FLOW INVARIANCE CONDITIONS FOR HOLOMORPHIC MAPPINGS IN BANACH SPACES

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Abstract

Let D be the open unit ball in a complex Banach space X. We provide necessary and sufficient conditions for a holomorphic mapping $f: D \to X$ to be semi-complete.

Let D be a convex subset of a Banach space X and let $f: \bar{D} \to X$ be a continuous mapping on $\bar{D}$, the closure of D. Then the following tangency condition of flow invariance

$$\lim_{h \to 0^+} \operatorname{dist}(x - hf(x), \bar{D})/h = 0, \quad x \in \bar{D} \tag{1}$$

is necessary to solve the evolution equation $du/dt + f(u) = 0,\ u(0) = x \in \bar{D}$. It is known that in the above setting (1) is equivalent to $F = I - f$ being weakly inward.

For the classes of monotone and accretive mappings, the flow invariance condition (1) (or equivalently, weak inwardness) was systematically used to study the fixed point set of the mapping $F = I - f$ (see, for example [5], [8], [17], [19], [26]).

Accretive mappings are of much interest because of their connection with the theory of semigroups of contraction mappings. For instance, a result of Martin [17] shows that if D is a convex subset of X, and $f: \bar{D} \to X$ is a continuous accretive mapping on $\bar{D}$, then (1) is also sufficient for the existence of solutions to the Cauchy problems

$$\begin{cases} \dfrac{du}{dt} + f(u) = 0 \\ u(0) = x \in \bar{D}, \end{cases} \tag{2}$$

which are defined on $\mathbb{R}^+ = [0, \infty)$. These solutions yield a continuous semigroup of contraction mappings on $\bar{D}$.

In this paper we will concentrate on another class of mappings. These mappings are holomorphic in the open unit ball D of a complex Banach space. Recall that a mapping $f\colon D \to \tilde{D} \subseteq Y$ (Y is another complex Banach space) is said to be holomorphic in D if it is Fréchet differentiable at each point of D. The class of such mappings will be denoted by $\mathrm{Hol}(D, \tilde{D})$.

Definition. Let D be the open unit ball in a complex Banach space X. A mapping $f \in \mathrm{Hol}(D, X)$ is said to be a semi-complete vector field on D if the Cauchy problem (2) has a solution $\{u(t, x)\} \subset D$, which is well-defined on $\mathbb{R}^+$ for each initial datum $x \in D$.

Note that since $f \in \mathrm{Hol}(D, X)$ is locally Lipschitzian, this solution is unique, and $F_t = u(t, \cdot)$ is a one-parameter semigroup (flow) of holomorphic self-mappings of D.

In other words, $f \in \mathrm{Hol}(D, X)$ is semi-complete if it is an infinitesimal generator of a flow on D. In the case where this flow consists of automorphisms of D it can be extended to a one-parameter group and the Cauchy problem (2) has a unique solution, $\{u(t, x)\} \subset D$, defined on all of $\mathbb{R} = (-\infty, \infty)$ for each $x \in D$. The converse is also true. In such a situation the mapping f is said to be a complete vector field on D (see [4], [6], [12], [24]).

For the class of holomorphic mappings an analog of Martin's theorem was given in [2]; namely, if $f \in \mathrm{Hol}(D, X)$ has a uniformly continuous extension to $\bar{D}$, then it is a semi-complete vector field if and only if it satisfies the boundary flow invariance condition (1).

Note that each holomorphic self-mapping of D is non-expansive with respect to any metric assigned to D by a Schwarz–Pick system (see, for example [6], [7], [9], [11]).

On the other hand, it does not seem sufficiently natural to consider a boundary condition to characterise semi-complete vector fields because the open unit ball itself is a complete metric space with respect to such a metric. In addition, there are many examples of semi-complete vector fields which have no continuous extension to $\bar{D}$ (see the examples below). In particular, if $F \in \mathrm{Hol}(D, D)$, then $f = I - F$ is semi-complete (see [20]).

Thus the first question which arises is:

> Is there an interior flow invariance condition which characterises the class of semi-complete vector fields?

In addition, it would be desirable to find a condition from which (1) could directly be derived in the case where $f \in \mathrm{Hol}(D, X)$ has a continuous extension to $\bar{D}$. The second question of interest is:

> If f is a semi-complete factor field, what additional condition will imply that f is actually complete?

