# On the aggregation of branching processes

Ph.D. Thesis

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# Chapter 1

# Introduction

## 1.1 Motivation, historical background, and an overview of our results and methods

During my master's studies I started working with branching processes with my supervisor at the time, Gábor Szűcs. I found this widely applicable class of processes really interesting while we performed change–point detection of their parameters. As I started my doctoral studies with the supervision of Professor Gyula Pap, he suggested that I could join him and his longtime coauthor Mátyás Barczy to study these processes from a different aspect, namely, the limit behavior of their aggregates. Our research arose from the work of Pilipauskaitė and Surgailis [38], who investigated random coefficient autoregressive processes of order 1 (AR(1)) in the same sense. We got really invested in this topic through the years and answered many related questions in the four published and two submitted papers that we wrote together. In this current work, I am going to present the results of four of these papers. These four share the assumption that the innovation (also called immigration) has finite second moment, while in the two other papers it does not. These two are left out due to the length limitations and the fact that the covered four form a unit together, with some unanswered questions remaining.

In general, we consider independent copies of a stationary branching process, we denote these by $(X_k^{(j)})_{k=1,2,\ldots}$, $j = 1, 2, \ldots$. We are interested in the limit behavior of the aggregate process $\left(\sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$, as both $n$, the time parameter, and $N$, the number of copies tend to infinity in some manner. If we take the limits in an iterated manner, i.e., first $n$ tends to infinity and then $N$ tends to infinity, or vice versa, then the resulting limit theorem is called an iterated one. If both converge to infinity at the same time, then it is called a simultaneous limit theorem. To achieve such limit theorems, we also consider the simple aggregates, $\sum_{k=1}^{n} X_k^{(j)}$, which is called temporal (or time-aggregated), and $\sum_{j=1}^{N} X_k^{(j)}$, which is called contemporaneous (or space-aggregated).

The aggregation problem is concerned with the relationship between individual (micro) and aggregate (macro) behavior. Random coefficient AR(1) models, where the coefficient of the autoregressive component is a random variable instead of a constant, appeared in Robinson [47]. However, the scheme of contemporaneous aggregation of random coefficient AR(1) models was firstly proposed by Granger [18] in order to obtain the long memory phenomena in aggregated time series. In Gonçalves and Gouriéroux [17], the contemporaneous aggregation of random coefficient AR(1) models is thoroughly studied. They examine whether the aggregated process inherits certain properties from the random coefficient AR(1) processes, such as stationarity and long memory. They also characterize the processes that can appear as the contemporaneous aggregates of such AR(1) processes. Zaffaroni [58] continues the previous research by examining the convergence of the previously discussed aggregates. Moreover, they extend the results by considering autoregressive moving averages (ARMA) models instead or the AR ones. Some economical and empirical examples are also provided in the paper. Oppenheim and Viano [36] deal with aggregates of discrete and continuous time stochastic processes as well. In particular, they give the limit of the contemporaneous aggregate of random parameter AR(p) processes and investigate the long memory property of the limit. Celov et al. [11] is about the so-called disaggregation problem. By observing an aggregated process, their aim is to determine the individual processes that lead to that aggregated one, if such exist. Again, they confine the search to random coefficient AR(1) processes as these individual processes. Beran et al. [8] is also about this disaggregation problem, more specifically, they consider a Beta mixing distribution for the squared coefficient of the random coefficient AR(1) processes, and estimate its parameters based on the aggregated process. Puplinskaitė and Surgailis [41, 42] discussed contemporaneous aggregation of random coefficient AR(1) processes with infinite variance and innovations in the domain of attraction of a stable law.

As this thesis is about discrete time stochastic processes, we are only going to provide a non-exhaustive list of resources for the aggregation of continuous time stochastic processes. Related problems for some network traffic models were studied in Willinger et al. [56], Taqqu et al. [53], Gaigalas and Kaj [16] and Dombry and Kaj [13], where independent and centered ON/OFF processes are aggregated, in Mikosch et al. [33], where aggregation of $M/G/\infty$ queues with heavy-tailed activity periods are investigated, in Pipiras et al. [40], where integrated renewal or renewal-reward processes are considered, or in Iglói and Terdik [20], where the limit behavior of the aggregate of certain random coefficient Ornstein–Uhlenbeck processes is examined. On page 521 in Jirak [23] one can find a lot of further references for papers dealing with the aggregation of continuous time stochastic processes.

As we will be working on similar limit theorems for our models, we recall some of the results in Pilipauskaitė and Surgailis [38], which describe the limit behavior of sums

$$S_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}, \qquad t \in [0, \infty), \qquad N, n \in \{1, 2, \ldots\}, \tag{1.1}$$

where $(X_k^{(j)})_{k\in\{0,1,\ldots\}}$, $j \in \{1, 2, \ldots\}$, are independent copies of a stationary random coefficient AR(1) process

$$X_k = aX_{k-1} + \varepsilon_k, \qquad k \in \{1, 2, \ldots\}, \tag{1.2}$$

with standardized independent and identically distributed (i.i.d.) innovations $(\varepsilon_k)_{k\in\{1,2,\ldots\}}$ having $\mathbb{E}(\varepsilon_1) = 0$ and $\operatorname{Var}(\varepsilon_1) = 1$, and a random coefficient $a$ with values in $(0, 1)$, being independent of $(\varepsilon_k)_{k\in\{1,2,\ldots\}}$ and admitting a probability density function of the form

$$\psi(x)(1 - x)^{\beta}, \qquad x \in (0, 1),$$

where $\beta \in (-1, \infty)$ and $\psi$ is an integrable function on $(0,1)$ with $\lim_{x\uparrow 1} \psi(x) := \psi_1 > 0$. Here the distribution of $X_0$ is chosen as the unique stationary distribution of the model (1.2). Its existence was shown in Puplinskaitė and Surgailis [41, Proposition 1]. We point out that they considered so-called idiosyncratic innovations, i.e., the innovations $(\varepsilon_k^{(j)})_{k\in\{1,2,\dots\}}$, $j = 1, 2, \dots$, belonging to $(X_k^{(j)})_{k\in\{0,1,\dots\}}$, $j = 1, 2, \dots$, are independent. In Pilipauskaitė and Surgailis [38], scaling limits of the finite dimensional distributions of $(A_{N,n}^{-1} S_t^{(N,n)})_{t\in[0,\infty)}$, are derived, where $A_{N,n}$ are some scaling factors and first $N \to \infty$ and then $n \to \infty$, or vice versa, or both $N$ and $n$ increase to infinity, possibly with different rates. Then, Pilipauskaitė and Surgailis [39] extended their results in [38] from the case of idiosyncratic innovations to the case of common innovations, i.e., when $(\varepsilon_k^{(j)})_{k\in\{1,2,\dots\}} = (\varepsilon_k^{(1)})_{k\in\{1,2,\dots\}}$, $j = 1, 2, \dots$. Very recently, Pilipauskaitė et al. [37] have extended their earlier results again in the sense that they released the assumptions on the idiosyncratic innovations. In this new paper the innovations can be in the domain of attraction of a stable law, while in Pilipauskaitė and Surgailis [38] they had to be in that of the normal law. For historical fidelity, we note that Theil [54] already considered contemporaneous aggregations of linear regression models with non-random coefficients, and later Zellner [59] investigated the case of random coefficients. They both examined the estimators of the coefficients and their properties.

Now let us introduce the integer-valued autoregressive (INAR) processes, which will have a very important role in this work. The theory and application of integer-valued time series models are rapidly developing and important topics, see, e.g., Steutel and van Harn [50] and Weiß [55]. The INAR(1) process is among the most fertile integer-valued time series models, and it was first introduced by McKenzie [32] and Al-Osh and Alzaid [1]. An INAR(1) time series model is a stochastic process $(X_k)_{k\in\{0,1,\dots\}}$ satisfying the recursive equation

$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \{1, 2, \dots\}, \tag{1.3}$$

where $(\varepsilon_k)_{k\in\{1,2,\dots\}}$ are i.i.d. non-negative integer-valued random variables, $(\xi_{k,j})_{k,j\in\{1,2,\dots\}}$ are i.i.d. Bernoulli random variables with mean $\alpha \in (0,1)$, and $X_0$ is a non-negative integer-valued random variable such that $X_0$, $(\xi_{k,j})_{k,j\in\{1,2,\dots\}}$, and $(\varepsilon_k)_{k\in\{1,2,\dots\}}$ are independent. By using the binomial thinning operator $\alpha\,\circ$ due to Steutel and van Harn [50], the INAR(1) model in (1.3) can be written as

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \{1, 2, \dots\}, \tag{1.4}$$

which form captures the resemblance with the AR model. We note that an INAR(1) process can also be considered as a special branching process with immigration having Bernoulli offspring distribution. Leonenko et al. [30] introduced the aggregate $\sum_{j=1}^{\infty} X^{(j)}$ of a sequence of independent stationary INAR(1) processes $X^{(j)}$, $j \in \{1, 2, \dots\}$, where $X_k^{(j)} = \alpha^{(j)} \circ X_{k-1}^{(j)} + \varepsilon_k^{(j)}$, $k, j \in \{1, 2, \dots\}$. Under appropriate conditions on $\alpha^{(j)}$, $j \in \{1, 2, \dots\}$, and on the distributions of $\varepsilon_k^{(j)}$, $k, j \in \{1, 2, \dots\}$, they showed that the process $\sum_{j=1}^{\infty} X^{(j)}$ is well-defined in $L^2$-sense and it has long memory. INAR(1) processes with Poisson innovations and generalized integer-valued autoregressive processes of order $p$ (GINAR($p$)) are the special cases of multitype Galton–Watson processes with immigration. Chapter

2 is devoted to the temporal and contemporaneous aggregates of stationary multitype Galton–Watson processes with immigration.

Then in Chapters 3 and 4 we will consider a certain randomized (also called random coefficient) INAR(1) process $(X_k)_{k\in\{0,1,\dots\}}$ with randomized thinning parameter $\alpha$, given formally by the recursive equation

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \{1,2,\dots\}, \tag{1.5}$$

where $\alpha$ is a random variable with values in $(0,1)$ and $X_0$ is some appropriate random variable. This means that, conditionally on $\alpha$, the process $(X_k)_{k\in\{0,1,\dots\}}$ is an INAR(1) process with thinning parameter $\alpha$. Conditionally on $\alpha$, the i.i.d. innovations $(\varepsilon_k)_{k\in\{1,2,\dots\}}$ are supposed to have a Poisson distribution with parameter $\lambda \in (0,\infty)$, and the conditional distribution of the initial value $X_0$ given $\alpha$ is supposed to be the unique stationary distribution, namely, a Poisson distribution with parameter $\lambda/(1-\alpha)$. For a rigorous construction of this process, see Section 3.1. Here we only note that $(X_k)_{k\in\{0,1,\dots\}}$ is a strictly stationary sequence, but it is not even a Markov chain (so it is not an INAR(1) process) if $\alpha$ is not degenerate, see Appendix A of Barczy et al. [5]. Let us also remark that the choice of Poisson distributed innovations serves a technical purpose. It allows us to calculate and use the explicit stationary distribution and the joint generator function given in (2.21). We are planning to try releasing this assumption and giving more general results in future research.

Note that there is another way of randomizing the INAR(1) model (1.4), a so-called random coefficient INAR(1) process (RCINAR(1)), proposed by Zheng et al. [60] and Leonenko et al. [30]. It differs from (1.5), namely, it is a stochastic process formally given by the recursive equation

$$X_k = \alpha_k \circ X_{k-1} + \varepsilon_k, \qquad k \in \{1,2,\dots\},$$

where $(\alpha_k)_{k\in\{1,2,\dots\}}$ is an i.i.d. sequence of random variables with values in $(0,1)$. An RCINAR(1) process can be considered as a special kind of branching process with immigration in a random environment, see Key [28], where a rigorous construction is given on the state space of the so-called genealogical trees.

In Chapters 3 and 4, we have similar limit theorems for randomized INAR(1) processes that Pilipauskaitė and Surgailis [38, Theorems 2.1, 2.2 and 2.3] have for random coefficient AR(1) processes. The techniques of our proofs differ from those of Pilipauskaitė and Surgailis [38] in many cases. Concerning the iterated limit theorems, for a somewhat detailed comparison, see Remark 3.15. For the case of the simultaneous limit theorems, see the beginning of Chapter 4 and Remark 4.4. Also, in our work, centralization has an important role, since, opposed to the AR(1) and random coefficient AR(1) processes, the considered branching processes do not have zero mean. We present limit theorems with three different centerings. In some cases, when possible, we apply the expected values, and for the randomized INAR(1) processes, the conditional expected values with respect to the random coefficient as centralization. Moreover, as both of these are theoretical values instead of observable ones, we also investigate the limit behavior of the aggregates centered with the average of the random variables in question. Our proofs rely heavily on the multidimensional central limit theorem, the functional martingale central limit theorem, and Lemmas 3.21 and 4.5. We developed the latter two for this research. We also use that in case of zero mean Gaussian processes, to prove their convergence in distribution, the convergence of their covariance functions has to be shown. Furthermore, both Theorem 4.3 of Beran et al. [7], which is about convergence of partial sums of a Hermite function of a stationary sequence of standard normal random variables, and Theorem 7.1 of Resnick [46], which is about weak convergence of partial sum processes for a triangular array towards a Lévy process, are used once. In a few cases, the proofs reduce to showing that some random variables are in the domain of attraction of a stable or normal distribution.

Pilipauskaitė and Surgailis [38, Page 1014] formulated an open problem that concerns the possible existence and description of the limit distribution of the double sum (1.1) for general i.i.d. processes $(X_t^{(j)})_{t\in[0,\infty)}$, $j = 1, 2, \ldots$. We partially solve this open problem for some randomized INAR(1) processes. The list of iterated limit theorems is complete, however, we only have some of the possible simultaneous limit theorems. The rest of them remain for future work, where we plan to follow the technique of Pilipauskaitė et al. [37]. (For more details, see the introduction of Chapter 4.) Since INAR(1) processes are special branching processes with immigration, based on our results, later on, one may proceed with general branching processes with immigration.

Note that the proofs are always presented in a dedicated section at the end of the respective chapter. The notations used in this work are the standard ones applied in the literature. In what follows we collect the general ones, while the special ones, for example the symbols denoting certain processes, are compiled in Table B.1 at the end of the thesis. Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers, and complex numbers, respectively. For all $d \in \mathbb{N}$, the $d \times d$ identity matrix is denoted by $\boldsymbol{I}_d$. The standard basis in $\mathbb{R}^d$ is denoted by $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$. For $\boldsymbol{v} \in \mathbb{R}^d$, the Euclidean norm is denoted by $\|\boldsymbol{v}\|$, and for $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, the induced matrix norm is denoted by $\|\boldsymbol{A}\|$ as well (with a little abuse of notation). As a reminder, it is defined as $\|\boldsymbol{A}\| := \sup\left\{\|\boldsymbol{A}\boldsymbol{v}\| : \boldsymbol{v} \in \mathbb{R}^d \text{ with } \|\boldsymbol{v}\| = 1\right\}$ for any matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$. Let $\varrho(\boldsymbol{A})$ denote the spectral radius of $\boldsymbol{A}$, i.e., the maximum of the absolute values of the eigenvalues of $\boldsymbol{A}$ for any matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$. The notations $\xrightarrow{\text{a.s.}}$, $\xrightarrow{\mathbb{P}}$, and $\xrightarrow{\mathcal{D}}$ denote convergence in almost sure sense, in probability, and in distribution, respectively, while $\xrightarrow{\mathcal{D}_{\mathrm{f}}}$ and $\mathcal{D}_{\mathrm{f}}$-lim mean the convergence of finite dimensional distributions. Furthermore, the notation $\stackrel{\mathcal{D}}{=}$ denotes the equality in distribution of random variables or stochastic processes. All the random variables will be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any set $B \subseteq \Omega$, we define $\mathbf{1}_B : \Omega \to \{0, 1\}$, the indicator function of $B$ as

$$\mathbf{1}_B(x) := \begin{cases} 1, & x \in B, \\ 0, & x \notin B. \end{cases}$$

## 1.2 Presentation overview

This thesis is about the limit behavior of the temporal and contemporaneous aggregates of certain branching processes. All of the chapters of this current work deal with this question, either for different processes, different manners regarding the convergence (iterated, simultaneous), or different centralizations.

More precisely, this work consists of the following parts. The introduction (first chapter) contains our motivation, the historical background, the essence of our main results, and the presentation overview.

The second chapter deals with the aggregation of stationary multitype Galton–Watson branching processes with immigration. Many properties of these processes are discussed, and limit theorems are presented concerning their aggregates, and those of two special cases: GINAR($p$) processes and INAR(1) processes with Poisson immigration.

The third and fourth chapters are devoted to stationary randomized INAR(1) processes with Poisson innovations. The third chapter contains the definition and properties of these processes, along with the limit theorems concerning simple aggregates and the iterated limit theorems related to the temporal and contemporaneous aggregates. The fourth chapter solely consists of the simultaneous limit theorems related to these temporal and contemporaneous aggregates. At the end of the thesis, in Appendix A, there is an overview of the many lengthy theorems of these two chapters. We present the essence of these limit theorems together to show their resemblance and differences as well.

## 1.3 Credits

All the proofs of this dissertation are joint work with my supervisors Gyula Pap, and with the exception of a few, with Mátyás Barczy.

The proofs of Chapter 2 are based on the paper, [3],

M. Barczy, F. K. Nedényi, and G. Pap. On aggregation of multitype Galton–Watson branching processes with immigration. *Mod. Stoch. Theory Appl.* 5(1):53–79, 2018.

In Chapter 3 the proofs of Theorems 3.11 and 3.13 are based on the paper, [35],

F. Nedényi and G. Pap. Iterated scaling limits for aggregation of random coefficient AR(1) and INAR(1) processes. *Statist. Probab. Lett.* 118:16–23, 2016.

The rest of the proofs of Chapter 3 is based on the paper, [6],

M. Barczy, F. Nedényi, and G. Pap. Iterated limits for aggregation of randomized INAR(1) processes with Poisson innovations. *J. Math. Anal. Appl.* 451(1):524–543, 2017.

The proofs of Chapter 4 are based on the paper, submitted to a journal, [4],

M. Barczy, F. K. Nedényi, and G. Pap. On simultaneous limits for aggregation of stationary randomized INAR(1) processes with Poisson innovations. *ArXiv* 2001.07127, 2020+.

# Chapter 2

# Limit theorems for the aggregation of multitype Galton–Watson branching processes with immigration

This chapter deals with the aggregation of multitype Galton–Watson branching processes with immigration. The proofs of this chapter are based on the paper Barczy et al. [3].

In Section 2.1 many properties of these processes are discussed. In Lemma 2.5 we state that for a subcritical, positively regular multitype Galton–Watson branching process with nontrivial immigration, its unique stationary distribution admits finite $\gamma^{\mathrm{th}}$ moments provided that the branching and immigration distributions have finite $\gamma^{\mathrm{th}}$ moments, where $\gamma \in \{1, 2, 3\}$. In case of $\gamma \in \{1, 2\}$, Quine [44] contains this result, however, in case of $\gamma = 3$, we have not found any precise published proof in the literature for it. The result is something like a folklore. In the unpublished work by Szűcs [52], under ergodicity assumptions, the existence of general moments of the stationary distribution were obtained for multitype Galton–Watson processes with immigration. In a recent work by Kevei and Wiandt [27], that is yet to be published, they show that for a subcritical multitype Galton–Watson process with immigration, the stationary distribution admits finite $\gamma^{\mathrm{th}}$ moments provided that the branching and immigration distributions have finite $\max\{\gamma, 1\}^{\mathrm{th}}$ and $\gamma^{\mathrm{th}}$ moments, respectively, for any $\gamma > 0$. In Barczy et al. [3, Proof of Lemma 1] one can find the direct proof of our Lemma 2.5. As a by-product, that proof gives an explicit formula for the third moment in question, which is not available in the above mentioned papers by the other authors. We note that these moments have been studied before, e.g., in the paper of Quine [43], a recursion is given for the central moments which expresses the considered central moment of the random variable at time $n$ with that and the lower moments of the variable at time $n - 1$.

As the main results of Section 2.1, limit theorems are presented concerning the aggregates of the considered stationary multitype Galton–Watson branching processes with immigration. In two further sections of this chapter two special cases of these processes are presented: GINAR($p$) processes in Section 2.2 and INAR(1) processes with Poisson immigration in Section 2.4. Section 2.3 is a preparation for the INAR(1) processes with Poisson innovations, containing some rather technical results. Let us note here that in the paper Barczy et al. [5], which is the extended ArXiv version of the paper Barczy et al. [6], direct proofs were presented for the results of Section 2.4. However, that paper preceded Barczy et al. [3], the one about the aggregation of general multitype Galton–Watson processes with immigration. Since the results of Section 2.4 are special cases of those derived for the multitype Galton–Watson processes with immigration, we omit their original proofs. In Section 2.5 we give two interesting and relevant examples on how the aggregates of branching processes can model some current phenomena. All of the proofs of this chapter are in Section 2.6.

## 2.1 Aggregation of multitype Galton–Watson branching processes

In this section we define the multitype Galton–Watson branching processes with immigration and examine some of their general properties. We also present the limit theorems corresponding to the temporal and contemporaneous aggregates of these processes. In the following sections, we will investigate the same for the following special Galton–Watson branching processes with immigration: GINAR(p) models and INAR(1) processes with Poisson innovations.

Let $(\boldsymbol{Y}_k = [Y_{k,1}, \ldots, Y_{k,p}]^\top)_{k\in\mathbb{Z}_+}$ be a $p$-type Galton–Watson branching process with immigration, where $p \in \mathbb{N}$. For each $k, \ell \in \mathbb{Z}_+$ and $i, j \in \{1, \ldots, p\}$, $Y_{k,j}$ denotes the number of $j$-type individuals in the $k^{\mathrm{th}}$ generation, and $\xi_{k,\ell}^{(i,j)}$ denotes the number of $j$-type offsprings produced by the $\ell^{\mathrm{th}}$ individual belonging to type $i$ of the $(k-1)^{\mathrm{th}}$ generation, while $\varepsilon_k^{(i)}$ denotes the number of immigrants of type $i$ in the $k^{\mathrm{th}}$ generation. Then we have

$$\boldsymbol{Y}_k = \sum_{\ell=1}^{Y_{k-1,1}} \begin{bmatrix} \xi_{k,\ell}^{(1,1)} \\ \vdots \\ \xi_{k,\ell}^{(1,p)} \end{bmatrix} + \cdots + \sum_{\ell=1}^{Y_{k-1,p}} \begin{bmatrix} \xi_{k,\ell}^{(p,1)} \\ \vdots \\ \xi_{k,\ell}^{(p,p)} \end{bmatrix} + \begin{bmatrix} \varepsilon_k^{(1)} \\ \vdots \\ \varepsilon_k^{(p)} \end{bmatrix} =: \sum_{i=1}^{p} \sum_{\ell=1}^{Y_{k-1,i}} \boldsymbol{\xi}_{k,\ell}^{(i)} + \boldsymbol{\varepsilon}_k \tag{2.1}$$

for every $k \in \mathbb{N}$, where we define $\sum_{\ell=1}^{0} := \boldsymbol{0}$. Here $\big\{\boldsymbol{Y}_0, \boldsymbol{\xi}_{k,\ell}^{(i)}, \boldsymbol{\varepsilon}_k : k, \ell \in \mathbb{N}, i \in \{1, \ldots, p\}\big\}$ are supposed to be independent $\mathbb{Z}_+^p$-valued random vectors. Note that we do not assume independence among the components of these vectors. Moreover, for all $i \in \{1, \ldots, p\}$, $\{\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}_{k,\ell}^{(i)} : k, \ell \in \mathbb{N}\}$ and $\{\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_k : k \in \mathbb{N}\}$ are supposed to consist of identically distributed random vectors, respectively.

Let us introduce the notations $\boldsymbol{m}_{\boldsymbol{\varepsilon}} := \mathbb{E}(\boldsymbol{\varepsilon}) \in \mathbb{R}_+^p$, $\boldsymbol{M}_{\boldsymbol{\xi}} := \mathbb{E}\big(\big[\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(p)}\big]\big) \in \mathbb{R}_+^{p\times p}$ and

$$\boldsymbol{v}_{(i,j)} := \big[\mathrm{Cov}(\xi^{(1,i)}, \xi^{(1,j)}), \ldots, \mathrm{Cov}(\xi^{(p,i)}, \xi^{(p,j)}), \mathrm{Cov}(\varepsilon^{(i)}, \varepsilon^{(j)})\big]^\top \in \mathbb{R}^{(p+1)\times 1} \tag{2.2}$$

for $i, j \in \{1, \ldots, p\}$, provided that the expectations and covariances in question are finite. Recall that $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}})$ denotes the spectral radius of $\boldsymbol{M}_{\boldsymbol{\xi}}$, i.e., the maximum of the absolute values of the eigenvalues of $\boldsymbol{M}_{\boldsymbol{\xi}}$. The process $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$ is called subcritical, critical or supercritical if $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}})$ is smaller than $1$, equal to $1$, or larger than $1$, respectively. This classification can be motivated by calculating the expected values of the process. First we observe that for each $n \in \mathbb{N}$, using (2.1), we obtain

$$\begin{aligned}\mathbb{E}(\boldsymbol{Y}_n \mid \mathcal{F}_{n-1}^{\boldsymbol{Y}}) &= \sum_{i=1}^{p} \sum_{\ell=1}^{Y_{n-1,i}} \mathbb{E}(\boldsymbol{\xi}_{n,\ell}^{(i)} \mid \mathcal{F}_{n-1}^{\boldsymbol{Y}}) + \mathbb{E}(\boldsymbol{\varepsilon}_n \mid \mathcal{F}_{n-1}^{\boldsymbol{Y}}) = \sum_{i=1}^{p} Y_{n-1,i}\, \mathbb{E}(\boldsymbol{\xi}^{(i)}) + \mathbb{E}(\boldsymbol{\varepsilon}) \\ &= \sum_{i=1}^{p} \mathbb{E}(\boldsymbol{\xi}^{(i)}) \boldsymbol{e}_i^\top \boldsymbol{Y}_{n-1} + \boldsymbol{m}_{\boldsymbol{\varepsilon}} = \boldsymbol{M}_{\boldsymbol{\xi}} \boldsymbol{Y}_{n-1} + \boldsymbol{m}_{\boldsymbol{\varepsilon}},\end{aligned} \tag{2.3}$$

where $\mathcal{F}_{n-1}^{\boldsymbol{Y}} := \sigma(\boldsymbol{Y}_0, \dots, \boldsymbol{Y}_{n-1})$, $n \in \mathbb{N}$, and $Y_{n-1,i} := \boldsymbol{e}_i^\top \boldsymbol{Y}_{n-1}$, $i \in \{1, \dots, p\}$. Then, by taking the expectation, and further iterating this step, we get

$$\begin{aligned}\mathbb{E}(\boldsymbol{Y}_n) &= \boldsymbol{M}_{\boldsymbol{\xi}}\, \mathbb{E}(\boldsymbol{Y}_{n-1}) + \boldsymbol{m}_{\boldsymbol{\varepsilon}} = \dots \\ &= \boldsymbol{M}_{\boldsymbol{\xi}}^n\, \mathbb{E}(\boldsymbol{Y}_0) + (\boldsymbol{M}_{\boldsymbol{\xi}}^{n-1} + \boldsymbol{M}_{\boldsymbol{\xi}}^{n-2} + \dots + \boldsymbol{M}_{\boldsymbol{\xi}} + \boldsymbol{I}_p)\boldsymbol{m}_{\boldsymbol{\varepsilon}} \\ &= \boldsymbol{M}_{\boldsymbol{\xi}}^n\, \mathbb{E}(\boldsymbol{Y}_0) + (\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{I}_p)^{-1}(\boldsymbol{M}_{\boldsymbol{\xi}}^n - \boldsymbol{I}_p)\boldsymbol{m}_{\boldsymbol{\varepsilon}}.\end{aligned}$$

This form clearly shows that the limit behavior of the expectation as $n \to \infty$ depends on that of $\boldsymbol{M}_{\boldsymbol{\xi}}^n$, which originates in the magnitude of the spectral radius of the matrix $\boldsymbol{M}_{\boldsymbol{\xi}}$. This classification is even further detailed for GINAR(p) processes in Remark 2.11.

The matrix $\boldsymbol{M}_{\boldsymbol{\xi}}$ is called primitive if there is a positive integer $n \in \mathbb{N}$ such that all the entries of $\boldsymbol{M}_{\boldsymbol{\xi}}^n$ are positive. The process $(\boldsymbol{Y}_k)_{k \in \mathbb{Z}_+}$ is called positively regular if $\boldsymbol{M}_{\boldsymbol{\xi}}$ is primitive. In what follows, we suppose that

$$\begin{gathered}\mathbb{E}(\boldsymbol{\xi}^{(i)}) \in \mathbb{R}_+^p, \quad i \in \{1, \dots, p\}, \qquad \boldsymbol{m}_{\boldsymbol{\varepsilon}} \in \mathbb{R}_+^p \setminus \{\boldsymbol{0}\}, \\ \varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1, \qquad \boldsymbol{M}_{\boldsymbol{\xi}} \text{ is primitive.}\end{gathered} \tag{2.4}$$

For further application, we define the matrix

$$\boldsymbol{V} := (V_{i,j})_{i,j=1}^p := \left( \boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{m}_{\boldsymbol{\varepsilon}} \\ 1 \end{bmatrix} \right)_{i,j=1}^p \in \mathbb{R}^{p \times p}, \tag{2.5}$$

provided that the covariances in question are finite.

**Remark 2.1.** Note that the matrix $(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}$, which appears in (2.5) and throughout the chapter, exists. Indeed, $\lambda \in \mathbb{C}$ is an eigenvalue of $\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}$ if and only if $1 - \lambda$ is that of $\boldsymbol{M}_{\boldsymbol{\xi}}$. Therefore, since $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1$, all eigenvalues of $\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}$ are non-zero. This means that $\det(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}) \neq 0$, so $(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}$ does exist. One could also refer to Corollary 5.6.16 and Lemma 5.6.10 in Horn and Johnson [19]. □

Now we recall the definitions of the reducibility and irreducibility of a matrix. See, e.g., Horn and Johnson [19, Definitions 6.2.21 and 6.2.22].

**Definition 2.2.** *A matrix $\boldsymbol{A} \in \mathbb{R}^{p \times p}$ is reducible if there is a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{p \times p}$ such that*

$$\boldsymbol{P}^\top \boldsymbol{A} \boldsymbol{P} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0}_{p-q,q} & \boldsymbol{D} \end{bmatrix}, \qquad 1 \leqslant q \leqslant p-1,$$

*where $\boldsymbol{0}_{p-q,q}$ is a matrix of size $(p-q) \times q$ with all zero entries. The matrix $\boldsymbol{A} \in \mathbb{R}^{p \times p}$ is irreducible if it is not reducible.*

Next we recall the definition of aperiodicity, see, e.g., Danka and Pap [12, Introduction].

**Definition 2.3.** *The types* $\{1,\ldots,p\}$ *can be partitioned according to communication of types, namely, into* $r$ *nonempty mutually disjoint subsets* $D_1,\ldots,D_r$ *such that an individual of type* $j$ *may not have offspring of type* $i$ *unless there exists* $\ell \in \{1,\ldots,r\}$ *with* $i \in D_{\ell-1}$ *and* $j \in D_\ell$, *where subscripts are considered modulo* $r$. *This partitioning is unique up to cyclic permutation of the subsets. The number* $r$ *is called the index of cyclicity of the matrix* $\boldsymbol{M}_{\boldsymbol{\xi}}$. *The matrix* $\boldsymbol{M}_{\boldsymbol{\xi}}$ *is called aperiodic if its index of cyclicity is* 1.

Under (2.4), by the Theorem in Quine [44], there is a unique stationary distribution $\boldsymbol{\pi}$ for $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$. Indeed, under (2.4), $\boldsymbol{M}_{\boldsymbol{\xi}}$ is irreducible (see Definition 2.2) following from the primitivity of $\boldsymbol{M}_{\boldsymbol{\xi}}$, see Definition 8.5.0 and Theorem 8.5.2 in Horn and Johnson [19]. Further, $\boldsymbol{M}_{\boldsymbol{\xi}}$ is aperiodic (see Definition 2.3), since this is equivalent to the primitivity of $\boldsymbol{M}_{\boldsymbol{\xi}}$, see Kesten and Stigum [26, page 314] and Kesten and Stigum [25, Section 3]. Finally, since $\boldsymbol{m}_{\boldsymbol{\varepsilon}} \in \mathbb{R}_+^p \setminus \{\boldsymbol{0}\}$, the probability generator function of $\boldsymbol{\varepsilon}$ at $\boldsymbol{0}$ is less than 1, and

$$\mathbb{E}\left(\log\left(\sum_{i=1}^p \varepsilon^{(i)}\right)\mathbb{1}_{\{\boldsymbol{\varepsilon}\neq\boldsymbol{0}\}}\right) \leqslant \mathbb{E}\left(\sum_{i=1}^p \varepsilon^{(i)}\mathbb{1}_{\{\boldsymbol{\varepsilon}\neq\boldsymbol{0}\}}\right) \leqslant \mathbb{E}\left(\sum_{i=1}^p \varepsilon^{(i)}\right) = \sum_{i=1}^p \mathbb{E}(\varepsilon^{(i)}) < \infty,$$

so one can apply the Theorem in Quine [44].

**Remark 2.4.** Note that $\boldsymbol{V}$ is symmetric and positive semidefinite, since $\boldsymbol{v}_{(i,j)} = \boldsymbol{v}_{(j,i)}$, $i,j \in \{1,\ldots,p\}$, and for all $\boldsymbol{x} \in \mathbb{R}^p$,

$$\boldsymbol{x}^\top \boldsymbol{V}\boldsymbol{x} = \sum_{i=1}^p\sum_{j=1}^p V_{i,j}x_ix_j = \left(\sum_{i=1}^p\sum_{j=1}^p x_ix_j\boldsymbol{v}_{(i,j)}^\top\right)\begin{bmatrix}(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{m}_{\boldsymbol{\varepsilon}}\\ 1\end{bmatrix},$$

where

$$\sum_{i=1}^p\sum_{j=1}^p x_ix_j\boldsymbol{v}_{(i,j)}^\top = \big[\boldsymbol{x}^\top \operatorname{Cov}(\boldsymbol{\xi}^{(1)},\boldsymbol{\xi}^{(1)})\boldsymbol{x},\ldots,\boldsymbol{x}^\top \operatorname{Cov}(\boldsymbol{\xi}^{(p)},\boldsymbol{\xi}^{(p)})\boldsymbol{x},\boldsymbol{x}^\top \operatorname{Cov}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon})\boldsymbol{x}\big].$$

Here $\boldsymbol{x}^\top \operatorname{Cov}(\boldsymbol{\xi}^{(i)},\boldsymbol{\xi}^{(i)})\boldsymbol{x} \geqslant 0$, $i \in \{1,\ldots,p\}$, $\boldsymbol{x}^\top \operatorname{Cov}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon})\boldsymbol{x} \geqslant 0$, and $(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{m}_{\boldsymbol{\varepsilon}} \in \mathbb{R}_+^p$ since $(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{m}_{\boldsymbol{\varepsilon}}$ is nothing else but the expectation vector of the unique stationary distribution of $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$, see 2.28 or formula (26) of Quine [44] (as all of the conditions of that formula are satisfied). Since the matrix $\boldsymbol{V}$ is symmetric and positive semidefinite, $\boldsymbol{V}^{1/2}$ exists, which we will need in the upcoming statements. □

For the proofs of some of our main results (Propositions 2.6 and 2.7), we are going to need a connection between the finiteness of the moments of the offspring and immigration distributions, and that of the stationary distribution. The following lemma captures this connection. On the possible alternatives of this lemma see the the introduction of this chapter.

