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Differential equations with state-dependent piecewise constant argument

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ABSTRACT

A new class of differential equations with state-dependent piecewise constant argument is introduced. It is an extension of systems with piecewise constant argument. Fundamental theoretical results for the equations—the existence and uniqueness of solutions, the existence of periodic solutions, and the stability of the zero solution—are obtained. Appropriate examples are constructed.

1. Introduction

The theory of differential equations with piecewise constant argument (EPCA) of type

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x([t])), \tag{1}$$

where [.] stands for the greatest integer function, was initiated in [1,2], and has been developed for the last three decades using the method of reduction to discrete equations by many researchers [3–15]. Applications of these equations for problems of biology, mechanics, and electronics can be seen in papers [1,16–18]. The theoretical depth of investigation of these equations was established by papers [1,2], where the reduction to discrete equations had been chosen as the main instrument of study.

In [19,20] we introduced differential equations with piecewise constant argument of generalized type (EPCAG) of type

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x(\beta(t))), \tag{2}$$

where $\beta(t) = \theta_i$ if $\theta_i \leq t < \theta_{i+1}$, i integer, is an identification function, and θ_i is a strictly increasing sequence of real numbers.

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The equations were extended in papers [21,22] to systems with advanced and delayed argument

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x(\gamma(t))), \tag{3}$$

where $\gamma(t) = \zeta_i$ if $t \in [\theta_i, \theta_{i+1})$ and ζ_i, $i \in \mathbb{Z}$, is a sequence such that $\theta_i \le \zeta_i \le \theta_{i+1}$.

Differential equations of another type were proposed in [23]:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + f(t, x(\theta_{\sigma(t)-p_1}), x(\theta_{\sigma(t)-p_2}), \ldots, x(\theta_{\sigma(t)-p_m})), \tag{4}$$

where $\sigma(t) = i$ if $\theta_i \le t < \theta_{i+1}$, $i = \cdots - 2, -1, 0, 1, 2, \ldots$, is an identification function, θ_i is a strictly ordered sequence of real numbers, unbounded on the left and on the right, p_j, $j = 1, 2, \ldots, m$, are fixed integers.

All of these equations are reduced to equivalent integral equations such that one can investigate many problems which have not been properly solvable by using discrete equations—for instance, existence and uniqueness of solutions, stability, differentiable and continuous dependence of solutions on parameters [19–26].

In this paper we generalize Eqs. (2) to a new type of system, differential equations with state-dependent piecewise constant argument (ESPA), where intervals of constancy of the independent argument are not prescribed and they depend on the present state of a motion. The method of analysis for the equations was initiated in [19–23]. We are confident that introduction of these equations will provide new opportunities for the development of theory of differential equations and for applications [27,16,28–34]. Eqs. (3) and (4) can also be extended to ESPA. One must say that the present results build on the rich experience accumulated for dynamical systems with discontinuities [35–41] and are strongly influenced by theoretical concepts developed for equations of a different type with discontinuities [42–46].

Our paper consists of two main parts. In Section 2 we introduce the most general—for the present time—form of the equations. Some basic properties of ordinary differential equations, *constancy switching surfaces*, are defined, which give us a start in the investigation. One of them is called the *extension property*. The definition of the solutions is given. In the rest of the work we realize the general concepts for equations of a particular type, namely, quasilinear systems. Existence and uniqueness theorems, periodicity, and stability of the zero solution are discussed.

2. Generalities

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$ be the sets of all natural and real numbers, and integers, respectively. Denote by $\|.\|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$.

Let $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ and $\mathcal{A} \subseteq \mathbb{Z}$ be nonempty intervals of real numbers and integers, correspondingly. Let $\mathcal{G} \subseteq \mathbb{R}^n$ be an open connected region. Denote by $C(\mathcal{G}, \mathcal{I})$ and $C^1(\mathcal{G}, \mathcal{I})$ the sets of all continuous and continuously differentiable functions from $\mathcal{G}$ to $\mathcal{I}$, respectively. Fix a sequence of real valued functions $\{\tau_i(x)\} \subset C(\mathcal{G}, \mathcal{I})$, where $i \in \mathcal{A}$.

We introduce the following assumption.

(A1) There exist two positive real numbers θ and $\bar{\theta}$ such that $\theta \le \tau_{i+1}(x) - \tau_i(y) \le \bar{\theta}$ for all $x, y \in \mathcal{G}$ and $i \in \mathcal{A}$.

Set the surfaces $S_i = \{(t, x) \in \mathcal{I} \times \mathcal{G} : t = \tau_i(x)\}$, $i \in \mathcal{A}$, in $\mathcal{I} \times \mathcal{G}$, and define the regions $D_i = \{(t, x) \in \mathcal{I} \times \mathcal{G} : \tau_i(x) \le t < \tau_{i+1}(x)\}$, $i \in \mathcal{A}$, and $D_r = \{(t, x) \in \mathcal{I} \times \mathcal{G} : \tau_r(x) \le t\}$ if $\max \mathcal{A} = r < \infty$. Because of (A1), one can see that the D_i's, $i \in \mathcal{A}$, are nonempty disjoint sets.

We consider the equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x(\beta(t, x))), \tag{5}$$

where $t \in \mathcal{I}$, $x \in \mathcal{G}$, and $\beta(t, x)$ is a functional such that if $x(t) : \mathcal{I} \to \mathcal{G}$ is a continuous function, and $(t, x(t)) \in D_i$ for some $i \in \mathcal{A}$, then $\beta(t, x) = \eta_i$, where η_i satisfies the equation $\eta = \tau_i(x(\eta))$. From the description of functions τ, this implies that one can call surfaces $t = \tau_i(x)$ *constancy switching surfaces*, since the solution's piecewise constant argument changes its value at the moment of meeting one of the surfaces.

We call system (5) *a system of differential equations with state-dependent piecewise constant argument, ESPA*.

Let us introduce the following conditions, which are necessary for defining a solution of Eq. (5) on $\mathcal{I}$.