In other words, when does the flow generated by f consist of automorphisms of D?

As we shall see below, such a condition exists. It has a purely local character and involves the derivative of f at the origin.

We begin with the following concept.

Let X be a Banach space and let X' be its dual. For $x \in X$ and $x' \in X'$ we use the pairing $\langle x, x'\rangle$ to denote $x'(x)$. The duality mapping $J\colon X \to 2^{X'}$ is defined by

$$J(x) := \{x' \in X'\colon \operatorname{Re}\langle x, x'\rangle = \|x\|^2 = \|x'\|^2\}$$

for each $x \in X$.

If D is the open unit ball in X and f maps $\bar{D}$ into X, then it follows by a result in [18] that (1) is equivalent to the condition

$$\inf_{x' \in J(x)} \operatorname{Re}\langle f(x), x'\rangle \geq 0, \quad x \in \partial D. \tag{1'}$$

Following Krasnoselskii [15] we will call conditions of such a type 'one-sided estimates'.

For the Euclidean ball D in $X = \mathbb{C}^n$, a certain condition in this direction was established by Abate [1]. He proved that $f \in \operatorname{Hol}(D, \mathbb{C}^n)$ is a semi-complete vector field if and only if it satisfies the estimate

$$2[\|g(x)\|^2 - |\langle g(x), x\rangle|^2]\operatorname{Re}\langle g(x), x\rangle + (1 - \|x\|^2)^2 \operatorname{Re}\langle f'(x)f(x), g(x)\rangle \geq 0, \tag{3}$$

where

$$g(x) = (1 - \|x\|^2)f(x) + \langle f(x), x\rangle x.$$

For $n = 1$ this condition becomes

$$\operatorname{Re} f(z)\bar{z} \geq -\frac{1}{2}\operatorname{Re} f'(z)(1 - |z|^2), \tag{3'}$$

where $z \in \Delta$, the open unit disk in the complex plane $\mathbb{C}$, and $f \in \operatorname{Hol}(\Delta, \mathbb{C})$.

Despite the simplicity of condition (3′) it is not clear how (1′) can be derived from (3′) when f has a continuous extension to $\bar{\Delta}$.

On the other hand, this condition may be useful in studying the behavior of the derivative of a semi-complete vector field in Δ. Therefore it is natural to ask the following question:

Can this condition be extended to a general Banach space in a form similar to (3′) (instead of condition (3))?

Note also that in the one-dimensional case it follows from the maximum principle for harmonic functions that (1′) implies the following interior condition:

$$\operatorname{Re} f(z)\bar{z} \geq \operatorname{Re} f(0)\bar{z}(1 - |z|^2), \quad z \in \Delta. \tag{1''}$$

Conversely, it is clear that (1′) results from (1″) if f has a continuous extension to all of $\bar{\Delta}$. Thus another question arises:

Are (1″) and its Banach space analog necessary and sufficient for f to be semi-complete?

We now state our main result. It provides affirmative answers, in any Banach space, to all the questions raised above.

Theorem. *Let D be the open unit ball in a complex Banach space X. Then* $f \in$

Hol(D, X) *is a semi-complete vector field on D if and only if it is bounded on each subset strictly inside D and one of the following conditions holds:*

(a) *For each* $x \in D$ *there exists* $x' \in J(x)$ *such that*

$$Re\,\langle f(x) - f(0)(1 - \|x\|^2), x'\rangle \geq 0;$$

(b) $\inf_{x' \in J(x)} Re\,\langle 2\|x\|^2 f(x) + (1 - \|x\|^2) f'(x)x, x'\rangle \geq 0 \quad x \in D;$

(c) *For each* $x \in D$ *and for each* $x' \in J(x)$,

$$Re\,\langle \frac{1 - \|x\|}{1 + \|x\|} f'(0)x + (1 - \|x\|^2) f(0), x'\rangle \leq Re\,\langle f(x), x'\rangle$$
$$\leq Re\,\langle \frac{1 + \|x\|}{1 - \|x\|} f'(0)x + (1 - \|x\|^2) f(0), x'\rangle\ .$$

Furthermore, equality in one of the conditions (a), (b) or (c) holds if and only if it holds in the other conditions and f *is complete.*

Note that in the one-dimensional case condition (a) reduces to ($1''$) and condition (b) becomes Abate's condition ($3'$). The following corollary is a direct consequence of the Theorem.