For each $\gamma \in \mathbb{N}$, we say that the $\gamma^{\text{th}}$ moment of a random vector is finite if all of its mixed moments of order $\gamma$ are finite.

**Lemma 2.5.** *Let us assume* (2.4). *For each* $\gamma \in \{1,2,3\}$, *the unique stationary distribution* $\boldsymbol{\pi}$ *has a finite* $\gamma^{\text{th}}$ *moment, provided that the* $\gamma^{\text{th}}$ *moments of* $\boldsymbol{\xi}^{(i)}$, $i \in \{1,\ldots,p\}$, *and* $\boldsymbol{\varepsilon}$ *are finite.*

For a more general version of this lemma, see the main theorem of Kevei and Wiandt [27]. We omit the lengthy, direct proof of this lemma in this thesis, however, it is presented in Barczy et al. [3, Proof of Lemma 1]. Note that the latter proof, as a by-product, gives also an explicit formula for the third moment in question.

In what follows, we suppose (2.4) and that the distribution of $\boldsymbol{Y}_0$ is the unique stationary distribution $\boldsymbol{\pi}$, hence the Markov chain $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$ is strictly stationary. Recall that, by (2.1) in Quine and Durham [45], for any measurable function $f:\mathbb{R}^p\to\mathbb{R}$ satisfying $\mathbb{E}(|f(\boldsymbol{Y}_0)|)<\infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n} f(\boldsymbol{Y}_k) \xrightarrow{\text{a.s.}} \mathbb{E}(f(\boldsymbol{Y}_0)) \qquad \text{as} \ \ n\to\infty. \tag{2.6}$$

First we consider a simple aggregation procedure. For each $N\in\mathbb{N}$, consider the stochastic process $\boldsymbol{S}^{(N)}=(\boldsymbol{S}^{(N)}_k)_{k\in\mathbb{Z}_+}$ given by

$$\boldsymbol{S}^{(N)}_k := \sum_{j=1}^{N}(\boldsymbol{Y}^{(j)}_k - \mathbb{E}(\boldsymbol{Y}^{(j)}_k)), \qquad k\in\mathbb{Z}_+, \tag{2.7}$$

where $\boldsymbol{Y}^{(j)}=(\boldsymbol{Y}^{(j)}_k)_{k\in\mathbb{Z}_+}$, $j\in\mathbb{N}$, is a sequence of independent copies of the strictly stationary $p$-type Galton–Watson process $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$ with immigration. Here we point out that we consider so-called idiosyncratic immigrations, i.e., the immigrations belonging to $\boldsymbol{Y}^{(j)}$, $j\in\mathbb{N}$, are independent. Note that throughout the thesis we will keep using the notation $\boldsymbol{S}^{(N)}$ for the same aggregate of the current processes considered in different parts of this work.

**Proposition 2.6.** *If all entries of the vectors* $\boldsymbol{\xi}^{(i)}$, $i\in\{1,\ldots,p\}$, *and* $\boldsymbol{\varepsilon}$ *have finite second moments, then*

$$N^{-\frac{1}{2}}\boldsymbol{S}^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \boldsymbol{\mathcal{Y}} \qquad \text{as} \ \ N\to\infty,$$

*where* $\boldsymbol{\mathcal{Y}}=(\boldsymbol{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ *is a stationary* $p$*-dimensional zero mean Gaussian process with covariances*

$$\mathbb{E}(\boldsymbol{\mathcal{Y}}_0\boldsymbol{\mathcal{Y}}_k^\top) = \operatorname{Cov}(\boldsymbol{Y}_0,\boldsymbol{Y}_k) = \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^k, \qquad k\in\mathbb{Z}_+, \tag{2.8}$$

*where*

$$\operatorname{Var}(\boldsymbol{Y}_0) = \sum_{k=0}^{\infty}\boldsymbol{M}_{\boldsymbol{\xi}}^k\boldsymbol{V}(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^k. \tag{2.9}$$

**Proposition 2.7.** *If all entries of the vectors* $\boldsymbol{\xi}^{(i)}$, $i\in\{1,\ldots,p\}$, *and* $\boldsymbol{\varepsilon}$ *have finite third moments, then*

$$\left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}\boldsymbol{S}^{(1)}_k\right)_{t\in\mathbb{R}_+} = \left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}(\boldsymbol{Y}^{(1)}_k - \mathbb{E}(\boldsymbol{Y}^{(1)}_k))\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{B}$$

*as* $n\to\infty$, *where* $\boldsymbol{B}=(\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ *is a* $p$*-dimensional standard Brownian motion, and the matrix* $\boldsymbol{V}$ *is defined in* (2.5). *(Note that, by Remark 2.4,* $\boldsymbol{V}^{1/2}$ *does exist.)*

Note that Propositions 2.6 and 2.7 are about the scalings of the space-aggregated process $\boldsymbol{S}^{(N)}$ and the time-aggregated process $\big(\sum_{k=1}^{\lfloor nt\rfloor}\boldsymbol{S}_k^{(1)}\big)_{t\in\mathbb{R}_+}$, respectively.

Now we turn to examine the space- and time-aggregated process. For each $N, n \in \mathbb{N}$, consider the stochastic process $\boldsymbol{S}^{(N,n)} = (\boldsymbol{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$\boldsymbol{S}_t^{(N,n)} := \sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})), \qquad t \in \mathbb{R}_+. \tag{2.10}$$

Note that throughout the thesis we will keep using the notation $\boldsymbol{S}^{(N,n)}$ for the same aggregate of the current processes considered in different parts of this work.

**Theorem 2.8.** *If all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1,\dots,p\}$, and $\boldsymbol{\varepsilon}$ have finite second moments, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\,\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\,(nN)^{-\frac{1}{2}}\boldsymbol{S}^{(N,n)} = (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{B}, \tag{2.11}$$

*where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a $p$-dimensional standard Brownian motion, and the matrix $\boldsymbol{V}$ is defined in* (2.5)*.*

*If all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1,\dots,p\}$, and $\boldsymbol{\varepsilon}$ have finite third moments, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\,\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\,(nN)^{-\frac{1}{2}}\boldsymbol{S}^{(N,n)} = (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{B}, \tag{2.12}$$

*where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a $p$-dimensional standard Brownian motion, and the matrix $\boldsymbol{V}$ is defined in* (2.5)*.*

**Theorem 2.9.** *If all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1,\dots,p\}$, and $\boldsymbol{\varepsilon}$ have finite third moments, then*

$$(nN)^{-\frac{1}{2}}\boldsymbol{S}^{(N,n)} \stackrel{\mathcal{D}}{\longrightarrow} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{B}, \tag{2.13}$$

*if both $n$ and $N$ converge to infinity (at any rate), where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a standard $p$-dimensional Brownian motion and the matrix $\boldsymbol{V}$ is defined in* (2.5)*.*

A key ingredient of the proofs is the fact that $(\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k))_{k\in\mathbb{Z}_+}$ can be rewritten as a subcritical first order vector autoregressive process with coefficient matrix $\boldsymbol{M}_{\boldsymbol{\xi}}$ and with heteroscedastic innovations, see (2.30).

In the following remark we investigate the meaning of taking iterated limits, and the notations $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}$ and $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}$, that appeared in Theorem 2.8, and which will be important in all of the iterated limit theorems of this thesis.

**Remark 2.10.** Let us begin with the understanding of (2.12). Taking the first limit ( $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}(nN)^{-\frac{1}{2}}\boldsymbol{S}^{(N,n)}$), by Proposition 2.7, means that for each $N \in \mathbb{N}$, every $m \in \mathbb{Z}_+$, and $0 = t_0 < t_1 < \cdots < t_m < \infty$, we have that

$$\left((nN)^{-\frac{1}{2}}\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt_0\rfloor}(\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})),\,\dots,\,(nN)^{-\frac{1}{2}}\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt_m\rfloor}(\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)}))\right)$$

converges in distribution as $n \to \infty$ to the following limit:

$$\left( N^{-\frac{1}{2}} \sum_{j=1}^{N} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{B}_{t_0}^{(j)}, \ldots, N^{-\frac{1}{2}} \sum_{j=1}^{N} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{B}_{t_m}^{(j)} \right).$$

The summations in the latter formula could be problematic since there is no guarantee that the terms of the sums are defined on the same probability space. However, since the processes $\boldsymbol{B}^{(j)}$, $j \in \mathbb{N}$, are independent of each other, they can be considered on a probability space that can be chosen as the infinite product probability space. Then we can investigate whether the latter $m+1$-dimensional vector converges in distribution as $N \to \infty$.

Now we turn to (2.11). Taking the first limit ($\mathcal{D}_\mathrm{f}$-$\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \boldsymbol{S}^{(N,n)}$), by Proposition 2.6, means that for each $n \in \mathbb{N}$, every $m \in \mathbb{Z}_+$, and $0 = t_0 < t_1 < \cdots < t_m < \infty$, we have that

$$\left( (nN)^{-\frac{1}{2}} \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt_0 \rfloor} (\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})), \ldots, (nN)^{-\frac{1}{2}} \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt_m \rfloor} (\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})) \right)$$

converges in distribution as $N \to \infty$ to the following limit:

$$\left( n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt_0 \rfloor} \boldsymbol{\mathcal{Y}}_k, \ldots, n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt_m \rfloor} \boldsymbol{\mathcal{Y}}_k \right).$$

Here the summations do not cause any issues as $\boldsymbol{\mathcal{Y}}_k$, $k \in \mathbb{N}$, is a stochastic process. Then it remains to show whether the latter $m+1$-dimensional vector converges in distribution as $n \to \infty$.

Note that similar arguments hold for the limit theorems presented in the forthcoming sections and chapters as well. □

## 2.2 A special case: aggregation of GINAR processes

We devote this section to the analysis of aggregation of generalized integer-valued autoregressive processes of order $p \in \mathbb{N}$ (GINAR($p$) processes), which are special cases of the $p$-type Galton–Watson branching processes with immigration introduced in (2.1). For historical fidelity, we note that it was Latour [29] who introduced GINAR($p$) processes as generalizations of INAR($p$) processes. This class of processes became popular in modeling integer-valued time series data such as the daily number of claims at an insurance company. In fact, a GINAR(1) process is a (general) single-type Galton–Watson branching process with immigration.

Let $(Z_k)_{k \geqslant -p+1}$ be a GINAR($p$) process. Namely, for each $k, \ell \in \mathbb{Z}_+$ and $i \in \{1, \ldots, p\}$, the number of individuals in the $k^{\mathrm{th}}$ generation will be denoted by $Z_k$, the number of offsprings produced by the $\ell^{\mathrm{th}}$ individual belonging to the $(k-i)^{\mathrm{th}}$ generation will be denoted by $\xi_{k,\ell}^{(i,1)}$, and the number of immigrants in the $k^{\mathrm{th}}$ generation will be denoted by $\varepsilon_k^{(1)}$. Here the 1-s in the superscripts of $\xi_{k,\ell}^{(i,1)}$ and $\varepsilon_k^{(1)}$ are displayed in order to have a better comparison with (2.1). Then we have

$$Z_k = \sum_{\ell=1}^{Z_{k-1}} \xi_{k,\ell}^{(1,1)} + \cdots + \sum_{\ell=1}^{Z_{k-p}} \xi_{k,\ell}^{(p,1)} + \varepsilon_k^{(1)}, \qquad k \in \mathbb{N}. \tag{2.14}$$

Here $\{Z_0, Z_{-1}, \ldots, Z_{-p+1}, \xi_{k,\ell}^{(i,1)}, \varepsilon_k^{(1)} : k, \ell \in \mathbb{N}, i \in \{1, \ldots, p\}\}$ are supposed to be independent nonnegative integer-valued random variables. Moreover, for all $i \in \{1, \ldots, p\}$, $\{\xi^{(i,1)}, \xi_{k,\ell}^{(i,1)} : k, \ell \in \mathbb{N}\}$ and $\{\varepsilon^{(1)}, \varepsilon_k^{(1)} : k \in \mathbb{N}\}$ are supposed to consist of identically distributed random variables, respectively.

A GINAR($p$) process can be embedded in a $p$-type Galton–Watson branching process with immigration $(\boldsymbol{Y}_k = [Z_k, \ldots, Z_{k-p+1}]^\top)_{k \in \mathbb{Z}_+}$ with the corresponding $p$-dimensional random vectors

$$\boldsymbol{\xi}_{k,\ell}^{(1)} = \begin{bmatrix} \xi_{k,\ell}^{(1,1)} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \boldsymbol{\xi}_{k,\ell}^{(p-1)} = \begin{bmatrix} \xi_{k,\ell}^{(p-1,1)} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{\xi}_{k,\ell}^{(p)} = \begin{bmatrix} \xi_{k,\ell}^{(p,1)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_k = \begin{bmatrix} \varepsilon_k^{(1)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for any $k, \ell \in \mathbb{N}$.

In what follows, we reformulate the classification that was introduced for multitype Galton–Watson processes for GINAR($p$) processes in terms of the expectations of the offspring distributions.

**Remark 2.11.** In case of a GINAR($p$) process, if $\mathbb{E}(\xi^{(p,1)}) > 0$, then by Proposition 2.2 in Barczy et al. [2],

$$\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) \begin{cases} < \\ = \\ > \end{cases} 1 \qquad \Longleftrightarrow \qquad \sum_{i=1}^{p} \mathbb{E}(\xi^{(i,1)}) \begin{cases} < \\ = \\ > \end{cases} 1.$$

□

Next we specialize the matrix $\boldsymbol{V}$, defined in (2.5), in case of a subcritical GINAR($p$) process.

**Remark 2.12.** In case of a GINAR($p$) process, the vectors

$$\boldsymbol{v}_{(i,j)} = \big[\mathrm{Cov}(\xi^{(1,i)}, \xi^{(1,j)}), \ldots, \mathrm{Cov}(\xi^{(p,i)}, \xi^{(p,j)}), \mathrm{Cov}(\varepsilon^{(i)}, \varepsilon^{(j)})\big]^\top \in \mathbb{R}^{(p+1)\times 1}$$

for $i, j \in \{1, \ldots, p\}$ are all zero vectors except for when $i = j = 1$. Therefore, in case of $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1$, the matrix $\boldsymbol{V}$, defined in (2.5), reduces to

$$\boldsymbol{V} = \boldsymbol{v}_{(1,1)}^\top \begin{bmatrix} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\, \mathbb{E}(\varepsilon^{(1)}) \boldsymbol{e}_1 \\ 1 \end{bmatrix} (\boldsymbol{e}_1 \boldsymbol{e}_1^\top). \tag{2.15}$$

□

Finally, we specialize the limit distribution in Theorems 2.8 and 2.9 in case of a subcritical GINAR(1) process.

**Remark 2.13.** Let us note that in case of $p = 1$ and $\mathbb{E}(\xi^{(1,1)}) < 1$ (yielding that the corresponding GINAR(1) process is subcritical), the limit process in Theorems 2.8 and 2.9 can be written as

$$\frac{1}{1 - \mathbb{E}(\xi^{(1,1)})} \sqrt{\frac{\mathbb{E}(\varepsilon^{(1)})\,\mathrm{Var}(\xi^{(1,1)}) + (1 - \mathbb{E}(\xi^{(1,1)}))\,\mathrm{Var}(\varepsilon^{(1)})}{1 - \mathbb{E}(\xi^{(1,1)})}}\, B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ is a standard one-dimensional Brownian motion. Indeed, this holds, since in this special case $\boldsymbol{M_\xi} = \mathbb{E}(\xi^{(1,1)})$, yielding that $(\boldsymbol{I}_p - \boldsymbol{M_\xi})^{-1} = (1 - \mathbb{E}(\xi^{(1,1)}))^{-1}$, and, by (2.15),

$$\boldsymbol{V} = \begin{bmatrix} \mathrm{Cov}(\xi^{(1,1)}, \xi^{(1,1)}) \\ \mathrm{Cov}(\varepsilon^{(1)}, \varepsilon^{(1)}) \end{bmatrix}^\top \begin{bmatrix} \frac{\mathbb{E}(\varepsilon^{(1)})}{1-\mathbb{E}(\xi^{(1,1)})} \\ 1 \end{bmatrix} = \frac{\mathrm{Var}(\xi^{(1,1)})\,\mathbb{E}(\varepsilon^{(1)})}{1 - \mathbb{E}(\xi^{(1,1)})} + \mathrm{Var}(\varepsilon^{(1)}). \qquad \square$$

## 2.3 Some properties of Galton–Watson and INAR(1) processes

This section is a technical one to support Section 2.4, which concerns the aggregation of INAR(1) processes with Poisson immigration.

Let us consider a single-type Galton–Watson process with immigration, $(Y_k)_{k\in\mathbb{Z}_+}$. For each $k, \ell \in \mathbb{Z}_+$, the number of individuals in the $k^{\mathrm{th}}$ generation will be denoted by $Y_k$, the number of offsprings produced by the $\ell^{\mathrm{th}}$ individual belonging to the $(k-1)^{\mathrm{th}}$ generation will be denoted by $\xi_{k,\ell}$, and the number of immigrants in the $k^{\mathrm{th}}$ generation will be denoted by $\varepsilon_k$. Then, as a special case of (2.1), we have

$$Y_k = \sum_{\ell=1}^{Y_{k-1}} \xi_{k,\ell} + \varepsilon_k, \qquad k \in \mathbb{N},$$

where we recall that $\sum_{\ell=1}^{0} = 0$. Here $\big\{Y_0,\, \xi_{k,\ell},\, \varepsilon_k : k, \ell \in \mathbb{N}\big\}$ are supposed to be independent nonnegative integer-valued random variables. Moreover, $\{\xi_{k,\ell} : k, \ell \in \mathbb{N}\}$ and $\{\varepsilon_k : k \in \mathbb{N}\}$ are supposed to consist of identically distributed random variables, respectively.

Let us introduce the generator functions

$$F_k(z) := \mathbb{E}(z^{Y_k}), \quad k \in \mathbb{Z}_+, \qquad G(z) := \mathbb{E}(z^{\xi_{1,1}}), \qquad H(z) := \mathbb{E}(z^{\varepsilon_1}) \tag{2.16}$$

for $z \in D := \{z \in \mathbb{C} : |z| \leqslant 1\}$. First we observe that for each $k \in \mathbb{N}$, the conditional generator function $\mathbb{E}(z_k^{Y_k} \,|\, Y_{k-1})$ of $Y_k$ given $Y_{k-1}$ takes the form

$$\mathbb{E}(z_k^{Y_k} \,|\, Y_{k-1}) = \mathbb{E}\Big(z_k^{\sum_{\ell=1}^{Y_{k-1}} \xi_{k,\ell} + \varepsilon_k} \,\Big|\, Y_{k-1}\Big) = \mathbb{E}(z_k^{\varepsilon_k}) \prod_{\ell=1}^{Y_{k-1}} \mathbb{E}(z_k^{\xi_{k,\ell}}) = H(z_k)\, G(z_k)^{Y_{k-1}} \tag{2.17}$$

for $z_k \in D$, where we define $\prod_{\ell=1}^{0} := 1$. The aim of the following discussion is to calculate the joint generator function of the finite dimensional distributions of $(Y_k)_{k\in\mathbb{Z}_+}$. Using (2.17), we also have the recursion

$$F_k(z) = \mathbb{E}(\mathbb{E}(z^{Y_k} \,|\, Y_{k-1})) = \mathbb{E}(H(z)\, G(z)^{Y_{k-1}}) = H(z)\,\mathbb{E}\big(G(z)^{Y_{k-1}}\big) = H(z)\, F_{k-1}(G(z))$$

for $z \in D$ and $k \in \mathbb{N}$. Put $G_{(0)}(z) := z$ and $G_{(1)}(z) := G(z)$ for $z \in D$, and introduce the iterates $G_{(k+1)}(z) := G_{(k)}(G(z))$, $z \in D$, $k \in \mathbb{N}$. The above recursion yields

$$F_k(z) = H(z)\, H(G(z)) \cdots H(G_{(k-1)}(z))\, F_0(G_{(k)}(z)) = F_0(G_{(k)}(z)) \prod_{j=0}^{k-1} H(G_{(j)}(z))$$

for $z \in D$ and $k \in \mathbb{N}$. Supposing that $\mathbb{E}(\xi_{1,1}) = G'(1-) < 1$, $0 < \mathbb{P}(\xi_{1,1} = 0) < 1$, $0 < \mathbb{P}(\xi_{1,1} = 1)$ and $0 < \mathbb{P}(\varepsilon_1 = 0) < 1$, the Markov chain $(Y_k)_{k\in\mathbb{Z}_+}$ is irreducible and aperiodic. Further, it is ergodic (positive recurrent) if and only if $\sum_{\ell=1}^{\infty} \log(\ell)\, \mathbb{P}(\varepsilon_1 = \ell) < \infty$, and in this case the unique stationary distribution has the generator function

$$\widetilde{F}(z) := \prod_{j=0}^{\infty} H(G_{(j)}(z)), \qquad z \in D, \tag{2.18}$$

see, e.g., Seneta [48, Chapter 5] and Foster and Williamson [15, Theorem, part (iii)].

Now we turn to the case of INAR(1) processes with Poisson immigration. From here on, we are going to denote the process in question with $X$ instead of the previous $Y$ to be in accordance with Chapter 3, where the randomized INAR(1) process is also denoted by $X$. We consider this special case with Bernoulli offspring and Poisson immigration distributions, namely,

$$\begin{aligned} &\mathbb{P}(\xi_{1,1} = 1) = \alpha = 1 - \mathbb{P}(\xi_{1,1} = 0), \\ &\mathbb{P}(\varepsilon_1 = \ell) = \frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda}, \qquad \ell \in \mathbb{Z}_+, \end{aligned} \tag{2.19}$$

with $\alpha \in (0, 1)$ and $\lambda \in (0, \infty)$. With the special choices (2.19), the single-type Galton–Watson process with immigration $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with Poisson innovation. Then

$$G(z) = 1 - \alpha + \alpha z, \qquad H(z) = \sum_{\ell=0}^{\infty} \frac{z^\ell \lambda^\ell}{\ell!}\mathrm{e}^{-\lambda} = \mathrm{e}^{\lambda(z-1)}, \qquad z \in \mathbb{C},$$

hence

$$G_{(j)}(z) = 1 - \alpha^j + \alpha^j z, \qquad z \in \mathbb{C}, \quad j \in \mathbb{N}.$$

Indeed, by induction, for all $j \in \mathbb{Z}_+$,

$$G_{(j+1)}(z) = G(G_{(j)}(z)) = \alpha G_{(j)}(z) + 1 - \alpha = \alpha(1 - \alpha^j + \alpha^j z) + 1 - \alpha = 1 - \alpha^{j+1} + \alpha^{j+1} z.$$

Since $\mathbb{E}(\xi_{1,1}) = G'(1-) = \alpha \in (0, 1)$, $\mathbb{P}(\xi_{1,1} = 0) = 1 - \alpha \in (0, 1)$, $\mathbb{P}(\xi_{1,1} = 1) = \alpha > 0$, $\mathbb{P}(\varepsilon_1 = 0) = \mathrm{e}^{-\lambda} \in (0, 1)$, and

$$\sum_{\ell=1}^{\infty} \log(\ell)\frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda} \leqslant \sum_{\ell=1}^{\infty} \ell\frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda} = \mathbb{E}(\varepsilon_1) = \lambda < \infty,$$

the Markov chain $(X_k)_{k\in\mathbb{Z}_+}$ has a unique stationary distribution admitting a generator function of the form

$$\widetilde{F}(z) = \prod_{j=0}^{\infty} \mathrm{e}^{\lambda(G_{(j)}(z)-1)} = \prod_{j=0}^{\infty} \mathrm{e}^{\alpha^j\lambda(z-1)} = \mathrm{e}^{(1-\alpha)^{-1}\lambda(z-1)}, \qquad z \in \mathbb{C},$$

thus it is a Poisson distribution with expectation $(1 - \alpha)^{-1}\lambda$.

Suppose now that the initial distribution is a Poisson distribution with expectation $(1 - \alpha)^{-1}\lambda$, hence the Markov chain $(X_k)_{k\in\mathbb{Z}_+}$ is strictly stationary and

$$F_0(z_0) = \mathbb{E}(z_0^{X_0}) = \mathrm{e}^{(1-\alpha)^{-1}\lambda(z_0-1)}, \qquad z_0 \in \mathbb{C}. \tag{2.20}$$

**Proposition 2.14.** *Under* (2.19) *and supposing that the distribution of* $X_0$ *is Poisson distribution with expectation* $(1-\alpha)^{-1}\lambda$, *the joint generator function of* $(X_0, X_1, \ldots, X_k)$, $k \in \mathbb{Z}_+$, *takes the form*

$$\begin{aligned} F_{0,\ldots,k}(z_0, \ldots, z_k) &:= \mathbb{E}(z_0^{X_0} z_1^{X_1} \cdots z_k^{X_k}) \\ &= \exp\left\{ \frac{\lambda}{1-\alpha} \sum_{0 \leqslant i \leqslant j \leqslant k} \alpha^{j-i}(z_i - 1) z_{i+1} \cdots z_{j-1}(z_j - 1) \right\} \end{aligned} \tag{2.21}$$

*for all* $k \in \mathbb{N}$ *and* $z_0, \ldots, z_k \in \mathbb{C}$, *where, for* $i = j$, *the term in the sum above is* $z_i - 1$. *Alternatively, one can write up the joint generator function as*

$$F_{0,\ldots,k}(z_0, \ldots, z_k) = \exp\left\{ \lambda \sum_{0 \leqslant i \leqslant j \leqslant k} (1-\alpha)^{K_{i,j,k}} \alpha^{j-i} (z_i z_{i+1} \cdots z_j - 1) \right\}, \tag{2.22}$$

*where*

$$K_{i,j,k} := \begin{cases} -1 & \text{if } i = 0 \text{ and } j = k, \\ 0 & \text{if } i = 0 \text{ and } 0 \leqslant j \leqslant k-1, \\ 0 & \text{if } 1 \leqslant i \leqslant k \text{ and } j = k, \\ 1 & \text{if } 1 \leqslant i \leqslant j \leqslant k-1. \end{cases}$$

**Remark 2.15.** Under the conditions of Proposition 2.14, the distribution of $(X_0, X_1)$ can be represented using independent Poisson distributed random variables. Namely, if $U$, $V$ and $W$ are independent Poisson distributed random variables with parameters $\lambda(1-\alpha)^{-1}\alpha$, $\lambda$ and $\lambda$, respectively, then $(X_0, X_1) \stackrel{\mathcal{D}}{=} (U+V, U+W)$. Indeed, for all $z_0, z_1 \in \mathbb{C}$,

$$\begin{aligned} \mathbb{E}(z_0^{U+V} z_1^{U+W}) &= \mathbb{E}((z_0 z_1)^U z_0^V z_1^W) = \mathbb{E}((z_0 z_1)^U)\, \mathbb{E}(z_0^V)\, \mathbb{E}(z_1^W) \\ &= \mathrm{e}^{\lambda(1-\alpha)^{-1}\alpha(z_0 z_1 - 1)} \mathrm{e}^{\lambda(z_0-1)} \mathrm{e}^{\lambda(z_1-1)}, \end{aligned}$$

as desired based on (2.22). Further, note that formula (2.22) shows that $(X_0, \ldots, X_k)$ has a $(k+1)$-variate Poisson distribution, see, e.g., Johnson et al. [24, (37.85)]. □

## 2.4 A special case: aggregation of INAR(1) processes

In this section we investigate the aggregation of stationary INAR(1) processes with Poisson innovation, which are special cases of the $p$-type Galton–Watson branching processes with immigration introduced in (2.1). This section also serves a basis for Chapter 3, which concerns the aggregation of certain randomized INAR(1) processes.

Let $(X_k)_{k\in\mathbb{Z}_+}$ be an INAR(1) process with offspring and immigration distributions given in (2.19) and with initial distribution given in (2.20), hence the process is strictly stationary. Let $X^{(j)} = (X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, be a sequence of independent copies of the stationary INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$.

First we consider a simple aggregation procedure. For each $N \in \mathbb{N}$, consider the stochastic process $S^{(N)} = (S_k^{(N)})_{k\in\mathbb{Z}_+}$ given by

$$S_k^{(N)} := \sum_{j=1}^{N} (X_k^{(j)} - \mathbb{E}(X_k^{(j)})), \qquad k \in \mathbb{Z}_+, \tag{2.23}$$

where $\mathbb{E}(X_k^{(j)}) = (1-\alpha)^{-1}\lambda$, $k \in \mathbb{Z}_+$, $j \in \mathbb{N}$, since the stationary distribution is Poisson with expectation $(1-\alpha)^{-1}\lambda$. The following propositions and theorems are the special cases of Propositions 2.6 and 2.7, and Theorems 2.8 and 2.9, since the offspring and innovation variables all have finite moments. In our special case, one can easily verify the limit process of Proposition 2.7 and Theorems 2.8 and 2.9 by Remark 2.13. Indeed, all we have to do is substitute $\mathbb{E}(\xi^{(1,1)}) = \alpha$, $\operatorname{Var}(\xi^{(1,1)}) = \alpha(1-\alpha)$ and $\mathbb{E}(\varepsilon^{(1)}) = \operatorname{Var}(\varepsilon^{(1)}) = \lambda$ into the formula of Remark 2.13.

**Proposition 2.16.** *We have*

$$N^{-\frac{1}{2}} S^{(N)} \xrightarrow{\mathcal{D}_f} \mathcal{X} \qquad \text{as } N \to \infty,$$

*where* $\mathcal{X} = (\mathcal{X}_k)_{k\in\mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\mathcal{X}_0\mathcal{X}_k) = \operatorname{Cov}(X_0, X_k) = \frac{\lambda\alpha^k}{1-\alpha}, \qquad k \in \mathbb{Z}_+. \tag{2.24}$$

The latter covariance is the special case of (2.8) since the variance of the stationary distribution (which is Poisson distribution with parameter $(1-\alpha)^{-1}\lambda$) is $\operatorname{Var}(X_0) = (1-\alpha)^{-1}\lambda$ and the matrix $\boldsymbol{M_\xi}$ reduces to $\alpha$. Note that formula (2.9) also results $\operatorname{Var}(X_0) = (1-\alpha)^{-1}\lambda$.

**Proposition 2.17.** *We have*

$$\left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt\rfloor} S_k^{(1)}\right)_{t\in\mathbb{R}_+} = \left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt\rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)}))\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B$$

*as* $n \to \infty$, *where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

Note that Propositions 2.16 and 2.17 are about the scaling of the space-aggregated process $S^{(N)}$ and the time-aggregated process $\big(\sum_{k=1}^{\lfloor nt\rfloor} S_k^{(1)}\big)_{t\in\mathbb{R}_+}$, respectively.

For each $N, n \in \mathbb{N}$, consider the stochastic process $S^{(N,n)} = (S_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$S_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt\rfloor} (X_k^{(j)} - \mathbb{E}(X_k^{(j)})), \qquad t \in \mathbb{R}_+. \tag{2.25}$$

**Theorem 2.18.** *We have*

$$\mathcal{D}_f\text{-}\lim_{n\to\infty} \mathcal{D}_f\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} S^{(N,n)} = \mathcal{D}_f\text{-}\lim_{N\to\infty} \mathcal{D}_f\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} S^{(N,n)} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B,$$

*where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

**Theorem 2.19.** *We have*

$$(nN)^{-\frac{1}{2}} S^{(N,n)} \xrightarrow{\mathcal{D}} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B,$$

*if both* $n$ *and* $N$ *converge to infinity (at any rate), where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

We point out that the process $S^{(N,n)}$ contains the expected values of the processes $X^{(j)}$, $j \in \mathbb{N}$. Therefore, in a statistical testing, it could not be used directly. So we consider a similar process

$$\widehat{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[ X_k^{(j)} - \frac{\sum_{\ell=1}^{n} X_\ell^{(j)}}{n} \right], \qquad t \in \mathbb{R}_+, \tag{2.26}$$

which does not require the knowledge of the expectation of the processes $X^{(j)}$, $j \in \mathbb{N}$. Note that the summands in $\widehat{S}_t^{(N,n)}$ have $0$ means, so we do not need any additional centering. Moreover, $\widehat{S}^{(N,n)}$ is related to the previously examined process in the following way:

$$\widehat{S}_t^{(N,n)} = \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[ X_k^{(j)} - \mathbb{E}(X_k^{(j)}) - \frac{\sum_{\ell=1}^{n} (X_\ell^{(j)} - \mathbb{E}(X_\ell^{(j)}))}{n} \right] = S_t^{(N,n)} - \frac{\lfloor nt \rfloor}{n} S_1^{(N,n)}.$$

Therefore, by Theorems 2.18 and 2.19, using Slutsky's lemma, the following limit theorems hold.

**Corollary 2.20.** *We have*

$$\begin{aligned} &\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} \\ &\quad = \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} (B_t - tB_1)_{t\in\mathbb{R}_+}, \end{aligned}$$

*where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

**Corollary 2.21.** *We have*

$$(nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} \xrightarrow{\mathcal{D}} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} (B_t - tB_1)_{t\in\mathbb{R}_+},$$

*if both* $n$ *and* $N$ *converge to infinity (at any rate), where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

## 2.5 Applications

In this section, to illustrate the fact that the aggregation of branching processes is an important topic from the point of view of applications as well, now we present two interesting and relevant examples, where the phenomena of aggregation of this kind of processes may come into play.