(A2) For a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$, there is an integer $j \in \mathcal{A}$ such that $t_0 \ge \tau_j(x_0)$, and $j \ge k$ if $t_0 \ge \tau_k(x_0)$, $k \in \mathcal{A}$.

One can see that the functional $\beta(t, x) \le t$ for all $t \in \mathcal{I}$, $x \in \mathcal{G}$. Indeed, to define system (5), the point (t, x) must be in D_j for some $j \in \mathcal{A}$.

Consider the ordinary differential equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = f(t, y(t), z), \tag{6}$$

where z is a constant vector in $\mathcal{G}$.

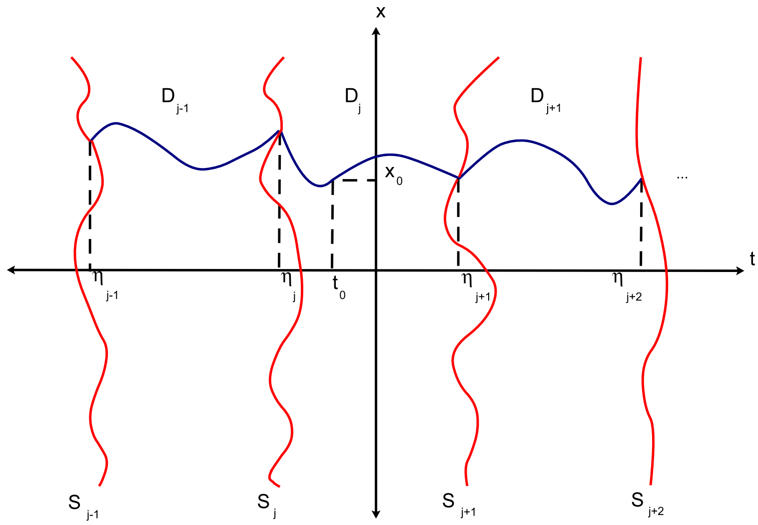

Fig. 1. A solution of a differential equation with state-dependent argument.

We impose the following assumption.

(B0) For a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$, solution $y(t) = y(t, t_0, x_0)$ of Eq. (6) exists and is unique in any interval of existence, and it has an open maximal interval of existence such that any limit point of the set $(t, y(t))$, as t tends to the endpoints of the maximal interval of existence, is a boundary point of $\mathcal{I} \times \mathcal{G}$.

Let us recall that condition (B0) is valid, if, for example, the function f is continuous in t, and satisfies the local Lipschitz condition in y.

We shall need the following conditions:

(A3) for a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$ satisfying (A2), there exists a solution $y(t) = y(t, t_0, x_0)$ of Eq. (6) such that $\eta_j = \tau_j(y(\eta_j))$ for some $j \in \mathcal{A}$, and $\eta_j \leq t_0$;
(A4) for each $z \in \mathcal{G}$ and $j \in \mathcal{A}$ the solution $y(t, \tau_j(z), z)$ of Eq. (6) does not meet the surface S_j if $t > \tau_j(z)$;
(A5) for a given $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$ belonging to $S_j, j \in \mathcal{A}$, there exist a surface $S_{j-1} \subset \mathcal{I} \times \mathcal{G}$ and a solution $y(t) = y(t, t_0, x_0)$ of Eq. (6) such that $\eta_{j-1} = \tau_{j-1}(y(\eta_{j-1}))$ for some $\eta_{j-1} < t_0$.

If a point $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$ satisfies (A2) and (A3), then we say that this point has the *extension property*.

Let us introduce the definition of a solution of (5).

Definition 2.1. A function $x(t)$ is said to be a solution of Eq. (5) on an interval $\mathcal{J} \subseteq \mathcal{I}$ if:

(i) it is continuous on $\mathcal{J}$,
(ii) the derivative $x'(t)$ exists at each point $t \in \mathcal{J}$ with the possible exception of the points η_i, $i \in \mathcal{A}$, for which the equation $\eta = \tau_i(x(\eta))$ is satisfied, where the right derivative exists,
(iii) the function $x(t)$ satisfies Eq. (5) on each interval (η_i, η_{i+1}), $i \in \mathcal{A}$, and it holds for the right derivative of $x(t)$ at the points η_i.

Fix $(t_0, x_0) \in \mathcal{I} \times \mathcal{G}$. Assume that it has the extension property. We shall consider the problem of global existence of a solution $x(t) = x(t, t_0, x_0)$ of (5).

Let us investigate the problem for increasing t. Either the point (t_0, x_0) is in S_j, or there is a ball $B((t_0, x_0); \epsilon) \subset D_j$ for some real number $\epsilon > 0$, and $j \in \mathcal{A}$. The solution $x(t)$ is defined on an interval $[\eta_j, t_0]$, $\eta_j \leq t_0$ by the extension property, and satisfies the initial value problem (IVP)

$$\begin{aligned} &y'(t) = f(t, y(t), y(\eta_i)),\\ &y(\eta_i) = x(\eta_i), \end{aligned} \tag{7}$$

such that $\eta_i = \tau_i(x(\eta_i))$ for $i = j$ (see Fig. 1). By using (A4) and (B0), there exists a solution $\psi(t) = \psi(t, \eta_j, x(\eta_j))$ of (7) defined on the right maximal interval of existence, $[t_0, \beta)$. If $\psi(t)$ does not intersect S_{j+1}, or the constancy switching surface S_{j+1} does not exist, then the right maximal interval of $x(t)$ is $[t_0, \beta)$, $\beta > t_0$. Otherwise, there is some $\xi \in \mathcal{I}$ such that $t_0 < \xi < \beta$, and $\xi = \tau_{j+1}(\psi(\xi))$. Then by defining $\eta_{j+1} = \xi$, we define the solution $x(t)$ as $\psi(t)$ on $[t_0, \eta_{j+1}]$. Now, one can apply the above discussion for (t_0, x_0) to the point $(\eta_{j+1}, x(\eta_{j+1}))$.