Corollary 1. *Let* $f \in$ Hol(D, X) *be a semi-complete vector field on* D*. Then the linear operator* $A = f'(0)$ *is (totally) accretive, that is*

$$\inf_{y' \in J(y)} Re\,\langle Ay, y'\rangle \geq 0, \quad y \in X. \tag{4}$$

Proof. Substitute $x = ty$, $x' = ty'$ in (b), where $\|y\| = \|y'\| = 1$, $y' \in J(y)$ and $t \in (0, 1)$. Letting t go to zero we get (4). ■

In turn (4) implies that the left-hand inequality in (c) is sharper than (a). Moreover, it implies that (a) holds, in fact, for all $x' \in J(x)$. Thus we have that if f has a continuous extension to $\bar{D}$, then (a) yields the flow invariance boundary condition ($1'$) (equivalently, (1)). Furthermore, condition (c) also leads to a characterisation for a semi-complete vector field to be complete.

Following Krein [16] (see also Vesentini [25]), we say that a linear operator $A\colon X \to X$ is conservative if, for all $x \in X$ and $x' \in J(x)$,

$$\mathrm{Re}\,\langle Ax, x'\rangle = 0.$$

Corollary 2. *Let* $f \in$ Hol(D, X) *be a semi-complete vector field. Then* f *is actually complete if and only if its derivative at zero,* $f'(0)$*, is a conservative linear operator.*

It is well known that a complete vector field g on the open unit ball D in a Banach space X is a polynomial of degree at most 2 (see, for example [24] and [6]).

More precisely, g has the form

$$g(x) = a + Ax + P_a(x), \tag{5}$$

where a is an element of X, A is a conservative operator on X, and P_a is a homogeneous form of the second degree such that $P_{ia} = iP_a$.

One of the consequences of this representation is an infinitesimal analog of Cartan's uniqueness theorem [6], [12]: If $g \in \mathrm{Hol}(D, X)$ is a complete vector field such that $g(0) = 0$ and $g'(0) = 0$, then $g \equiv 0$.

Applying Corollary 2, we obtain the following extension of this theorem.

Corollary 3. *If $f \in \mathrm{Hol}(D, X)$ is a semi-complete vector field on D, such that $f(0) = 0$ and $f'(0) = 0$, then $f \equiv 0$.*

It was shown in [20] that if $f \in \mathrm{Hol}(D, X)$ has the form

$$f = I - F, \tag{6}$$

where F is a self-mapping of D, then f is semi-complete. Thus Corollary 3 is also a generalisation of Cartan's uniqueness theorem (sometimes this theorem is also called the Generalised Schwarz Lemma (see [10]).

Note also that the linear space $H_{1,\infty}(D, X)$ of all holomorphic mappings on D with radius of boundedness equal to 1 ($f \in H_{1,\infty}(D, X)$ if and only if it is bounded on each subset strictly inside D; see, for example [7]) is a locally convex space with the topology of local uniform convergence over D (see also [12]).

By using the above Theorem, we obtain the following corollary (cf. [21]).

Corollary 4. *The set $\mathcal{G}$ of all semi-complete vector fields on D is a closed real cone in $H_{1,\infty}(D, X)$. Actually, $\mathcal{G}$ is the closure in the topology of local uniform convergence over D of the set $\{f \in \mathrm{Hol}(D, X)\colon f = \alpha g,\ \alpha \geq 0\}$, where g is of the form* (6).

Suppose now that a complex Banach space X is a so-called JB^* triple system. This is the same as saying that its open unit ball D is a homogeneous domain, that is for each pair $x, y \in D$ there exists a holomorphic automorphism of D such that $F(x) = y$ (see, for example [6], [24]). Then it is well-known that for each $a \in X$ there exists a homogeneous polynomial $P_a(x)$ such that $P_{ia} = iP_a$ and the mapping $g\colon D \to X$ defined by

$$g(x) = a - P_a(x) \tag{7}$$

is a complete vector field on D, which is called a transvection of D. Using this fact and Corollary 4 we get the following representation theorem (cf. [13]).