**Example 2.22.** A usual INAR(1) process with immigration, $(X_k)_{k\in\mathbb{Z}_+}$, can be used to model migration, which is an important task nowadays all over the world. More precisely, given a camp, for all $k \in \mathbb{Z}_+$, the random variable $X_k$ can be interpreted as the number of migrants present in the camp at time $k$, and every migrant will stay in the camp with probability $\alpha \in (0,1)$ independently of each other (i.e., with probability $1-\alpha$ each migrant leaves the camp) and, at any time $k \in \mathbb{N}$, new migrants may come to the camp. Given several camps in a country, we may suppose that the corresponding INAR(1) processes share the same parameter $\alpha$ and they are independent. So, the temporal and contemporaneous aggregates of these INAR(1) processes is the total usage of the camps in terms of the number of migrants in the given country in a given time period, and this quantity may be worth studying. $\Box$

**Example 2.23.** As of 2020, modeling the COVID-19 contamination of the population of a certain region or country is of great importance. Multitype Galton–Watson processes with immigration have been frequently used to model the spreading of a number of diseases, and they can be applied for this new disease as well. For example, Yanev et al. [57] applied a two-type Galton–Watson process with immigration to model the number of undetected, COVID-19-infected, and detected, COVID-19-infected people in a population. The temporal and contemporaneous aggregation of the first coordinate process of the two-type branching process in question would mean the total number of undetected, infected people up to some given time point across several regions, which is of great importance. The aim of their paper is to use the daily statistics (which is the number of detected infected people) to estimate the expected value of the non-observed number of undetected contaminated individuals. $\Box$

## 2.6 Proofs

**Proof of Proposition 2.6.** Similarly as (2.3), we have

$$\mathbb{E}(\boldsymbol{Y}_k \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) = \boldsymbol{M}_{\boldsymbol{\xi}} \boldsymbol{Y}_{k-1} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k \in \mathbb{N},$$

where $\mathcal{F}_k^{\boldsymbol{Y}} = \sigma(\boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_k)$, $k \in \mathbb{Z}_+$. Consequently,

$$\mathbb{E}(\boldsymbol{Y}_k) = \boldsymbol{M}_{\boldsymbol{\xi}} \, \mathbb{E}(\boldsymbol{Y}_{k-1}) + \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k \in \mathbb{N}, \tag{2.27}$$

and, taking into account the fact that $\mathbb{E}(\boldsymbol{Y}_k) = \mathbb{E}(\boldsymbol{Y}_0)$, $k \in \mathbb{Z}_+$ (following from the strict stationarity of $(\boldsymbol{Y}_k)_{k \in \mathbb{Z}_+}$), we have

$$\mathbb{E}(\boldsymbol{Y}_0) = (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{m}_{\boldsymbol{\varepsilon}}. \tag{2.28}$$

Note that this also follows by formula (26) of Quine [44] (as all of the conditions of that formula are satisfied). Put

$$\begin{aligned} \boldsymbol{U}_k :&= \boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) = \boldsymbol{Y}_k - (\boldsymbol{M}_{\boldsymbol{\xi}} \boldsymbol{Y}_{k-1} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}) \\ &= \sum_{i=1}^{p} \sum_{\ell=1}^{Y_{k-1,i}} (\boldsymbol{\xi}_{k,\ell}^{(i)} - \mathbb{E}(\boldsymbol{\xi}_{k,\ell}^{(i)})) + (\boldsymbol{\varepsilon}_k - \mathbb{E}(\boldsymbol{\varepsilon}_k)), \qquad k \in \mathbb{N}. \end{aligned}$$

Then $\mathbb{E}(\boldsymbol{U}_k \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) = \boldsymbol{0}$, $k \in \mathbb{N}$, and using the independence of $\big\{\boldsymbol{\xi}_{k,\ell}^{(i)}, \boldsymbol{\varepsilon}_k : k, \ell \in \mathbb{N}, i \in \{1, \ldots, p\}\big\}$, we have

$$\mathbb{E}(U_{k,i} U_{k,j} \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) = \sum_{q=1}^{p} Y_{k-1,q} \operatorname{Cov}(\xi_{k,1}^{(q,i)}, \xi_{k,1}^{(q,j)}) + \operatorname{Cov}(\varepsilon_k^{(i)}, \varepsilon_k^{(j)}) = \boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} \boldsymbol{Y}_{k-1} \\ 1 \end{bmatrix} \tag{2.29}$$

for $i, j \in \{1, \ldots, p\}$ and $k \in \mathbb{N}$, where $[U_{k,1}, \ldots, U_{k,p}]^\top := \boldsymbol{U}_k$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, using $\boldsymbol{Y}_k = \boldsymbol{M}_{\boldsymbol{\xi}} \boldsymbol{Y}_{k-1} + \boldsymbol{m}_\varepsilon + \boldsymbol{U}_k$ and (2.27), we obtain

$$\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k) = \boldsymbol{M}_{\boldsymbol{\xi}} (\boldsymbol{Y}_{k-1} - \mathbb{E}(\boldsymbol{Y}_{k-1})) + \boldsymbol{U}_k, \qquad k \in \mathbb{N}. \tag{2.30}$$

Consequently,

$$
\begin{aligned}
&\mathbb{E}((\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k))(\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k))^\top \mid \mathcal{F}_{k-1}^{\boldsymbol{Y}}) \\
&= \mathbb{E}((\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{Y}_{k-1} - \mathbb{E}(\boldsymbol{Y}_{k-1})) + \boldsymbol{U}_k)(\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{Y}_{k-1} - \mathbb{E}(\boldsymbol{Y}_{k-1})) + \boldsymbol{U}_k)^\top \mid \mathcal{F}_{k-1}^{\boldsymbol{Y}}) \\
&= \mathbb{E}(\boldsymbol{U}_k \boldsymbol{U}_k^\top \mid \mathcal{F}_{k-1}^{\boldsymbol{Y}}) + \boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{Y}_{k-1} - \mathbb{E}(\boldsymbol{Y}_{k-1}))(\boldsymbol{Y}_{k-1} - \mathbb{E}(\boldsymbol{Y}_{k-1}))^\top \boldsymbol{M}_{\boldsymbol{\xi}}^\top
\end{aligned}
$$

for all $k \in \mathbb{N}$. Taking the expectation, by (2.28) and (2.29), we conclude

$$
\operatorname{Var}(\boldsymbol{Y}_k) = \mathbb{E}(\boldsymbol{U}_k \boldsymbol{U}_k^\top) + \boldsymbol{M}_{\boldsymbol{\xi}} \operatorname{Var}(\boldsymbol{Y}_{k-1}) \boldsymbol{M}_{\boldsymbol{\xi}}^\top = \boldsymbol{V} + \boldsymbol{M}_{\boldsymbol{\xi}} \operatorname{Var}(\boldsymbol{Y}_{k-1}) \boldsymbol{M}_{\boldsymbol{\xi}}^\top, \qquad k \in \mathbb{N}.
$$

Under the conditions of the proposition, by Lemma 2.5, the unique stationary distribution $\boldsymbol{\pi}$ has a finite second moment, hence, using again the stationarity of $(\boldsymbol{Y}_k)_{k \in \mathbb{Z}_+}$, for each $N \in \mathbb{N}$, we get

$$
\operatorname{Var}(\boldsymbol{Y}_0) = \boldsymbol{V} + \boldsymbol{M}_{\boldsymbol{\xi}} \operatorname{Var}(\boldsymbol{Y}_0) \boldsymbol{M}_{\boldsymbol{\xi}}^\top = \sum_{k=0}^{N-1} \boldsymbol{M}_{\boldsymbol{\xi}}^k \boldsymbol{V} (\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^k + \boldsymbol{M}_{\boldsymbol{\xi}}^N \operatorname{Var}(\boldsymbol{Y}_0) (\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^N. \tag{2.31}
$$

Here $\lim_{N\to\infty} \boldsymbol{M}_{\boldsymbol{\xi}}^N \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^N = \boldsymbol{0} \in \mathbb{R}^{p\times p}$. Indeed, $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) = \lim_{k\to\infty} \|\boldsymbol{M}_{\boldsymbol{\xi}}^k\|^{1/k}$ by the Gelfand formula, see, e.g., Horn and Johnson [19, Corollary 5.6.14]. Hence there exists $k_0 \in \mathbb{N}$ such that

$$
\|\boldsymbol{M}_{\boldsymbol{\xi}}^k\|^{1/k} \leqslant \varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) + \frac{1 - \varrho(\boldsymbol{M}_{\boldsymbol{\xi}})}{2} = \frac{1 + \varrho(\boldsymbol{M}_{\boldsymbol{\xi}})}{2} < 1 \qquad \text{for all } k \geqslant k_0, \tag{2.32}
$$

since $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1$. Thus, for all $N \geqslant k_0$,

$$
\begin{aligned}
\|\boldsymbol{M}_{\boldsymbol{\xi}}^N \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^N\| &\leqslant \|\boldsymbol{M}_{\boldsymbol{\xi}}^N\| \|\operatorname{Var}(\boldsymbol{Y}_0)\| \|(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^N\| = \|\boldsymbol{M}_{\boldsymbol{\xi}}^N\| \|\operatorname{Var}(\boldsymbol{Y}_0)\| \|\boldsymbol{M}_{\boldsymbol{\xi}}^N\| \\
&\leqslant \left(\frac{1 + \varrho(\boldsymbol{M}_{\boldsymbol{\xi}})}{2}\right)^{2N} \|\operatorname{Var}(\boldsymbol{Y}_0)\|,
\end{aligned}
$$

hence $\|\boldsymbol{M}_{\boldsymbol{\xi}}^N \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^N\| \to 0$ as $N \to \infty$. Consequently,

$$
\operatorname{Var}(\boldsymbol{Y}_0) = \sum_{k=0}^{\infty} \boldsymbol{M}_{\boldsymbol{\xi}}^k \boldsymbol{V} (\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^k,
$$

yielding (2.9). Moreover, by (2.30),

$$
\begin{aligned}
&\mathbb{E}((\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0))(\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k))^\top \mid \mathcal{F}_{k-1}^{\boldsymbol{Y}}) = (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0)) \mathbb{E}((\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k))^\top \mid \mathcal{F}_{k-1}^{\boldsymbol{Y}}) \\
&= (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0))(\boldsymbol{Y}_{k-1} - \mathbb{E}(\boldsymbol{Y}_{k-1}))^\top \boldsymbol{M}_{\boldsymbol{\xi}}^\top, \qquad k \in \mathbb{N}.
\end{aligned}
$$

Taking the expectation, we conclude

$$
\operatorname{Cov}(\boldsymbol{Y}_0, \boldsymbol{Y}_k) = \operatorname{Cov}(\boldsymbol{Y}_0, \boldsymbol{Y}_{k-1}) \boldsymbol{M}_{\boldsymbol{\xi}}^\top, \qquad k \in \mathbb{N}.
$$

Hence, by induction, we obtain the formula for $\operatorname{Cov}(\boldsymbol{Y}_0, \boldsymbol{Y}_k)$. The statement will follow from the multidimensional central limit theorem. Due to the continuous mapping theorem, it is sufficient to show the convergence $N^{-1/2}(\boldsymbol{S}_0^{(N)}, \boldsymbol{S}_1^{(N)}, \ldots, \boldsymbol{S}_k^{(N)}) \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{Y}}_0, \boldsymbol{\mathcal{Y}}_1, \ldots, \boldsymbol{\mathcal{Y}}_k)$ as $N \to \infty$ for all $k \in \mathbb{Z}_+$. For all $k \in \mathbb{Z}_+$, the random vectors $\big((\boldsymbol{Y}_0^{(j)} -$

$\mathbb{E}(\boldsymbol{Y}_0^{(j)}))^\top, (\boldsymbol{Y}_1^{(j)} - \mathbb{E}(\boldsymbol{Y}_1^{(j)}))^\top, \ldots, (\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)}))^\top\big)^\top$, $j \in \mathbb{N}$, are independent, identically distributed having zero mean vector and covariances

$$\operatorname{Cov}(\boldsymbol{Y}_{\ell_1}^{(j)}, \boldsymbol{Y}_{\ell_2}^{(j)}) = \operatorname{Cov}(\boldsymbol{Y}_0^{(j)}, \boldsymbol{Y}_{\ell_2-\ell_1}^{(j)}) = \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\ell_2-\ell_1}$$

for $j \in \mathbb{N}$, $\ell_1, \ell_2 \in \{0, 1, \ldots, k\}$, $\ell_1 \leqslant \ell_2$, following from the strict stationarity of $\boldsymbol{Y}^{(j)}$ and from (2.8). □

**Proof of Proposition 2.7.** It is known that

$$\boldsymbol{U}_k = \boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) = \boldsymbol{Y}_k - \boldsymbol{M}_{\boldsymbol{\xi}} \boldsymbol{Y}_{k-1} - \boldsymbol{m}_\varepsilon, \qquad k \in \mathbb{N},$$

are martingale differences with respect to the filtration $(\mathcal{F}_k^{\boldsymbol{Y}})_{k\in\mathbb{Z}_+}$. The functional martingale central limit theorem can be applied, see, e.g., Jacod and Shiryaev [21, Theorem VIII.3.33]. Indeed, using (2.29) and the fact that the first moment of $\boldsymbol{Y}_0$ exists and is finite, by (2.6), for each $t \in \mathbb{R}_+$, and $i, j \in \{1, \ldots, p\}$, we have

$$\frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(U_{k,i} U_{k,j} \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) \xrightarrow{\text{a.s.}} \boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} \mathbb{E}(\boldsymbol{Y}_0) \\ 1 \end{bmatrix} t = V_{i,j} t \qquad \text{as } n \to \infty,$$

and hence the convergence holds in probability as well. Moreover, the conditional Lindeberg condition holds, namely, for all $\delta > 0$,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big(\|\boldsymbol{U}_k\|^2 \mathbf{1}_{\{\|\boldsymbol{U}_k\| > \delta\sqrt{n}\}} \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}\big) &\leqslant \frac{1}{\delta n^{3/2}} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(\|\boldsymbol{U}_k\|^3 \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}}) \\ &\leqslant \frac{C_3 (p+1)^3}{\delta n^{3/2}} \sum_{k=1}^{\lfloor nt \rfloor} \left\| \begin{bmatrix} \boldsymbol{Y}_{k-1} \\ 1 \end{bmatrix} \right\|^3 \xrightarrow{\text{a.s.}} 0 \end{aligned} \tag{2.33}$$

with $C_3 := \max\{\mathbb{E}(\|\boldsymbol{\xi}^{(i)} - \mathbb{E}(\boldsymbol{\xi}^{(i)})\|^3),\ i \in \{1, \ldots, p\},\ \mathbb{E}(\|\boldsymbol{\varepsilon} - \mathbb{E}(\boldsymbol{\varepsilon})\|^3)\}$, where the last inequality follows by Proposition 3.3 of Nedényi [34], and the almost sure convergence is a consequence of (2.6), since, under the third order moment assumptions in Proposition 2.7, by Lemma 2.5 and (2.6),

$$\frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \left\| \begin{bmatrix} \boldsymbol{Y}_{k-1} \\ 1 \end{bmatrix} \right\|^3 \xrightarrow{\text{a.s.}} t\, \mathbb{E}\left( \left\| \begin{bmatrix} \boldsymbol{Y}_0 \\ 1 \end{bmatrix} \right\|^3 \right) \qquad \text{as } n \to \infty.$$

Hence we obtain

$$\left( \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{U}_k \right)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} \widetilde{\boldsymbol{B}} \qquad \text{as } n \to \infty,$$

where $\widetilde{\boldsymbol{B}} = (\widetilde{\boldsymbol{B}}_t)_{t\in\mathbb{R}_+}$ is a $p$-dimensional zero mean Brownian motion such that $\operatorname{Var}(\widetilde{\boldsymbol{B}}_1) = \boldsymbol{V}$. Using (2.30), we have

$$\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k) = \boldsymbol{M}_{\boldsymbol{\xi}}^k (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0)) + \sum_{j=1}^k \boldsymbol{M}_{\boldsymbol{\xi}}^{k-j} \boldsymbol{U}_j, \qquad k \in \mathbb{N}.$$

Consequently, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k)) \\
&= \frac{1}{\sqrt{n}} \left[ \left( \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{M}_{\boldsymbol{\xi}}^k \right) (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0)) + \sum_{k=1}^{\lfloor nt \rfloor} \sum_{j=1}^{k} \boldsymbol{M}_{\boldsymbol{\xi}}^{k-j} \boldsymbol{U}_j \right] \\
&= \frac{1}{\sqrt{n}} \left[ (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} (\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor + 1}) (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0)) + \sum_{j=1}^{\lfloor nt \rfloor} \left( \sum_{k=j}^{\lfloor nt \rfloor} \boldsymbol{M}_{\boldsymbol{\xi}}^{k-j} \right) \boldsymbol{U}_j \right] \\
&= \frac{1}{\sqrt{n}} \left[ (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \left( (\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor + 1}) (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0)) + \sum_{j=1}^{\lfloor nt \rfloor} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - j + 1}) \boldsymbol{U}_j \right) \right],
\end{aligned} \tag{2.34}$$

implying the statement using Slutsky's lemma since $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1$. Indeed,

$$\frac{1}{\sqrt{n}} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} (\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor + 1}) (\boldsymbol{Y}_0 - \mathbb{E}(\boldsymbol{Y}_0)) \xrightarrow{\text{a.s.}} \boldsymbol{0} \qquad \text{as} \quad n \to \infty,$$

since $\lim_{n\to\infty} \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor + 1} = \boldsymbol{0}$ by (2.32). Moreover, $n^{-1/2} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - j + 1} \boldsymbol{U}_j$ converges in $L_1$ and hence in probability to $\boldsymbol{0}$ as $n \to \infty$, since by (2.29),

$$\mathbb{E}(|U_{k,j}|) \leqslant \sqrt{\mathbb{E}(U_{k,j}^2)} = \sqrt{\boldsymbol{v}_{(j,j)}^\top \begin{bmatrix} \mathbb{E}(\boldsymbol{Y}_0) \\ 1 \end{bmatrix}} = \sqrt{V_{j,j}}, \qquad j \in \{1, \ldots, p\}, \qquad k \in \mathbb{N}, \tag{2.35}$$

and hence

$$\begin{aligned}
&\mathbb{E}\left( \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - k + 1} \boldsymbol{U}_k \right\| \right) \leqslant \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(\| \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - k + 1} \boldsymbol{U}_k \|) \\
&\leqslant \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \| \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - k + 1} \| \, \mathbb{E}(\| \boldsymbol{U}_k \|) \leqslant \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \| \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - k + 1} \| \sum_{j=1}^{p} \mathbb{E}(|U_{k,j}|) \\
&\leqslant \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \| \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - k + 1} \| \sum_{j=1}^{p} \sqrt{V_{j,j}} \to 0 \qquad \text{as} \quad n \to \infty,
\end{aligned} \tag{2.36}$$

since, applying (2.32) for $\lfloor nt \rfloor \geqslant k_0$, we have

$$\begin{aligned}
&\sum_{k=1}^{\lfloor nt \rfloor} \| \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor - k + 1} \| = \sum_{k=1}^{\lfloor nt \rfloor} \| \boldsymbol{M}_{\boldsymbol{\xi}}^k \| = \sum_{k=1}^{k_0 - 1} \| \boldsymbol{M}_{\boldsymbol{\xi}}^k \| + \sum_{k=k_0}^{\lfloor nt \rfloor} \| \boldsymbol{M}_{\boldsymbol{\xi}}^k \| \\
&\leqslant \sum_{k=1}^{k_0 - 1} \| \boldsymbol{M}_{\boldsymbol{\xi}}^k \| + \sum_{k=k_0}^{\lfloor nt \rfloor} \left( \frac{1 + \varrho(\boldsymbol{M}_{\boldsymbol{\xi}})}{2} \right)^k \leqslant \sum_{k=1}^{k_0 - 1} \| \boldsymbol{M}_{\boldsymbol{\xi}}^k \| + \sum_{k=k_0}^{\infty} \left( \frac{1 + \varrho(\boldsymbol{M}_{\boldsymbol{\xi}})}{2} \right)^k < \infty.
\end{aligned}$$

Consequently, by Slutsky's lemma,

$$\left( n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} (\boldsymbol{Y}_k - \mathbb{E}(\boldsymbol{Y}_k)) \right)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \widetilde{\boldsymbol{B}} \qquad \text{as} \quad n \to \infty,$$

where $\widetilde{\boldsymbol{B}} = (\widetilde{\boldsymbol{B}}_t)_{t\in\mathbb{R}_+}$ is a $p$-dimensional zero mean Brownian motion such that $\operatorname{Var}(\widetilde{\boldsymbol{B}}_1) = \boldsymbol{V}$, as desired. Furthermore, $\boldsymbol{V}^{1/2}\boldsymbol{B} \stackrel{\mathcal{D}}{=} \widetilde{\boldsymbol{B}}$, which completes the proof. $\square$

**Proof of Theorem 2.8.** First, we prove (2.12). For all $N, m \in \mathbb{N}$ and all $t_1, \ldots, t_m \in \mathbb{R}_+$, by Proposition 2.7 and the continuity theorem, we have

$$\frac{1}{\sqrt{n}}(\boldsymbol{S}_{t_1}^{(N,n)}, \ldots, \boldsymbol{S}_{t_m}^{(N,n)}) \xrightarrow{\mathcal{D}} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}^{\frac{1}{2}} \sum_{\ell=1}^{N} (\boldsymbol{B}_{t_1}^{(\ell)}, \ldots, \boldsymbol{B}_{t_m}^{(\ell)})$$

as $n \to \infty$, where $\boldsymbol{B}^{(\ell)} = (\boldsymbol{B}_t^{(\ell)})_{t\in\mathbb{R}_+}$, $\ell \in \{1, \ldots, N\}$, are independent $p$-dimensional standard Brownian motions. Since

$$\frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} (\boldsymbol{B}_{t_1}^{(\ell)}, \ldots, \boldsymbol{B}_{t_m}^{(\ell)}) \stackrel{\mathcal{D}}{=} (\boldsymbol{B}_{t_1}, \ldots, \boldsymbol{B}_{t_m}), \qquad N \in \mathbb{N}, \quad m \in \mathbb{N},$$

we obtain the convergence (2.12).

Now, we turn to prove (2.11). For all $n \in \mathbb{N}$ and for all $t_1, \ldots, t_m \in \mathbb{R}_+$ with $t_1 < \ldots < t_m$, $m \in \mathbb{N}$, by Proposition 2.6 and by the continuous mapping theorem, we have

$$\begin{aligned}
\frac{1}{\sqrt{N}}\big((\boldsymbol{S}_{t_1}^{(N,n)})^\top, \ldots, (\boldsymbol{S}_{t_m}^{(N,n)})^\top\big)^\top &\xrightarrow{\mathcal{D}} \left(\sum_{k=1}^{\lfloor nt_1\rfloor} \boldsymbol{\mathcal{Y}}_k^\top, \ldots, \sum_{k=1}^{\lfloor nt_m\rfloor} \boldsymbol{\mathcal{Y}}_k^\top\right)^\top \\
&\stackrel{\mathcal{D}}{=} \mathcal{N}_{pm}\left(\boldsymbol{0}, \operatorname{Var}\left(\left(\sum_{k=1}^{\lfloor nt_1\rfloor} \boldsymbol{\mathcal{Y}}_k^\top, \ldots, \sum_{k=1}^{\lfloor nt_m\rfloor} \boldsymbol{\mathcal{Y}}_k^\top\right)^\top\right)\right)
\end{aligned}$$

as $N \to \infty$, where $(\boldsymbol{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is the $p$-dimensional zero mean stationary Gaussian process given in Proposition 2.6 and, by (2.8),

$$\begin{aligned}
&\operatorname{Var}\left(\left(\sum_{k=1}^{\lfloor nt_1\rfloor} \boldsymbol{\mathcal{Y}}_k^\top, \ldots, \sum_{k=1}^{\lfloor nt_m\rfloor} \boldsymbol{\mathcal{Y}}_k^\top\right)^\top\right) = \left(\operatorname{Cov}\left(\sum_{k=1}^{\lfloor nt_i\rfloor} \boldsymbol{\mathcal{Y}}_k, \sum_{k=1}^{\lfloor nt_j\rfloor} \boldsymbol{\mathcal{Y}}_k\right)\right)_{i,j=1}^{m} \\
&= \left(\sum_{k=1}^{\lfloor nt_i\rfloor} \sum_{\ell=1}^{\lfloor nt_j\rfloor} \operatorname{Cov}(\boldsymbol{\mathcal{Y}}_k, \boldsymbol{\mathcal{Y}}_\ell)\right)_{i,j=1}^{m} \\
&= \Bigg(\sum_{k=1}^{\lfloor nt_i\rfloor} \sum_{\ell=1}^{(k-1)\wedge\lfloor nt_j\rfloor} \boldsymbol{M}_{\boldsymbol{\xi}}^{k-\ell} \operatorname{Var}(\boldsymbol{Y}_0) + (\lfloor nt_i\rfloor \wedge \lfloor nt_j\rfloor) \operatorname{Var}(\boldsymbol{Y}_0) \\
&\qquad + \operatorname{Var}(\boldsymbol{Y}_0) \sum_{k=1}^{\lfloor nt_i\rfloor} \sum_{\ell=k+1}^{\lfloor nt_j\rfloor} (\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\ell-k}\Bigg)_{i,j=1}^{m},
\end{aligned}$$

where $\sum_{\ell=q_1}^{q_2} := 0$ for all $q_2 < q_1$, $q_1, q_2 \in \mathbb{N}$. By the continuity theorem, for all $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m \in \mathbb{R}^p$, $m \in \mathbb{N}$, we conclude

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left(\exp\left\{\mathrm{i} \sum_{j=1}^{m} \boldsymbol{\theta}_j^\top n^{-\frac{1}{2}} N^{-\frac{1}{2}} \boldsymbol{S}_{t_j}^{(N,n)}\right\}\right) \\
&= \exp\left\{-\frac{1}{2n} \sum_{i=1}^{m} \sum_{j=1}^{m} \boldsymbol{\theta}_i^\top \left[\sum_{k=1}^{\lfloor nt_i\rfloor} \sum_{\ell=1}^{\lfloor nt_j\rfloor} \operatorname{Cov}(\boldsymbol{\mathcal{Y}}_k, \boldsymbol{\mathcal{Y}}_\ell)\right] \boldsymbol{\theta}_j\right\}
\end{aligned}$$

$$\to \exp\Big\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(t_i\wedge t_j)\boldsymbol{\theta}_i^\top\Big[\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)$$
$$+\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\boldsymbol{M}_{\boldsymbol{\xi}}^\top\Big]\boldsymbol{\theta}_j\Big\}\qquad \text{as}\quad n\to\infty.$$

Indeed, for all $s,t\in\mathbb{R}_+$ with $s<t$, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\sum_{\ell=1}^{\lfloor nt\rfloor}\operatorname{Cov}(\boldsymbol{\mathcal{Y}}_k,\boldsymbol{\mathcal{Y}}_\ell)\\
&=\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\sum_{\ell=1}^{k-1}\boldsymbol{M}_{\boldsymbol{\xi}}^{k-\ell}\operatorname{Var}(\boldsymbol{Y}_0)+\frac{\lfloor ns\rfloor}{n}\operatorname{Var}(\boldsymbol{Y}_0)+\frac{1}{n}\operatorname{Var}(\boldsymbol{Y}_0)\sum_{k=1}^{\lfloor ns\rfloor}\sum_{\ell=k+1}^{\lfloor nt\rfloor}(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\ell-k}\\
&=\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}(\boldsymbol{M}_{\boldsymbol{\xi}}-\boldsymbol{M}_{\boldsymbol{\xi}}^{k})(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)+\frac{\lfloor ns\rfloor}{n}\operatorname{Var}(\boldsymbol{Y}_0)\\
&\quad+\frac{1}{n}\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\sum_{k=1}^{\lfloor ns\rfloor}(\boldsymbol{M}_{\boldsymbol{\xi}}^\top-(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor nt\rfloor-k+1})\\
&=\frac{1}{n}\Big(\lfloor ns\rfloor\boldsymbol{M}_{\boldsymbol{\xi}}-\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor ns\rfloor})(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\Big)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)\\
&\quad+\frac{\lfloor ns\rfloor}{n}\operatorname{Var}(\boldsymbol{Y}_0)+\frac{1}{n}\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\\
&\qquad\qquad\times\Big(\lfloor ns\rfloor\boldsymbol{M}_{\boldsymbol{\xi}}^\top-(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}(\boldsymbol{I}_p-(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor ns\rfloor})(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor nt\rfloor-\lfloor ns\rfloor+1}\Big)\\
&=\frac{\lfloor ns\rfloor}{n}\Big(\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\boldsymbol{M}_{\boldsymbol{\xi}}^\top\Big)\\
&\quad-\frac{1}{n}\Big(\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor ns\rfloor})(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-2}\operatorname{Var}(\boldsymbol{Y}_0)\\
&\qquad\quad+\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-2}(\boldsymbol{I}_p-(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor ns\rfloor})(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor nt\rfloor-\lfloor ns\rfloor+1}\Big)\\
&\to s\Big(\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\boldsymbol{M}_{\boldsymbol{\xi}}^\top\Big)\quad \text{as}\quad n\to\infty,
\end{aligned}$$

since $\lim_{n\to\infty}\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor ns\rfloor}=\boldsymbol{0}$, $\lim_{n\to\infty}(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor ns\rfloor}=\boldsymbol{0}$ and $\lim_{n\to\infty}(\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{\lfloor nt\rfloor-\lfloor ns\rfloor+1}=\boldsymbol{0}$ by (2.32). It remains to show that

$$\begin{aligned}
&\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\boldsymbol{M}_{\boldsymbol{\xi}}^\top\\
&=(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}.
\end{aligned}\tag{2.37}$$

We have

$$\boldsymbol{M}_{\boldsymbol{\xi}}(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}=(\boldsymbol{I}_p-(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}))(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}=(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}-\boldsymbol{I}_p,\tag{2.38}$$

and hence $(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}\boldsymbol{M}_{\boldsymbol{\xi}}^\top=(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}-\boldsymbol{I}_p$, thus the left-hand side of equation (2.37) can be written as

$$\begin{aligned}
&((\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}-\boldsymbol{I}_p)\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)((\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}-\boldsymbol{I}_p)\\
&=(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\operatorname{Var}(\boldsymbol{Y}_0)-\operatorname{Var}(\boldsymbol{Y}_0)+\operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p-\boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1}.
\end{aligned}$$

By (2.31), we have $\boldsymbol{V} = \operatorname{Var}(\boldsymbol{Y}_0) - \boldsymbol{M}_{\boldsymbol{\xi}} \operatorname{Var}(\boldsymbol{Y}_0)\boldsymbol{M}_{\boldsymbol{\xi}}^\top$, hence, by (2.38), the right-hand side of the equation (2.37) can be written as

$$\begin{aligned}
&(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}(\operatorname{Var}(\boldsymbol{Y}_0) - \boldsymbol{M}_{\boldsymbol{\xi}} \operatorname{Var}(\boldsymbol{Y}_0)\boldsymbol{M}_{\boldsymbol{\xi}}^\top)(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1} \\
&= (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1} - (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{M}_{\boldsymbol{\xi}} \operatorname{Var}(\boldsymbol{Y}_0)\boldsymbol{M}_{\boldsymbol{\xi}}^\top(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1} \\
&= (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1} \\
&\qquad - ((\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} - \boldsymbol{I}_p) \operatorname{Var}(\boldsymbol{Y}_0)((\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1} - \boldsymbol{I}_p) \\
&= (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \operatorname{Var}(\boldsymbol{Y}_0) - \operatorname{Var}(\boldsymbol{Y}_0) + \operatorname{Var}(\boldsymbol{Y}_0)(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}}^\top)^{-1},
\end{aligned}$$

and we conclude (2.37). This implies the convergence (2.11). □

**Proof of Theorem 2.9.** As $n$ and $N$ converge to infinity simultaneously, (2.13) is equivalent to $(nN_n)^{-\frac{1}{2}}\boldsymbol{S}^{(N_n,n)} \xrightarrow{\mathcal{D}} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{B}$ as $n \to \infty$ for any sequence $(N_n)_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty} N_n = \infty$. As we have seen in the proof of Proposition 2.7, for each $j \in \mathbb{N}$,

$$\boldsymbol{U}_k^{(j)} := \boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)} \,|\, \mathcal{F}_{k-1}^{\boldsymbol{Y}^{(j)}}) = \boldsymbol{Y}_k^{(j)} - \boldsymbol{M}_{\boldsymbol{\xi}}\boldsymbol{Y}_{k-1}^{(j)} - \boldsymbol{m}_\varepsilon, \qquad k \in \mathbb{N},$$

are martingale differences with respect to the filtration $(\mathcal{F}_k^{\boldsymbol{Y}^{(j)}})_{k\in\mathbb{Z}_+}$. We are going to apply the functional martingale central limit theorem, see, e.g., Jacod and Shiryaev [21, Theorem VIII.3.33], for the triangular array consisting of the random vectors

$$(\boldsymbol{V}_k^{(n)})_{k\in\mathbb{N}} := (nN_n)^{-\frac{1}{2}}\big(\boldsymbol{U}_1^{(1)}, \ldots, \boldsymbol{U}_1^{(N_n)}, \boldsymbol{U}_2^{(1)}, \ldots, \boldsymbol{U}_2^{(N_n)}, \boldsymbol{U}_3^{(1)}, \ldots, \boldsymbol{U}_3^{(N_n)}, \ldots\big)$$

in the $n^{\text{th}}$ row for each $n \in \mathbb{N}$ with the filtration $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$ given by $\mathcal{F}_k^{(n)} := \mathcal{F}_k^{\overline{\boldsymbol{Y}}^{(n)}} = \sigma(\overline{\boldsymbol{Y}}_0^{(n)}, \ldots, \overline{\boldsymbol{Y}}_k^{(n)})$, where

$$(\overline{\boldsymbol{Y}}_k^{(n)})_{k\in\mathbb{Z}_+} := \big((\boldsymbol{Y}_0^{(1)}, \ldots, \boldsymbol{Y}_0^{(N_n)}), \boldsymbol{Y}_1^{(1)}, \ldots, \boldsymbol{Y}_1^{(N_n)}, \boldsymbol{Y}_2^{(1)}, \ldots, \boldsymbol{Y}_2^{(N_n)}, \ldots\big).$$

Hence $\mathcal{F}_0^{(n)} = \sigma(\boldsymbol{Y}_0^{(1)}, \ldots, \boldsymbol{Y}_0^{(N_n)})$, and for each $k = \ell N_n + r$ with $\ell \in \mathbb{Z}_+$ and $r \in \{1, \ldots, N_n\}$, we have

$$\mathcal{F}_k^{(n)} = \sigma\big(\big(\cup_{j=1}^r \mathcal{F}_{\ell+1}^{\boldsymbol{Y}^{(j)}}\big) \cup \big(\cup_{j=r+1}^{N_n} \mathcal{F}_\ell^{\boldsymbol{Y}^{(j)}}\big)\big),$$

where $\cup_{j=N_n+1}^{N_n} := \emptyset$. Moreover, $\overline{\boldsymbol{Y}}_0^{(n)} = (\boldsymbol{Y}_0^{(1)}, \ldots, \boldsymbol{Y}_0^{(N_n)})$, and for $k = \ell N_n + r$ with $\ell \in \mathbb{Z}_+$ and $r \in \{1, \ldots, N_n\}$, we have $\overline{\boldsymbol{Y}}_k^{(n)} = \boldsymbol{Y}_{\ell+1}^{(r)}$ and $\boldsymbol{V}_k^{(n)} = (nN_n)^{-\frac{1}{2}}\boldsymbol{U}_{\ell+1}^{(r)}$.