Proceeding in this way, we may come to the case where for some $k \in \mathcal{A}$, $k > j$, the solution $\psi(t) = \psi(t, \eta_k, x(\eta_k))$ has a right maximal interval $[\eta_k, \gamma)$ and this solution does not meet S_{k+1}, and then $[t_0, \gamma)$, $\gamma > \eta_k$, is the right maximal interval of existence of $x(t)$. If there is no such k, then either $x(t)$ is continuable to $+\infty$ if the set $\mathcal{A}$ is unbounded from above, or the solution achieves the point $(\eta_r, x(\eta_r))$, $\eta_r = \tau_r(x(\eta_r))$ and then $x(t)$ has the right maximal interval $[t_0, \kappa)$, $\kappa > \eta_r$ where $[\eta_r, \kappa)$ is the right maximal interval of solution $\psi(t)$ of Eq. (7) for $i = r$.

On the basis of the above discussion we can conclude that if the extension property for (t_0, x_0) and conditions (A4) and (B0) are valid, then solution $x(t, t_0, x_0)$ of Eq. (5) has a right maximal interval of existence, and it is open from the right.

Now consider decreasing t. Assume, again, that (t_0, x_0) satisfies the extension property. Let us consider first $(t_0, x_0) \in S_j$. If condition (A5) is not valid, then the solution $x(t, t_0, x_0)$ does not exist for $t \le t_0$. Otherwise, it is continuable to η_{j-1} such that $\eta_{j-1} = \tau_{j-1}(x(\eta_{j-1}))$, and satisfies Eq. (7) for $i = j - 1$. Then, again, as for $(\eta_j, x(\eta_j))$, we may have the same discussion for the point $(\eta_{j-1}, x(\eta_{j-1}))$. Finally, we may conclude that either there exists η_k, $k \le j$, such that the left maximal interval of $x(t)$ is $[\eta_k, t_0]$ (it is true also if there exists $k = \min \mathcal{A}$) or the solution is continuable to $-\infty$. Let us now consider the case where (t_0, x_0) is an interior point of D_j, and satisfies the extension property. Then, it is continuable to the left up to S_j, and then, we can repeat the above discussion. So, we can draw the conclusion that the left maximal interval of existence of $x(t)$ is either a closed interval $[\eta_k, t_0]$, $k \in \mathcal{A}$, or an infinite interval $(-\infty, t_0]$. By combining the left and right maximal intervals, we define the solution $x(t)$ on the maximal interval of existence.

3. Quasilinear systems

In this section, we investigate the existence and uniqueness of solutions of quasilinear ESPA.

Let $\mathcal{I} = \mathbb{R}$, $\mathcal{G} = \mathbb{R}^n$ and $\mathcal{A} = \mathbb{Z}$. Fix a sequence of real numbers $\{\theta_i\} \subset \mathbb{R}$ such that $\theta_i < \theta_{i+1}$ for all $i \in \mathbb{Z}$. Take a sequence of functions $\xi_i(x) \in C(\mathbb{R}^n, \mathbb{R})$. Set $\tau_i(x) = \theta_i + \xi_i(x)$. Define the constancy switching surfaces $S_i = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = \theta_i + \xi_i(x)\}$, $i \in \mathbb{Z}$, and the regions $D_i = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \theta_i + \xi_i(x) \le t < \theta_{i+1} + \xi_{i+1}(x)\}$, $i \in \mathbb{Z}$.

Let us now consider the following quasilinear differential equation:

$$x'(t) = A(t)x(t) + F(t, x(t), x(\beta(t, x))), \tag{8}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\beta(t, x)$ is a functional such that if $x(t) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function, and $(t, x(t)) \in D_i$ for some $i \in \mathbb{Z}$, then $\beta(t, x) = \eta_i$, where η_i satisfies the equation $\eta = \theta_i + \xi_i(x(\eta))$.

Fix $H \in \mathbb{R}$, $H > 0$, and define $K_H = \{x \in \mathbb{R}^n : \|x\| < H\}$. We introduce the following assumptions:

(Q1) there exist positive real numbers c, d such that $c \le \theta_{i+1} - \theta_i \le d$, $i \in \mathbb{Z}$;
(Q2) there exists $l \in \mathbb{R}$, $0 \le 2l < c$, such that $|\xi_i(x)| \le l$, $i \in \mathbb{Z}$, for all $x \in K_H$;
(Q3) the functions $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous;
(Q4) there exists a Lipschitz constant $L_1 > 0$ such that

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\| \le L_1 [\|x_1 - x_2\| + \|y_1 - y_2\|]$$

for $t \in \mathbb{R}$ and $x_1, y_1, x_2, y_2 \in K_H$;
(Q5) $\sup_{t\in\mathbb{R}} \|A(t)\| = \kappa < \infty$;
(Q6) $\sup_{t\in\mathbb{R}} \|F(t, 0, 0)\| = N < \infty$;
(Q7) there exists a Lipschitz constant $L_2 > 0$ such that

$$|\xi_i(x) - \xi_i(y)| \le L_2\|x - y\|$$

for all $x, y \in K_H$ and $i \in \mathbb{Z}$.

One can see that conditions (Q1) and (Q2) imply (A1) with $\theta = c - 2l$ and $\bar{\theta} = d + 2l$. Also, Eq. (6) for system (8) has the form

$$y'(t) = A(t)y(t) + F(t, y(t), z), \tag{9}$$

where $z \in \mathbb{R}^n$ is a constant vector. Hence, under conditions (Q1)–(Q4), it is not difficult to see that (A2) and (B0) are valid for the last equation.

Let $X(t)$ be a fundamental matrix solution of the homogeneous system, corresponding to Eq. (9),

$$x'(t) = A(t)x(t), \tag{10}$$

such that $X(0) = I$, where I is an $n \times n$ identity matrix. Denote by $X(t, s) = X(t)X^{-1}(s)$, $t, s \in \mathbb{R}$, the transition matrix of (10). For the transition matrix $X(t, s)$, one can obtain the following inequalities:

$$m \le X(t, s) \le M, \tag{11}$$

$$\|X(t, s) - X(\bar{t}, s)\| \le \kappa M|t - \bar{t}|, \tag{12}$$

where $m = \exp(-\kappa\bar{\theta})$ and $M = \exp(\kappa\bar{\theta})$ if $t, \bar{t}, s \in [\theta_j - l, \theta_{j+1} + l]$ for some $j \in \mathbb{Z}$.