Corollary 5. *Let X be a JB^* triple system and let D be its open unit ball. Then the cone $\mathcal{G}$ of semi-complete vector fields on D admits the decomposition*

$$\mathcal{G} = \mathcal{G}_0 \bigoplus \mathcal{G}_+, \tag{8}$$

where $\mathcal{G}_0$ is the real Banach subspace of $H_{1,\infty}(D,X)$ consisting of transvections and $\mathcal{G}_+$ is the subcone of $\mathcal{G}$ such that for each $h \in \mathcal{G}_+$,

$$\inf_{x' \in J(x)} Re\,\langle h(x), x'\rangle \geq 0, \quad \textit{for all } x \in D.$$

In other words, $f \in \mathcal{G}$ admits a unique representation

$$f = g + h, \tag{9}$$

where $g = f(0) - P_{f(0)}(x)$ is complete, $h \in \mathcal{G}_+$ and $h(0) = 0$.

The natural examples of JB^* triple systems are a complex Hilbert space H, the space of bounded linear operators $L(H)$ on H, and its subspaces J such that $A \in J$ if and only if $AA^*A \in J$ (such subspaces are usually called J^*-algebras). In the latter case the general form of transvections on D is

$$g(x) = a - xa^*x,$$

where $a \in J$ and a^* is its conjugate. Thus each semi-complete vector field on the open unit ball of a J^*-algebra has the form

$$f(x) = f(0) - xf^*(0)x + h(x), \tag{10}$$

where $h \in \mathcal{G}_+$ and $h(0) = 0$.

In particular, when $X = \mathbb{C}$ is the complex plane and $D = \Delta$ is the open unit disk in $\mathbb{C}$, (10) becomes

$$f(z) = f(0) - \overline{f(0)}z^2 + zp(z), \tag{10$'$}$$

where $p(z) \in \mathrm{Hol}(\Delta, \mathbb{C})$ and

$$\mathrm{Re}\, p(z) \geq 0, \qquad z \in \Delta. \tag{11}$$

That is, $p(z)$ is a function in the class of Carathéodory. Using the Riesz–Herglotz integral characterisation of this class (see, for example [3]) we deduce the following conclusion:

$f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is a semi-complete vector field if and only if it admits the representation

$$f(z) = a + ibz - \bar{a}z^2 + z\int_{\partial\Delta} \frac{1 + \bar{\zeta}z}{1 - \bar{\zeta}z}\, d\mu(\zeta), \tag{12}$$

where $a \in \mathbb{C}$, $b \in \mathbb{R}$ and μ is a positive measure on $\partial\Delta$.

As a matter of fact, this representation is the key to proving our theorem because we will partially use a reduction to the one-dimensional case. We will also get (10$'$) with (11) (respectively (12)) by an independent method.

We will also need two preliminary results for a general Banach space.

Proposition 1. [22]. *Let D be a bounded domain in a complex Banach space X and let a family $\{F_t\}_{t\geq 0}$ be a continuous one-parameter semigroup of holomorphic self-mappings of D, that is $F_t \in \mathrm{Hol}(D, D)$, $t \geq 0$, $F_t \circ F_s = F_{t+s}$, $t, s \geq 0$, and the strong limit*

$$\lim_{t\to 0^+} F_t(x) = x, \quad x \in D. \tag{13}$$

Then the following assertions are equivalent:

(i) *The convergence in* (13) *is locally uniform;*

(ii) *For each $x \in D$ the vector-valued function $u(t, x) = F_t(x)\colon \mathbb{R}^+ \to D$ is right-differentiable at zero, that is there exists the strong limit*

$$\lim_{t\to 0^+} \frac{1}{t}(x - u(t, x)) = f(x), \ x \in D;$$

(iii) *The family $\{f_t\}_{t>0}$ of difference approximations $f_t = \frac{1}{t}(I - F_t) \in \mathrm{Hol}(D, X)$ is uniformly bounded on each subset strictly inside D.*

Again let D be the open unit ball in a Banach space X (real or complex). Following Kobayashi and Oharu [14] we will say that a mapping $f\colon D \to X$ is locally (totally) accretive on D if for each $r \in (0, 1)$ there exists $\omega = \omega(r) \in \mathbb{R}$, such that

$$\inf\{\mathrm{Re}\,\langle f(x) - f(y), (x - y)'\rangle\colon (x - y)' \in J(x - y)\} \geq \omega\|x - y\|^2 \tag{14}$$

whenever x and y belong to the level set $\bar{D}_r = \{x \in X\colon \|x\| \leq r < 1\}$.