Next we check that for each $n \in \mathbb{N}$, $(\boldsymbol{V}_k^{(n)})_{k\in\mathbb{N}}$ is a sequence of martingale differences with respect to $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$. We will use that $\mathbb{E}(\boldsymbol{\xi} \,|\, \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)) = \mathbb{E}(\boldsymbol{\xi} \,|\, \mathcal{G}_1)$ for a random vector $\boldsymbol{\xi}$ and for $\sigma$-algebras $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ such that $\sigma(\sigma(\boldsymbol{\xi}) \cup \mathcal{G}_1)$ and $\mathcal{G}_2$ are independent and $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$. For each $k = \ell N_n + 1$ with $\ell \in \mathbb{Z}_+$, we have $\mathbb{E}(\boldsymbol{V}_k^{(n)} \,|\, \mathcal{F}_{k-1}^{(n)}) = (nN_n)^{-\frac{1}{2}}\, \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(1)} \,|\, \mathcal{F}_\ell^{\boldsymbol{Y}^{(1)}}) = \boldsymbol{0}$, since

$$\mathbb{E}(\boldsymbol{U}_{\ell+1}^{(1)} \,|\, \mathcal{F}_{k-1}^{(n)}) = \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(1)} \,|\, \sigma(\cup_{j=1}^{N_n} \mathcal{F}_\ell^{\boldsymbol{Y}^{(j)}})) = \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(1)} \,|\, \mathcal{F}_\ell^{\boldsymbol{Y}^{(1)}}) = \boldsymbol{0}.$$

In a similar way, for each $k = \ell N_n + r$ with $\ell \in \mathbb{Z}_+$ and $r \in \{2, \ldots, N_n\}$, we have $\mathbb{E}(\boldsymbol{V}_k^{(n)} \,|\, \mathcal{F}_{k-1}^{(n)}) = (nN_n)^{-\frac{1}{2}}\, \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(r)} \,|\, \mathcal{F}_\ell^{\boldsymbol{Y}^{(r)}}) = \boldsymbol{0}$, since

$$\mathbb{E}(\boldsymbol{U}_{\ell+1}^{(r)} \,|\, \mathcal{F}_{k-1}^{(n)}) = \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(r)} \,|\, \sigma((\cup_{j=1}^{r-1} \mathcal{F}_{\ell+1}^{\boldsymbol{Y}^{(j)}}) \cup (\cup_{j=r}^{N_n} \mathcal{F}_\ell^{\boldsymbol{Y}^{(j)}}))) = \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(r)} \,|\, \mathcal{F}_\ell^{\boldsymbol{Y}^{(r)}}) = \boldsymbol{0}.$$

We want to obtain a functional central limit theorem for the sequence

$$\left(\sum_{k=1}^{\lfloor nt\rfloor N_n} \boldsymbol{V}_k^{(n)}\right)_{t\in\mathbb{R}_+} = \left(\frac{1}{\sqrt{nN_n}} \sum_{\ell=1}^{\lfloor nt\rfloor} \sum_{r=1}^{N_n} \boldsymbol{U}_\ell^{(r)}\right)_{t\in\mathbb{R}_+}, \qquad n \in \mathbb{N}.$$

First, we calculate the conditional variance matrix of $\boldsymbol{V}_k^{(n)}$. If $k = \ell N_n + 1$ with $\ell \in \mathbb{Z}_+$, then

$$\begin{aligned}
\mathbb{E}(\boldsymbol{V}_k^{(n)}(\boldsymbol{V}_k^{(n)})^\top \,|\, \mathcal{F}_{k-1}^{(n)}) &= (nN_n)^{-1}\, \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(1)}(\boldsymbol{U}_{\ell+1}^{(1)})^\top \,|\, \sigma(\cup_{j=1}^{N_n} \mathcal{F}_\ell^{\boldsymbol{Y}^{(j)}})) \\
&= (nN_n)^{-1}\, \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(1)}(\boldsymbol{U}_{\ell+1}^{(1)})^\top \,|\, \mathcal{F}_\ell^{\boldsymbol{Y}^{(1)}}).
\end{aligned}$$

In a similar way, if $k = \ell N_n + r$ with $\ell \in \mathbb{Z}_+$ and $r \in \{2, \ldots, N_n\}$, then

$$\begin{aligned}
\mathbb{E}(\boldsymbol{V}_k^{(n)}(\boldsymbol{V}_k^{(n)})^\top \,|\, \mathcal{F}_{k-1}^{(n)}) &= (nN_n)^{-1}\, \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(r)}(\boldsymbol{U}_{\ell+1}^{(r)})^\top \,|\, \sigma((\cup_{j=1}^{r-1} \mathcal{F}_{\ell+1}^{\boldsymbol{Y}^{(j)}}) \cup (\cup_{j=r}^{N_n} \mathcal{F}_\ell^{\boldsymbol{Y}^{(j)}}))) \\
&= (nN_n)^{-1}\, \mathbb{E}(\boldsymbol{U}_{\ell+1}^{(r)}(\boldsymbol{U}_{\ell+1}^{(r)})^\top \,|\, \mathcal{F}_\ell^{\boldsymbol{Y}^{(r)}}).
\end{aligned}$$

Consequently, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
\sum_{k=1}^{\lfloor nt\rfloor N_n} \mathbb{E}(\boldsymbol{V}_k^{(n)}(\boldsymbol{V}_k^{(n)})^\top \,|\, \mathcal{F}_{k-1}^{(n)}) &= \sum_{\ell=1}^{\lfloor nt\rfloor} \sum_{r=1}^{N_n} \mathbb{E}(\boldsymbol{V}_{(\ell-1)N_n+r}^{(n)}(\boldsymbol{V}_{(\ell-1)N_n+r}^{(n)})^\top \,|\, \mathcal{F}_{(\ell-1)N_n+r-1}^{(n)}) \\
&= \frac{1}{nN_n} \sum_{\ell=1}^{\lfloor nt\rfloor} \sum_{r=1}^{N_n} \mathbb{E}(\boldsymbol{U}_\ell^{(r)}(\boldsymbol{U}_\ell^{(r)})^\top \,|\, \mathcal{F}_{\ell-1}^{\boldsymbol{Y}^{(r)}}).
\end{aligned}$$

Next, we show that for each $t \in \mathbb{R}_+$ and $i, j \in \{1, \ldots, p\}$, we have

$$\frac{1}{nN_n} \sum_{\ell=1}^{\lfloor nt\rfloor} \sum_{r=1}^{N_n} \mathbb{E}(U_{\ell,i}^{(r)} U_{\ell,j}^{(r)} \,|\, \mathcal{F}_{\ell-1}^{\boldsymbol{Y}^{(r)}}) = \frac{1}{nN_n} \sum_{\ell=1}^{\lfloor nt\rfloor} \sum_{r=1}^{N_n} \boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} \boldsymbol{Y}_{\ell-1}^{(r)} \\ 1 \end{bmatrix} \xrightarrow{\mathbb{P}} \boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} \mathbb{E}(\boldsymbol{Y}_0) \\ 1 \end{bmatrix} t = V_{i,j} t$$

as $n \to \infty$. Indeed, the equality follows by (2.29), and for the convergence in probability, note that $\lim_{n\to\infty} \frac{\lfloor nt\rfloor}{n} = t$, $t \in \mathbb{R}_+$, and, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\left(\left(\frac{1}{\lfloor nt\rfloor N_n} \sum_{\ell=1}^{\lfloor nt\rfloor} \sum_{r=1}^{N_n} \boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} \boldsymbol{Y}_{\ell-1}^{(r)} - \mathbb{E}(\boldsymbol{Y}_0) \\ 0 \end{bmatrix}\right)^2\right) \\
&= \frac{1}{\lfloor nt\rfloor^2 N_n^2}\, \mathbb{E}\left(\left(\boldsymbol{v}_{(i,j)}^\top \sum_{\ell_1=1}^{\lfloor nt\rfloor} \sum_{r_1=1}^{N_n} \begin{bmatrix} \boldsymbol{Y}_{\ell_1-1}^{(r_1)} - \mathbb{E}(\boldsymbol{Y}_0) \\ 0 \end{bmatrix}\right)\right. \\
&\qquad\qquad \times \left.\left(\sum_{\ell_2=1}^{\lfloor nt\rfloor} \sum_{r_2=1}^{N_n} \begin{bmatrix} \boldsymbol{Y}_{\ell_2-1}^{(r_2)} - \mathbb{E}(\boldsymbol{Y}_0) \\ 0 \end{bmatrix}^\top \boldsymbol{v}_{(i,j)}\right)\right) \\
&= \frac{1}{\lfloor nt\rfloor^2 N_n^2} \boldsymbol{v}_{(i,j)}^\top \sum_{\ell_1=1}^{\lfloor nt\rfloor} \sum_{\ell_2=1}^{\lfloor nt\rfloor} \sum_{r_1=1}^{N_n} \sum_{r_2=1}^{N_n} \begin{bmatrix} \mathbb{E}((\boldsymbol{Y}_{\ell_1-1}^{(r_1)} - \mathbb{E}(\boldsymbol{Y}_0))(\boldsymbol{Y}_{\ell_2-1}^{(r_2)} - \mathbb{E}(\boldsymbol{Y}_0))^\top) & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} \boldsymbol{v}_{(i,j)} \\
&= \frac{1}{\lfloor nt\rfloor^2 N_n} \boldsymbol{v}_{(i,j)}^\top \sum_{\ell_1=1}^{\lfloor nt\rfloor} \sum_{\ell_2=1}^{\lfloor nt\rfloor} \begin{bmatrix} \mathbb{E}((\boldsymbol{Y}_{\ell_1-1} - \mathbb{E}(\boldsymbol{Y}_0))(\boldsymbol{Y}_{\ell_2-1} - \mathbb{E}(\boldsymbol{Y}_0))^\top) & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} \boldsymbol{v}_{(i,j)} \\
&\leqslant \frac{1}{\lfloor nt\rfloor^2 N_n} \|\boldsymbol{v}_{(i,j)}\|^2 \sum_{\ell_1=1}^{\lfloor nt\rfloor} \sum_{\ell_2=1}^{\lfloor nt\rfloor} \mathbb{E}\left(\|(\boldsymbol{Y}_{\ell_1-1} - \mathbb{E}(\boldsymbol{Y}_0))(\boldsymbol{Y}_{\ell_2-1} - \mathbb{E}(\boldsymbol{Y}_0))^\top\|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{\lfloor nt\rfloor^2 N_n}\|\boldsymbol{v}_{(i,j)}\|^2 \sum_{\ell_1=1}^{\lfloor nt\rfloor}\sum_{\ell_2=1}^{\lfloor nt\rfloor}\sum_{m_1=1}^{p}\sum_{m_2=1}^{p} \mathbb{E}(|(Y_{\ell_1-1,m_1}-\mathbb{E}(Y_{0,m_1}))(Y_{\ell_2-1,m_2}-\mathbb{E}(Y_{0,m_2}))|)\\
&\leqslant \frac{1}{\lfloor nt\rfloor^2 N_n}\|\boldsymbol{v}_{(i,j)}\|^2 \sum_{\ell_1=1}^{\lfloor nt\rfloor}\sum_{\ell_2=1}^{\lfloor nt\rfloor}\sum_{m_1=1}^{p}\sum_{m_2=1}^{p} \sqrt{\operatorname{Var}(Y_{\ell_1-1,m_1})\operatorname{Var}(Y_{\ell_2-1,m_2})}\\
&= \frac{1}{N_n}\|\boldsymbol{v}_{(i,j)}\|^2 \sum_{m_1=1}^{p}\sum_{m_2=1}^{p} \sqrt{\operatorname{Var}(Y_{0,m_1})\operatorname{Var}(Y_{0,m_2})} \to 0 \qquad \text{as } n\to\infty,
\end{aligned}
$$

where we used that $\|\boldsymbol{Q}\| \leqslant \sum_{i=1}^p \sum_{j=1}^p |q_{i,j}|$ for every matrix $\boldsymbol{Q} = (q_{i,j})_{i,j=1}^p \in \mathbb{R}^{p\times p}$.

Moreover, in a similar way, the conditional Lindeberg condition holds, namely, for all $\delta > 0$,

$$
\begin{aligned}
\sum_{k=1}^{\lfloor nt\rfloor N_n} \mathbb{E}(\|\boldsymbol{V}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{V}_k^{(n)}\|>\delta\}} \mid \mathcal{F}_{k-1}^{(n)}) &= \frac{1}{nN_n}\sum_{\ell=1}^{\lfloor nt\rfloor}\sum_{r=1}^{N_n} \mathbb{E}(\|\boldsymbol{U}_\ell^{(r)}\|^2 \mathbb{1}_{\{\|\boldsymbol{U}_\ell^{(r)}\|>\delta\sqrt{nN_n}\}} \mid \mathcal{F}_{\ell-1}^{\boldsymbol{Y}^{(r)}})\\
&\leqslant \frac{1}{\delta n^{3/2} N_n^{1/2}} \sum_{\ell=1}^{\lfloor nt\rfloor} \mathbb{E}(\|\boldsymbol{U}_\ell^{(1)}\|^3 \mid \mathcal{F}_{\ell-1}^{\boldsymbol{Y}^{(1)}}) \xrightarrow{\text{a.s.}} 0
\end{aligned}
$$

as $n\to\infty$, where the almost sure convergence follows by (2.33). Hence we obtain

$$
\left(\frac{1}{\sqrt{nN_n}}\sum_{\ell=1}^{\lfloor nt\rfloor}\sum_{r=1}^{N_n}\boldsymbol{U}_\ell^{(r)}\right)_{t\in\mathbb{R}_+} = \left(\sum_{k=1}^{\lfloor nt\rfloor N_n}\boldsymbol{V}_k^{(n)}\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \boldsymbol{V}^{\frac{1}{2}}\boldsymbol{B} \qquad \text{as } n\to\infty,
$$

where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a $p$-dimensional standard Brownian motion. Using (2.34), for each $n\in\mathbb{N}$ and $t\in\mathbb{R}_+$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{nN_n}}\sum_{\ell=1}^{\lfloor nt\rfloor}\sum_{r=1}^{N_n}(\boldsymbol{Y}_\ell^{(r)} - \mathbb{E}(\boldsymbol{Y}_\ell^{(r)}))\\
&= \frac{1}{\sqrt{n}}\left[(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}(\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor+1})\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}(\boldsymbol{Y}_0^{(r)} - \mathbb{E}(\boldsymbol{Y}_0^{(r)}))\right]\\
&\quad - \frac{1}{\sqrt{n}}\left[(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\sum_{m=1}^{\lfloor nt\rfloor}\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}\boldsymbol{U}_m^{(r)}\right]\\
&\quad + (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\frac{1}{\sqrt{nN_n}}\sum_{m=1}^{\lfloor nt\rfloor}\sum_{r=1}^{N_n}\boldsymbol{U}_m^{(r)},
\end{aligned}
$$

implying the statement using Slutsky's lemma, since $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1$. Indeed, by (2.32), $\lim_{n\to\infty}\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor+1} = \boldsymbol{0}$, thus

$$
\lim_{n\to\infty}(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}(\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor+1}) = (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}\boldsymbol{M}_{\boldsymbol{\xi}},
$$

and, by Proposition 2.6,

$$
\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}(\boldsymbol{Y}_0^{(r)} - \mathbb{E}(\boldsymbol{Y}_0^{(r)})) \xrightarrow{\mathcal{D}} \mathcal{N}_p(\boldsymbol{0}, \operatorname{Var}(\boldsymbol{Y}_0)) \qquad \text{as } n\to\infty,
$$

where $\mathcal{N}_p(\mathbf{0}, \operatorname{Var}(\boldsymbol{Y}_0))$ denotes a $p$-dimensional normal distribution with zero mean and with covariance matrix $\operatorname{Var}(\boldsymbol{Y}_0)$, and then Slutsky's lemma yields that

$$\frac{1}{\sqrt{n}}\left[(\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1}(\boldsymbol{M}_{\boldsymbol{\xi}} - \boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor+1})\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}(\boldsymbol{Y}_0^{(r)} - \mathbb{E}(\boldsymbol{Y}_0^{(r)}))\right] \xrightarrow{\mathbb{P}} \mathbf{0} \qquad \text{as } n \to \infty.$$

Further,

$$\begin{aligned}
&\left\|\mathbb{E}\left(\frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}\boldsymbol{U}_m^{(r)}\right)\right\| \leqslant \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\mathbb{E}\left(\left\|\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}\boldsymbol{U}_m^{(r)}\right\|\right)\\
&\leqslant \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\|\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\|\,\mathbb{E}\left(\left\|\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}\boldsymbol{U}_m^{(r)}\right\|\right)\\
&\leqslant \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\|\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\|\sum_{j=1}^{p}\mathbb{E}\left(\left|\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}U_{m,j}^{(r)}\right|\right)\\
&\leqslant \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\|\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\|\sum_{j=1}^{p}\sqrt{\mathbb{E}\left(\left(\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n}U_{m,j}^{(r)}\right)^2\right)}\\
&= \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\|\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\|\sum_{j=1}^{p}\sqrt{\mathbb{E}((U_{m,j}^{(1)})^2)}\\
&\leqslant \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor nt\rfloor}\|\boldsymbol{M}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-m+1}\|\sum_{j=1}^{p}\sqrt{V_{j,j}} \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

by (2.36), where for the last inequality we used (2.35). This completes the proof. $\square$

**Proof of Proposition 2.14.** First we prove (2.21) by induction. Note that by (2.20) the statement holds for $k = 0$. We suppose that it holds for $0, \ldots, k$, and show that it is also true for $k + 1$. Using (2.17) it is easy to see that

$$\begin{aligned}
&F_{0,\ldots,k,k+1}(z_0, \ldots, z_k, z_{k+1}) = \mathbb{E}\left(z_0^{X_0}\cdots z_k^{X_k} z_{k+1}^{X_{k+1}}\right)\\
&= \mathbb{E}\left(z_0^{X_0}\cdots z_k^{X_k}\,\mathbb{E}\left(z_{k+1}^{X_{k+1}} \mid X_0, \ldots, X_k\right)\right) = \mathbb{E}\left(z_0^{X_0}\cdots z_k^{X_k}\,\mathbb{E}\left(z_{k+1}^{X_{k+1}} \mid X_k\right)\right)\\
&= \mathbb{E}\left(z_0^{X_0}\cdots z_k^{X_k}\mathrm{e}^{\lambda(z_{k+1}-1)}(1 - \alpha + \alpha z_{k+1})^{X_k}\right).
\end{aligned}$$

On the one hand, for any $z_0, \ldots, z_{k+1} \in \mathbb{C}$, by the assumption of the induction,

$$\begin{aligned}
&F_{0,\ldots,k,k+1}(z_0, \ldots, z_k, z_{k+1}) = \mathrm{e}^{\lambda(z_{k+1}-1)}F_{0,\ldots,k}(z_0, \ldots, z_{k-1}, z_k(1 - \alpha + \alpha z_{k+1}))\\
&= \exp\Bigg\{\frac{\lambda}{1-\alpha}\Bigg[(1-\alpha)(z_{k+1} - 1) + \sum_{0\leqslant i\leqslant j\leqslant k-1}\alpha^{j-i}(z_i - 1)z_{i+1}\cdots z_{j-1}(z_j - 1)\\
&\qquad\qquad + \mathrm{Sum}_1 + z_k(1 - \alpha + \alpha z_{k+1}) - 1\Bigg]\Bigg\},
\end{aligned}$$

with

$$\mathrm{Sum}_1 := \sum_{0\leqslant i\leqslant k-1}\alpha^{k-i}(z_i - 1)z_{i+1}\cdots z_{k-1}[z_k(1 - \alpha + \alpha z_{k+1}) - 1].$$

On the other hand, the right hand side of (2.21) for $k+1$ has the form

$$\exp\left\{\frac{\lambda}{1-\alpha}\left[\sum_{0\leqslant i\leqslant j\leqslant k-1}\alpha^{j-i}(z_i-1)z_{i+1}\cdots z_{j-1}(z_j-1)+\mathrm{Sum}_2+\mathrm{Sum}_3\right]\right\},$$

where

$$\begin{aligned}\mathrm{Sum}_2 &= \sum_{0\leqslant i\leqslant k}\alpha^{k-i}(z_i-1)z_{i+1}\cdots z_{k-1}(z_k-1)\\ &= (z_k-1)+\sum_{0\leqslant i\leqslant k-1}\alpha^{k-i}(z_i-1)z_{i+1}\cdots z_{k-1}(z_k-1),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sum}_3 &= \sum_{0\leqslant i\leqslant k+1}\alpha^{k+1-i}(z_i-1)z_{i+1}\cdots z_k(z_{k+1}-1)\\ &= (z_{k+1}-1)+\alpha(z_k-1)(z_{k+1}-1)+\sum_{0\leqslant i\leqslant k-1}\alpha^{k+1-i}(z_i-1)z_{i+1}\cdots z_k(z_{k+1}-1).\end{aligned}$$

Since

$$\mathrm{Sum}_1=\sum_{0\leqslant i\leqslant k-1}\alpha^{k-i}(z_i-1)z_{i+1}\cdots z_{k-1}(z_k-1)+\sum_{0\leqslant i\leqslant k-1}\alpha^{k+1-i}(z_i-1)z_{i+1}\cdots z_k(z_{k+1}-1),$$

in order to show (2.21) for $k+1$, it is enough to check that

$$(1-\alpha)(z_{k+1}-1)+z_k(1-\alpha+\alpha z_{k+1})-1=(z_k-1)+(z_{k+1}-1)+\alpha(z_k-1)(z_{k+1}-1),$$

which holds trivially.

Now we prove (2.22). In formula (2.21), for fixed indices $0\leqslant i\leqslant j\leqslant k$ the term in the sum gives

$$\begin{aligned}&(z_i-1)z_{i+1}\cdots z_{j-1}(z_j-1)\\ &\quad=(z_i\cdots z_j-1)-(z_i\cdots z_{j-1}-1)-(z_{i+1}\cdots z_j-1)+(z_{i+1}\cdots z_{j-1}-1),\end{aligned}$$

meaning that the sum consists of similar terms as in (2.22). We only have to show that the coefficients coincide in the formulas (2.22) and (2.21). In (2.22) the coefficient of $z_i\cdots z_j-1$ is $\lambda(1-\alpha)^{K_{i,j,k}}\alpha^{j-i}$. In (2.21) this term may appear multiple times, depending on the indices $i$ and $j$. If $i=0$ and $j=k$, then it only appears once, with coefficient $\lambda\alpha^{j-i}/(1-\alpha)$, that is the same as in (2.22). However, if $i=0$ and $0\leqslant j\leqslant k-1$ in (2.22), then the term also appears when the indices are $i$ and $j+1$ in (2.21), meaning that the coefficient is

$$\lambda\left(\frac{\alpha^{j-i}}{1-\alpha}-\frac{\alpha^{j+1-i}}{1-\alpha}\right)=\lambda\alpha^{j-i},$$

which is the same as in (2.22). Similarly, if $1\leqslant i\leqslant k$ and $j=k$ in (2.22), then the term also appears when the indices are $i-1$ and $j$ in (2.21), meaning that the coefficient is

$$\lambda\left(\frac{\alpha^{j-i}}{1-\alpha}-\frac{\alpha^{j-(i-1)}}{1-\alpha}\right)=\lambda\alpha^{j-i},$$

which is the same as in (2.22). If $1 \leqslant i \leqslant j \leqslant k-1$ in (2.22), then the term appears three more times, for the index pairs $(i-1, j)$, $(i, j+1)$, $(i-1, j+1)$ in (2.21), resulting the coefficient

$$\lambda \left( \frac{\alpha^{j-i}}{1-\alpha} - \frac{\alpha^{j-(i-1)}}{1-\alpha} - \frac{\alpha^{(j+1)-i}}{1-\alpha} + \frac{\alpha^{(j+1)-(i-1)}}{1-\alpha} \right) = \lambda \alpha^{j-i} \frac{1-2\alpha+\alpha^2}{1-\alpha} = \lambda \alpha^{j-i}(1-\alpha),$$

which is the same as in (2.22). This completes the proof. $\square$

# Chapter 3

# Iterated limit theorems for the aggregation of randomized INAR(1) processes with Poisson innovations

This chapter is about the introduction of certain randomized (or random coefficient) INAR(1) processes with Poisson innovations, the behavior of the simple (temporal or contemporaneous) aggregates of these processes, and the iterated limit theorems belonging to the temporal and contemporaneous aggregates. The proofs of this chapter are based on the paper Barczy et al. [6], except for those of Theorems 3.11 and 3.13, which were published in Nedényi and Pap [35]. These two are the iterated theorems corresponding to $\beta = 1$, where $\beta$ is a parameter of the mixing distribution. The latter paper also contains the corresponding iterated limit theorems of the random coefficient AR(1) processes, which were left out of the work of Pilipauskaitė and Surgailis as they require a different proof technique. Theorem 3.17, and the two results for the case $\beta = 1$ in Corollary 3.19 were developed for this thesis, these are not contained in any of our papers. However, these complete the results of the chapter, as here all the iterated limit theorems are presented with the three considered centerings (the expectation, the conditional expectation, the empirical expectation), when applicable.

As it was already stated in the Introduction, in this chapter, and the forthcoming one, we will consider a certain strictly stationary randomized (also called random coefficient) INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$ with randomized thinning parameter $\alpha$, given formally by the recursive equation

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

where $\alpha$ is a random variable with values in $(0, 1)$ and $X_0$ is some appropriate random variable. This means that, conditionally on $\alpha$, the process $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with thinning parameter $\alpha$. Conditionally on $\alpha$, the i.i.d. innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ are supposed to have a Poisson distribution with parameter $\lambda \in (0, \infty)$, and the conditional distribution of the initial value $X_0$ given $\alpha$ is supposed to be the unique stationary distribution, namely, a Poisson distribution with parameter $\lambda/(1 - \alpha)$. In Section 3.1 a rigorous proof is presented verifying that such a process exists. Section 3.2 is about the simple aggregates of the independent copies of the defined random coefficient process, without specifying the distribution of the random coefficient (called the mixing distribution). Then we introduce a specific type of mixing distribution and start investigating the temporal and contemporaneous aggregates of the processes. The iterated limit theorems (when first the time scale $n \to \infty$ and then the number of independent copies $N \to \infty$ or vice versa) are presented in Section 3.3. We have similar limit theorems for randomized INAR(1) processes that Pilipauskaitė and Surgailis [38, Theorems 2.1 and 2.3] have for random coefficient AR(1) processes. However, the techniques of our proofs differ from theirs in many cases, for a somewhat detailed comparison, see Remark 3.15. Section 3.4 contains some technical results that are necessary for the proofs, which can be found in Section 3.5.

## 3.1 Random coefficient INAR(1) processes with Poisson innovations

Let $\lambda \in (0, \infty)$, and let $\mathbb{P}_\alpha$ be a probability measure on $(0, 1)$. Then there exist a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a random variable $\alpha$ with distribution $\mathbb{P}_\alpha$, and random variables $\{X_0, \xi_{k,\ell}, \varepsilon_k : k, \ell \in \mathbb{N}\}$, conditionally independent given $\alpha$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$\mathbb{P}(\xi_{k,\ell} = 1 \,|\, \alpha) = \alpha = 1 - \mathbb{P}(\xi_{k,\ell} = 0 \,|\, \alpha), \qquad k, \ell \in \mathbb{N}, \tag{3.1}$$

$$\mathbb{P}(\varepsilon_k = j \,|\, \alpha) = \frac{\lambda^j}{j!} \mathrm{e}^{-\lambda}, \qquad j \in \mathbb{Z}_+, \qquad k \in \mathbb{N}, \tag{3.2}$$

$$\mathbb{P}(X_0 = j \,|\, \alpha) = \frac{\lambda^j}{j!(1-\alpha)^j} \mathrm{e}^{-(1-\alpha)^{-1}\lambda}, \qquad j \in \mathbb{Z}_+. \tag{3.3}$$

(Note that the conditional distribution of $\varepsilon_k$ does not depend on $\alpha$.) Indeed, for each $n \in \mathbb{N}$, by Ionescu Tulcea's theorem (see, e.g., Shiryaev [49, II. § 9, Theorem 2]), there exist a probability space $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ and random variables $\alpha^{(n)}$, $X_0^{(n)}$, $\varepsilon_k^{(n)}$ and $\xi_{k,\ell}^{(n)}$ for $k, \ell \in \{1, \ldots, n\}$ on $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ such that

$$\begin{aligned}
&\mathbb{P}_n(\alpha^{(n)} \in B, \ X_0^{(n)} = x_0, \ \varepsilon_k^{(n)} = j_k, \ \xi_{k,\ell}^{(n)} = x_{k,\ell} \ \text{for all} \ k, \ell \in \{1, \ldots, n\}) \\
&= \int_B p_n\left(a, x_0, (j_k)_{k=1}^n, (x_{k,\ell})_{k,\ell=1}^n\right) \mathbb{P}_\alpha(\mathrm{d}a)
\end{aligned}$$

for all $B \in \mathcal{B}(\mathbb{R})$, $x_0 \in \mathbb{Z}_+$, $(j_k)_{k=1}^n \in \mathbb{Z}_+^n$, $(x_{k,\ell})_{k,\ell=1}^n \in \{0,1\}^{n \times n}$, with

$$p_n\left(a, x_0, (j_k)_{k=1}^n, (x_{k,\ell})_{k,\ell=1}^n\right) := \frac{\lambda^{x_0}}{x_0!(1-a)^{x_0}} \mathrm{e}^{-(1-a)^{-1}\lambda} \prod_{k=1}^n \frac{\lambda^{j_k}}{j_k!} \mathrm{e}^{-\lambda} \prod_{k,\ell=1}^n a^{x_{k,\ell}} (1-a)^{1-x_{k,\ell}},$$

since the mapping $(0,1) \ni a \mapsto p_n\left(a, x_0, (j_k)_{k=1}^n, (x_{k,\ell})_{k,\ell=1}^n\right)$ is Borel measurable for all $x_0 \in \mathbb{Z}_+$, $(j_k)_{k=1}^n \in \mathbb{Z}_+^n$, $(x_{k,\ell})_{k,\ell=1}^n \in \{0,1\}^{n \times n}$, and

$$\sum \left\{ p_n\left(a, x_0, (j_k)_{k=1}^n, (x_{k,\ell})_{k,\ell=1}^n\right) : x_0 \in \mathbb{Z}_+, \, (j_k)_{k=1}^n \in \mathbb{Z}_+^n, \, (x_{k,\ell})_{k,\ell=1}^n \in \{0,1\}^{n \times n} \right\} = 1$$

for all $a \in (0,1)$. Then the Kolmogorov consistency theorem implies the existence of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and random variables $\alpha$, $X_0$, $\varepsilon_k$ and $\xi_{k,\ell}$ for $k, \ell \in \mathbb{N}$ on

$(\Omega, \mathcal{A}, \mathbb{P})$ with the desired properties (3.1), (3.2) and (3.3), since for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum \bigl\{ p_{n+1}\bigl(a, x_0, (j_k)_{k=1}^{n+1}, (x_{k,\ell})_{k,\ell=1}^{n+1}\bigr) \\
&\qquad : j_{n+1} \in \mathbb{Z}_+, \, (x_{n+1,\ell})_{\ell=1}^{n}, (x_{k,n+1})_{k=1}^{n} \in \{0,1\}^n, \, x_{n+1,n+1} \in \{0,1\} \bigr\} \\
&= p_n\bigl(a, x_0, (j_k)_{k=1}^{n}, (x_{k,\ell})_{k,\ell=1}^{n}\bigr).
\end{aligned}$$

Define a process $(X_k)_{k \in \mathbb{Z}_+}$ by

$$X_k = \sum_{\ell=1}^{X_{k-1}} \xi_{k,\ell} + \varepsilon_k, \qquad k \in \mathbb{N}.$$

By Section 2.3, conditionally on $\alpha$, the process $(X_k)_{k \in \mathbb{Z}_+}$ is a strictly stationary INAR(1) process with thinning parameter $\alpha$ and with Poisson innovations. Moreover, by the law of total probability, it is also (unconditionally) strictly stationary. However, it was shown in Barczy et al. [5, Appendix A] that the process $(X_k)_{k \in \mathbb{Z}_+}$ is not a Markov chain (so it is not an INAR(1) process) if $\alpha$ is not degenerate.

The process $(X_k)_{k \in \mathbb{Z}_+}$ can be called a randomized INAR(1) process with Poisson innovations, and the distribution of $\alpha$ is the so-called mixing distribution of the model. The conditional expectation of $X_0$ given $\alpha$ is $\mathbb{E}(X_0 \,|\, \alpha) = (1-\alpha)^{-1}\lambda$. Here and in the sequel conditional expectations like $\mathbb{E}(X_0 \,|\, \alpha)$ are meant in the generalized sense, see, e.g., in Stroock [51, § 5.1.1]. Then, as the negative part of $X_0$ is $0$, which is integrable, the conditional expectation does exist in this generalized sense.

Let $\alpha^{(j)}$, $j \in \mathbb{N}$, be a sequence of independent copies of the random variable $\alpha$, and let $(X_k^{(j)})_{k \in \mathbb{Z}_+}$, $j \in \mathbb{N}$, be a sequence of independent copies of the process $(X_k)_{k \in \mathbb{Z}_+}$ with idiosyncratic innovations (i.e., the innovations $(\varepsilon_k^{(j)})_{k \in \mathbb{Z}_+}, j \in \mathbb{N}$, belonging to $(X_k^{(j)})_{k \in \mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent) such that $(X_k^{(j)})_{k \in \mathbb{Z}_+}$ conditionally on $\alpha^{(j)}$ is a strictly stationary INAR(1) process with thinning parameter $\alpha^{(j)}$ and with Poisson innovations for all $j \in \mathbb{N}$.

## 3.2 Limit theorems with general mixing distribution

First we consider a simple aggregation procedure. For each $N \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N)} = (\widetilde{S}_k^{(N)})_{k \in \mathbb{Z}_+}$ given by

$$\widetilde{S}_k^{(N)} := \sum_{j=1}^{N} (X_k^{(j)} - \mathbb{E}(X_k^{(j)} \,|\, \alpha^{(j)})) = \sum_{j=1}^{N} \Bigl( X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}} \Bigr), \qquad k \in \mathbb{Z}_+. \tag{3.4}$$

**Proposition 3.1.** *If* $\mathbb{E}\left((1-\alpha)^{-1}\right) < \infty$, *then*

$$N^{-\frac{1}{2}} \widetilde{S}^{(N)} \stackrel{\mathcal{D}_\mathrm{f}}{\longrightarrow} \widetilde{\mathcal{Y}} \qquad \text{as} \quad N \to \infty,$$

*where* $(\widetilde{\mathcal{Y}}_k)_{k \in \mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}\left( X_0 - \frac{\lambda}{1-\alpha}, X_k - \frac{\lambda}{1-\alpha} \right) = \lambda \, \mathbb{E}\Bigl( \frac{\alpha^k}{1-\alpha} \Bigr), \qquad k \in \mathbb{Z}_+. \tag{3.5}$$

**Proposition 3.2.** *We have*

$$\left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{S}_k^{(1)}\right)_{t\in\mathbb{R}_+} = \left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(1)} - \mathbb{E}(X_k^{(1)} \mid \alpha^{(1)}))\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_\mathrm{f}} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B$$

*as* $n\to\infty$, *where* $B=(B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion that is independent of* $\alpha$.

In the next two propositions, which are counterparts of Propositions 2.16 and 2.17, we point out that the usual centralization leads to limit theorems similar to Propositions 3.1 and 3.2, but with occasionally different scaling and with different limit processes. We use again the notation $S^{(N)}=(S_k^{(N)})_{k\in\mathbb{Z}_+}$ given in (2.23) for the simple aggregation (with the usual centralization) of the randomized process.