Let us fix $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. The following lemma is the main auxiliary result of this paper. A similar assertion for EPCAG is proved in [20].

Lemma 3.1. *Suppose that* (Q1)–(Q3) *are fulfilled. Then, $x(t)$ is a solution of Eq.* (8) *with $x(t_0) = x_0$ for $t \geq t_0$, if and only if it satisfies the equation*

$$x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s, x(s), x(\beta(s, x)))\mathrm{d}s. \tag{13}$$

Proof. *Necessity*. Assume that $x(t)$ is a solution of Eq. (8) such that $x(t_0) = x_0$, $(t_0, x_0) \in D_j$ for some $j \in \mathbb{Z}$. Define

$$\phi(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s, x(s), x(\beta(s, x)))\mathrm{d}s. \tag{14}$$

Assume that $(t, x(t)) \in D_j \setminus S_j$. Then, there exists a moment $\eta_j \in \mathbb{R}$ such that $\beta(s, x) = \eta_j$ for all $(s, x(s)) \in D_j$. Also, we have

$$\phi'(t) = A(t)\phi(t) + F(t, x(t), x(\eta_j)),$$

and

$$x'(t) = A(t)x(t) + F(t, x(t), x(\eta_j)).$$

Hence,

$$[\phi(t) - x(t)]' = A(t)[\phi(t) - x(t)].$$

Calculating the limit values at $\eta_j, j \in \mathbb{Z}$, we can find that

$$\phi'(\eta_j \pm 0) = A(\eta_j \pm 0)\phi(\eta_j \pm 0) + F(\eta_j \pm 0, x(\eta_j \pm 0), x(\beta(\eta_j \pm 0, x(\eta_j \pm 0)))),$$
$$x'(\eta_j \pm 0) = A(\eta_j \pm 0)x(\eta_j \pm 0) + F(\eta_j \pm 0, x(\eta_j \pm 0), x(\beta(\eta_j \pm 0, x(\eta_j \pm 0)))).$$

Consequently,

$$[\phi(t) - x(t)]'|_{t=\eta_j+0} = [\phi(t) - x(t)]'|_{t=\eta_j-0}.$$

Thus, $[\phi(t) - x(t)]$ is a continuously differentiable function defined for $t \geq t_0$ satisfying (10) with the initial condition $\phi(t_0) - x(t_0) = 0$. Using uniqueness of solutions of Eq. (10) we conclude that $\phi(t) - x(t) \equiv 0$ for $t \geq t_0$.

Sufficiency. Suppose that $x(t)$ is a solution of (13) for $t \geq t_0$. Fix $j \in \mathbb{Z}$ and consider the region D_j. If $(t, x(t)) \in D_j \setminus S_j$, then by differentiating (13) one can see that $x(t)$ satisfies Eq. (8). Moreover, considering $(t, x(t)) \to S_j$, and taking into account that $x(\beta(t, x))$ is a right-continuous function, we find that $x(t)$ satisfies Eq. (8) in D_j. The lemma is proved. □

The following example shows that for even simple linear ESPA we have difficulties with the uniqueness of solutions.

Example 3.1. Consider the equation

$$x'(t) = -2x(\beta(t, x)), \tag{15}$$

where $\beta(t, x)$ is defined by using the sequences $\theta_j = 2j$ and $\xi_j(x) = \cos x/4$, $j \in \mathbb{Z}$. Fix $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, that satisfies the equation $t = (\cos x)/4$. Then solution $x(t)$ of (15) with $x(t_0) = x_0$ is of the form $x(t) = (1-2(t-\cos x_0/4))x_0$ for $t \in [t_0, 5/4)$. In particular, for $(t_0, x_0) = (1/4, 0)$ and $(1/4, 2\pi)$, the corresponding solutions are $x_1(t) = 0$ and $x_2(t) = \pi(3 - 4t)$, each of which passes through the point $(3/4, 0)$. Hence, uniqueness does not hold.

Denote $\tilde{M} = 2L_1H + N$. From now on we need the following assumption:

(Q8) $2M\bar{\theta}L_1 < \min\{1 - 2(\kappa H + M\tilde{M})L_2, 1 - N\bar{\theta}M/H\}$.

Let $h \in \mathbb{R}, 0 < h < \left(\frac{1-2ML_1\bar{\theta}}{M}H - N\bar{\theta}\right)$. The following lemma imposes sufficient conditions for Eq. (8) to satisfy the extension property.

Lemma 3.2. *Suppose that conditions* (Q1)–(Q8) *are fulfilled, and $(t_0, x_0) \in D_j$ for some $j \in \mathbb{Z}$ such that $\|x_0\| < h$. Then there exists a solution $y(t) = y(t, t_0, x_0)$ of Eq.* (8) *such that $\eta_j = \theta_j + \xi_j(y(\eta_j))$ for some $\eta_j \leq t_0$, and $y(t) \in K_H$ for all $t \in [\theta_j - l, \theta_{j+1} + l]$.*

Proof. If $(t_0, x_0) \in S_j$, then by taking $\eta_j = t_0$ we can conclude the result directly. Suppose that $(t_0, x_0) \in D_j \setminus S_j$. Let us construct the following sequences. Take $\eta^0 = \theta_j$, $y_0(t) = X(t, t_0)x_0$, and define

$$\eta^{k+1} = \theta_j + \xi_j(y_k(\eta^k)), \tag{16}$$

$$y_{k+1}(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s, y_k(s), y_k(\eta^k))\mathrm{d}s \tag{17}$$

for all $k \in \mathbb{Z}, k \geq 0$.