Proposition 2. [*23*]. *Let D be the open unit ball in a Banach space X, and let $f\colon D \to X$ be a continuous locally (totally) accretive mapping. Suppose that there is an increasing continuous function α on the interval $[0, 1]$ such that $\alpha(0) \cdot \alpha(1) \leq 0$, α is left differentiable at 1, and for all $x \in D$, the following condition holds:*

$$Re\,\langle f(x), x'\rangle \geq \alpha(\|x\|) \cdot \|x\|, \tag{15}$$

where x' is an element of $J(x)$. Then f generates a one-parameter semigroup $\{F_t\}$, $t \geq 0$, which is locally Lipschitzian on D:

$$\|F_t(x) - F_t(y)\| \leq e^{\omega t}\|x - y\|$$

whenever $x, y \in \bar{D}_r$.

PROOF OF THE THEOREM. Let f be a semi-complete vector field on the open unit ball D of a complex Banach space X. Then the Cauchy problem (2) has a unique solution $\{u(t, x)\} \subset D$ on $\mathbb{R}^+ \times D$. The uniqueness of this solution implies that the family, $\{F_t\}_{t\geq 0}$, $F_t = u(t, \cdot)$, is a one-parameter continuous semigroup of holomorphic self-mappings of D. In addition, it is obvious that condition (ii) of Proposition 1 holds. Then by (iii), $f_t = (I - F_t)/t$, and hence f is uniformly bounded on each subset strictly inside D.

Now we will prove that condition (a) of the theorem is satisfied. Indeed, fix any $x \in D$ and $x' \in J(x)$, and set $u = x/\|x\|$, $u' = x'/\|x\|$. Consider the holomorphic function $\hat{f}$ on the unit disk $\Delta \subset \mathbb{C}$ defined as follows:

$$\hat{f}(\lambda) = \langle f(\lambda u), u' \rangle, \qquad \lambda \in \Delta. \tag{16}$$

Similarly we define a family $\{\hat{F}_t\}_{t\geq 0}$ of holomorphic self-mappings of Δ:

$$\hat{F}_t(\lambda) = \langle F_t(\lambda u), u' \rangle, \qquad \lambda \in \Delta,\ t \geq 0. \tag{17}$$

It is clear that $\hat{F}_0(\lambda) = \lambda$ and that there exists the limit

$$\lim_{t\to 0^+} \frac{1}{t}(\lambda - \hat{F}_t(\lambda)) = \hat{f}(\lambda). \tag{18}$$

Now let $\mathscr{M}_b$ denote the Möbius transformation on Δ defined by

$$\mathscr{M}_b(\lambda) = \frac{\lambda - b}{1 - \lambda\bar{b}}, \quad b \in \Delta.$$

Consider the family $\{\mathscr{H}_t\}_{t\geq 0}$ of holomorphic self-mappings of Δ defined by:

$$\mathscr{H}_t(\lambda) = \mathscr{M}_{\hat{F}_t(0)}(\hat{F}_t(\lambda)), \qquad \lambda \in \Delta,\ t \geq 0.$$

Note that since $\mathscr{H}_t(0) = 0$ for all t, it follows by the Schwarz Lemma that

$$|\mathscr{H}_t(\lambda)| \leq |\lambda| \text{ for all } \lambda \in \Delta \text{ and } t \geq 0. \tag{19}$$

Now by simple calculations and (18) one can conclude that for each $\lambda \in \Delta$ the curve $\mathscr{H}_t(\lambda)\colon \mathbb{R}^+ \to \Delta$ is right-differentiable at zero and

$$\lim_{t\to 0^+} \frac{\lambda - \mathscr{H}_t(\lambda)}{t} = \hat{f}(\lambda) - \hat{f}(0) + \overline{\hat{f}(0)}\lambda^2 := h(\lambda), \qquad \lambda \in \Delta. \tag{20}$$

It follows now by (19) and (20) that for all $\lambda \in \Delta$:

$$\operatorname{Re} h(\lambda)\bar{\lambda} \geq 0 \tag{21}$$

and

$$h(0) = 0. \tag{22}$$

But (21) means that

$$\operatorname{Re}\hat{f}(\lambda)\bar{\lambda} \geq \operatorname{Re}\hat{f}(0)\bar{\lambda}(1 - |\lambda|^2). \tag{23}$$

Setting $\lambda = \|x\|$ we have by (16)

$$\operatorname{Re}\langle f(x), x' \rangle \geq \operatorname{Re}\langle f(0), x' \rangle (1 - \|x\|^2).$$

Since x and $x' \in J(x)$ are arbitrary, this proves the following implication: if $f \in \mathcal{G}$, then f satisfies (a).