**Proposition 3.3.** *If* $\mathbb{E}\left((1-\alpha)^{-2}\right)<\infty$, *then*

$$N^{-\frac{1}{2}}S^{(N)} \xrightarrow{\mathcal{D}_\mathrm{f}} \mathcal{Y} \qquad \text{as } N\to\infty,$$

*where* $\mathcal{Y}=(\mathcal{Y}_k)_{k\in\mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\mathcal{Y}_0\mathcal{Y}_k)=\mathrm{Cov}(X_0,X_k)=\lambda\,\mathbb{E}\Big(\frac{\alpha^k}{1-\alpha}\Big)+\lambda^2\,\mathrm{Var}\left(\frac{1}{1-\alpha}\right), \qquad k\in\mathbb{Z}_+.$$

**Proposition 3.4.** *If* $\mathbb{E}\left((1-\alpha)^{-1}\right)<\infty$, *then*

$$\left(n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}S_k^{(1)}\right)_{t\in\mathbb{R}_+} = \left(n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(1)} - \mathbb{E}(X_k^{(1)}))\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_\mathrm{f}} \left(\Big(\frac{\lambda}{1-\alpha}-\mathbb{E}\Big(\frac{\lambda}{1-\alpha}\Big)\Big)t\right)_{t\in\mathbb{R}_+}$$

*as* $n\to\infty$.

In Proposition 3.4 the limit process is simply a line with a random slope.

## 3.3 Iterated limit theorems with specific mixing distribution

First we recall the definition of regularly varying sequences and functions, which we will need in the forthcoming paragraph and during a proof of this chapter. For the definition, see, e.g., Bojanic and Seneta [9, Corollary 1 and Formula (1.4)].

**Definition 3.5.** *A sequence of positive numbers* $a_n$, $n\in\mathbb{N}$, *is regularly varying with some finite index* $A$ *if for every* $b>0$ *we have*

$$\lim_{n\to\infty}\frac{a_{\lfloor bn\rfloor}}{a_n}=b^A.$$

*Similarly,* $f$, *a positive and measurable function on* $[0,\infty)$ *is regularly varying with some finite index* $A$ *if for every* $b>0$ *we have*

$$\lim_{x\to\infty}\frac{f(bx)}{f(x)}=b^A.$$

In the forthcoming results we assume that the distribution of the random variable $\alpha$, i.e., the mixing distribution, has a probability density of the form

$$\psi(x)(1-x)^{\beta}, \qquad x \in (0,1), \tag{3.6}$$

where $\psi$ is a function on $(0,1)$ having a limit $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$. Note that necessarily $\beta \in (-1,\infty)$ (otherwise $\int_0^1 \psi(x)(1-x)^{\beta}\,\mathrm{d}x = \infty$), the function $(0,1) \ni x \mapsto \psi(x)$ is integrable on $(0,1)$, and the function $(0,1) \ni x \mapsto \psi(x)(1-x)^{\beta}$ is regularly varying at the point 1 (i.e., $(0,\infty) \ni x \mapsto \psi(1-\frac{1}{x})\,x^{-\beta}$ is regularly varying at infinity). For the definition of a regularly varying function see Definition 3.5. Further, in case of $\psi(x) = \frac{\Gamma(a+\beta+2)}{\Gamma(a+1)\Gamma(\beta+1)}x^{a}$, $x \in (0,1)$, with some $a \in (-1,\infty)$, the random variable $\alpha$ is Beta distributed with parameters $a+1$ and $\beta+1$. Certain $\circ$ operators, where the summands are random parameter Bernoulli distributions with a parameter having Beta distribution, appear in catastrophe models. Moreover, the Nobel prize winner Clive W. J. Granger used the square root of a Beta distribution as a mixing distribution for random coefficient AR(1) processes, see Granger [18].

**Remark 3.6.** Under the condition (3.6), for each $\ell \in \mathbb{N}$, the expectation $\mathbb{E}\big((1-\alpha)^{-\ell}\big)$ is finite if and only if $\beta > \ell - 1$. Indeed, if $\beta > \ell - 1$, then, by choosing $\varepsilon \in (0,1)$ with $\sup_{a\in(1-\varepsilon,1)}\psi(a) \leqslant 2\psi_1$, we have $\mathbb{E}\big((1-\alpha)^{-\ell}\big) = I_1(\varepsilon) + I_2(\varepsilon)$, where

$$\begin{aligned}
I_1(\varepsilon) &:= \int_0^{1-\varepsilon} \psi(a)(1-a)^{\beta-\ell}\,\mathrm{d}a \leqslant \max\{\varepsilon^{\beta-\ell}, 1\} \int_0^{1-\varepsilon} \psi(a)\,\mathrm{d}a < \infty, \\
I_2(\varepsilon) &:= \int_{1-\varepsilon}^{1} \psi(a)(1-a)^{\beta-\ell}\,\mathrm{d}a \leqslant 2\psi_1 \int_{1-\varepsilon}^{1} (1-a)^{\beta-\ell}\,\mathrm{d}a = \frac{2\psi_1\varepsilon^{\beta-\ell+1}}{\beta-\ell+1} < \infty.
\end{aligned}$$

Conversely, if $\beta \leqslant \ell - 1$, then, by choosing $\varepsilon \in (0,1)$ with $\sup_{a\in(1-\varepsilon,1)}\psi(a) \geqslant \psi_1/2$, we have

$$\mathbb{E}\left((1-\alpha)^{-\ell}\right) \geqslant \int_{1-\varepsilon}^{1} \psi(a)(1-a)^{\beta-\ell}\,\mathrm{d}a \geqslant \frac{\psi_1}{2}\int_{1-\varepsilon}^{1}(1-a)^{\beta-\ell}\,\mathrm{d}a = \infty.$$

This means that in case of $\beta \in (-1,0]$, the processes $S^{(N,n)} = (S_t^{(N,n)})_{t\in\mathbb{R}_+}$, $N,n \in \mathbb{N}$, given in (2.25) are not defined for the randomized INAR(1) process introduced in this section with mixing distribution given in (3.6). Moreover, with this mixing distribution, Propositions 3.1, 3.2, 3.3 and 3.4 are valid in case of $\beta > 0$, $\beta > -1$, $\beta > 1$ and $\beta > 0$, respectively. □

For each $N,n \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N,n)} = (\widetilde{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$\widetilde{S}_t^{(N,n)} := \sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)} \mid \alpha^{(j)})), \qquad t \in \mathbb{R}_+. \tag{3.7}$$

**Remark 3.7.** If $\beta > 0$, then the covariances of the strictly stationary process $(X_k - \mathbb{E}(X_k \mid \alpha))_{k\in\mathbb{Z}_+} = (X_k - \frac{\lambda}{1-\alpha})_{k\in\mathbb{Z}_+}$ exist and take the form

$$\operatorname{Cov}\big(X_0 - \mathbb{E}(X_0 \mid \alpha), X_k - \mathbb{E}(X_k \mid \alpha)\big) = \mathbb{E}\left(\frac{\lambda\alpha^k}{1-\alpha}\right), \qquad k \in \mathbb{Z}_+,$$

see (3.15). Further,

$$\sum_{k=0}^{\infty} \Big| \operatorname{Cov}(X_0 - \mathbb{E}(X_0 \,|\, \alpha), X_k - \mathbb{E}(X_k \,|\, \alpha)) \Big| = \sum_{k=0}^{\infty} \mathbb{E}\left(\frac{\lambda\alpha^k}{1-\alpha}\right) = \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\sum_{k=0}^{\infty}\alpha^k\right)$$
$$= \lambda\,\mathbb{E}\left(\frac{1}{(1-\alpha)^2}\right),$$

which is finite if and only if $\beta > 1$, see Remark 3.6. This means that the strictly stationary process $(X_k - \mathbb{E}(X_k \,|\, \alpha))_{k\in\mathbb{Z}_+}$ has short memory (i.e., it has summable covariances) if $\beta > 1$, and long memory if $\beta \in (0,1]$ (i.e., it has non-summable covariances). □

For $\beta \in (0,2)$, let $(\mathcal{B}_{1-\frac{\beta}{2}}(t))_{t\in\mathbb{R}_+}$ denote a fractional Brownian motion with parameter $1-\beta/2$, that is a Gaussian process with zero mean and covariance function

$$\operatorname{Cov}(\mathcal{B}_{1-\frac{\beta}{2}}(t_1), \mathcal{B}_{1-\frac{\beta}{2}}(t_2)) = \frac{t_1^{2-\beta} + t_2^{2-\beta} - |t_2 - t_1|^{2-\beta}}{2}, \qquad t_1, t_2 \in \mathbb{R}_+. \tag{3.8}$$

The next four results are limit theorems for appropriately scaled versions of $\widetilde{S}^{(N,n)}$, first taking the limit $N \to \infty$ and then $n \to \infty$ in the case $\beta \in (-1,1]$. The first three are counterparts of (2.8), (2.9) and (2.7) of Theorem 2.1 in Pilipauskaitė and Surgailis [38], respectively. The counterpart of the fourth (for $\beta = 1$), however, was not done by Pilipauskaitė and Surgailis, as it could not be handled by the same proof technique as the other ones. It was published in Nedényi and Pap [35, Theorem 3.3] along with the result of our Theorem 3.11.

**Theorem 3.8.** *If* $\beta \in (-1,0)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2(1+\beta)}}\, \widetilde{S}^{(N,n)} = (V_{2(1+\beta)} t)_{t\in\mathbb{R}_+},$$

*where* $V_{2(1+\beta)}$ *is a symmetric* $2(1+\beta)$*-stable random variable (not depending on* $t$*) with characteristic function*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta V_{2(1+\beta)}}) = \mathrm{e}^{-K_\beta|\theta|^{2(1+\beta)}}, \qquad \theta \in \mathbb{R},$$

*where*

$$K_\beta := \psi_1 \left(\frac{\lambda}{2}\right)^{1+\beta} \frac{\Gamma(-\beta)}{1+\beta}.$$

**Theorem 3.9.** *If* $\beta = 0$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1} (N\log N)^{-\frac{1}{2}}\, \widetilde{S}^{(N,n)} = (W_{\lambda\psi_1} t)_{t\in\mathbb{R}_+},$$

*where* $W_{\lambda\psi_1}$ *is a normally distributed random variable with mean zero and with variance* $\lambda\psi_1$.

**Theorem 3.10.** *If* $\beta \in (0,1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1+\frac{\beta}{2}} N^{-\frac{1}{2}}\, \widetilde{S}^{(N,n)} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}}\,\mathcal{B}_{1-\frac{\beta}{2}}.$$

**Theorem 3.11.** *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (n \log n)^{-\frac{1}{2}} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sqrt{2\lambda\psi_1} B,$$

*where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

The next two results are limit theorems for an appropriately scaled version of $\widetilde{S}^{(N,n)}$, first taking the limit $n \to \infty$ and then $N \to \infty$ in the case $\beta \in (-1, 1]$. The first is a counterpart of (2.10) of Theorem 2.1 in Pilipauskaitė and Surgailis [38]. The second one is the counterpart of Nedényi and Pap [35, Theorem 3.4], which is about the respective aggregate of the random coefficient AR(1) processes. Again, this case was not covered in Pilipauskaitė and Surgailis [38] due to requiring a different proof technique.

**Theorem 3.12.** *If* $\beta \in (-1, 1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} N^{-\frac{1}{1+\beta}} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \mathcal{Y}_{1+\beta},$$

*where* $\mathcal{Y}_{1+\beta} = \big(\mathcal{Y}_{1+\beta}(t) := \sqrt{Y_{(1+\beta)/2}}\, B_t\big)_{t\in\mathbb{R}_+}$, *and* $Y_{(1+\beta)/2}$ *is a positive* $\frac{1+\beta}{2}$*-stable random variable with Laplace transform* $\mathbb{E}(\mathrm{e}^{-\theta Y_{(1+\beta)/2}}) = \mathrm{e}^{-k_\beta \theta^{\frac{1+\beta}{2}}}$, $\theta \in \mathbb{R}_+$, *with*

$$k_\beta := \frac{(2\lambda)^{\frac{1+\beta}{2}} \psi_1}{1+\beta} \Gamma\left(\frac{1-\beta}{2}\right),$$

*and* $(B_t)_{t\in\mathbb{R}_+}$ *is an independent standard Wiener process. The process* $\mathcal{Y}_{1+\beta}$ *has* $(1+\beta)$*-stable one-dimensional distributions and stationary increments.*

Let us note that in Barczy et al. [5, Second proof of Theorem 4.10], which is the extended ArXiv version of the paper Barczy et al. [6], by an additional proof, we showed that the characteristic function of $Y_{(1+\beta)/2}$ introduced in Theorem 3.12 is

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Y_{(1+\beta)/2}}) = \exp\left\{-k_\beta |\theta|^{\frac{1+\beta}{2}} \mathrm{e}^{-\mathrm{i}\,\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{4}}\right\}, \qquad \theta \in \mathbb{R},$$

where $k_\beta$ is given in Theorem 3.12.

**Theorem 3.13.** *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} (N \log N)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sqrt{\lambda\psi_1} B,$$

*where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

Next we show an iterated scaling limit theorem where the order of the iteration can be arbitrary in the case $\beta \in (1, \infty)$, which is a counterpart of Theorem 2.3 in Pilipauskaitė and Surgailis [38].

**Theorem 3.14.** *If* $\beta \in (1, \infty)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sigma B,$$

*where* $\sigma^2 := \lambda\, \mathbb{E}((1+\alpha)(1-\alpha)^{-2})$ *and* $(B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

By Remark 3.6, if $\beta > 1$, then $\mathbb{E}\big((1-\alpha)^{-2}\big) < \infty$, and hence $\sigma^2 < \infty$, where $\sigma^2$ is given in Theorem 3.14.

In the following remark we compare our results to those of Pilipauskaitė and Surgailis [38], which paper motivated our work. Moreover, we summarize the proof techniques of our results.

**Remark 3.15.** Theorems 3.8, 3.9 and 3.12 are counterparts of (2.8), (2.9) and (2.10) of Theorem 2.1 in Pilipauskaitė and Surgailis [38]. The proofs of these theorems use the same technique, namely, expansions of characteristic functions, and we provide all the technical details. Theorem 3.10 is a counterpart of (2.7) of Theorem 2.1 in Pilipauskaitė and Surgailis [38]. Our proof of Theorem 3.10 is completely different from the proof of its counterpart as we apply Theorem 4.3 of Beran et al. [7], which is about the convergence of partial sums of a Hermite function of a stationary sequence of standard normal random variables. The proof of Theorem 3.13, which result has no counterpart in Pilipauskaitė and Surgailis [38], uses the well-known theorem about weak convergence of partial sum processes for a triangular array towards a Lévy process due to Resnick [46, Theorem 7.1]. Theorem 3.14 is a counterpart of Theorem 2.3 in Pilipauskaitė and Surgailis [38]. The proofs of the first convergence of Theorem 3.14 and that of Theorem 3.11, which result, again, does not have a counterpart in Pilipauskaitė and Surgailis [38], rely on checking the convergence of the covariances of some Gaussian processes. The proof of the second convergence of Theorem 3.14 is based on the multidimensional central limit theorem. □

In the next theorems we consider the usual centralization with $\mathbb{E}(X_k^{(j)})$, when applicable. These are the counterparts of Theorems 3.10, 3.11, 3.12, 3.13 and 3.14. Recall that, due to Remark 3.6, the expectation $\mathbb{E}(X_0) = \mathbb{E}\big(\frac{\lambda}{1-\alpha}\big)$ is finite if and only if $\beta > 0$, so Theorems 3.8 and 3.9 can not have counterparts in this sense.

**Theorem 3.16.** *If $\beta \in (0,1)$, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1}N^{-\frac{1}{1+\beta}} S^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1}N^{-\frac{1}{1+\beta}} S^{(N,n)} = \big(Z_{1+\beta}\, t\big)_{t\in\mathbb{R}_+},$$

*where $Z_{1+\beta}$ is a $(1+\beta)$-stable random variable with characteristic function $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Z_{1+\beta}}) = \mathrm{e}^{-|\theta|^{1+\beta}\omega_\beta(\theta)}$, $\theta \in \mathbb{R}$, where*

$$\omega_\beta(\theta) := \frac{\psi_1\Gamma(1-\beta)\lambda^{1+\beta}}{-\beta(1+\beta)}\mathrm{e}^{-\mathrm{i}\pi\,\mathrm{sign}(\theta)(1+\beta)/2}, \qquad \theta \in \mathbb{R}.$$

Note that the following theorem was developed for this thesis, it has not appeared in any of our papers.

**Theorem 3.17.** *If $\beta = 1$, then there exists a sequence $(a_N)_{N\in\mathbb{N}}$ such that $\sqrt{N}/a_N = \mathrm{o}(1)$ as $N \to \infty$ (meaning that $\lim_{N\to\infty} \sqrt{N}/a_N = 0$) and*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1}a_N^{-1} S^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1}a_N^{-1} S^{(N,n)} = (W\,t)_{t\in\mathbb{R}_+},$$

*where $W$ is a standard normally distributed random variable.*

**Theorem 3.18.** *If* $\beta \in (1, \infty)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2}} S^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} S^{(N,n)}$$
$$= (W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}\, t)_{t\in\mathbb{R}_+},$$

*where* $W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}$ *is a normally distributed random variable with mean zero and with variance* $\lambda^2 \operatorname{Var}((1-\alpha)^{-1})$.

In case of Theorems 3.8, 3.9, 3.16, 3.17 and 3.18 the limit processes are lines with random slopes. Let us note that the theorems of this section are summarized in some tables in Appendix A.

We point out that the processes of doubly indexed partial sums, $S^{(N,n)}$ and $\widetilde{S}^{(N,n)}$ contain the expected or conditional expected values of the processes $X^{(j)}$, $j \in \mathbb{N}$. Therefore, in a statistical testing, they could not be used directly. So we consider a similar process

$$\widehat{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[ X_k^{(j)} - \frac{\sum_{\ell=1}^{n} X_\ell^{(j)}}{n} \right], \qquad t \in \mathbb{R}_+, \tag{3.9}$$

which does not require the knowledge of the expectation or conditional expectation of the processes $X^{(j)}$, $j \in \mathbb{N}$. Note that the summands in $\widehat{S}_t^{(N,n)}$ have $0$ conditional means with respect to $\alpha$, so we do not need any additional centering. Moreover, $\widehat{S}^{(N,n)}$ is related to the two previously examined processes in the following way: in case of $\beta \in (0, \infty)$ (which ensures the existence of $\mathbb{E}(X_k^{(j)})$, $k \in \mathbb{Z}_+$), we have

$$\widehat{S}_t^{(N,n)} = \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[ X_k^{(j)} - \mathbb{E}(X_k^{(j)}) - \frac{\sum_{\ell=1}^{n} (X_\ell^{(j)} - \mathbb{E}(X_\ell^{(j)}))}{n} \right] = S_t^{(N,n)} - \frac{\lfloor nt \rfloor}{n} S_1^{(N,n)},$$

and in case of $\beta \in (-1, \infty)$,

$$\widehat{S}_t^{(N,n)} = \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[ X_k^{(j)} - \mathbb{E}(X_k^{(j)} \,|\, \alpha^{(j)}) - \frac{\sum_{\ell=1}^{n} (X_\ell^{(j)} - \mathbb{E}(X_\ell^{(j)} \,|\, \alpha^{(j)}))}{n} \right]$$
$$= \widetilde{S}_t^{(N,n)} - \frac{\lfloor nt \rfloor}{n} \widetilde{S}_1^{(N,n)}$$

for every $t \in \mathbb{R}_+$. Therefore, by Theorems 3.10, 3.12, 3.11, 3.13, and 3.14, using Slutsky's lemma, the following limit theorems hold. Note that the two results of the following corollary which hold when $\beta = 1$ have not been published before.

**Corollary 3.19.** *If* $\beta \in (0, 1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1+\frac{\beta}{2}} N^{-\frac{1}{2}}\, \widehat{S}^{(N,n)} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}} \left( \mathcal{B}_{1-\frac{\beta}{2}}(t) - t\mathcal{B}_{1-\frac{\beta}{2}}(1) \right)_{t\in\mathbb{R}_+},$$

*where the process* $\mathcal{B}_{1-\frac{\beta}{2}}$ *is given by* (3.8).

*If* $\beta \in (-1, 1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} N^{-\frac{1}{1+\beta}} n^{-\frac{1}{2}}\, \widehat{S}^{(N,n)} = (\mathcal{Y}_{1+\beta}(t) - t\mathcal{Y}_{1+\beta}(1))_{t\in\mathbb{R}_+},$$

*where the process $\mathcal{Y}_{1+\beta}$ is given in Theorem 3.12.*

*If $\beta = 1$, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (n\log n)^{-\frac{1}{2}} N^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sqrt{2\lambda\psi_1}(B_t - tB_1)_{t\in\mathbb{R}_+},$$

*moreover,*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} (N\log N)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sqrt{\lambda\psi_1}(B_t - tB_1)_{t\in\mathbb{R}_+},$$

*where $B = (B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.*

*If $\beta \in (1, \infty)$, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sigma(B_t - tB_1)_{t\in\mathbb{R}_+},$$

*where $\sigma^2$ is given in Theorem 3.14 and $B = (B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.*

In Corollary 3.19, the limit processes restricted on the time interval $[0, 1]$ are bridges in the sense that they take the same value (namely, 0) at the time points 0 and 1, and especially, in case of $\beta \in (1, \infty)$, it is a Wiener bridge. We note that no counterparts appear for the rest of the theorems because in those cases the limit processes are lines with random slopes, which result the constant zero process in this alternative case. In case of $\beta \in (-1, 0]$, by applying some smaller scaling factors, one could try to achieve a non-degenerate weak limit of $\widehat{S}^{(N,n)}$ by first taking the limit $N \to \infty$ and then that of $n \to \infty$.

Let us point out that Example 2.22, which was discussed in Chapter 2, is also relevant for the aggregation of randomized INAR(1) processes with Poisson innovations. Indeed, in this case, the camps in question can each have independent parameters $\alpha$ coming from a certain distribution.

## 3.4 Some technical results

In this section we gather some technical results that are used in the upcoming proofs. We will frequently use the following well-known inequalities:

$$1 - \mathrm{e}^{-x} \leqslant x, \qquad x \in \mathbb{R}, \tag{3.10}$$

$$|\mathrm{e}^{\mathrm{i}u} - 1| \leqslant |u|, \qquad |\mathrm{e}^{\mathrm{i}u} - 1 - \mathrm{i}u| \leqslant u^2/2, \qquad u \in \mathbb{R}. \tag{3.11}$$

The next lemma is about how the inequalities in (3.11) change if we replace $u \in \mathbb{R}$ with an arbitrary complex number.

**Lemma 3.20.** *We have*

$$|\mathrm{e}^{z} - 1| \leqslant |z|\mathrm{e}^{|z|}, \qquad z \in \mathbb{C}, \tag{3.12}$$

$$|\mathrm{e}^{z} - 1 - z| \leqslant \frac{|z|^2}{2}\mathrm{e}^{|z|}, \qquad z \in \mathbb{C}. \tag{3.13}$$

The next lemma gives a set of sufficient conditions for the convergence of the integral $N\int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) \psi(a)(1-a)^\beta \,\mathrm{d}a$ as $N \to \infty$, where $(z_N(a))_{N\in\mathbb{N}}$ is a sequence of complex numbers. We use this lemma in the proofs of Theorems 3.8, 3.9, and 3.12.

**Lemma 3.21.** *Suppose that* $(0,1) \ni x \mapsto \psi(x)(1-x)^\beta$ *is a probability density, where* $\psi$ *is a function on* $(0,1)$ *having a limit* $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$ *(and necessarily* $\beta \in (-1,\infty)$*). For all* $a \in (0,1)$, *let* $(z_N(a))_{N\in\mathbb{N}}$ *be a sequence of complex numbers such that*

$$\lim_{N\to\infty} \sup_{a\in(0,1-\varepsilon)} |N z_N(a)| = 0 \qquad \text{for all } \varepsilon \in (0,1), \tag{3.14}$$

$$\limsup_{N\to\infty} N \int_{1-\varepsilon_0}^1 \left|1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right| (1-a)^\beta \, \mathrm{d}a < \infty \qquad \text{for some } \varepsilon_0 \in (0,1),$$

$$\lim_{\varepsilon\downarrow 0} \limsup_{N\to\infty} \left| N \int_{1-\varepsilon}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) (1-a)^\beta \, \mathrm{d}a - I \right| = 0$$

*with some* $I \in \mathbb{C}$. *Then*

$$\lim_{N\to\infty} N \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) \psi(a)(1-a)^\beta \, \mathrm{d}a = \psi_1 I.$$

## 3.5 Proofs

**Proof of Proposition 3.1.** We have

$$\mathbb{E}\Big(X_k - \frac{\lambda}{1-\alpha}\Big) = \mathbb{E}\Big[\mathbb{E}\Big(X_k - \frac{\lambda}{1-\alpha} \,\Big|\, \alpha\Big)\Big] = 0, \qquad k \in \mathbb{Z}_+,$$

and hence, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}
&\operatorname{Cov}\Big(X_0 - \frac{\lambda}{1-\alpha}, X_k - \frac{\lambda}{1-\alpha}\Big) = \mathbb{E}\Big[\Big(X_0 - \frac{\lambda}{1-\alpha}\Big)\Big(X_k - \frac{\lambda}{1-\alpha}\Big)\Big] \\
&= \mathbb{E}\Big\{\mathbb{E}\Big[\Big(X_0 - \frac{\lambda}{1-\alpha}\Big)\Big(X_k - \frac{\lambda}{1-\alpha}\Big) \,\Big|\, \alpha\Big]\Big\} = \mathbb{E}\Big(\frac{\lambda\alpha^k}{1-\alpha}\Big),
\end{aligned} \tag{3.15}$$

where we applied (2.24). Now the statement follows from the multidimensional central limit theorem. Due to the continuous mapping theorem, it is sufficient to show the convergence $N^{-1/2}(\widetilde{S}_0^{(N)}, \widetilde{S}_1^{(N)}, \dots, \widetilde{S}_k^{(N)}) \xrightarrow{\mathcal{D}} (\mathcal{Y}_0, \mathcal{Y}_1, \dots, \mathcal{Y}_k)$ as $N \to \infty$ for all $k \in \mathbb{Z}_+$. For all $k \in \mathbb{Z}_+$, the random vectors $\big(X_0^{(j)} - \frac{\lambda}{1-\alpha}, X_1^{(j)} - \frac{\lambda}{1-\alpha}, \dots, X_k^{(j)} - \frac{\lambda}{1-\alpha}\big)$, $j \in \mathbb{N}$, are independent, identically distributed having zero expectation vector and covariances

$$\operatorname{Cov}(X_{\ell_1}^{(j)}, X_{\ell_2}^{(j)}) = \operatorname{Cov}(X_0^{(j)}, X_{|\ell_2-\ell_1|}^{(j)}) = \lambda\, \mathbb{E}\left(\frac{\alpha^{|\ell_2-\ell_1|}}{1-\alpha}\right), \quad j \in \mathbb{N}, \quad \ell_1, \ell_2 \in \{0, 1, \dots, k\},$$

following from the strict stationarity of $X^{(j)}$ and from the form of $\operatorname{Cov}(X_0, X_k)$. $\square$

**Proof of Proposition 3.2.** For each $n \in \mathbb{N}$ and each $t \in \mathbb{R}_+$, put

$$\widetilde{T}_t^{(n)} := n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{S}_k^{(1)}.$$

For each $m \in \mathbb{N}$, each $t_1, \dots, t_m \in \mathbb{R}_+$, and each bounded continuous function $g : \mathbb{R}^m \to \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}(g(\widetilde{T}_{t_1}^{(n)}, \dots, \widetilde{T}_{t_m}^{(n)})) = \int_0^1 \mathbb{E}(g(\widetilde{T}_{t_1}^{(n)}, \dots, \widetilde{T}_{t_m}^{(n)}) \,|\, \alpha = a)\, \mathbb{P}_\alpha(\mathrm{d}a) \\
&= \int_0^1 \mathbb{E}\bigg(g\bigg(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt_1 \rfloor}\Big(X_k - \frac{\lambda}{1-a}\Big), \dots, n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt_m \rfloor}\Big(X_k - \frac{\lambda}{1-a}\Big)\bigg) \,\bigg|\, \alpha = a\bigg)\, \mathbb{P}_\alpha(\mathrm{d}a).
\end{aligned}$$

Proposition 2.17, the portmanteau theorem and the boundedness of $g$ justify the usage of the dominated convergence theorem, and we get

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(g(\widetilde{T}_{t_1}^{(n)}, \dots, \widetilde{T}_{t_m}^{(n)})) = \int_0^1 \mathbb{E}\biggl(g\biggl(\frac{\sqrt{\lambda(1+a)}}{1-a}B_{t_1}, \dots, \frac{\sqrt{\lambda(1+a)}}{1-a}B_{t_m}\biggr)\biggr)\,\mathbb{P}_\alpha(\mathrm{d}a)\\
&= \int_0^1 \mathbb{E}\biggl(g\biggl(\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B_{t_1}, \dots, \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B_{t_m}\biggr)\bigg|\,\alpha = a\biggr)\,\mathbb{P}_\alpha(\mathrm{d}a)\\
&= \mathbb{E}\biggl(g\biggl(\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B_{t_1}, \dots, \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B_{t_m}\biggr)\biggr),
\end{aligned}$$

hence we obtain the statement by the portmanteau theorem. □

**Proof of Proposition 3.3.** For all $k \in \mathbb{Z}_+$, by the strict stationarity of $(X_k)_{k\in\mathbb{Z}_+}$ and (3.15), we have

$$\begin{aligned}
\operatorname{Cov}(X_0, X_k) &= \mathbb{E}\left[\left(X_0 - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)\left(X_k - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)\right]\\
&= \mathbb{E}\left[\left(X_0 - \frac{\lambda}{1-\alpha}\right)\left(X_k - \frac{\lambda}{1-\alpha}\right)\right] + \mathbb{E}\left[\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)^2\right]\\
&= \lambda\,\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right) + \lambda^2 \operatorname{Var}\left(\frac{1}{1-\alpha}\right),
\end{aligned} \tag{3.16}$$

since

$$\begin{aligned}
&\mathbb{E}\left[\left(X_k - \frac{\lambda}{1-\alpha}\right)\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)\right]\\
&= \mathbb{E}\left\{\mathbb{E}\left[\left(X_k - \frac{\lambda}{1-\alpha}\right)\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)\,\bigg|\,\alpha\right]\right\}\\
&= \mathbb{E}\left\{\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)\mathbb{E}\left(X_k - \frac{\lambda}{1-\alpha}\,\bigg|\,\alpha\right)\right\} = 0
\end{aligned}$$

for all $k \in \mathbb{Z}_+$.

The statement follows from the multidimensional central limit theorem as in the proof of Proposition 3.1. Indeed, for all $k \in \mathbb{Z}_+$, the random vectors

$$\left(X_0^{(j)} - \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right), X_1^{(j)} - \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right), \dots, X_k^{(j)} - \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right)\right), \quad j \in \mathbb{N},$$

are independent, identically distributed having zero expectation vector and covariances

$$\operatorname{Cov}(X_{\ell_1}^{(j)}, X_{\ell_2}^{(j)}) = \operatorname{Cov}(X_0^{(j)}, X_{|\ell_2-\ell_1|}^{(j)}) = \lambda\,\mathbb{E}\left(\frac{\alpha^{|\ell_2-\ell_1|}}{1-\alpha}\right) + \lambda^2 \operatorname{Var}\left(\frac{1}{1-\alpha}\right)$$

for $j \in \mathbb{N}$ and $\ell_1, \ell_2 \in \{0, 1, \dots, k\}$, following from the strict stationarity of $X^{(j)}$ and from the form of $\operatorname{Cov}(X_0, X_k)$ given in (3.16). □

**Proof of Proposition 3.4.** We have a decomposition $S_k^{(1)} = \widetilde{S}_k^{(1)} + R_k^{(1)}$, $k \in \mathbb{Z}_+$, with

$$R_k^{(1)} := \mathbb{E}(X_k^{(1)} \mid \alpha^{(1)}) - \mathbb{E}(X_k^{(1)}) = \frac{\lambda}{1-\alpha^{(1)}} - \mathbb{E}\biggl(\frac{\lambda}{1-\alpha^{(1)}}\biggr), \qquad k \in \mathbb{Z}_+.$$

We have

$$\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} R_k^{(1)}\right)_{t\in\mathbb{R}_+} = \left(\frac{\lfloor nt\rfloor}{n}\left(\frac{\lambda}{1-\alpha^{(1)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(1)}}\right)\right)\right)_{t\in\mathbb{R}_+}$$
$$\xrightarrow{\mathcal{D}_\mathrm{f}} \left(\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)t\right)_{t\in\mathbb{R}_+}$$

as $n\to\infty$. Moreover, by Proposition 3.2, $\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty}\big(n^{-1/2}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{S}_k^{(1)}\big)_{t\in\mathbb{R}_+}$ exists, hence

$$\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{S}_k^{(1)}\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_\mathrm{f}} 0 \qquad \text{as } n\to\infty,$$

implying that for all $m\in\mathbb{N}$ and all $t_1,\ldots,t_m\in\mathbb{R}_+$, we have

$$\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt_1\rfloor}\widetilde{S}_k^{(1)},\ldots,\frac{1}{n}\sum_{k=1}^{\lfloor nt_m\rfloor}\widetilde{S}_k^{(1)}\right) \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty.$$

By Slutsky's lemma we conclude the statement. □

In the following two proofs, the notations $\mathrm{O}(1)$ and $|\,\mathrm{O}(1)|$ stand for a possibly complex and respectively real sequence $(a_k)_{k\in\mathbb{N}}$ that is bounded and can only depend on the parameters $\lambda$, $\psi_1$, $\beta$, and on some fixed $m\in\mathbb{N}$ and $\theta_1,\ldots,\theta_m\in\mathbb{R}$. Further, we call the attention that several $\mathrm{O}(1)$-s (respectively $|\,\mathrm{O}(1)|$-s) in the same formula do not necessarily mean the same bounded sequence.

In the forthcoming proof (and in Chapter 4) we are going to need some conditional generator functions. The conditional generator function of $X_0$ given $\alpha\in(0,1)$ has the form

$$F_0(z_0\,|\,\alpha) := \mathbb{E}(z_0^{X_0}\,|\,\alpha) = \mathrm{e}^{(1-\alpha)^{-1}\lambda(z_0-1)} \tag{3.17}$$

for $z_0\in D := \{z\in\mathbb{C} : |z|\leqslant 1\}$. The joint conditional generator function of $X_0, X_1,\ldots,X_k$ given $\alpha$ will be denoted by $F_{0,\ldots,k}(z_0,\ldots,z_k\,|\,\alpha)$, $z_0,\ldots,z_k\in D$.