Let $\|.\|_0 = \max_{t\in[\theta_j-l,\theta_{j+1}+l]} \|.\|$. It is straightforward to see that

$$\begin{aligned}\|y_{k+1}\|_0 &\le M\|x_0\| + \left\|\int_{t_0}^{t} \|X(t,s)\|\|F(s,y_k(s),y_k(\eta^k))\|\mathrm{d}s\right\|_0\\ &\le Mh + NM\bar{\theta} + 2ML_1\bar{\theta}\|y_k\|_0\\ &\le \frac{1-(2ML_1\bar{\theta})^{k+2}}{1-2ML_1\bar{\theta}}(Mh+NM\bar{\theta}).\end{aligned}$$

Using (Q8), we see that $y_k(t) \in K_H$ for all $t \in [\theta_j - l, \theta_{j+1} + l]$, $k \in \mathbb{Z}$, $k \ge 0$.

Now, we will show that the sequence $\{y_k(t)\}$ is uniformly convergent. Eqs. (16) and (17) imply that

$$\begin{aligned}|\eta^{k+1}-\eta^k| &= |\xi_j(y_k(\eta^k)) - \xi_j(y_{k-1}(\eta^{k-1}))|\\ &\le L_2\|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|,\\ \|y_{k+1}-y_k\|_0 &\le \max_{t\in[\theta_j-l,\theta_{j+1}+l]}\left|\int_{t_0}^{t} M\|F(s,y_k(s),y_k(\eta^k)) - F(s,y_{k-1}(s),y_{k-1}(\eta^{k-1}))\|\mathrm{d}s\right|\\ &\le ML_1\bar{\theta}\left[\|y_k-y_{k-1}\|_0 + \|y_k(\eta^k) - y_{k-1}(\eta^{k-1})\|\right],\\ \|y_{k+1}(\eta^{k+1}) - y_k(\eta^k)\| &\le \left\|X(\eta^{k+1},t_0) - X(\eta^k,t_0)\right\|\|x_0\| + \left|\int_{\eta_k}^{\eta_{k+1}} \|X(\eta^{k+1},s)F(s,y_k(s),y_k(\eta^k))\|\mathrm{d}s\right|\\ &\quad + \left|\int_{t_0}^{\eta_k} \|X(\eta^{k+1},s)F(s,y_k(s),y_k(\eta^k)) - X(\eta^k,s)F(s,y_{k-1}(s),y_{k-1}(\eta^{k-1}))\|\mathrm{d}s\right|\\ &\le (\kappa h + \tilde{M}(1+\kappa\bar{\theta}))M|\eta_{k+1}-\eta_k| + ML_1\bar{\theta}\left[\|y_k-y_{k-1}\|_0 + \|y_k(\eta^k)-y_{k-1}(\eta^{k-1})\|\right]\\ &\le M\big(L_2(\kappa h + \tilde{M}(1+\kappa\bar{\theta})) + L_1\bar{\theta}\big)[\|y_k-y_{k-1}\|_0 + \|y_k(\eta^k)-y_{k-1}(\eta^{k-1})\|]\\ &\le \big(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta}\big)[\|y_k-y_{k-1}\|_0 + \|y_k(\eta^k)-y_{k-1}(\eta^{k-1})\|].\end{aligned}$$

Then,

$$|\eta^{k+1}-\eta^k| \le \left[2\big(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta}\big)\right]^{k-1}\bar{\theta}M\tilde{M}, \tag{18}$$

$$\|y_{k+1}(\eta^{k+1}) - y_k(\eta^k)\| \le \left[2\big(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta}\big)\right]^{k}\bar{\theta}M\tilde{M}, \tag{19}$$

$$\|y_{k+1}-y_k\|_0 \le \left[2\big(L_2(\kappa H + M\tilde{M}) + ML_1\bar{\theta}\big)\right]^{k}\bar{\theta}M\tilde{M}. \tag{20}$$

Thus, there exist a unique moment η_j and a solution $y(t)$ of Eq. (8) with $y(t_0) = x_0$ such that $\eta_j = \theta_j + \xi_j(y(\eta_j))$, and η^k and y_k converge as $k \to \infty$, respectively. The lemma is proved. □

In what follows, we will consider differential equations of type (8) such that the solutions intersect each constancy switching surface not more than once. In the previous section this assumption coincides with (A4). The following lemma defines the sufficient condition for this property.

From now on we shall need the following condition:

(Q9) $L_2\left[\kappa MH + M\tilde{M}\right] < 1$.

Lemma 3.3. *Suppose that* (Q1)–(Q7), (Q9) *hold. Then every solution $x(t) \in K_H$ of Eq.* (8) *meets any constancy switching surface not more than once.*

Proof. Suppose the contrary. Then, there exist a solution $x(t) \in K_H$ of (8) and a surface S_j, $j \in \mathbb{Z}$, such that $x(t)$ meets this surface more than once. Let the first intersection be at $t = t_0$ and another intersection at $t = t^*$, so we have $t_0 = \theta_j + \xi_j(x(t_0))$ and $t^* = \theta_j + \xi_j(x(t^*))$ for $t_0 < t^*$. Then, we have

$$\begin{aligned}|t^*-t_0| &\le L_2\|X(t^*,t_0)x(t_0) + \int_{t_0}^{t^*} X(t,s)F(s,x(s),x(\beta(s,x)))\mathrm{d}s - x(t_0)\|\\ &\le L_2\left[\kappa MH + M\tilde{M}\right]|t^*-t_0|,\end{aligned}$$

which contradicts (Q9). The lemma is proved. □

From the above lemmas we conclude the following theorem.

Theorem 3.1. *Assume that conditions* (Q1)–(Q9) *are fulfilled, and* $(t_0, x_0) \in D_j$ *for some* $j \in \mathbb{Z}$ *such that* $\|x_0\| < h$. *Then there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of* (8) *on* $[\eta_j, \eta_{j+1}]$ *such that* $\eta_j = \theta_j + \xi_j(x(\eta_j))$, $\eta_{j+1} = \theta_{j+1} + \xi_{j+1}(x(\eta_{j+1}))$, *and* $x(t) \in K_H$.

4. Periodic solutions

In this section, we investigate periodic solutions of quasilinear ESPA of type (8).