Now we will show that (a) is equivalent to (b). To do this we return again to the function

$$\hat{f}(\lambda) = \hat{f}(0) - \overline{\hat{f}(0)}\lambda^2 + h(\lambda) \tag{24}$$

defined by (16), where $h(\lambda)$ satisfies (20) and (21). But these conditions are equivalent to the conditions

$$h(\lambda) = \lambda \cdot p(\lambda), \qquad \lambda \in \Delta, \tag{25}$$

with

$$\operatorname{Re} p(\lambda) \geq 0. \tag{26}$$

So (23) is equivalent to (25) for $\hat{f}$ of the form (24). Now, in the same terms, we translate condition (b). If we define $\hat{f}$ as above by (16), then (b) implies

$$\operatorname{Re}\,[2\hat{f}(\lambda)\bar{\lambda} + \hat{f}'(\lambda)(1 - |\lambda|^2)] \geq 0. \tag{27}$$

If we substitute $\hat{f}$ here in the form (24) with $h(\lambda) = \lambda p(\lambda)$ we see that (27) is equivalent to the condition:

$$\operatorname{Re}\,[\lambda p'(\lambda) + \frac{1 + |\lambda|^2}{1 - |\lambda|^2} p(\lambda)] \geq 0, \qquad \lambda \in \Delta. \tag{28}$$

We intend to show that (28) is equivalent to (26). Again setting $\lambda = \|x\|$ in (16) and (27) and noting that $\hat{f}'(\lambda) = \langle f'(\lambda u)u, u'\rangle = \langle f'(x)x, x'\rangle \frac{1}{\|x\|^2}$ for $x \neq 0$, we will then get, by continuity, the equivalence of conditions (a) and (b) of the Theorem.

So, let $p \in \operatorname{Hol}(\Delta, \mathbb{C})$ satisfy (26). Define $F = (p-1)(p+1)^{-1}$. Since the mapping $w = \frac{z-1}{z+1}$ maps the right half-plane into Δ, F is a self-mapping of Δ. Applying the Schwarz–Pick Lemma to F, we obtain

$$\left|\left(\frac{p-1}{p+1}\right)'\right| = \frac{2|p'|}{|1+p|^2} \leq \frac{|p+1|^2 - |p-1|^2}{|p+1|^2(1 - |\lambda|^2)}$$

or

$$|p'(\lambda)| \leq \frac{2\operatorname{Re} p(\lambda)}{1 - |\lambda|^2}.$$

This implies

$$\operatorname{Re}\,(-\lambda p'(\lambda)) \leq |\lambda p'(\lambda)| \leq \frac{2|\lambda| \operatorname{Re} p(\lambda)}{1 - |\lambda|^2} \leq \frac{1 + |\lambda|^2}{1 - |\lambda|^2} \operatorname{Re} p(\lambda),$$

which is equivalent to (28).

In the opposite direction, (28) implies (26), we prove the following more general fact:

Let $p \in \mathrm{Hol}(\Delta, \mathbb{C})$ and suppose that there is a positive function ψ: $[0,1) \to \mathbb{R}^+$ such that the following condition holds:

$$\mathrm{Re}\,(\lambda p'(\lambda) + \psi(|\lambda|)p(\lambda)) \geq 0, \qquad \lambda \in \Delta. \tag{28'}$$

Then $\mathrm{Re}\, p(\lambda) \geq 0$ everywhere on Δ. Setting $\lambda = \mathrm{r}e^{i\theta}$ we have

$$\lambda p'(\lambda) = r\frac{\partial p}{\partial r},$$

and (28') becomes

$$\mathrm{Re}\,(r\frac{\partial p}{\partial r}) + \psi(r)\mathrm{Re}\, p(\lambda) \geq 0, \qquad \lambda = re^{i\theta} \in \Delta.$$

