. Then $\|z\Box c\| < 1$ holds for all $z \in D$ and there exists an automorphism $g \in \mathsf{Aut}(D)$ such that

$$(*) \qquad g(z) = c + \lambda(1 + z\Box c)^{-1} z$$

for $\lambda = dg(0) \in \mathsf{GL}(F)$ and all $z \in D$, compare [**15**] p. 132. Therefore, if E is an inner ideal in F, the function $f(z) := (1 + z\Box c)^{-1} z$ maps B into E and hence $g(B) \subset (A \cap D)$ fo rthe affine subspace $A := c + \lambda(E)$. By the implicit function theorem $g(B)$ is a neighbourhood of c in A, therefore g^{-1} maps the domain $(A \cap D) \subset A$ into E, i.e. $g(B) = A \cap D$. In particular, $g(B)$ is convex in F. Any other automorphism $\tilde{g} \in \mathsf{Aut}(D)$ with $\tilde{g}(0) = c$ is of the form $\tilde{g} = gk$ for some $k \in \mathsf{Aut}(D) \cap \mathsf{GL}(F)$. Then k respects the triple product and hence $\tilde{E} := k(E)$ is also an inner ideal of F, i.e. also $\tilde{g}(B)$ is convex. On the contrary, suppose that $g(B)$ is convex in F for all $g \in \mathsf{Aut}(D)$. Fix an arbitrary $c \in D$ and choose g as in $(*)$. Then $f(z) = \lambda^{-1}(g(z) - c) = z - \{zcz\} + o(\|z\|^2)$ defines a holomorphic mapping $f : D \to F$ with $f(0) = 0$, $df(0) = \mathsf{id}$ and $f(B)$ convex. By the implicit function theorem $f(B)$ must be contained in E. This implies $\{zcz\} = -\lim_{t\to 0} t^{-2}(f(tz) - tz)) \in E$ for all $z \in B$ and all $c \in D$. Therefore E is an inner ideal in F.

A JB*-triple F is called a JBW*-triple if F as a Banach space is the dual of another Banach space, compare [**14**] and [**10**]. This predual is uniquely determined by F and is denoted by F_*. It is known [**2**] that the triple product on every JBW*-triple is separately w^*-continuous. For every JB*-triple E the bidual E^{**} is a JBW*-triple with triple product extending the one of $E \subset E^{**}$, compare [**5**]. The advantage of JBW*-triples is that they contain many tripotents. By [**8**] a linear subspace E of the JBW*-triple F is a w^*-closed inner ideal if and only if E is the range of a structural projection P in F (structural means $Q(P(a)) = P \circ Q(a) \circ P$ for all $a \in F$—such a projection is automatically contractive and w^*-continuous).

EXAMPLE 7.13. Let $F := \mathcal{L}(H, K)$ for complex Hilbert spaces H, K_1, K_2 and $K := K_1 \oplus_2 K_2$. Then every $z \in F$ can be realized as a pair $z = (z_1, z_2)$ with $z_k \in \mathcal{L}(H, K_k)$ for $k = 1, 2$ and $P(z_1, z_2) = (z_1, 0)$ defines a structural projection onto a w^*-closed inner ideal E with $E^\perp = 0$. In general, the projection P is not almost neutral and also E is not the Peirce-1-space of a tripotent, compare the matrix example at the end.

THEOREM 7.14. *Let E be a w^*-closed ideal in the JBW*-triple F and let P*

be the corresponding structural projection from F onto E. Then

$$T_E(F) = \ker(P) = \{z \in F : Q(E)z = 0\} \text{ and}$$
$$\Theta_E(F) = E^{\perp} = \{z \in F : (E\Box E)z = 0\}.$$

Furthermore, E is rigid in F if and only if $\Theta_E(F) = 0$.

Proof. Fix $v \in T_E(F)$ and $a \in E$. Then there is a tripotent $e \in E$ with $a \in F_1(e)$. This implies $Q(a)v = 0$ because of $v \in F^0(e)$ and hence $T_E(F) \subset \mathsf{Ker}(E)$. The opposite inclusion follows with 7.7. But $\mathsf{Ker}(E) = \ker(P)$, compare [**8**]. The statement concerning $\Theta_E(F)$ follows by a similar argument. Finally, $E^{\perp} = 0$ is necessary for E being rigid in F. Let us therefore assume conversely, that $E^{\perp} = 0$ holds. Consider a homogeneous polynomial $f : E \to F$ of degree $m \geqslant 2$ with $z + f(z) \in D$ for all $z \in B$ and fix $c \in B$. We have to show that $f(c) = 0$. Because of 7.4 it is enough to show that $f(c) \in \Theta_c(F)$. Choose a tripotent $e \in E$ with $c \in U := F_1(e)$ and put $Z := F_0(e)$. Then $\|e + tf(e)\| \leqslant 1$ for all $t \in \Delta$ implies $f(e) \in Z$ by 7.1. Now let the selfadjoint part V of the JB*-algebra U and $A = \exp(iV)$ be as in the proof of Proposition 7.11. Every $a \in A$ has the same Peirce spaces as e and therefore also $f(a) \in Z$ by the above reasoning. Since A is a set of determinacy in U this implies $f(c) \in Z \subset \Theta_c(F)$ as a consequence of 2.2 and 7.9.

As an example consider the case of an arbitrary W*-algebra F. Then F is also a JBW*-triple and a w^*-closed linear subspace $E \subset F$ is an inner ideal if and only if $E = eFc$ for (Hermitian) projections $e, c \in F$ having the same central support, compare [**8**] p. 59. Then $T_E(F) = (1-e)F + F(1-c)$ and $\Theta_E(F) = (1-e)F(1-c)$.

We close with a finite dimensional illustration of Theorem 7.14: For fixed integers $1 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant m$ with $n \leqslant m$ the matrix space $F := \mathbb{C}^{n\times m} = \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ is a JBW*-triple of dimension nm and rank n. Write every matrix $z \in F$ in block form $z = \binom{ab}{cd}$, where a, b, c, d are rectangular matrices of sizes $p\times q$, $p\times(m-q)$, $(n-p)\times q$ and $(n-p)\times(m-q)$ respectively. $P(z) = \binom{a0}{00}$ defines a structural projection onto an inner ideal E of F. Then $E \approx \mathbb{C}^{p\times q}$ is the Peirce-1-space of a tripotent $e \in F$ if and only if E has square size, i.e. $p = q$. Under the assumption $E \neq F$ the projection P is neutral if and only if $n = 1$, that is, if F is a Hilbert space. Furthermore, P is neutral if and only if $q \geqslant p = n$ holds, that is, if E and F have the same rank. E is rigid in F if and only if $p = n$ or $q = m$. Finally, for the set A of Proposition 7.2 the following conditions are equivalent: (i) $A \neq \{0\}$, (ii) A is a set of determinacy in E, (iii) E and F have the same rank. Also, A is dense in E if and only if $E = F$ or F is a complex Hilbert space. In particular, E may be rigid in F in the case $A = \{0\}$.

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