Let ω and p be a fixed positive real number and integer, respectively. We shall introduce the following assumptions:

(Q10) the functions $A(t)$ and $F(t,x,y)$ are ω-periodic in t;
(Q11) the sequence $\theta_i+\xi_i(x)$ satisfies (ω,p)-periodicity, i.e. $\theta_{i+p}=\theta_i+\omega$ and $\xi_{i+p}(x)=\xi_i(x)$ for all $i\in\mathbb{Z}$ and $x\in\mathbb{R}^n$;
(Q12) $\det(I-X(\omega))\neq 0$; that is, system (10) does not have any ω-periodic solution.

We define, if (Q12) is fulfilled, the function

$$G(t,s)=\begin{cases}X(t)(I-X(\omega))^{-1}X^{-1}(s), & 0\le s\le t\le\omega,\\ X(t+\omega)(I-X(\omega))^{-1}X^{-1}(s), & 0\le t<s\le\omega,\end{cases} \tag{21}$$

which is called *Green's function* [47]. Let $\max_{t,s\in[0,\omega]}\|G(t,s)\|=K$.

We need the following lemma to prove the main theorem. This lemma can be proved using Lemma 3.1.

Lemma 4.1. *Suppose that* (Q1)–(Q12) *are fulfilled. Then the solution $x(t)$ of Eq.* (8) *is ω-periodic if and only if it satisfies the integral equation*

$$x(t)=\int_0^{\omega}G(t,s)F(s,x(s),x(\beta(s,x)))\mathrm{d}s. \tag{22}$$

Let $\|.\|_\omega=\max_{t\in[0,\omega]}\|.\|$. Denote by Φ the set of all continuous and piecewise continuously differentiable ω-periodic functions on $\mathbb{R}$ such that if $\phi\in\Phi$ then $\|\phi(t)\|_\omega<H$, $\|\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\|_\omega<N+(2L_1+\kappa)H$.

We introduce the following assumption to prove the next theorem:

(Q13)

$$\begin{aligned}&(2KL_1\omega-1)H+NK\omega<0;\\ &L_2(N+(2L_1+\kappa)H)<1;\\ &KL_1(2-L_2(N+(2L_1+\kappa)H))\omega+2KHL_1L_2p+L_2(N+(2L_1+\kappa)H)<1.\end{aligned}$$

Theorem 4.1. *Suppose that* (Q1)–(Q13) *hold. Then Eq.* (8) *has a unique ω-periodic solution $\phi(t)$ such that $\phi(t)\in K_H$.*

Proof. Suppose that for all $x\in K_H$ and $k=j,\dots,j+p-1$, for some $j\in\mathbb{Z}$ and $p>1$, we have $0\le\theta_k+\xi_k(x)\le\omega$. The other cases are similar. Define an operator T on Φ as

$$T[\phi]=\int_0^{\omega}G(t,s)F(s,\phi(s),\phi(\beta(s,\phi)))\mathrm{d}s. \tag{23}$$

Using (Q13), it is easy to see that $\|T[\phi]\|_\omega<H$ and $\|\frac{dT[\phi]}{dt}\|_\omega<N+(2L_1+\kappa)H$. That is, $T[\phi]\in\Phi$.

Now, we will show that the operator T is contractive on Φ. Let $\phi_1,\phi_2\in\Phi$. One can see that using (Q13), the function $\phi_i(t)$ intersects any constancy switching surface S_k exactly once at $t=\eta_k^i$ for all $i=1,2$ and $k=j,\dots,j+p-1$. Without loss of generality suppose that $\eta_k^1\le\eta_k^2$.

Also, one can show, using the mean value theorem and (Q13), that the inequality

$$\|\phi_1(\eta_k^1)-\phi_2(\eta_k^2)\|\le\frac{1}{1-L_2(N+(2L_1+\kappa)H)}\|\phi_1-\phi_2\|_\omega \tag{24}$$

is satisfied.

Using (23) and (Q11), we write

$$\begin{aligned}T[\phi_i(t)]=&\int_0^{\eta_j^i}G(t,s)F(s,\phi_i(s),\phi_i(\eta_{j+p-1}^i))\mathrm{d}s+\sum_{k=j}^{j+p-2}\int_{\eta_k^i}^{\eta_{k+1}^i}G(t,s)F(s,\phi_i(s),\phi_i(\eta_k^i))\mathrm{d}s\\ &+\int_{\eta_{j+p-1}^i}^{\omega}G(t,s)F(s,\phi_i(s),\phi_i(\eta_{j+p-1}^i))\mathrm{d}s\end{aligned}$$

for $i=1,2$.

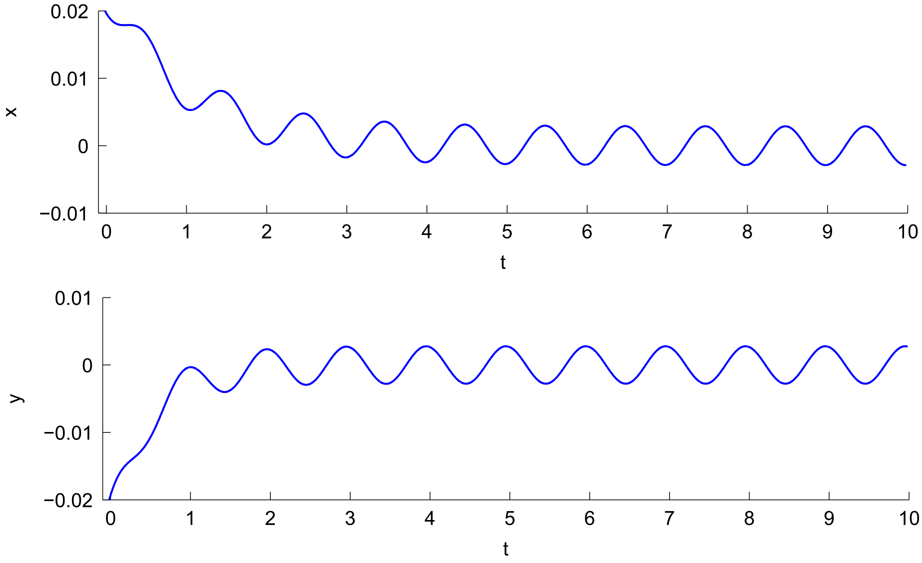

Fig. 2. A solution $(x(t), y(t))$ of ESPA that approaches the 1-periodic solution as time increases.