Assume now that there exists $\lambda_0 = r_0 e^{i\theta_0}$ in Δ such that

$$\mathrm{Re}\, p(\lambda_0) < 0.$$

Since (28′) implies that $\mathrm{Re}\, p(0) \geq 0$, there exist r_1 and r_0 with $0 \leq r_1 < r_0$ such that $\mathrm{Re}\, p(r_1 e^{i\theta_0}) = 0$ and $\mathrm{Re}\, p(r_0 e^{i\theta_0}) < 0$. Thus one can find $r_2 \in (r_1, r_0)$ such that

$$\mathrm{Re}\, p(r_2 e^{i\theta_0}) < 0$$

and

$$\mathrm{Re}\frac{\partial p}{\partial r}(r_2 e^{i\theta_0}) < 0.$$

This implies

$$\mathrm{Re}\,(r_2\frac{\partial p}{\partial r}) + \psi(r_2)\mathrm{Re}\, p(r_2 e^{i\theta_0}) < 0,$$

a contradiction. Thus $\mathrm{Re}\, p(\lambda) \geq 0$ for all $\lambda \in \Delta$ and we are done. In other words, conditions (a) and (b) are equivalent.

To obtain (c) we now represent $\hat{f}(\lambda) = \langle f(\lambda u), u' \rangle$ in the form (12):

$$\hat{f}(\lambda) = a + ib\lambda - \bar{a}\lambda^2 + \lambda\int_{\partial\Delta}\frac{1+\bar{\zeta}\lambda}{1-\bar{\zeta}\lambda}d\mu(\zeta),$$

and we calculate:

$$\mathrm{Re}\,\frac{1+\bar{\zeta}\lambda}{1-\bar{\zeta}\lambda} = \frac{1-|\lambda|^2}{|1-\bar{\zeta}\lambda|^2} = \frac{1-|\lambda|}{1+|\lambda|}\frac{(1+|\lambda|)^2}{|1-\bar{\zeta}\lambda|^2} = \frac{1+|\lambda|}{1-|\lambda|}\cdot\frac{(1-|\lambda|)^2}{|1-\bar{\zeta}\lambda|^2}.$$

Since $|\zeta| = 1$, this equality shows that

$$\frac{1-|\lambda|}{1+|\lambda|} \leq \mathrm{Re}\,\frac{1+\bar{\zeta}\lambda}{1-\bar{\zeta}\lambda} \leq \frac{1+|\lambda|}{1-|\lambda|}. \tag{29}$$

Noting that $p(0) = \hat{f}'(0)$ (see (24) and (25)), we obtain from (12) and (29):

$$\operatorname{Re} \hat{f}(0)\bar{\lambda}(1-|\lambda|^2) + |\lambda|^2 \frac{1+|\lambda|}{1-|\lambda|}\hat{f}'(0) \geq \operatorname{Re}\hat{f}(\lambda)\bar{\lambda} \geq \operatorname{Re}\hat{f}(0)\bar{\lambda}(1-|\lambda|^2) + |\lambda|^2 \frac{1-|\lambda|}{1+|\lambda|}\hat{f}'(0).$$

Once again, setting $\lambda = \|x\|$ we get from the last inequality condition (c).

Since by (b), $\operatorname{Re}\langle f'(0)x, x'\rangle \geq 0$, it is clear that (c) is stronger than (a).

Now let $f \in \operatorname{Hol}(D, X)$ be bounded on each subset strictly inside D and satisfy condition (a) of the Theorem. Then on each level set, $\bar{D}_r = \{x \in X : \|x\| \leq r < 1\}$, f is Lipschitzian, that is

$$\|f(x) - f(y)\| \leq L_r \|x - y\|,$$

where

$$x, y \in \bar{D}_r, \quad L_r \geq \sup_{x \in \bar{D}_r} \|f'(x)\|.$$

Therefore f is locally (totally) accretive with $\omega = -L_r$ (see (14)).

In addition, the function $\alpha(r) = -\|f(0)\|(1-r^2)$ is increasing and differentiable on $[0, 1]$, and satisfies all the other conditions of Proposition 2. Hence f is semi-complete. ■

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