**Proof of Theorem 3.8.** To prove this limit theorem it is enough to show that for any $n\in\mathbb{N}$,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2(1+\beta)}}\,\widetilde{S}^{(N,n)} = (\lfloor nt\rfloor V_{2(1+\beta)})_{t\in\mathbb{R}_+}.$$

For this, by the continuous mapping theorem, it is enough to verify that for any $m\in\mathbb{N}$,

$$N^{-\frac{1}{2(1+\beta)}}\sum_{j=1}^{N}\left(X_1^{(j)} - \frac{\lambda}{1-\alpha^{(j)}},\ldots,X_m^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right) \xrightarrow{\mathcal{D}} V_{2(1+\beta)}(1,\ldots,1)$$

as $N\to\infty$. So, by the continuity theorem, we have to check that for any $m\in\mathbb{N}$ and $\theta_1,\ldots,\theta_m\in\mathbb{R}$ the convergence

$$\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{m}\theta_k\left(N^{-\frac{1}{2(1+\beta)}}\sum_{j=1}^{N}\left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right)\right)\right\}\right)$$
$$= \mathbb{E}\left(\exp\left\{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\sum_{j=1}^{N}\sum_{k=1}^{m}\theta_k\left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right)\right\}\right)$$

$$= \left[\mathbb{E}\left(\exp\left\{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\sum_{k=1}^{m}\theta_k\left(X_k - \frac{\lambda}{1-\alpha}\right)\right\}\right)\right]^N$$

$$\to \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\sum_{k=1}^{m}\theta_k V_{2(1+\beta)}}\right) = \mathrm{e}^{-K_\beta|\sum_{k=1}^{m}\theta_k|^{2(1+\beta)}} \qquad \text{as} \quad N \to \infty$$

holds. Note that it suffices to show

$$\Theta_N := N\left[1 - \mathbb{E}\left(\exp\left\{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\sum_{k=1}^{m}\theta_k\left(X_k - \frac{\lambda}{1-\alpha}\right)\right\}\right)\right] \to K_\beta\left|\sum_{k=1}^{m}\theta_k\right|^{2(1+\beta)}$$

as $N \to \infty$, since it implies that $(1 - \Theta_N/N)^N \to \mathrm{e}^{-K_\beta|\sum_{k=1}^{m}\theta_k|^{2(1+\beta)}}$ as $N \to \infty$. By applying (2.21) to the left hand side, we get

$$\Theta_N = N\,\mathbb{E}\Big[1 - F_{0,\dots,m-1}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_m}\,\Big|\,\alpha\Big)\mathrm{e}^{-\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\frac{\lambda}{1-\alpha}\sum_{k=1}^{m}\theta_k}\Big]$$

$$= N\,\mathbb{E}\left[1 - \mathrm{e}^{\frac{\lambda}{1-\alpha}A_N(\alpha)}\right] = N\int_0^1\left(1 - \mathrm{e}^{\frac{\lambda}{1-a}A_N(a)}\right)\psi(a)(1-a)^\beta\,\mathrm{d}a,$$

where $F_{0,\dots,m-1}(z_0,\dots,z_{m-1}\,|\,\alpha) := \mathbb{E}(z_0^{X_0}z_1^{X_1}\cdots z_{m-1}^{X_{m-1}}\,|\,\alpha)$, $z_0,\dots,z_{m-1} \in D$, and

$$A_N(a) := -\frac{\mathrm{i}(\theta_1 + \cdots + \theta_m)}{N^{\frac{1}{2(1+\beta)}}}$$
$$+ \sum_{1\leqslant \ell\leqslant j\leqslant m} a^{j-\ell}\big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1\big)\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}(\theta_{\ell+1}+\cdots+\theta_{j-1})}\big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_j} - 1\big)$$

for $a \in [0,1]$. Let us show that for any $\varepsilon \in (0,1)$ we have $\sup_{a\in(0,1-\varepsilon)}|NA_N(a)| \to 0$ as $N \to \infty$. Using (3.11), for any $\varepsilon \in (0,1)$ we get

$$\sup_{a\in(0,1-\varepsilon)} N|A_N(a)| = \sup_{a\in(0,1-\varepsilon)} N\Bigg|\sum_{k=1}^{m}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k} - 1 - \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k\Big)$$
$$+ \sum_{1\leqslant\ell<j\leqslant m} a^{j-\ell}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1\Big)\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}(\theta_{\ell+1}+\cdots+\theta_{j-1})}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_j} - 1\Big)\Bigg|$$
$$\leqslant N\left(\sum_{k=1}^{m} N^{-\frac{1}{1+\beta}}\frac{\theta_k^2}{2} + \sum_{1\leqslant\ell<j\leqslant m} N^{-\frac{1}{1+\beta}}|\theta_\ell||\theta_j|\right) = N^{\frac{\beta}{1+\beta}}\frac{\left(\sum_{k=1}^{m}|\theta_k|\right)^2}{2} \to 0 \tag{3.18}$$

as $N \to \infty$, since $\beta/(1+\beta) < 0$. Therefore, by Lemma 3.21, substituting $a = 1 - z^{-1}N^{-\frac{1}{1+\beta}}$, the statement of the theorem will follow from

$$\begin{aligned}
&\limsup_{N\to\infty} N\int_{1-\varepsilon}^{1}\left|1 - \mathrm{e}^{\frac{\lambda}{1-a}A_N(a)}\right|(1-a)^\beta\,\mathrm{d}a \\
&= \limsup_{N\to\infty}\int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^{\infty}\left|1 - \mathrm{e}^{\lambda zN^{\frac{1}{1+\beta}}A_N\left(1-z^{-1}N^{-\frac{1}{1+\beta}}\right)}\right|z^{-(2+\beta)}\,\mathrm{d}z < \infty
\end{aligned} \tag{3.19}$$

for all $\varepsilon \in (0,1)$ and

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\limsup_{N\to\infty}\left|N\int_{1-\varepsilon}^{1}\left(1 - \mathrm{e}^{\frac{\lambda}{1-a}A_N(a)}\right)(1-a)^\beta\,\mathrm{d}a - I\right| \\
&= \lim_{\varepsilon\downarrow 0}\limsup_{N\to\infty}\left|\int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^{\infty}\left(1 - \mathrm{e}^{\lambda zN^{\frac{1}{1+\beta}}A_N\left(1-z^{-1}N^{-\frac{1}{1+\beta}}\right)}\right)z^{-(2+\beta)}\,\mathrm{d}z - I\right| = 0
\end{aligned} \tag{3.20}$$

with

$$I := \int_0^\infty \left(1 - \mathrm{e}^{-\frac{\lambda z}{2}\left(\sum_{k=1}^m \theta_k\right)^2}\right) z^{-(2+\beta)}\,\mathrm{d}z$$
$$= \left(\frac{\lambda}{2}\left|\sum_{k=1}^m \theta_k\right|^2\right)^{1+\beta} \int_0^\infty (1-\mathrm{e}^{-z}) z^{-(2+\beta)}\,\mathrm{d}z = \psi_1^{-1} K_\beta \left|\sum_{k=1}^m \theta_k\right|^{2(1+\beta)},$$

where the last equality is justified by Li [31, formula (1.28)]. Next we check (3.19) and (3.20).

By Taylor expansion,

$$\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1 = \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell + N^{-\frac{1}{1+\beta}}\,\mathrm{O}(1) = N^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1),$$
$$\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1 - \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell = -N^{-\frac{1}{1+\beta}}\frac{\theta_\ell^2}{2} + N^{-\frac{3}{2(1+\beta)}}\,\mathrm{O}(1)$$

for all $\ell \in \{1, \ldots, m\}$, resulting

$$\lambda z N^{\frac{1}{1+\beta}} A_N\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right) = -\frac{\lambda z\left(\sum_{k=1}^m \theta_k\right)^2}{2} + \frac{z\,\mathrm{O}(1)}{N^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}} \tag{3.21}$$

for $z > N^{-\frac{1}{1+\beta}}$. Indeed, for $z > N^{-\frac{1}{1+\beta}}$, we have

$$\begin{aligned}
&A_N\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right) \\
&= \sum_{k=1}^m \left(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k} - 1 - \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k\right) \\
&\quad + \sum_{1\leqslant \ell<j\leqslant m} \left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)^{j-\ell} \left(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1\right) \mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}(\theta_{\ell+1}+\cdots+\theta_{j-1})} \\
&\qquad\qquad \times \left(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_j} - 1\right) \\
&= \sum_{k=1}^m \left(-\frac{\theta_k^2}{2N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}}\right) \\
&\quad + \sum_{1\leqslant \ell<j\leqslant m} \left(1 + \frac{\mathrm{O}(1)}{zN^{\frac{1}{1+\beta}}}\right)\left(\frac{\mathrm{i}\theta_\ell}{N^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}}\right)\left(1 + \frac{\mathrm{O}(1)}{N^{\frac{1}{2(1+\beta)}}}\right)\left(\frac{\mathrm{i}\theta_j}{N^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}}\right) \\
&= -\frac{\sum_{k=1}^m \theta_k^2}{2N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}} - \frac{\sum_{1\leqslant\ell<j\leqslant m}\theta_\ell\theta_j}{N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{zN^{\frac{2}{1+\beta}}} \\
&= -\frac{\left(\sum_{k=1}^m \theta_k\right)^2}{2N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{zN^{\frac{2}{1+\beta}}},
\end{aligned}$$

since by Bernoulli's inequality

$$\left|\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)^{j-\ell} - 1\right| \leqslant \frac{j-\ell}{zN^{\frac{1}{1+\beta}}} \leqslant \frac{m}{zN^{\frac{1}{1+\beta}}},$$

yielding that

$$\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)^{j-\ell} = 1 + \frac{\mathrm{O}(1)}{zN^{\frac{1}{1+\beta}}}.$$

By (3.21), for $z \in [1, \infty)$ and for large enough $N$ we have

$$\lambda z N^{\frac{1}{1+\beta}} \operatorname{Re} A_N\big(1 - z^{-1} N^{-\frac{1}{1+\beta}}\big) = -\frac{\lambda z (\sum_{k=1}^m \theta_k)^2}{2} \left(1 - \frac{\operatorname{Re} \mathrm{O}(1)}{N^{\frac{1}{2(1+\beta)}}}\right) + \frac{\operatorname{Re} \mathrm{O}(1)}{N^{\frac{1}{1+\beta}}}$$
$$\leqslant -\frac{\lambda z (\sum_{k=1}^m \theta_k)^2}{4} + \frac{|\,\mathrm{O}(1)|}{N^{\frac{1}{1+\beta}}} \leqslant 0,$$

hence we obtain

$$\begin{aligned} &\int_1^\infty \left|1 - \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} A_N(1 - z^{-1} N^{-\frac{1}{1+\beta}})}\right| z^{-(\beta+2)} \,\mathrm{d}z \\ &\leqslant \int_1^\infty \left(1 + \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} \operatorname{Re} A_N(1 - z^{-1} N^{-\frac{1}{1+\beta}})}\right) z^{-(\beta+2)} \,\mathrm{d}z \leqslant 2 \int_1^\infty z^{-(\beta+2)} \,\mathrm{d}z < \infty. \end{aligned} \tag{3.22}$$

Again by (3.21), for $\varepsilon \in (0, 1)$, $z \in \big(\varepsilon^{-1} N^{-\frac{1}{1+\beta}}, 1\big]$ and for large enough $N$, we have

$$\left|\lambda z N^{\frac{1}{1+\beta}} A_N\Big(1 - z^{-1} N^{-\frac{1}{1+\beta}}\Big)\right| \leqslant \frac{\lambda z (\sum_{k=1}^m \theta_k)^2}{2} + \frac{z|\,\mathrm{O}(1)|}{N^{\frac{1}{2(1+\beta)}}} + \frac{|\,\mathrm{O}(1)|}{N^{\frac{1}{1+\beta}}}$$
$$\leqslant z\left(\frac{\lambda (\sum_{k=1}^m \theta_k)^2}{2} + \frac{|\,\mathrm{O}(1)|}{N^{\frac{1}{2(1+\beta)}}} + \varepsilon|\,\mathrm{O}(1)|\right) \leqslant z|\,\mathrm{O}(1)| \leqslant |\,\mathrm{O}(1)|,$$

since $N^{-\frac{1}{1+\beta}} < z\varepsilon$. Hence, using (3.12), we obtain

$$\begin{aligned} &\int_{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}}^1 \left|1 - \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} A_N\left(1 - z^{-1} N^{-\frac{1}{1+\beta}}\right)}\right| z^{-(2+\beta)} \,\mathrm{d}z \\ &\leqslant \int_{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}}^1 \left|\lambda z N^{\frac{1}{1+\beta}} A_N\big(1 - z^{-1} N^{-\frac{1}{1+\beta}}\big)\right| \mathrm{e}^{\left|\lambda z N^{\frac{1}{1+\beta}} A_N\left(1 - z^{-1} N^{-\frac{1}{1+\beta}}\right)\right|} z^{-(2+\beta)} \,\mathrm{d}z \\ &\leqslant |\,\mathrm{O}(1)| \mathrm{e}^{|\,\mathrm{O}(1)|} \int_0^1 z^{-(1+\beta)} \,\mathrm{d}z < \infty, \end{aligned}$$

which, together with (3.22), imply (3.19).

Now we turn to prove (3.20). By (3.10), we have

$$\left|\int_0^{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-\frac{\lambda z}{2} (\sum_{k=1}^m \theta_k)^2}\Big) z^{-(2+\beta)} \,\mathrm{d}z\right| \leqslant \int_0^{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}} \frac{\lambda z (\sum_{k=1}^m \theta_k)^2}{2} z^{-(2+\beta)} \,\mathrm{d}z$$
$$= \frac{\lambda (\sum_{k=1}^m \theta_k)^2}{2} \int_0^{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}} z^{-(1+\beta)} \,\mathrm{d}z = \frac{\lambda (\sum_{k=1}^m \theta_k)^2}{2(-\beta)} \left(\frac{1}{\varepsilon N^{\frac{1}{1+\beta}}}\right)^{-\beta} \to 0$$

as $N \to \infty$, hence (3.20) reduces to check that $\lim_{\varepsilon \downarrow 0} \limsup_{N \to \infty} I_{N,\varepsilon} = 0$, where

$$I_{N,\varepsilon} := \int_{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}}^\infty \left[\mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} A_N(1 - z^{-1} N^{-\frac{1}{1+\beta}})} - \mathrm{e}^{-\frac{\lambda z}{2} (\sum_{k=1}^m \theta_k)^2}\right] z^{-(2+\beta)} \,\mathrm{d}z.$$

Applying again (3.21), we obtain

$$|I_{N,\varepsilon}| \leqslant \int_{\varepsilon^{-1} N^{-\frac{1}{1+\beta}}}^\infty \mathrm{e}^{-\frac{\lambda z}{2} (\sum_{k=1}^m \theta_k)^2} \left|\mathrm{e}^{z N^{-\frac{1}{2(1+\beta)}} \mathrm{O}(1) + N^{-\frac{1}{1+\beta}} \mathrm{O}(1)} - 1\right| z^{-(2+\beta)} \,\mathrm{d}z.$$

Here, for $\varepsilon \in (0, 1)$ and $z \in (\varepsilon^{-1} N^{-\frac{1}{1+\beta}}, \infty)$, we have

$$\left|z N^{-\frac{1}{2(1+\beta)}} \mathrm{O}(1) + N^{-\frac{1}{1+\beta}} \mathrm{O}(1)\right| \leqslant z\big(N^{-\frac{1}{2(1+\beta)}} + \varepsilon\big)|\,\mathrm{O}(1)|,$$

and hence, by (3.12), we get

$$\begin{aligned}
&\left|\mathrm{e}^{zN^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1)+N^{-\frac{1}{1+\beta}}\,\mathrm{O}(1)}-1\right| \\
&\qquad\leqslant \left|zN^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1)+N^{-\frac{1}{1+\beta}}\,\mathrm{O}(1)\right|\mathrm{e}^{\left|zN^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1)+N^{-\frac{1}{1+\beta}}\,\mathrm{O}(1)\right|} \\
&\qquad\leqslant z\big(N^{-\frac{1}{2(1+\beta)}}+\varepsilon\big)|\,\mathrm{O}(1)|\,\mathrm{e}^{z\left(N^{-\frac{1}{2(1+\beta)}}+\varepsilon\right)|\,\mathrm{O}(1)|}.
\end{aligned}$$

Consequently, for large enough $N$ and small enough $\varepsilon\in(0,1)$,

$$\begin{aligned}
|I_{N,\varepsilon}| &\leqslant \big(N^{-\frac{1}{2(1+\beta)}}+\varepsilon\big)|\,\mathrm{O}(1)|\int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^{\infty}\mathrm{e}^{-\frac{\lambda z}{2}(\sum_{k=1}^m\theta_k)^2+z\left(N^{-\frac{1}{2(1+\beta)}}+\varepsilon\right)|\,\mathrm{O}(1)|}z^{-(1+\beta)}\,\mathrm{d}z \\
&\leqslant \big(N^{-\frac{1}{2(1+\beta)}}+\varepsilon\big)|\,\mathrm{O}(1)|\int_0^{\infty}\mathrm{e}^{-\frac{\lambda z}{4}(\sum_{k=1}^m\theta_k)^2}z^{-(1+\beta)}\,\mathrm{d}z,
\end{aligned}$$

that gets arbitrarily close to zero as $N$ approaches infinity and $\varepsilon$ tends to $0$, since the integral is finite due to the fact that

$$\frac{1}{\Gamma(-\beta)}\left(\frac{\lambda}{4}\left(\sum_{k=1}^m\theta_k\right)^2\right)^{-\beta}\mathrm{e}^{-\lambda z(\sum_{k=1}^m\theta_k)^2/4}\,z^{-(1+\beta)},\qquad z>0,$$

is the density function of a Gamma distributed random variable with parameters $-\beta$ and $\lambda(\sum_{k=1}^m\theta_k)^2/4$. This yields (3.20) completing the proof. □

**Proof of Theorem 3.9.** Similarly as in the proof of Theorem 3.8, it suffices to show that for any $m\in\mathbb{N}$ and $\theta_1,\ldots,\theta_m\in\mathbb{R}$ we have the convergence

$$N\left[1-\mathbb{E}\left(\exp\left\{\frac{\mathrm{i}}{\sqrt{N\log N}}\sum_{k=1}^m\theta_k\left(X_k-\frac{\lambda}{1-\alpha}\right)\right\}\right)\right]\to\frac{\lambda\psi_1}{2}\left(\sum_{k=1}^m\theta_k\right)^2$$

as $N\to\infty$. By applying (2.21), the left hand side equals

$$\begin{aligned}
&N\,\mathbb{E}\left[1-F_{0,\ldots,m-1}\left(\mathrm{e}^{\frac{\mathrm{i}\theta_1}{\sqrt{N\log N}}},\ldots,\mathrm{e}^{\frac{\mathrm{i}\theta_m}{\sqrt{N\log N}}}\;\Big|\;\alpha\right)\mathrm{e}^{-\frac{\mathrm{i}\lambda(\theta_1+\cdots+\theta_m)}{(1-\alpha)\sqrt{N\log N}}}\right] \\
&=N\,\mathbb{E}\left[1-\mathrm{e}^{\frac{\lambda}{1-\alpha}B_N(\alpha)}\right]=N\int_0^1\left(1-\mathrm{e}^{\frac{\lambda}{1-a}B_N(a)}\right)\psi(a)\,\mathrm{d}a
\end{aligned}$$

with

$$\begin{aligned}
B_N(a):=&\sum_{k=1}^m\left(\mathrm{e}^{\frac{\mathrm{i}\theta_k}{\sqrt{N\log N}}}-1-\frac{\mathrm{i}\theta_k}{\sqrt{N\log N}}\right) \\
&+\sum_{1\leqslant\ell<j\leqslant m}a^{j-\ell}\big(\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{\sqrt{N\log N}}}-1\big)\mathrm{e}^{\frac{\mathrm{i}(\theta_{\ell+1}+\cdots+\theta_{j-1})}{\sqrt{N\log N}}}\big(\mathrm{e}^{\frac{\mathrm{i}\theta_j}{\sqrt{N\log N}}}-1\big),\qquad a\in[0,1].
\end{aligned}$$

Similarly as in (3.18), for any $\varepsilon\in(0,1)$ we have

$$\sup_{a\in(0,1-\varepsilon)}|NB_N(a)|\leqslant\frac{(\sum_{k=1}^m\theta_k)^2}{2\log N}\to0$$

as $N \to \infty$. Therefore, by Lemma 3.21, substituting $a = 1 - z/N$, the statement of the theorem will follow from

$$\limsup_{N\to\infty} N \int_{1-\varepsilon}^{1} \left|1 - \mathrm{e}^{\frac{\lambda}{1-a} B_N(a)}\right| \mathrm{d}a = \limsup_{N\to\infty} \int_0^{\varepsilon N} \left|1 - \mathrm{e}^{\frac{\lambda N}{z} B_N(1-\frac{z}{N})}\right| \mathrm{d}z < \infty, \tag{3.23}$$

and

$$\lim_{N\to\infty} N \int_{1-\varepsilon}^{1} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} B_N(a)}\right) \mathrm{d}a = \lim_{N\to\infty} \int_0^{\varepsilon N} \left(1 - \mathrm{e}^{\frac{\lambda N}{z} B_N(1-\frac{z}{N})}\right) \mathrm{d}z = \frac{\lambda}{2} \left(\sum_{k=1}^{m} \theta_k\right)^2 \tag{3.24}$$

for all $\varepsilon \in (0, 1)$. Next we check (3.23) and (3.24).

Using Taylor expansions, similarly as in the proof of Theorem 3.8, we get

$$\frac{\lambda N}{z} B_N\Big(1 - \frac{z}{N}\Big) = -\frac{\lambda(\sum_{k=1}^m \theta_k)^2}{2z \log N} + \frac{\mathrm{O}(1)}{zN^{1/2}(\log N)^{3/2}} + \frac{\mathrm{O}(1)}{N \log N}. \tag{3.25}$$

Indeed, for $z \in [0, N]$ we have

$$\begin{aligned}
B_N\Big(1 - \frac{z}{N}\Big) &= \sum_{k=1}^{m} \left(\mathrm{e}^{\frac{\mathrm{i}\theta_k}{\sqrt{N \log N}}} - 1 - \frac{\mathrm{i}\theta_k}{\sqrt{N \log N}}\right) \\
&\quad + \sum_{1 \leqslant \ell < j \leqslant m} \Big(1 - \frac{z}{N}\Big)^{j-\ell} \left(\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{\sqrt{N \log N}}} - 1\right) \mathrm{e}^{\frac{\mathrm{i}(\theta_{\ell+1}+\cdots+\theta_{j-1})}{\sqrt{N \log N}}} \left(\mathrm{e}^{\frac{\mathrm{i}\theta_j}{\sqrt{N \log N}}} - 1\right) \\
&= \sum_{k=1}^{m} \left(-\frac{\theta_k^2}{2N \log N} + \frac{\mathrm{O}(1)}{(N \log N)^{3/2}}\right) \\
&\quad + \sum_{1 \leqslant \ell < j \leqslant m} \left(1 + \frac{z\,\mathrm{O}(1)}{N}\right) \left(\frac{\mathrm{i}\theta_\ell}{\sqrt{N \log N}} + \frac{\mathrm{O}(1)}{N \log N}\right) \\
&\qquad\qquad \times \left(1 + \frac{\mathrm{O}(1)}{\sqrt{N \log N}}\right) \left(\frac{\mathrm{i}\theta_j}{\sqrt{N \log N}} + \frac{\mathrm{O}(1)}{N \log N}\right) \\
&= -\frac{\sum_{k=1}^m \theta_k^2}{2N \log N} + \frac{\mathrm{O}(1)}{(N \log N)^{3/2}} - \frac{\sum_{1 \leqslant \ell < j \leqslant m} \theta_\ell \theta_j}{N \log N} + \frac{\mathrm{O}(1)}{(N \log N)^{3/2}} + \frac{z\,\mathrm{O}(1)}{N^2 \log N} \\
&= -\frac{(\sum_{k=1}^m \theta_k)^2}{2N \log N} + \frac{\mathrm{O}(1)}{(N \log N)^{3/2}} + \frac{z\,\mathrm{O}(1)}{N^2 \log N},
\end{aligned}$$

since, by Bernoulli's inequality,

$$\left|\Big(1 - \frac{z}{N}\Big)^{j-\ell} - 1\right| \leqslant (j - \ell)\frac{z}{N} \leqslant m\frac{z}{N},$$

yielding that

$$\Big(1 - \frac{z}{N}\Big)^{j-\ell} = 1 + \frac{z}{N}\,\mathrm{O}(1).$$

By (3.25), for $z \in \big(0, (\log N)^{-1}\big)$ and for large enough $N$ we have

$$\begin{aligned}
\frac{\lambda N}{z} \operatorname{Re} B_N\Big(1 - \frac{z}{N}\Big) &= -\frac{\lambda(\sum_{k=1}^m \theta_k)^2}{2z \log N} \left(1 - \frac{\operatorname{Re}\mathrm{O}(1)}{\sqrt{N \log N}}\right) + \frac{\operatorname{Re}\mathrm{O}(1)}{N \log N} \\
&\leqslant -\frac{\lambda(\sum_{k=1}^m \theta_k)^2}{4z \log N} + \frac{|\,\mathrm{O}(1)|}{N \log N} \leqslant -\frac{\lambda(\sum_{k=1}^m \theta_k)^2}{4} + \frac{|\,\mathrm{O}(1)|}{N \log N},
\end{aligned}$$

hence we obtain

$$\begin{aligned}&\int_0^{\frac{1}{\log N}}\left|1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right|\mathrm{d}z\leqslant\int_0^{\frac{1}{\log N}}\left(1+\mathrm{e}^{\frac{\lambda N}{z}\operatorname{Re}B_N(1-\frac{z}{N})}\right)\mathrm{d}z\\&\leqslant\frac{1}{\log N}\left(1+\exp\left\{-\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{4}+\frac{|\operatorname{O}(1)|}{N\log N}\right\}\right)\to 0\qquad\text{as}\quad N\to\infty.\end{aligned}\tag{3.26}$$

Note that for every $\varepsilon\in(0,1)$

$$\frac{1}{\log N}\int_{\frac{1}{\log N}}^{\varepsilon N}\frac{1}{z}\,\mathrm{d}z=\frac{\log\varepsilon+\log N+\log\log N}{\log N}\to 1\qquad\text{as}\quad N\to\infty,\tag{3.27}$$

$$\frac{1}{\log N}\int_{\frac{1}{\log N}}^{\varepsilon N}\frac{1}{z^2}\,\mathrm{d}z=\frac{\varepsilon N\log N-1}{\varepsilon N\log N}\to 1\qquad\text{as}\quad N\to\infty.\tag{3.28}$$

By (3.25), for all $z\in\big((\log N)^{-1},\varepsilon N\big)$, we have

$$\left|\frac{\lambda N}{z}B_N\left(1-\frac{z}{N}\right)\right|\leqslant\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{2z\log N}+\frac{|\operatorname{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\operatorname{O}(1)|}{N\log N}=|\operatorname{O}(1)|.$$

Thus, by (3.12), (3.25) and (3.27), we get

$$\begin{aligned}&\limsup_{N\to\infty}\int_{\frac{1}{\log N}}^{\varepsilon N}\left|1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right|\mathrm{d}z\\&\leqslant\limsup_{N\to\infty}\int_{\frac{1}{\log N}}^{\varepsilon N}\left|\frac{\lambda N}{z}B_N(1-\frac{z}{N})\right|\mathrm{e}^{\left|\frac{\lambda N}{z}B_N(1-\frac{z}{N})\right|}\,\mathrm{d}z\\&\leqslant\limsup_{N\to\infty}\mathrm{e}^{|\operatorname{O}(1)|}\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{2z\log N}+\frac{|\operatorname{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\operatorname{O}(1)|}{N\log N}\right]\mathrm{d}z<\infty,\end{aligned}$$

which, together with (3.26), imply (3.23).

Now we turn to prove (3.24). By (3.26), the convergence (3.24) reduces to prove that

$$\left|\int_{\frac{1}{\log N}}^{\varepsilon N}\left(1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right)\mathrm{d}z-\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2}\right|\to 0\qquad\text{as}\quad N\to\infty.$$

Using (3.27), it is enough to check that

$$\left|\int_{\frac{1}{\log N}}^{\varepsilon N}\left(\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}-1+\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}\right)\mathrm{d}z\right|\to 0\qquad\text{as}\quad N\to\infty.$$

By applying (3.13), (3.25) and (3.5), for large enough $N$ we get

$$\begin{aligned}&\left|\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\left(\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}-1\right)+\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}\right]\mathrm{d}z\right|\\&\leqslant\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{1}{2}\left|\frac{\lambda N}{z}B_N\Big(1-\frac{z}{N}\Big)\right|^2\mathrm{e}^{\left|\frac{\lambda N}{z}B_N(1-\frac{z}{N})\right|}+\left|\frac{\lambda N}{z}B_N\Big(1-\frac{z}{N}\Big)+\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}\right|\right]\mathrm{d}z\\&\leqslant\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{1}{2}\left(\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}+\frac{|\operatorname{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\operatorname{O}(1)|}{N\log N}\right)^2\mathrm{e}^{|\operatorname{O}(1)|}\right.\\&\qquad\left.+\frac{|\operatorname{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\operatorname{O}(1)|}{N\log N}\right]\mathrm{d}z\end{aligned}$$

$$\leqslant \int_{\frac{1}{\log N}}^{\varepsilon N} \Bigg[\frac{3}{2}\left(\frac{|\operatorname{O}(1)|}{z^2(\log N)^2} + \frac{|\operatorname{O}(1)|}{z^2 N(\log N)^3} + \frac{|\operatorname{O}(1)|}{N^2(\log N)^2}\right) + \frac{|\operatorname{O}(1)|}{zN^{1/2}(\log N)^{3/2}} + \frac{|\operatorname{O}(1)|}{N\log N}\Bigg]\,\mathrm{d}z,$$

which converges to $0$ as $N\to\infty$ using (3.27) and (3.28). This yields (3.24) completing the proof. $\square$

**Proof of Theorem 3.10.** By Remark 3.6, condition $\beta\in(0,1)$ implies $\mathbb{E}\left((1-\alpha)^{-1}\right)<\infty$. Hence, by Proposition 3.1 and the continuous mapping theorem, it suffices to show that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\left(\frac{1}{n^{1-\frac{\beta}{2}}}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{\mathcal{Y}}_k\right)_{t\in\mathbb{R}_+} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}}\,\mathcal{B}_{1-\frac{\beta}{2}}.$$

We are going to apply Theorem 4.3 in Beran et al. [7] with $m=1$ for the strictly stationary Gaussian process $\left(\widetilde{\mathcal{Y}}_k/\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}\right)_{k\in\mathbb{Z}_+}$, where, by (3.5),

$$\operatorname{Var}(\widetilde{\mathcal{Y}}_0)=\lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right),\qquad \operatorname{Cov}(\widetilde{\mathcal{Y}}_0,\widetilde{\mathcal{Y}}_k)=\lambda\,\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right),\qquad k\in\mathbb{Z}_+,$$

hence

$$\operatorname{Cov}\left(\frac{\widetilde{\mathcal{Y}}_0}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}},\frac{\widetilde{\mathcal{Y}}_k}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}}\right)=\frac{\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)},\qquad k\in\mathbb{Z}_+.$$

In order to check the conditions of Theorem 4.3 in Beran et al. [7], first we show that

$$k^\beta\,\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right)=k^\beta\int_0^1 a^k(1-a)^{\beta-1}\psi(a)\,\mathrm{d}a\to\psi_1\Gamma(\beta)\qquad\text{as }\ k\to\infty, \tag{3.29}$$

meaning that the covariance function of the process $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is regularly varying with index $-\beta$. For the definition of a regularly varying sequence see Definition 3.5. First note that, by Stirling's formula,

$$\begin{aligned}\lim_{k\to\infty}k^\beta\int_0^1 a^k(1-a)^{\beta-1}\psi_1\,\mathrm{d}a&=\lim_{k\to\infty}\psi_1\frac{k^\beta\Gamma(k+1)}{\Gamma(k+\beta+1)}\Gamma(\beta)\\&=\psi_1\Gamma(\beta)\lim_{k\to\infty}\sqrt{\frac{k}{k+\beta}}\left(\frac{k}{k+\beta}\right)^{k+\beta}\mathrm{e}^\beta=\psi_1\Gamma(\beta).\end{aligned}$$

Next, for arbitrary $\delta\in(0,\psi_1)$, there exists $\varepsilon\in(0,1)$ such that $|\psi(a)-\psi_1|\leqslant\delta$ for all $a\in[1-\varepsilon,1)$, and hence

$$k^\beta\int_{1-\varepsilon}^1 a^k(1-a)^{\beta-1}|\psi(a)-\psi_1|\,\mathrm{d}a\leqslant\delta\sup_{k\in\mathbb{N}}k^\beta\int_0^1 a^k(1-a)^{\beta-1}\,\mathrm{d}a$$

can be arbitrary small. Further, observe

$$\begin{aligned}k^\beta\int_0^{1-\varepsilon}a^k(1-a)^{\beta-1}\psi(a)\,\mathrm{d}a&\leqslant\frac{k^\beta(1-\varepsilon)^k}{\varepsilon}\int_0^{1-\varepsilon}(1-a)^\beta\psi(a)\,\mathrm{d}a\\&\leqslant\frac{k^\beta(1-\varepsilon)^k}{\varepsilon}\int_0^1(1-a)^\beta\psi(a)\,\mathrm{d}a=\frac{k^\beta(1-\varepsilon)^k}{\varepsilon}\to0\qquad\text{as }\ k\to\infty.\end{aligned}$$

In a similar way, we have

$$
\begin{aligned}
k^{\beta} \int_0^{1-\varepsilon} a^k (1-a)^{\beta-1} \psi_1 \, \mathrm{d}a &\leqslant \psi_1 k^{\beta} (1-\varepsilon)^k \int_0^{1-\varepsilon} (1-a)^{\beta-1} \, \mathrm{d}a \\
&\leqslant \psi_1 k^{\beta} (1-\varepsilon)^k \int_0^1 (1-a)^{\beta-1} \, \mathrm{d}a = \psi_1 \, \frac{k^{\beta}(1-\varepsilon)^k}{\beta} \to 0 \qquad \text{as} \ \ k \to \infty,
\end{aligned}
$$

hence

$$
k^{\beta} \int_0^{1-\varepsilon} a^k (1-a)^{\beta-1} |\psi(a) - \psi_1| \, \mathrm{d}a \to 0 \qquad \text{as} \ \ k \to \infty,
$$

implying (3.29). Applying (3.29), we conclude

$$
k^{\beta} \operatorname{Cov}\left( \frac{\widetilde{\mathcal{Y}}_0}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}}, \frac{\widetilde{\mathcal{Y}}_k}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}} \right) = k^{\beta} \frac{\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)} \to \frac{\psi_1 \Gamma(\beta)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)}
$$

as $k \to \infty$. Consequently, by Theorem 4.3 in Beran et al. [7],

$$
\left( \frac{1}{n^{1-\frac{\beta}{2}} L_1(n)^{1/2}} \sum_{k=1}^{\lfloor nt \rfloor} \frac{\widetilde{\mathcal{Y}}_k}{\sqrt{\lambda \, \mathbb{E}\left(\frac{1}{1-\alpha}\right)}} \right)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} \mathcal{Z}_{1,1-\frac{\beta}{2}} \stackrel{\mathcal{D}}{=} \mathcal{B}_{1-\frac{\beta}{2}}, \qquad \text{as} \ \ n \to \infty,
$$

where $\mathcal{Z}_{1,1-\frac{\beta}{2}}$ is the Hermite-Rosenblatt process defined in Definition 3.24 of Beran et al. [7], and

$$
L_1(n) = \frac{\psi_1 \Gamma(\beta)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)} \, C_1, \qquad n \in \mathbb{N}, \qquad \text{with} \qquad C_1 = \frac{2}{(1-\beta)(2-\beta)}.
$$