Then, using (24), we obtain

$$\begin{aligned}\|T[\phi_1]-T[\phi_2]\|_\omega &\le K\Bigg[\int_0^{\eta_j^1}\|F(s,\phi_1(s),\phi_1(\eta_{j+p-1}^1))-F(s,\phi_2(s),\phi_2(\eta_{j+p-1}^2))\|\mathrm{d}s\\ &\quad+\sum_{k=j}^{j+p-2}\int_{\eta_k^2}^{\eta_{k+1}^1}\|F(s,\phi_1(s),\phi_1(\eta_k^1))-F(s,\phi_2(s),\phi_2(\eta_k^2))\|\mathrm{d}s\\ &\quad+\int_{\eta_{j+p-1}^2}^{\omega}\|F(s,\phi_1(s),\phi_1(\eta_{j+p-1}^1))-F(s,\phi_2(s),\phi_2(\eta_{j+p-1}^2))\|\mathrm{d}s\\ &\quad+\sum_{k=j}^{j+p-1}\int_{\eta_k^1}^{\eta_k^2}\|F(s,\phi_1(s),\phi_1(\beta(s,\phi_1)))-F(s,\phi_2(s),\phi_2(\beta(s,\phi_2)))\|\mathrm{d}s\Bigg]\\ &\le\left[\frac{KL_1\omega(2-L_2(N+(2L_1+\kappa)H))+2KHL_1L_2p}{1-L_2(N+(2L_1+\kappa)H)}\right]\|\phi_1-\phi_2\|_\omega.\end{aligned}$$

Hence, T is contractive. Because of Lemma 4.1, we see that the fixed point is an ω-periodic solution of Eq. (8). The theorem is proved. □

Let us apply the last theorem to the following example.

Example 4.1. Consider the equation

$$\begin{aligned}x'(t)&=-x(t)-a\sin(2\pi t+y(\beta(t,x,y)))\\ y'(t)&=-2y(t)+a\sin(2\pi t+x(\beta(t,x,y))),\end{aligned}\tag{25}$$

where $t, x, y \in \mathbb{R}$, and a is a positive real number. Here, $\beta(t, x, y)$ is defined by $\theta_j = j, \xi_j(x, y) = -a\cos(x + y)$. The corresponding parameters in the conditions of Theorem 4.1 are $L_1 = a\sqrt{2}, L_2 = a, \bar{\theta} = 1 + 2a, \kappa = 2, N = a\sqrt{2}, M = e^{2+4a}, \tilde{M} = (2H + 1)a\sqrt{2}, \omega = 1, p = 1, K = e^2(1 - e^{-1})^{-1}$. One can show that conditions (Q1)–(Q13) are satisfied for $a = e^{-4}, H = 1$. Hence, by Theorem 4.1, we ensure that there is a 1-periodic solution of (25). Fig. 2 shows a solution $(x(t), y(t))$ of (25) with an initial condition $(x(-e^{-4}), y(-e^{-4})) = (0.02, -0.02)$ that approaches this periodic solution.

5. Stability of the zero solution

In this section we give sufficient conditions for stability of the zero solution.
Let us introduce the following conditions:

(Q14) $F(t, 0, 0) = 0$ for all $t \in \mathbb{R}$;
(Q15) $M\left[(1 + \bar{\theta}L_1)(e^{ML_1\bar{\theta}} - 1) + L_1\bar{\theta}\right] < 1$.

Define

$$K(L_1,\bar{\theta}) = \frac{M}{1 - M\left[(1+\bar{\theta}L_1)(e^{ML_1\bar{\theta}}-1) + L_1\bar{\theta}\right]}.$$

The following lemma plays a significant role in this paper. Using the technique in [20] and like in [24, Lemma 1.2], the following lemma can be proved.

Lemma 5.1. *Suppose that* (Q1)–(Q9), (Q14), (Q15) *are fulfilled. Then, every solution $x(t)$ of Eq. (8) satisfies the inequality*

$$\|x(\beta(t,x))\| \le K(L_1,\bar{\theta})\|x(t)\| \tag{26}$$

for all $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. Let $x(t)$ be a solution of (8). Then, there are $k \in \mathbb{Z}$ and $\eta_k \in \mathbb{R}$ such that $(t, x(t)) \in D_k$ and $\beta(t,x) = \eta_k$. Using Lemma 3.1, we have

$$x(t) = X(t,\eta_k)x(\eta_k) + \int_{\eta_k}^{t} X(t,s)F(s,x(s),x(\eta_k))\mathrm{d}s.$$

Then,

$$\begin{aligned}\|x(t)\| &\le M\|x(\eta_k)\| + ML_1\int_{\eta_k}^{t}(\|x(s)\| + \|x(\eta_k)\|)\,\mathrm{d}s\\ &\le M(1+\bar{\theta}L_1)\|x(\eta_k)\| + ML_1\int_{\eta_k}^{t}\|x(s)\|\mathrm{d}s.\end{aligned}$$

Hence, using the Gronwall–Bellman Lemma, we obtain

$$\|x(t)\| \le M(1+\bar{\theta}L_1)e^{ML_1(t-\eta_k)}\|x(\eta_k)\|.$$

Moreover,

$$x(\eta_k) = X(\eta_k,t)x(t) - \int_{\eta_k}^{t} X(\eta_k,s)F(s,x(s),x(\eta_k))\mathrm{d}s.$$

Then,

$$\begin{aligned}\|x(\eta_k)\| &\le M\|x(t)\| + ML_1\int_{\eta_k}^{t}(\|x(s)\| + \|x(\eta_k)\|)\mathrm{d}s\\ &\le M\|x(t)\| + M\left[(1+\bar{\theta}L_1)(e^{ML_1\bar{\theta}}-1) + L_1\bar{\theta}\right]\|x(\eta_k)\|.\end{aligned}$$

Thus, for $(t,x(t)) \in D_k$, we have $\|x(\eta_k)\| \le K(L_1,\bar{\theta})\|x(t)\|$. The lemma is proved. □

Definition 5.1. The zero solution of (8) is said to be uniformly stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x(t,t_0,x_0)\| < \epsilon$ whenever $\|x_0\| < \delta$ for $t \ge t_0$.