The fact that the Hermite-Rosenblatt process $\mathcal{Z}_{1,1-\frac{\beta}{2}}$ coincides in law with $\mathcal{B}_{1-\frac{\beta}{2}}$ is shown in Beran et al. [7], see Definition 3.23, the representation in formula (3.111), and page 195 of [7] for details. Hence we obtain the statement. $\square$

**Proof of Theorem 3.11.** Since, by Remark 3.6, $\mathbb{E}((1-\alpha)^{-1}) < \infty$, the condition in Proposition 3.1 is satisfied, meaning that

$$
N^{-\frac{1}{2}} \widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \widetilde{\mathcal{Y}} \qquad \text{as} \ \ N \to \infty,
$$

where $(\widetilde{\mathcal{Y}}_k)_{k \in \mathbb{Z}_+}$ is a stationary Gaussian process with zero mean and covariances

$$
\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}\left( X_0 - \frac{\lambda}{1-\alpha}, X_k - \frac{\lambda}{1-\alpha} \right) = \lambda \, \mathbb{E}\Big(\frac{\alpha^k}{1-\alpha}\Big), \qquad k \in \mathbb{Z}_+. \tag{3.30}
$$

Therefore, it suffices to show that

$$
\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n \to \infty} \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k = \sqrt{2\lambda\psi_1} B,
$$

where $B = (B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process. Since the two processes in question, $\big((n \log n)^{-1/2} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k\big)_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, and $\sqrt{2\lambda\psi_1} B$ are zero mean Gaussian processes, it suffices to show that the covariance function of $\big((n \log n)^{-1/2} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k\big)_{t \in \mathbb{R}_+}$ converges pointwise to that of $\sqrt{2\lambda\psi_1}B$ as $n \to \infty$. Therefore, it remains to show that for all $t_1, t_2 \in \mathbb{R}_+$ we have

$$\operatorname{Cov}\left(\frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt_1\rfloor}\widetilde{\mathcal{Y}}_k, \frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt_2\rfloor}\widetilde{\mathcal{Y}}_k\right) \to 2\lambda\psi_1\min(t_1,t_2), \tag{3.31}$$

as $n \to \infty$. By (3.30) we have

$$\operatorname{Cov}\left(\frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt_1\rfloor}\widetilde{\mathcal{Y}}_k, \frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt_2\rfloor}\widetilde{\mathcal{Y}}_k\right) = \frac{\lambda}{n\log n}\mathbb{E}\left(\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{\alpha^{|k-\ell|}}{1-\alpha}\right)$$

$$= \frac{\lambda}{n\log n}\int_0^1\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{a^{|k-\ell|}}{1-a}\psi(a)(1-a)\,\mathrm{d}a.$$

First we derive

$$\frac{1}{n\log n}\int_0^1\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}a^{|k-\ell|}\,\mathrm{d}a \to 2\min(t_1,t_2), \tag{3.32}$$

as $n \to \infty$. Suppose that $0 \leqslant t_1 < t_2$, then

$$\int_0^1\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}a^{|k-\ell|}\,\mathrm{d}a = \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{1}{|k-\ell|+1}$$

$$= \sum_{k=1}^{\lfloor nt_1\rfloor}\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+1}+\cdots+\frac{1}{\lfloor nt_2\rfloor-k+1}\right)$$

$$= \lfloor nt_1\rfloor\frac{1}{1}+(\lfloor nt_1\rfloor-1)\frac{1}{2}+\cdots+(\lfloor nt_1\rfloor-(\lfloor nt_1\rfloor-1))\frac{1}{\lfloor nt_1\rfloor}$$

$$+\lfloor nt_1\rfloor\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+1}\right)+(\lfloor nt_1\rfloor-1)\frac{1}{\lfloor nt_2\rfloor-(\lfloor nt_1\rfloor-1)+1}$$

$$+\cdots+\Big(\lfloor nt_1\rfloor-\big(\lfloor nt_1\rfloor-1\big)\Big)\frac{1}{\lfloor nt_2\rfloor-1+1}$$

$$= \lfloor nt_1\rfloor+\lfloor nt_1\rfloor\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\lfloor nt_1\rfloor}\right)-\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{\lfloor nt_1\rfloor-1}{\lfloor nt_1\rfloor}\right)$$

$$+\lfloor nt_1\rfloor\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\lfloor nt_2\rfloor}\right)$$

$$-\left(\frac{1}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+2}+\frac{2}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+3}+\cdots+\frac{\lfloor nt_1\rfloor-1}{\lfloor nt_2\rfloor}\right).$$

Applying that for any $\ell \in \mathbb{N}$ we have

$$\frac{\ell}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+\ell+1} = 1-\frac{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+1}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+\ell+1},$$

we get that

$$\int_0^1\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}a^{|k-\ell|}\,\mathrm{d}a = \lfloor nt_1\rfloor+(\lfloor nt_1\rfloor+1)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\lfloor nt_1\rfloor}\right)-(\lfloor nt_1\rfloor-1)$$

$$+\lfloor nt_1\rfloor\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\lfloor nt_2\rfloor}\right)-(\lfloor nt_1\rfloor-1)$$

$$+\big(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+1\big)\left(\frac{1}{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor+2}+\cdots+\frac{1}{\lfloor nt_2\rfloor}\right)$$

$$
\begin{aligned}
&= (\lfloor nt_1\rfloor + 1)(\widetilde{H}(\lfloor nt_1\rfloor) - 1) + 2 - \lfloor nt_1\rfloor + \lfloor nt_1\rfloor(\widetilde{H}(\lfloor nt_2\rfloor) - 1) \\
&\quad + \big(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1\big)\Big(\widetilde{H}(\lfloor nt_2\rfloor) - \widetilde{H}(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1)\Big) \\
&= (\lfloor nt_1\rfloor + 1)(\log(\lfloor nt_1\rfloor) + O(1)) + 2 - \lfloor nt_1\rfloor + \lfloor nt_1\rfloor(\log\lfloor nt_2\rfloor + O(1)) \\
&\quad + \big(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1\big)\left(\log(\lfloor nt_2\rfloor) - \log(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1) + O(1)\right),
\end{aligned}
$$

where

$$
\widetilde{H}(n) := \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}
$$

denotes the $n$-th harmonic number, and we use that $\widetilde{H}(n) = \log n + O(1)$ for every $n \in \mathbb{N}$. Therefore, convergence (3.32) holds. Consequently, (3.31) will follow from

$$
I_n := \frac{1}{n\log n}\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}|\psi(a) - \psi_1|\,\mathrm{d}a \to 0
$$

as $n \to \infty$. Note that for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for every $a \in (1-\delta_\varepsilon, 1)$ it holds that $|\psi(a) - \psi_1| < \varepsilon$. Since for every $a \in (0, 1-\delta_\varepsilon)$ it holds that $\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|} \leqslant \lfloor nt_1\rfloor \sum_{\ell=-\infty}^{\infty} a^{|\ell|} \leqslant 2\lfloor nt_1\rfloor(1-a)^{-1} \leqslant 2\lfloor nt_1\rfloor{\delta_\varepsilon}^{-1}$, we get

$$
\begin{aligned}
n\log n\, I_n &\leqslant \int_0^{1-\delta_\varepsilon} \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}(\psi(a) + \psi_1)\,\mathrm{d}a + \int_{1-\delta_\varepsilon}^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}|\psi(a) - \psi_1|\,\mathrm{d}a \\
&\leqslant \int_0^{1-\delta_\varepsilon} \frac{2\lfloor nt_1\rfloor}{\delta_\varepsilon}(\psi(a) + \psi_1)\,\mathrm{d}a + \varepsilon\int_{1-\delta_\varepsilon}^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}\,\mathrm{d}a.
\end{aligned}
$$

Then since $\psi$ is integrable on $(0,1)$, we have $\limsup_{n\to\infty} I_n \leqslant 0 + \varepsilon 4\min(t_1, t_2)$ for every $\varepsilon > 0$ by (3.32), resulting that $\lim_{n\to\infty} I_n = 0$. This completes the proof. $\Box$

**Proof of Theorem 3.12.** By Proposition 3.2, we have

$$
\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty}\left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(1)} - \mathbb{E}(X_k^{(1)} \mid \alpha^{(1)}))\right)_{t\in\mathbb{R}_+} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B,
$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and $\alpha$ is a random variable having a density function of the form (3.6) with $\beta \in (-1,1)$ and $\psi_1 \in (0,\infty)$, and being independent of $B$. Let $\mathcal{W}_t := \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B_t$, $t \in \mathbb{R}_+$, and $(\mathcal{W}_t^{(i)})_{t\in\mathbb{R}_+}$, $i \in \mathbb{N}$, be its independent copies. It remains to prove that

$$
\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty}\left(N^{-\frac{1}{1+\beta}}\sum_{i=1}^N \mathcal{W}_t^{(i)}\right)_{t\in\mathbb{R}_+} = \mathcal{Y}_{1+\beta}.
$$

Using the continuity theorem and the continuous mapping theorem, it is enough to prove that for all $m \in \mathbb{N}$, $\theta_1, \ldots, \theta_m \in \mathbb{R}$ and $0 =: t_0 < t_1 < t_2 < \cdots < t_m$,

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{j=1}^m \theta_j\left(N^{-\frac{1}{1+\beta}}\sum_{i=1}^N(\mathcal{W}_{t_j}^{(i)} - \mathcal{W}_{t_{j-1}}^{(i)})\right)\right\}\right) \\
&= \left[\mathbb{E}\left(\exp\left\{\mathrm{i}N^{-\frac{1}{1+\beta}}\sum_{j=1}^m \theta_j(\mathcal{W}_{t_j} - \mathcal{W}_{t_{j-1}})\right\}\right)\right]^N
\end{aligned}
$$

$$
\begin{aligned}
&\to \mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}\sum_{j=1}^m \theta_j(\mathcal{Y}_{1+\beta}(t_j)-\mathcal{Y}_{1+\beta}(t_{j-1}))\Bigg\}\Bigg)\\
&= \mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}\sum_{j=1}^m \theta_j\sqrt{Y_{(1+\beta)/2}}(B_{t_j}-B_{t_{j-1}})\Bigg\}\Bigg)\\
&= \mathbb{E}\Bigg(\exp\Bigg\{-\frac{1}{2}Y_{(1+\beta)/2}\sum_{j=1}^m \theta_j^2(t_j-t_{j-1})\Bigg\}\Bigg)\\
&= \exp\Bigg\{-k_\beta\Bigg(\frac{1}{2}\sum_{j=1}^m \theta_j^2(t_j-t_{j-1})\Bigg)^{\frac{1+\beta}{2}}\Bigg\} = \mathrm{e}^{-k_\beta\,\omega^{\frac{1+\beta}{2}}}
\end{aligned}
$$

as $N\to\infty$, where $\omega := \frac{1}{2}\sum_{j=1}^m \theta_j^2(t_j-t_{j-1})$. Note that, using the independence of $\alpha$ and $B$, it suffices to show

$$
\begin{aligned}
\Psi_N &:= N\Bigg[1-\mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}N^{-\frac{1}{1+\beta}}\sum_{j=1}^m \theta_j(\mathcal{W}_{t_j}-\mathcal{W}_{t_{j-1}})\Bigg\}\Bigg)\Bigg]\\
&= N\Bigg[1-\mathbb{E}\Bigg(\exp\Bigg\{-\frac{1}{2}N^{-\frac{2}{1+\beta}}\lambda(1+\alpha)(1-\alpha)^{-2}\sum_{j=1}^m \theta_j^2(t_j-t_{j-1})\Bigg\}\Bigg)\Bigg]\\
&= N\int_0^1\Big(1-\mathrm{e}^{-\omega N^{-\frac{2}{1+\beta}}\lambda(1+a)(1-a)^{-2}}\Big)\psi(a)(1-a)^\beta\,\mathrm{d}a \to k_\beta\,\omega^{\frac{1+\beta}{2}}
\end{aligned}
$$

as $N\to\infty$, since it implies that $(1-\Psi_N/N)^N \to \mathrm{e}^{-k_\beta\,\omega^{\frac{1+\beta}{2}}}$ as $N\to\infty$. For all $\varepsilon\in(0,1)$,

$$
\sup_{a\in(0,1-\varepsilon)}\big|-N\omega N^{-\frac{2}{1+\beta}}(1+a)(1-a)^{-1}\big| = \omega N^{\frac{-1+\beta}{1+\beta}}(2-\varepsilon)\varepsilon^{-1}\to 0
$$

as $N\to\infty$. Therefore, by Lemma 3.21, substituting $a = 1-N^{-\frac{1}{1+\beta}}y$, the statement of the theorem will follow from

$$
\begin{aligned}
&\limsup_{N\to\infty} N\int_{1-\varepsilon}^1\Big|1-\mathrm{e}^{-\omega N^{-\frac{2}{1+\beta}}\lambda(1+a)(1-a)^{-2}}\Big|(1-a)^\beta\,\mathrm{d}a\\
&= \limsup_{N\to\infty}\int_0^{\varepsilon N^{\frac{1}{1+\beta}}}\Big|1-\mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\Big|y^\beta\,\mathrm{d}y<\infty,
\end{aligned}
\tag{3.33}
$$

and

$$
\begin{aligned}
&\lim_{N\to\infty} N\int_{1-\varepsilon}^1\Big(1-\mathrm{e}^{-\omega N^{-\frac{2}{1+\beta}}\lambda(1+a)(1-a)^{-2}}\Big)(1-a)^\beta\,\mathrm{d}a\\
&= \lim_{N\to\infty}\int_0^{\varepsilon N^{\frac{1}{1+\beta}}}\Big(1-\mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\Big)y^\beta\,\mathrm{d}y = \psi_1^{-1}k_\beta\,\omega^{\frac{1+\beta}{2}}
\end{aligned}
\tag{3.34}
$$

for all $\varepsilon\in(0,1)$. Next we prove (3.33) and (3.34).

For all $N\in\mathbb{N}$ and $\varepsilon\in(0,1)$, using (3.10), we have

$$
\begin{aligned}
\int_0^{\varepsilon N^{\frac{1}{1+\beta}}}\Big|1-\mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\Big|y^\beta\,\mathrm{d}y &\leqslant \int_0^\infty\Big|1-\mathrm{e}^{-2\omega\lambda y^{-2}}\Big|y^\beta\,\mathrm{d}y\\
&\leqslant \int_0^1 y^\beta\,\mathrm{d}y + 2\omega\lambda\int_1^\infty y^{\beta-2}\,\mathrm{d}y<\infty,
\end{aligned}
$$

hence we obtain (3.33).

Now we turn to prove (3.34). For all $\varepsilon \in (0,1)$, we have

$$\left|\int_{\varepsilon N^{\frac{1}{1+\beta}}}^{\infty} \Big(1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\Big) y^{\beta}\,\mathrm{d}y\right| \leqslant 2\omega\lambda \int_{\varepsilon N^{\frac{1}{1+\beta}}}^{\infty} y^{\beta-2}\,\mathrm{d}y = \frac{2\omega\lambda}{1-\beta}\big(\varepsilon N^{\frac{1}{1+\beta}}\big)^{\beta-1} \to 0 \tag{3.35}$$

as $N \to \infty$. Further, using (3.12),

$$\begin{aligned}
&\left|\int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\Big) y^{\beta}\,\mathrm{d}y - \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\Big) y^{\beta}\,\mathrm{d}y\right| \\
&\leqslant \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big|\mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}} - \mathrm{e}^{-2\omega\lambda y^{-2}}\Big|\, y^{\beta}\,\mathrm{d}y \\
&= \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \mathrm{e}^{-2\omega\lambda y^{-2}} \Big|\mathrm{e}^{\omega\lambda N^{-\frac{1}{1+\beta}}{}^{a} y^{-1}} - 1\Big|\, y^{\beta}\,\mathrm{d}y \\
&\leqslant \omega\lambda N^{-\frac{1}{1+\beta}} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \mathrm{e}^{-2\omega\lambda y^{-2}} \mathrm{e}^{\omega\lambda N^{-\frac{1}{1+\beta}} y^{-1}} y^{\beta-1}\,\mathrm{d}y \\
&\leqslant \omega\lambda N^{-\frac{1}{1+\beta}} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \mathrm{e}^{-(2-\varepsilon)\omega\lambda y^{-2}} y^{\beta-1}\,\mathrm{d}y \leqslant \omega\lambda N^{-\frac{1}{1+\beta}} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} y^{\beta-1}\,\mathrm{d}y \\
&= \omega\lambda N^{-\frac{1}{1+\beta}} \frac{(\varepsilon N^{\frac{1}{1+\beta}})^{\beta}}{\beta} = \omega\lambda\,\frac{\varepsilon^{\beta} N^{\frac{\beta-1}{1+\beta}}}{\beta} \to 0 \qquad \text{as } N \to \infty,
\end{aligned}$$

hence, using (3.35), we conclude

$$\begin{aligned}
&\lim_{N\to\infty} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\Big) y^{\beta}\,\mathrm{d}y = \int_0^{\infty} \Big(1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\Big) y^{\beta}\,\mathrm{d}y \\
&= \frac{1}{2}\,(2\omega\lambda)^{\frac{1+\beta}{2}} \int_0^{\infty} (1 - \mathrm{e}^{-u}) u^{-\frac{3+\beta}{2}}\,\mathrm{d}u = \psi_1^{-1} k_{\beta}\,\omega^{\frac{1+\beta}{2}},
\end{aligned}$$

where the last equality follows by Li [31, formula (1.28)], thus we obtain (3.34). By Theorem 3.12 of Janson [22], a distribution with the Laplace transform given in the theorem is positive $\frac{1+\beta}{2}$-stable.

Now it remains to show that the process $\mathcal{Y}_{1+\beta}$ has stationary increments and $(1+\beta)$-stable one-dimensional distributions. Indeed, by the independence of $Y_{\frac{1+\beta}{2}}$ and $B$, for every $0 \leqslant s \leqslant t$ we have

$$\mathcal{Y}_{1+\beta}(t) - \mathcal{Y}_{1+\beta}(s) = \sqrt{Y_{\frac{1+\beta}{2}}}\,(B_t - B_s) \stackrel{\mathcal{D}}{=} \sqrt{Y_{\frac{1+\beta}{2}}}\,B_{t-s} = \mathcal{Y}_{1+\beta}(t-s).$$

Moreover, for every $t \geqslant 0$, we get

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{\mathrm{i}u\mathcal{Y}_{1+\beta}(t)}) &= \mathbb{E}\left(\mathbb{E}\left(\mathrm{e}^{\mathrm{i}u\sqrt{Y_{\frac{1+\beta}{2}}}\,B_t} \,\Big|\, Y_{\frac{1+\beta}{2}}\right)\right) = \mathbb{E}\left(\mathrm{e}^{-\frac{u^2 t}{2} Y_{\frac{1+\beta}{2}}}\right) = \mathrm{e}^{-k_{\beta}\left(\frac{u^2 t}{2}\right)^{\frac{1+\beta}{2}}} \\
&= \mathrm{e}^{-k_{\beta}\left(\frac{t}{2}\right)^{\frac{1+\beta}{2}} |u|^{1+\beta}}, \qquad u \in \mathbb{R},
\end{aligned}$$

therefore the one-dimensional distributions of $\mathcal{Y}_{1+\beta}$ are $(1+\beta)$-stable. $\square$

**Proof of Theorem 3.13.** By Proposition 3.2, we have

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \left( n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)} \,|\, \alpha^{(1)})) \right)_{t\in\mathbb{R}_+} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and $\alpha$ is a random variable having a density function of the form (3.6) with $\beta = 1$ and $\psi_1 \in (0,\infty)$, and being independent of $B$. Hence it remains to prove that

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} (T_t^{(N)})_{t\in\mathbb{R}_+} = \big(\sqrt{\lambda\psi_1}\, B_t\big)_{t\in\mathbb{R}_+},$$

where

$$T_t^{(N)} := \frac{1}{\sqrt{N\log N}} \sum_{j=1}^N \frac{\sqrt{\lambda(1+\alpha^{(j)})}}{1-\alpha^{(j)}} B_t^{(j)}, \qquad t\in\mathbb{R}_+, \qquad N\in\mathbb{N},$$

and $\alpha^{(j)}$, $j\in\mathbb{N}$, and $B^{(j)}$, $j\in\mathbb{N}$, are independent copies of $\alpha$ and $B$, respectively, being independent of each other as well. By the continuous mapping theorem, it is enough to show that for all $m\in\mathbb{N}$ and $0 =: t_0 \leqslant t_1 < t_2 < \cdots < t_m$,

$$\begin{aligned} &\Big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\Big) \\ &\quad\xrightarrow{\mathcal{D}} \Big(\sqrt{\lambda\psi_1}(B_{t_1} - B_{t_0}), \ldots, \sqrt{\lambda\psi_1}(B_{t_m} - B_{t_{m-1}})\Big) \end{aligned}$$

as $N\to\infty$. By the portmanteau theorem, it is enough to check that for all $m\in\mathbb{N}$, $0 = t_0 \leqslant t_1 < t_2 < \cdots < t_m$, and for all bounded and continuous functions $g:\mathbb{R}^m\to\mathbb{R}$,

$$\begin{aligned} &\mathbb{E}\Big(g\big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\big)\Big) \\ &\quad\to \mathbb{E}\left(g\left(\sqrt{\lambda\psi_1}(B_{t_1} - B_{t_0}), \ldots, \sqrt{\lambda\psi_1}(B_{t_m} - B_{t_{m-1}})\right)\right) \end{aligned}$$

as $N\to\infty$. By the properties of standard Wiener processes and their variances

$$\begin{aligned} &\mathbb{E}\Big(g\big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\big)\Big) \\ &= \mathbb{E}\Big[\mathbb{E}\Big[g\big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\big) \,\Big|\, \alpha^{(j)},\, j\in\mathbb{N}\Big]\Big] \\ &= \mathbb{E}\left[g\left(\sqrt{(N\log N)^{-1}\sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}}(\widetilde{B}_{t_1} - \widetilde{B}_{t_0}),\right.\right. \\ &\qquad\qquad \left.\left. \ldots, \sqrt{(N\log N)^{-1}\sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}}(\widetilde{B}_{t_m} - \widetilde{B}_{t_{m-1}})\right)\right] \\ &= \mathbb{E}\left[h\left((N\log N)^{-1}\sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}, \widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_m}\right)\right], \end{aligned}$$

where $(\widetilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process independent of $\alpha^{(j)}$, $j\in\mathbb{N}$, and $h:\mathbb{R}^{m+1}\to\mathbb{R}$ is an appropriate bounded and continuous function. Hence it suffices to show that

$$\frac{1}{N\log N}\sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2} \xrightarrow{\mathcal{D}} \lambda\psi_1, \qquad N\to\infty.$$

Let us apply a well-known theorem about weak convergence of partial sum processes for a triangular array towards a Lévy process due to Resnick [46, Theorem 7.1] with

$$X_{N,j} := \frac{1}{N}\frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}.$$

Then

$$N\,\mathbb{P}(X_{N,1} > x) = N\,\mathbb{P}\left(\frac{\lambda(1+\alpha)}{(1-\alpha)^2} > Nx\right) = N\int_{\max\{0,1-\widetilde{h}(\lambda,Nx)\}}^{1} \psi(a)(1-a)\mathrm{d}a,$$

where $\widetilde{h}(\lambda,x) = (1/4+\sqrt{1/16+x/(2\lambda)})^{-1}$. Indeed, by solving the quadratic equation $\lambda(1+\alpha)(1-\alpha)^{-2} = x$ for $\alpha$, one gets two roots, with exactly one of them being smaller than one: $1+\lambda/(2x)-\sqrt{\lambda^2/(4x^2)+2\lambda/x}$. After a simple rationalisation, we get that this root equals $1-\widetilde{h}(\lambda,x)$. In the rest of the proof all statements are understood for large enough $N$ values, which lets us write $1-\widetilde{h}(\lambda,Nx)$ instead of $\max\{0,1-\widetilde{h}(\lambda,Nx)\}$ in the integral above. Note that for every $\varepsilon > 0$ there is a $\delta_\varepsilon \in (0,1)$ such that for every $a \in (1-\delta_\varepsilon, 1)$ it holds that $|\psi(a)-\psi_1| < \varepsilon$ and $\psi(a) \leqslant 2\psi_1$. Then,

$$N\int_{1-\widetilde{h}(\lambda,Nx)}^{1} |\psi(a)-\psi_1|(1-a)\mathrm{d}a \leqslant N\varepsilon\frac{(\widetilde{h}(\lambda,Nx))^2}{2} \leqslant \frac{\varepsilon\lambda}{x}$$

for every $x > 0$ and large enough $N$, meaning that the limit is 0. Therefore, for every $x > 0$ we have

$$\begin{aligned}
\lim_{N\to\infty} N\,\mathbb{P}(X_{N,1} > x) &= \lim_{N\to\infty} N\int_{1-\widetilde{h}(\lambda,Nx)}^{1} \psi_1(1-a)\mathrm{d}a \\
&= \lim_{N\to\infty} N\psi_1\frac{(\widetilde{h}(\lambda,Nx))^2}{2} = \lim_{N\to\infty}\frac{\psi_1}{2}\frac{N}{\left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{Nx}{2\lambda}}\right)^2} = \frac{\psi_1\lambda}{x} =: \nu([x,\infty)),
\end{aligned}$$

where, since $\nu(\mathrm{d}x) = x^{-2}\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x$ holds (thus $\min\{1,x^2\}$ is $\nu$-integrable), $\nu$ is obviously a Lévy measure. We note that instead of the vague convergence required in formula (7.5) of Resnick [46], we verified convergence in distribution, which is a stronger condition. Furthermore, by the decomposition that holds for large enough $N$,

$$N\,\mathbb{E}\left(X_{N,1}^2\mathbb{1}_{\{|X_{N,1}|\leqslant\varepsilon\}}\right) = N\int_0^{1-\widetilde{h}(\lambda,N\varepsilon)}\left(\frac{\lambda(1+a)}{N(1-a)^2}\right)^2\psi(a)(1-a)\mathrm{d}a = I_N^{(1)} + I_N^{(2)},$$

where

$$I_N^{(1)} := N\int_0^{1-\delta_\varepsilon}\left(\frac{\lambda(1+a)}{N(1-a)^2}\right)^2\psi(a)(1-a)\mathrm{d}a \leqslant \frac{1}{N}\lambda^2\frac{2^2}{\delta_\varepsilon^4}1 \to 0$$

as $N\to\infty$, and

$$\begin{aligned}
I_N^{(2)} &:= N\int_{1-\delta_\varepsilon}^{1-\widetilde{h}(\lambda,N\varepsilon)}\left(\frac{\lambda(1+a)}{N(1-a)^2}\right)^2\psi(a)(1-a)\mathrm{d}a \\
&\leqslant \frac{8\psi_1\lambda^2}{N}\int_{1-\delta_\varepsilon}^{1-\widetilde{h}(\lambda,N\varepsilon)}\frac{\mathrm{d}a}{(1-a)^3} = \frac{4\psi_1\lambda^2}{N}\left[\widetilde{h}(\lambda,N\varepsilon)^{-2}-\delta_\varepsilon^{-2}\right] \\
&= \frac{4\psi_1\lambda^2}{N}\left[\left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{N\varepsilon}{2\lambda}}\right)^2-\delta_\varepsilon^{-2}\right] \leqslant 2\psi_1\lambda\varepsilon + 4\psi_1\lambda^2\left(\frac{1}{8N}+\frac{1}{2N}\sqrt{\frac{1}{16}+\frac{N\varepsilon}{2\lambda}}\right)
\end{aligned}$$

for large enough $N$ values, so it follows that

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} N\,\mathbb{E}\left(X_{N,1}^2\mathbb{1}_{\{|X_{N,1}|\leqslant\varepsilon\}}\right)=0.$$

Therefore, by applying Theorem 7.1 of Resnick [46], for $t=1$ we get that

$$\sum_{j=1}^{N}\left[\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2}-\mathbb{E}\left(\frac{\lambda(1+\alpha)}{N(1-\alpha)^2}\mathbb{1}_{\left\{\frac{\lambda(1+\alpha)}{N(1-\alpha)^2}\leqslant 1\right\}}\right)\right]\xrightarrow{\mathcal{D}} X_0,$$

where by (5.37) of Resnick [46]

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_0})=\exp\left\{\int_1^\infty(\mathrm{e}^{\mathrm{i}\theta x}-1)\frac{\psi_1\lambda\mathrm{d}x}{x^2}+\int_0^1(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x)\frac{\psi_1\lambda\mathrm{d}x}{x^2}\right\},\qquad\theta\in\mathbb{R}.$$

Let us consider the following decomposition:

$$\sum_{j=1}^{N}\left[\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2}-\mathbb{E}\left(\frac{\lambda(1+\alpha)}{N(1-\alpha)^2}\mathbb{1}_{\left\{\frac{\lambda(1+\alpha)}{N(1-\alpha)^2}\leqslant 1\right\}}\right)\right]=:\frac{\lambda}{N}\sum_{j=1}^{N}J_{j,N}^{(0)}+\lambda J_N^{(1)}+\lambda J_N^{(2)}+\lambda J_N^{(3)},$$

where

$$\begin{aligned}
J_{j,N}^{(0)}&:=\frac{(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}-\psi_1\int_0^{1-\sqrt{\frac{2\lambda}{N}}}\frac{2}{(1-a)^2}(1-a)\mathrm{d}a,\\
J_N^{(1)}&:=\psi_1\int_0^{1-\sqrt{\frac{2\lambda}{N}}}\frac{2}{(1-a)^2}(1-a)\mathrm{d}a-\psi_1\int_0^{1-\widetilde{h}(\lambda,N)}\frac{2}{(1-a)^2}(1-a)\mathrm{d}a,\\
J_N^{(2)}&:=\psi_1\int_0^{1-\widetilde{h}(\lambda,N)}\frac{2}{(1-a)^2}(1-a)\mathrm{d}a-\psi_1\int_0^{1-\widetilde{h}(\lambda,N)}\frac{1+a}{(1-a)^2}(1-a)\mathrm{d}a,\\
J_N^{(3)}&:=\psi_1\int_0^{1-\widetilde{h}(\lambda,N)}\frac{1+a}{(1-a)^2}(1-a)\mathrm{d}a-\int_0^{1-\widetilde{h}(\lambda,N)}\frac{1+a}{(1-a)^2}\psi(a)(1-a)\mathrm{d}a.
\end{aligned}$$

We show that

$$\frac{|J_N^{(1)}|+|J_N^{(2)}|+|J_N^{(3)}|}{\log N}\to 0,\qquad N\to\infty,$$

resulting

$$\begin{aligned}
\frac{1}{\log N}\sum_{j=1}^{N}\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2}&=\frac{1}{\log N}\sum_{j=1}^{N}\left[\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2}-\frac{\lambda\psi_1}{N}\int_0^{1-\sqrt{\frac{2\lambda}{N}}}\frac{2}{1-a}\mathrm{d}a\right]\\
&\quad+\frac{2\lambda\psi_1}{\log N}\left(-\log\left(\sqrt{\frac{2\lambda}{N}}\right)\right)\xrightarrow{\mathcal{D}} 0\cdot X_0+\lambda\psi_1=\lambda\psi_1,\qquad N\to\infty.
\end{aligned}$$

Indeed,

$$\frac{J_N^{(1)}}{\log N}=-\frac{\psi_1}{\log N}\int_{1-\sqrt{\frac{2\lambda}{N}}}^{1-\widetilde{h}(\lambda,N)}\frac{2}{1-a}\mathrm{d}a=-\frac{2\psi_1}{\log N}\log\left(\sqrt{\frac{2\lambda}{N}}\left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{N}{2\lambda}}\right)\right)$$

converges to 0 as $N\to\infty$. Moreover,

$$\frac{J_N^{(2)}}{\log N}=\frac{\psi_1}{\log N}\int_0^{1-\widetilde{h}(\lambda,N)}\frac{1-a}{(1-a)^2}(1-a)\mathrm{d}a=\frac{\psi_1}{\log N}\left(1-\frac{1}{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{N}{2\lambda}}}\right)$$

converges to 0 as $N \to \infty$. Finally,

$$\begin{aligned}
\left|\frac{J_N^{(3)}}{\log N}\right| &= \left|\frac{1}{\log N}\int_0^{1-\widetilde{h}(\lambda,N)} \frac{1+a}{1-a}(\psi_1 - \psi(a))\mathrm{d}a\right| \\
&\leqslant \frac{1}{\log N}\int_0^{1-\delta_\varepsilon} \frac{2}{\delta_\varepsilon}(\psi_1 + \psi(a))\mathrm{d}a + \frac{1}{\log N}\int_{1-\delta_\varepsilon}^{1-\widetilde{h}(\lambda,N)} \frac{2}{1-a}\,\varepsilon\,\mathrm{d}a \\
&\leqslant \frac{1}{\log N}\frac{2}{\delta_\varepsilon}\left(\psi_1 + \int_0^1 \psi(a)\mathrm{d}a\right) + \frac{2\varepsilon}{\log N}\left[\log\delta_\varepsilon + \log\left(\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{N}{2\lambda}}\right)\right].
\end{aligned}$$

One can easily see that, since $\psi$ is integrable on $(0,1)$, then for all $\varepsilon > 0$, we get $\limsup_{N\to\infty} |J_N^{(3)}/\log N| \leqslant 0 + \varepsilon$. This means that $\lim_{N\to\infty} J_N^{(3)}/\log N = 0$, which completes the proof. □

**Proof of Theorem 3.14.** Let us start with the proof of the first convergence of Theorem 3.14. Since $\mathbb{E}\left((1-\alpha)^{-1}\right) < \infty$, by Proposition 3.1, we have

$$\frac{1}{\sqrt{N}}\widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_\mathrm{f}} \widetilde{\mathcal{Y}} \qquad \text{as } N \to \infty,$$

where the strictly stationary Gaussian process $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is given in Proposition 3.1. Consequently, by the continuous mapping theorem, for all $n \in \mathbb{N}$, we get

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}}\widetilde{S}^{(N,n)} = \left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{\mathcal{Y}}_k\right)_{t\in\mathbb{R}_+},$$

hence it remains to prove that

$$\left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{\mathcal{Y}}_k\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_\mathrm{f}} \sigma B \qquad \text{as } n \to \infty.$$