Definition 5.2. The zero solution of (8) is said to be uniformly asymptotically stable if it is uniformly stable, and there is a number $b > 0$ such that for every $\zeta > 0$ there exists $T(\zeta) > 0$ such that $\|x_0\| < b$ implies that $\|x(t,t_0,x_0)\| < \zeta$ if $t > t_0 + T(\zeta)$.

Theorem 5.1. *Suppose that* (Q1)–(Q9), (Q14), (Q15) *hold. If the zero solution of Eq. (10) is uniformly asymptotically stable, then for sufficiently small Lipschitz constant L_1, the zero solution of Eq. (8) is uniformly asymptotically stable.*

Proof. Suppose that the zero solution of (10) is uniformly asymptotically stable. Then, there exist positive real numbers α and σ such that for $t > s$,

$$\|X(t,s)\| \le \alpha e^{-\sigma(t-s)}. \tag{27}$$

Let $x(t)$ be a solution of (8) with the initial condition $x(t_0) = x_0$ such that $\|x_0\| \le h$. We have for $t \ge t_0$,

$$\begin{aligned}\|x(t)\| &= \left\|X(t,t_0)x(t_0) + \int_{t_0}^{t} X(t,s)F(s,x(s),x(\beta(s,x)))\mathrm{d}s\right\|\\ &\le \alpha e^{-\sigma(t-t_0)}\|x_0\| + L_1\int_{t_0}^{t}\alpha e^{-\sigma(t-s)}(1+K(L_1,\bar{\theta}))\|x(s)\|\mathrm{d}s.\end{aligned}$$

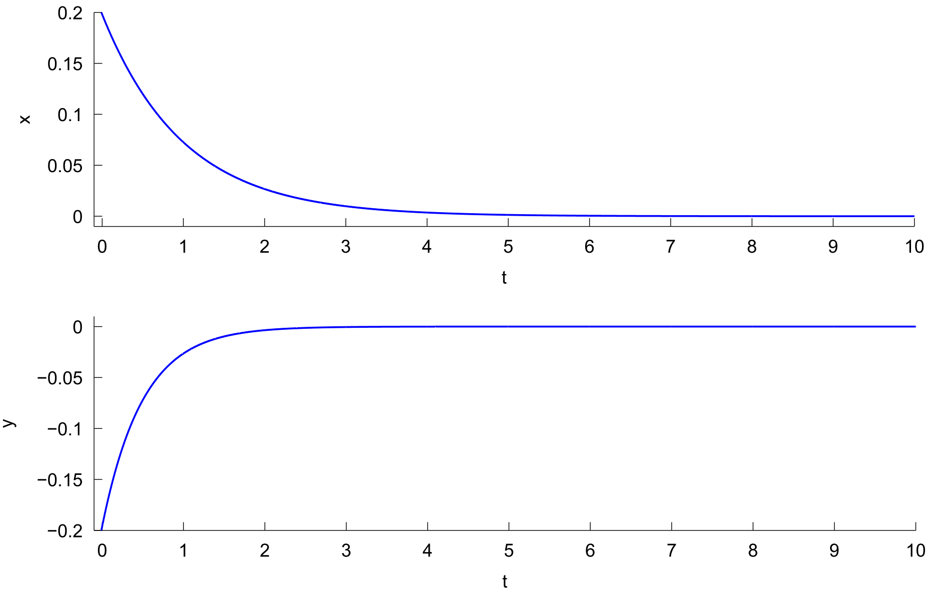

Fig. 3. A solution $(x(t), y(t))$ of ESPA that approaches the zero solution as time increases.

Then,

$$e^{\sigma t}\|x(t)\| \le \alpha e^{\sigma t_0}\|x_0\| + \alpha L_1(1 + K(L_1, \bar{\theta})) \int_{t_0}^{t} e^{\sigma s}\|x(s)\| ds.$$

Hence, using the Gronwall–Bellman Lemma, we have

$$\|x(t)\| \le \alpha e^{(\alpha L_1(1+K(L_1,\bar{\theta}))-\sigma)(t-t_0)}\|x_0\|.$$

Since for sufficiently small L_1, we have $\alpha L_1(1 + K(L_1, \bar{\theta})) - \sigma < 0$, the theorem is proved. □

The following example validates the last result.

Example 5.1. Consider the equation

$$\begin{aligned} x'(t) &= -x(t) - a\sin^2(y(\beta(t, x, y))) \\ y'(t) &= -2y(t) + a\sin^2(x(\beta(t, x, y))) \end{aligned} \tag{28}$$

where $t, x, y \in \mathbb{R}$, and a is a positive real number. Here, $\beta(t, x, y)$ is defined by $\theta_j = j$ and $\xi_j(x, y) = -a\cos(x + y)$. The corresponding parameters in the conditions of Theorem 5.1 are $L_1 = 2\sqrt{2}a$, $L_2 = a$, $\bar{\theta} = 1 + 2a$, $\kappa = 2$, $N = 0$, $M = e^{2+4a}$, $\tilde{M} = 4\sqrt{2}aH$. One can show that conditions (Q1)–(Q9), (Q14), (Q15) are satisfied for $a = e^{-4}$, $H = 1$. Hence, by Theorem 5.1, the zero solution is uniformly asymptotically stable. Fig. 3 shows a solution $(x(t), y(t))$ of (28) with initial condition $(x(-e^{-4}), y(-e^{-4})) = (0.02, -0.02)$ that approaches the zero solution.

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