Since the processes $\left(n^{-1/2}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{\mathcal{Y}}_k\right)_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, and $\sigma B$ are zero mean Gaussian processes, it suffices to show that the covariance function of $\left(n^{-1/2}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{\mathcal{Y}}_k\right)_{t\in\mathbb{R}_+}$ converges pointwise to that of $\sigma B$ as $n \to \infty$. For all $0 \leqslant t_1 \leqslant t_2$,

$$\begin{aligned}
\operatorname{Cov}\left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt_1\rfloor}\widetilde{\mathcal{Y}}_k, n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt_2\rfloor}\widetilde{\mathcal{Y}}_k\right) &= \frac{\lambda}{n}\,\mathbb{E}\left(\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{\alpha^{|k-\ell|}}{1-\alpha}\right) \\
&\to \lambda\,\mathbb{E}\left(\frac{1+\alpha}{(1-\alpha)^2}\right)\min(t_1,t_2) = \operatorname{Cov}(\sigma B_{t_1}, \sigma B_{t_2}) \qquad \text{as } n \to \infty,
\end{aligned}$$

since by simple calculations

$$\begin{aligned}
\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{\alpha^{|k-\ell|}}{1-\alpha} &= \frac{(1-\alpha^2)\lfloor nt_1\rfloor - \alpha\left(1-\alpha^{\lfloor nt_2\rfloor} - \alpha^{\lfloor nt_1\rfloor} + \alpha^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor}\right)}{(1-\alpha)^3} \\
&= \frac{\alpha(\alpha^{\lfloor nt_2\rfloor}-1) + \lfloor nt_2\rfloor(1-\alpha^2)/2}{(1-\alpha)^3} + \frac{\alpha(\alpha^{\lfloor nt_1\rfloor}-1) + \lfloor nt_1\rfloor(1-\alpha^2)/2}{(1-\alpha)^3} \\
&\quad - \frac{\alpha(\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1) + (\lfloor nt_2\rfloor - \lfloor nt_1\rfloor)(1-\alpha^2)/2}{(1-\alpha)^3},
\end{aligned}$$

and

$$\frac{1}{n}\,\mathbb{E}\biggl(\frac{\alpha(\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1)+(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)(1-\alpha^2)/2}{(1-\alpha)^3}\biggr)\to(t_2-t_1)\,\mathbb{E}\left(\frac{1+\alpha}{2(1-\alpha)^2}\right)$$

as $n\to\infty$. Indeed, by the dominated convergence theorem,

$$\frac{1}{n}\,\mathbb{E}\biggl(\frac{\alpha(\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1)}{(1-\alpha)^3}\biggr)\to 0\qquad\text{as } n\to\infty,$$

where the pointwise convergence follows by

$$\left|\frac{\alpha(\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1)}{(1-\alpha)^3}\right|\leqslant\frac{1}{(1-\alpha)^3},$$

and $(t_2-t_1+1)\frac{\alpha}{(1-\alpha)^2}$ serves as an integrable dominating function, since, by Remark 3.6, $\mathbb{E}\left(\frac{\alpha}{(1-\alpha)^2}\right)<\infty$, and

$$\begin{aligned}\frac{1}{n}\left|\frac{\alpha(\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1)}{(1-\alpha)^3}\right|&=\frac{\alpha(1+\alpha+\alpha^2+\cdots+\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor-1})}{n(1-\alpha)^2}\\&\leqslant\frac{\alpha(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)}{n(1-\alpha)^2}\leqslant(t_2-t_1+1)\frac{\alpha}{(1-\alpha)^2}.\end{aligned}$$

For the second convergence, first note that, by Proposition 3.2, we have

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(1)}-\mathbb{E}(X_k^{(1)}\,|\,\alpha^{(1)}))\right)_{t\in\mathbb{R}_+}=\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and $\alpha$ is a random variable having a density function of the form (3.6) with $\beta\in(1,\infty)$ and $\psi_1\in(0,\infty)$, and being independent of $B$. Hence it remains to prove that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\frac{\sqrt{\lambda(1+\alpha^{(j)})}}{1-\alpha^{(j)}}B^{(j)}=\sigma B,$$

where $\alpha^{(j)}$, $j\in\mathbb{N}$, and $B^{(j)}$, $j\in\mathbb{N}$, are independent copies of $\alpha$ and $B$, respectively, being independent of each other as well. Similarly to the proof of Theorem 3.13, it is enough to show that

$$\frac{1}{N}\sum_{j=1}^{N}\frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}\xrightarrow{\mathcal{D}}\sigma^2\qquad\text{as } N\to\infty.$$

This readily follows by the strong law of large numbers, since $\mathbb{E}\bigl(\frac{\lambda(1+\alpha)}{(1-\alpha)^2}\bigr)<\infty$ due to Remark 3.6. □

**Proof of Theorem 3.16.** We introduce the decomposition

$$S_t^{(N,n)}=R_t^{(N,n)}+\widetilde{S}_t^{(N,n)},\qquad t\in\mathbb{R}_+,\tag{3.36}$$

with

$$R_t^{(N,n)} := \sum_{j=1}^N \sum_{k=1}^{\lfloor nt\rfloor} (\mathbb{E}(X_k^{(j)} \mid \alpha^{(j)}) - \mathbb{E}(X_k^{(j)})) = \lfloor nt\rfloor \sum_{j=1}^N \left( \frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}}\right)\right)$$

for $t \in \mathbb{R}_+$. Since $\mathbb{E}\big((1-\alpha)^{-1}\big) < \infty$ for $\beta \in (0,1)$ by Remark 3.6, by Proposition 3.1, for each $n \in \mathbb{N}$, $\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)}$ exists, hence

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{-\frac{1}{1+\beta}} \widetilde{S}^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{\frac{\beta-1}{2(1+\beta)}} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0. \tag{3.37}$$

The distribution of the random variable $\lambda(1-\alpha)^{-1} - \mathbb{E}(\lambda(1-\alpha)^{-1})$ belongs to the domain of attraction of an $(1+\beta)$-stable distribution. Indeed, we have

$$\begin{aligned}
&\lim_{x\to\infty} x^{1+\beta}\, \mathbb{P}\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right) > x\right) \\
&= \lim_{x\to\infty} x^{1+\beta}\, \mathbb{P}\left(\alpha > 1 - \frac{1}{\lambda^{-1}x + \mathbb{E}((1-\alpha)^{-1})}\right) \\
&= \lim_{x\to\infty} \frac{1}{x^{-(1+\beta)}} \int_{1-(\lambda^{-1}x+\mathbb{E}((1-\alpha)^{-1}))^{-1}}^1 \psi(a)(1-a)^\beta \,\mathrm{d}a \\
&= \lim_{x\to\infty} \frac{-\psi(1-(\lambda^{-1}x+\mathbb{E}((1-\alpha)^{-1}))^{-1})(\lambda^{-1}x+\mathbb{E}((1-\alpha)^{-1}))^{-\beta-2}\lambda^{-1}}{-(1+\beta)x^{-(1+\beta)-1}} \\
&= \frac{\psi_1 \lambda^{1+\beta}}{1+\beta}
\end{aligned} \tag{3.38}$$

by L'Hôpital's rule. Further, using that $\mathbb{P}(\lambda(1-\alpha)^{-1} > 0) = 1$,

$$\lim_{x\to-\infty} |x|^{1+\beta}\, \mathbb{P}\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right) \leqslant x\right) = \lim_{x\to-\infty} |x|^{1+\beta} \cdot 0 = 0. \tag{3.39}$$

Consequently, for each $n \in \mathbb{N}$,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{-\frac{1}{1+\beta}} R^{(N,n)} = \big(\lfloor nt\rfloor Z_{1+\beta}\big)_{t\in\mathbb{R}_+},$$

see, e.g., Puplinskaitė and Surgailis [42, Remark 2.1]. Indeed, the characteristic function of the random variable $Z_{1+\beta}$ takes the form

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Z_{1+\beta}}) \\
&= \exp\left\{-|\theta|^{1+\beta} \frac{\Gamma(2-(1+\beta))}{1-(1+\beta)} \frac{\psi_1\lambda^{1+\beta}}{1+\beta} \left(\cos\left(\frac{\pi(1+\beta)}{2}\right) - \mathrm{i}\,\mathrm{sign}(\theta)\sin\left(\frac{\pi(1+\beta)}{2}\right)\right)\right\} \\
&= \exp\left\{ - |\theta|^{1+\beta} \frac{\Gamma(1-\beta)}{-\beta} \frac{\psi_1\lambda^{1+\beta}}{1+\beta} \mathrm{e}^{-\mathrm{i}\,\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{2}}\right\} \\
&= \exp\left\{ - |\theta|^{1+\beta} \omega_\beta(\theta)\right\}, \qquad \theta \in \mathbb{R}.
\end{aligned}$$

Together with (3.37), we obtain the first convergence.

By Proposition 3.2, for each $N \in \mathbb{N}$, $\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)}$ exists and hence

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} \widetilde{S}^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0,$$

and

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1}R^{(N,n)} = \left(t\sum_{j=1}^{N}\left(\frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}}\right)\right)\right)_{t\in\mathbb{R}_+}.$$

Based on the above considerations, using the decomposition (3.36) as well, we obtain the second convergence. □

**Proof of Theorem 3.17.** We commence with the proof of the first convergence. Using the decomposition in (3.36), it will suffice to show that there exists a sequence $(a_N)_{N\in\mathbb{N}}$, satisfying $\sqrt{N}/a_N = \mathrm{o}(1)$ such that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} a_N^{-1}\sum_{j=1}^{N}\left(\frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}}\right)\right) = (W\,t)_{t\in\mathbb{R}_+}, \tag{3.40}$$

where $W$ is a standard normally distributed random variable. Indeed, by Remark 3.6, $\mathbb{E}\left((1-\alpha)^{-1}\right) < \infty$ for $\beta = 1$. Then for each $n \in \mathbb{N}$, $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2}}\widetilde{S}^{(N,n)}$ exists by Proposition 3.1, hence for such a sequence $(a_N)_{N\in\mathbb{N}}$ we have

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} a_N^{-1}\widetilde{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\left(\sqrt{N}/a_N\right) N^{-\frac{1}{2}}\widetilde{S}^{(N,n)} = 0.$$

First we prove that (3.40) holds with some sequence $(a_N)_{N\in\mathbb{N}}$, then we confirm that this sequence satisfies $\sqrt{N}/a_N = \mathrm{o}(1)$.

To show (3.40) we are applying Theorem 1 in Bradley [10] to the sequence $\lambda(1-\alpha^{(j)})^{-1} - \mathbb{E}(\lambda(1-\alpha)^{-1})$, $j \in \mathbb{N}$, consisting of i.i.d. random variables with zero mean. Note that since these variables are independent, all the correlations are zero (i.e., $\rho = 0$ with the notation of Bradley [10]). Thus, in order to prove that the random variable $\lambda(1-\alpha)^{-1} - \mathbb{E}(\lambda(1-\alpha)^{-1})$ belongs to the domain of attraction of a normal distribution, we have to show that

$$\lim_{x\to\infty}\frac{x^2\,\mathbb{P}\left(\left|\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right| > x\right)}{\mathbb{E}\left((\lambda(1-\alpha)^{-1} - \mathbb{E}(\lambda(1-\alpha)^{-1}))^2\,\mathbf{1}_{\{|\lambda(1-\alpha)^{-1}-\mathbb{E}(\lambda(1-\alpha)^{-1})|\leqslant x\}}\right)} = 0.$$

Note that the calculations in (3.38) and (3.39) are valid for $\beta = 1$ as well, meaning that the numerator of the fraction converges to $\psi_1\lambda^2/2$. Furthermore, since by Remark 3.6 the second moment of the random variable $\lambda(1-\alpha)^{-1}$ is infinite, then by the monotone convergence theorem the denominator of the fraction converges to infinity as $x \to \infty$. Therefore, the limit of the fraction is indeed 0. Consequently, by Theorem 1 in Bradley [10], for each $n \in \mathbb{N}$,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} a_N^{-1}R^{(N,n)} = \left(\lfloor nt\rfloor W\right)_{t\in\mathbb{R}_+}$$

with some sequence $(a_N)_{N\in\mathbb{N}}$ that converges to infinity as $N \to \infty$. Now the convergence 3.40 that we aimed to prove easily follows after a division with $n$ and taking the limit as $n \to \infty$.

It remains to show that $\sqrt{N}/a_N = \mathrm{o}(1)$ as $N \to \infty$, which we do by applying formula (5.23) on page 579 of Feller [14]. It states that the sequence $(a_N)_{N\in\mathbb{N}}$ satisfies

$$\lim_{N\to\infty}\frac{N\,\mathbb{E}\left(\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)^2\mathbf{1}_{\{|\frac{\lambda}{1-\alpha}-\mathbb{E}(\frac{\lambda}{1-\alpha})|<a_N\}}\right)}{a_N^2} = C$$

with some constant $C \in (0, \infty)$. However, since the second moment of $(1-\alpha)^{-1}$ is infinite, $\mathbb{E}\left(\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)^2 \mathbf{1}_{\{|\frac{\lambda}{1-\alpha} - \mathbb{E}(\frac{\lambda}{1-\alpha})| < a_N\}}\right) \to \infty$ as $N \to \infty$, meaning that $N/a_N^2$ must converge to zero as $N \to \infty$.

Next we prove the second convergence. By Proposition 3.2, for each $N \in \mathbb{N}$, the limit $\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)}$ exists and hence

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} \widetilde{S}^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0,$$

and

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} R^{(N,n)} = \left( t \sum_{j=1}^{N} \left( \frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}}\right)\right)\right)_{t\in\mathbb{R}_+}.$$

Based on the above considerations, using the decomposition (3.36) as well, we obtain the second convergence with the same sequence $(a_N)_{N\in\mathbb{N}}$. □

**Proof of Theorem 3.18.** First note that, since $\beta > 1$, by Remark 3.6, $\operatorname{Var}((1-\alpha)^{-1}) < \infty$. Hence, by the central limit theorem, for each $n \in \mathbb{N}$,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2}} R^{(N,n)} = \left(\lfloor nt \rfloor W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}\right)_{t\in\mathbb{R}_+}.$$

Consequently,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2}} R^{(N,n)} = (W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})} t)_{t\in\mathbb{R}_+}.$$

By Theorem 3.14, $\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)}$ exists, hence

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-\frac{1}{2}} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0.$$

Using the decomposition (3.36), we have the first convergence.

Similarly, for each $N \in \mathbb{N}$,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} R^{(N,n)} = \left( \sum_{j=1}^{N} \left( \frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}}\right)\right) t\right)_{t\in\mathbb{R}_+},$$

and, by the central limit theorem,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} R^{(N,n)} = (W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})} t)_{t\in\mathbb{R}_+}.$$

By Theorem 3.14, we also have

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0,$$

which yields the second convergence using the decomposition (3.36) as well. □

**Proof of Lemma 3.20.** For any $z \in \mathbb{C}$ we have

$$|\mathrm{e}^z - 1| = \left| z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right| \leqslant |z| \left( 1 + \frac{|z|}{2!} + \frac{|z|^2}{3!} + \dots \right)$$
$$\leqslant |z| \left( 1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \dots \right) = |z| \mathrm{e}^{|z|},$$

$$
\begin{aligned}
|\mathrm{e}^z - 1 - z| = \left| \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right| &\leqslant \frac{|z|^2}{2} \left( 1 + \frac{|z|}{3} + \frac{|z|^2}{3 \cdot 4} + \dots \right) \\
&\leqslant \frac{|z|^2}{2} \left( 1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \dots \right) = \frac{|z|^2}{2} \mathrm{e}^{|z|},
\end{aligned}
$$

since $3 \cdot 4 \cdots (n+2) \geqslant n!$ for any $n \in \mathbb{N}$. $\square$

**Proof of Lemma 3.21** Using the dominated convergence theorem, first we check that

$$
\lim_{N \to \infty} N \int_0^{1-\varepsilon} \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a = 0 \tag{3.41}
$$

for all $\varepsilon \in (0,1)$. By applying (3.12) and using (3.14), for any $\varepsilon \in (0,1)$ and $a \in (0, 1-\varepsilon)$, we get

$$
\left| N \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \right| \leqslant N \left| \frac{\lambda}{1-a} z_N(a) \right| \mathrm{e}^{\left| \frac{\lambda}{1-a} z_N(a) \right|} \to 0 \tag{3.42}
$$

as $N \to \infty$. Further, if $\varepsilon \in (0,1)$ and $a \in (0, 1-\varepsilon)$, then

$$
\left| N \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \right| \leqslant \frac{\lambda}{\varepsilon} \sup_{N \in \mathbb{N}} \sup_{a \in (0,1-\varepsilon)} |N z_N(a)| \, \mathrm{e}^{\frac{\lambda}{\varepsilon} \sup_{N \in \mathbb{N}} \sup_{a \in (0,1-\varepsilon)} |z_N(a)|} =: C_\varepsilon,
$$

where $C_\varepsilon \in \mathbb{R}_+$. Since $\int_0^1 \psi(a)(1-a)^\beta \, \mathrm{d}a = 1$, we have

$$
\left| N \int_0^{1-\varepsilon} \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a \right| \leqslant \int_0^{1-\varepsilon} C_\varepsilon \psi(a)(1-a)^\beta \, \mathrm{d}a < \infty.
$$

Therefore, $(0, 1-\varepsilon) \ni a \mapsto C_\varepsilon \psi(a)(1-a)^\beta$ serves as a dominating integrable function. Thus the pointwise convergence in (3.42) results (3.41). Moreover, for all $\varepsilon \in (0,1)$, we have

$$
\begin{aligned}
&\left| N \int_0^1 \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a - \psi_1 I \right| \\
&\qquad \leqslant \left| N \int_0^{1-\varepsilon} \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a \right| \\
&\qquad\quad + \left| N \int_{1-\varepsilon}^1 \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) (\psi(a) - \psi_1)(1-a)^\beta \, \mathrm{d}a \right| \\
&\qquad\quad + \psi_1 \left| N \int_{1-\varepsilon}^1 \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) (1-a)^\beta \, \mathrm{d}a - I \right|,
\end{aligned}
$$

where

$$
\begin{aligned}
&\left| N \int_{1-\varepsilon}^1 \left( 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) (\psi(a) - \psi_1)(1-a)^\beta \, \mathrm{d}a \right| \\
&\leqslant N \sup_{a \in [1-\varepsilon, 1)} |\psi(a) - \psi_1| \int_{1-\varepsilon}^1 \left| 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right| (1-a)^\beta \, \mathrm{d}a,
\end{aligned}
$$

with $\sup_{a \in [1-\varepsilon,1)} |\psi(a) - \psi_1| \to 0$ as $\varepsilon \downarrow 0$, by the assumption. First taking $\limsup_{N \to \infty}$ and then $\varepsilon \downarrow 0$, using (3.41), we obtain the statement. $\square$

# Chapter 4

# Simultaneous limit theorems for the aggregation of randomized INAR(1) processes with Poisson innovations

This chapter is about the simultaneous limit theorems for the aggregation of randomized INAR(1) processes with Poisson innovations. The proofs of this chapter are based on the paper Barczy et al. [4].

We continue to investigate the temporal and contemporaneous aggregates of the randomized INAR(1) processes with Poisson innovations with the mixing distribution given in (3.6). In Section 4.1 two simultaneous limit theorems are given, meaning that the time scale $n$ and number of independent copies $N$ tend to infinity together, at some given rate. To represent the connection between the two parameters, throughout this chapter, we will use a sequence $N_n$, $n \in \mathbb{N}$, instead of the parameter $N$, and the theorems will be given as $n$ tends to infinity, which will always imply that $N_n$, $n \in \mathbb{N}$, tends to infinity as well. The two limit theorems in question cover the cases when $\beta \in (-1, 0)$ with $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$, and when $\beta = 0$ with $(\log N_n)^2 n^{-1} \to \infty$. This section also contains a technical result that is needed for the proofs of these theorems, which are given in Section 4.2. For the proofs, a new tool, namely Lemma 4.5 had to be developed, that is the counterpart of Lemma 3.21, which was applied to prove some of the iterated limit theorems in Chapter 3.

We note that the rest of the cases ($\beta > 0$, and $\beta \in (-1, 0]$ with a different rate of $n$ and $N$ than those in the previous paragraph) remain for future work. The proof technique that we applied could not be used for these cases. To be more precise, in the rest of the cases we could not find an $\varepsilon_n$, $n \in \mathbb{N}$, sequence to satisfy all three conditions of Lemma 4.5. However, very recently, for randomized autoregressive processes of order 1, Pilipauskaitė et al. [37] have found a somewhat new approach for studying simultaneous limits. Namely, they used an infinite series representation of the stationary distribution of their model for calculating the characteristic function of the finite dimensional distributions in question. In our case, i.e., in case of randomized INAR(1) processes, we also derived such a formula given in Barczy et al. [4, Formula (1.4)], and it is much more complicated. As a future work, using it, we plan to handle the remaining cases, which are left open: $\beta \in (0, \infty)$, and $\beta \in (-1, 0]$ with different proportions of the time scale $n$ and the number of independent copies $N$.

## 4.1 Simultaneous limit theorems

**Theorem 4.1.** *If* $\beta \in (-1, 0)$, *then*

$$n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \widetilde{S}^{(N_n,n)} \stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow} (V_{2(1+\beta)} t)_{t \in \mathbb{R}_+} \qquad \text{as } n \to \infty \text{ and } N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty,$$

*where* $V_{2(1+\beta)}$ *is a symmetric* $2(1+\beta)$*-stable random variable (not depending on* $t$*) with characteristic function*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta V_{2(1+\beta)}}) = \mathrm{e}^{-K_\beta |\theta|^{2(1+\beta)}}, \qquad \theta \in \mathbb{R},$$

*where* $K_\beta = \psi_1(\frac{\lambda}{2})^{1+\beta} \frac{\Gamma(-\beta)}{1+\beta}$.

We note that Theorem 4.1 can be considered as a counterpart of Theorem 3.8. The scaling factors and the limit processes coincide in these two theorems.

**Theorem 4.2.** *If* $\beta = 0$, *then*

$$n^{-1}(N_n \log N_n)^{-\frac{1}{2}} \widetilde{S}^{(N_n,n)} \stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow} (W_{\lambda\psi_1} t)_{t \in \mathbb{R}_+} \qquad \text{as } n \to \infty \text{ and } (\log N_n)^2 n^{-1} \to \infty,$$

*where* $W_{\lambda\psi_1}$ *has a normal distribution with mean* $0$ *and variance* $\lambda\psi_1$.

We note that Theorem 4.2 can be considered as a counterpart of Theorem 3.9. The scaling factors and the limit processes coincide in these two theorems. Let us note that the theorems of this section are summarized in a table in Appendix A.

In the next remark we compare our assumptions in Theorems 4.1 and 4.2 with the corresponding assumptions in Pilipauskaitė and Surgailis [38] for analogous results about simultaneous aggregation of random coefficient AR(1) processes.

**Remark 4.3.** In Theorem 4.1 (where $\beta \in (-1, 0)$), the condition $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$ as $n \to \infty$ yields that $N_n \to \infty$ as $n \to \infty$ and

$$N_n^{\frac{1}{1+\beta}} n^{-1} = N_n N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty \qquad \text{as } n \to \infty,$$

which is the form of the condition in Pilipauskaitė and Surgailis [38] for their convergence (2.12) for simultaneous aggregation of random coefficient AR(1) with the same mixing distribution given in (3.6). However, in case of $\beta \in (-1, 0)$, the condition $N_n^{\frac{1}{1+\beta}} n^{-1} \to \infty$ as $n \to \infty$ does not imply that $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$ as $n \to \infty$ in general. Indeed, for example, if $N_n := \lfloor n^\gamma \ln n \rfloor$ with some $\gamma \in (1+\beta, -1-\frac{1}{\beta})$, then $N_n^{\frac{1}{1+\beta}} n^{-1} \sim n^{-1+\frac{\gamma}{1+\beta}} (\ln n)^{\frac{1}{1+\beta}} \to \infty$ as $n \to \infty$, since $-1 + \frac{\gamma}{1+\beta} > 0$, but $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \sim n^{\frac{-1-\beta-\gamma\beta}{1+\beta}} (\ln n)^{\frac{-\beta}{1+\beta}} \to 0$ as $n \to \infty$, since $\frac{-1-\beta-\gamma\beta}{1+\beta} < 0$. We note that the condition $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$ as $n \to \infty$ in Theorem 4.1 might be replaced by $N_n^{\frac{1}{1+\beta}} n^{-1}$ as $n \to \infty$. However, a new proof technique would be needed, since our present one uses effectively that $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$ as $n \to \infty$, for example, in the proof of Theorem 4.1 we argue that for large enough $n$ and for any $z \in (N_n^{-1}, 1]$, we have $z^{-1} n N_n^{\frac{-1}{1+\beta}} |O(1)| \leqslant |O(1)|$ (see (4.8)).

In Theorem 4.2 (where $\beta = 0$), the condition $(\log N_n)^2 n^{-1} \to \infty$ as $n \to \infty$ yields that $N_n \to \infty$ as $n \to \infty$ and $N_n n^{-1} = n^{-1}(\log N_n)^2 \frac{N_n}{(\log N_n)^2} \to \infty \cdot \infty = \infty$ as $n \to \infty$, which is the form of the condition (1.6) in Pilipauskaitė and Surgailis [38] for their convergence (2.13). However, the condition $N_n n^{-1} \to \infty$ as $n \to \infty$ does not imply that $(\log N_n)^2 n^{-1} \to \infty$ as $n \to \infty$ in general. Indeed, for example, if $N_n := n^2$, then $N_n n^{-1} = n \to \infty$ as $n \to \infty$, but $(\log N_n)^2 n^{-1} = 4n^{-1}(\ln n)^2 \to 0$ as $n \to \infty$. Further, one can check that

$$\lim_{n\to\infty} \frac{n^{-1}(N_n \log N_n)^{-\frac{1}{2}}}{n^{-1}(N_n \log(N_n/n))^{-\frac{1}{2}}} = 1,$$

where $n^{-1}(N_n \log(N_n/n))^{-\frac{1}{2}}$ is the scaling factor in (2.13) in Pilipauskaitė and Surgailis [38]. Indeed,

$$\begin{aligned}\lim_{n\to\infty} \frac{n^{-1}(N_n \log N_n)^{-\frac{1}{2}}}{n^{-1}(N_n \log(N_n/n))^{-\frac{1}{2}}} &= \lim_{n\to\infty} \left( \frac{N_n \log N_n}{N_n \log N_n - N_n \log n} \right)^{-\frac{1}{2}} \\ &= \lim_{n\to\infty} \left( \frac{1}{1 - \log n / \log N_n} \right)^{-\frac{1}{2}} = 1,\end{aligned}$$

since $(\log n / \log N_n)^2 = [(\log N_n)^{-2} n] n^{-1} (\log n)^2 \to 0 \cdot 0 = 0$ as $n \to \infty$ under the condition $(\log N_n)^2 n^{-1} \to \infty$ as $n \to \infty$. We note that the condition $(\log N_n)^2 n^{-1} \to \infty$ as $n \to \infty$ in Theorem 4.2 might be replaced by $N_n n^{-1}$ as $n \to \infty$. However, a new proof technique would be needed, since our present one uses effectively that $(\log N_n)^2 n^{-1} \to \infty$ as $n \to \infty$. For example, in the proof of Theorem 4.2 we argue that $\frac{n}{N_n \log N_n} \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} 1 \, \mathrm{d}z = \frac{n}{(\log N_n)^2} \left(1 - \frac{1}{N_n}\right) \to 0$ as $n \to \infty$ (see (4.15)). □

**Remark 4.4.** The proofs of Theorems 4.1 and 4.2 are based on the explicit formula of the joint generator function of $(X_1, \ldots, X_k)$ given in (2.21), where $k \in \mathbb{N}$, and an auxiliary Lemma 4.5, which gives a set of sufficient conditions for the convergence of the integral $N_n \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \psi(a)(1-a)^\beta \, \mathrm{d}a$ as $n \to \infty$, where $(z_n(a))_{n\in\mathbb{N}}$ is a sequence of complex numbers. We were not able to use this proof technique for the cases $\beta \in (0, \infty)$, and $\beta \in (-1, 0]$ with different proportions of the time scale $n$ and the number of independent copies $N$. However, as it was explained in the beginning of this chapter, as future work, using a different technique, we plan to handle these cases as well. □

The next lemma is a variant of Lemma 3.21, and we use it in the proofs of Theorems 4.1 and 4.2.

**Lemma 4.5.** *Suppose that $(0, 1) \ni x \mapsto \psi(x)(1 - x)^\beta$ is a probability density, where $\psi$ is a function on $(0, 1)$ having a limit $\lim_{x\uparrow 1} \psi(x) = \psi_1 \in (0, \infty)$ (and then necessarily $\beta \in (-1, \infty)$). For all $a \in (0, 1)$, let $(z_n(a))_{n\in\mathbb{N}}$ be a sequence of complex numbers, let $n_0 \in \mathbb{N}$, $(\varepsilon_n)_{n\geqslant n_0}$ be a sequence in $(0, 1)$ with $\lim_{n\to\infty} \varepsilon_n = 0$, and let $(N_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that*

$$\sup_{n\geqslant n_0} \varepsilon_n^{-1} N_n \sup_{a\in(0,1-\varepsilon_n)} |z_n(a)| < \infty, \tag{4.1}$$

$$\limsup_{n\to\infty} N_n \int_{1-\varepsilon_n}^1 \left|1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right| (1-a)^\beta \, \mathrm{d}a < \infty,$$

$$\lim_{n\to\infty} \left| N_n \int_{1-\varepsilon_n}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) (1-a)^\beta \, \mathrm{d}a - I \right| = 0$$

*with some* $I \in \mathbb{C}$. *Then*

$$\lim_{n\to\infty} N_n \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \psi(a)(1-a)^\beta \,\mathrm{d}a = \psi_1 I.$$

## 4.2 Proofs

In the following two proofs, the notations $\mathrm{O}(1)$ and $|\,\mathrm{O}(1)|$ stand for a possibly complex and respectively real sequence $(a_k)_{k\in\mathbb{N}}$ that is bounded and can only depend on the parameters $\lambda$, $\psi_1$, $\beta$, and on some fixed $m \in \mathbb{N}$ and $\theta_1, \dots, \theta_m \in \mathbb{R}$. Further, we call the attention that several $\mathrm{O}(1)$-s (respectively $|\,\mathrm{O}(1)|$-s) in the same formula do not necessarily mean the same bounded sequence.

**Proof of Theorem 4.1.** To prove this limit theorem we have to show that for any sequence $(N_n)_{n\in\mathbb{N}}$ of positive integers with $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$, we have

$$n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \widetilde{S}^{(N_n,n)} \stackrel{\mathcal{D}_\mathrm{f}}{\longrightarrow} (V_{2(1+\beta)} t)_{t\in\mathbb{R}_+} \qquad \text{as } n\to\infty.$$

For this, by continuous mapping theorem, it is enough to verify that for any $m \in \mathbb{N}$ and $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ with $0 =: t_0 < t_1 < \ldots < t_m$, we have

$$\begin{aligned}
n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \sum_{j=1}^{N_n} \Bigg( &\sum_{k=1}^{\lfloor nt_1\rfloor} \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right), \sum_{k=\lfloor nt_1\rfloor+1}^{\lfloor nt_2\rfloor} \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right), \\
&\ldots, \sum_{k=\lfloor nt_{m-1}\rfloor+1}^{\lfloor nt_m\rfloor} \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right)\Bigg) \\
\stackrel{\mathcal{D}}{\longrightarrow}\; &V_{2(1+\beta)}(t_1, t_2 - t_1, \dots, t_m - t_{m-1}) \qquad \text{as } n\to\infty.
\end{aligned}$$

So, by continuity theorem, we have to check that for any $m \in \mathbb{N}$, $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\theta_1, \dots, \theta_m \in \mathbb{R}$ the convergence

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{\ell=1}^m \theta_\ell n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \sum_{j=1}^{N_n} \sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor} \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right)\right\}\right) \\
&\quad = \mathbb{E}\left(\exp\left\{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \sum_{j=1}^{N_n} \sum_{\ell=1}^m \theta_\ell \sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor} \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\right)\right\}\right) \\
&\quad = \left[\mathbb{E}\left(\exp\left\{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \sum_{\ell=1}^m \theta_\ell \sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor} \left(X_k - \frac{\lambda}{1-\alpha}\right)\right\}\right)\right]^{N_n} \\
&\quad \to \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}) V_{2(1+\beta)}}\right) = \mathrm{e}^{-K_\beta |\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1})|^{2(1+\beta)}} \qquad \text{as } n\to\infty
\end{aligned}$$

holds. Note that it suffices to show

$$\begin{aligned}
\Theta_n &:= N_n \left[1 - \mathbb{E}\left(\exp\left\{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \sum_{\ell=1}^m \theta_\ell \sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor} \left(X_k - \frac{\lambda}{1-\alpha}\right)\right\}\right)\right] \\
&\to K_\beta \left|\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1})\right|^{2(1+\beta)} \qquad \text{as } n\to\infty,
\end{aligned}$$

since it implies that for every $k \in \mathbb{N}$ $(1 - \Theta_n/N_n)^{N_n} \to \mathrm{e}^{-K_\beta |\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1})|^{2(1+\beta)}}$ as $n \to \infty$, as desired. Let us recall that, as it was introduced before the proof of Theorem 3.8, the joint conditional generator function of $X_0, X_1, \ldots, X_k$ given $\alpha$ is denoted by $F_{0,\ldots,k}(z_0, \ldots, z_k \,|\, \alpha)$, $z_0, \ldots, z_k \in D = \{z \in \mathbb{C} : |z| \leqslant 1\}$. Then, by applying (2.21) to the left hand side, we get

$$\begin{aligned}
\Theta_n = N_n \, \mathbb{E}\Bigg[ & 1 - \mathrm{e}^{-\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \frac{\lambda}{1-\alpha} \sum_{\ell=1}^m \theta_\ell (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor)} \\
& \times F_{0,\ldots,\lfloor n t_m \rfloor - 1}\Big( \underbrace{\mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_1}, \ldots, \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_1}}_{\lfloor n t_1 \rfloor \text{ items}}, \\
& \qquad \ldots, \underbrace{\mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_m}, \ldots, \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_m}}_{\lfloor n t_m \rfloor - \lfloor n t_{m-1} \rfloor \text{ items}} \,\Big|\, \alpha \Big) \Bigg] \\
= N_n \, \mathbb{E}\Big[ & 1 - \mathrm{e}^{\frac{\lambda}{1-\alpha} A_n(\alpha)} \Big] = N_n \int_0^1 \Big( 1 - \mathrm{e}^{\frac{\lambda}{1-a} A_n(a)} \Big) \psi(a) (1-a)^\beta \, \mathrm{d}a
\end{aligned}$$

with

$$\begin{aligned}
A_n(a) := & \sum_{\ell=1}^m \Big( \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell} - 1 - \mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell \Big) (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor) \\
& + \sum_{1 \leqslant \ell_1 < \ell_2 \leqslant m} \sum_{k_1 = \lfloor n t_{\ell_1 - 1} \rfloor + 1}^{\lfloor n t_{\ell_1} \rfloor} \sum_{k_2 = \lfloor n t_{\ell_2 - 1} \rfloor + 1}^{\lfloor n t_{\ell_2} \rfloor} a^{k_2 - k_1} \big( \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_{\ell_1}} - 1 \big) \big( \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_{\ell_2}} - 1 \big) \\
& \qquad \times \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \big( (\lfloor n t_{\ell_1} \rfloor - k_1) \theta_{\ell_1} + \sum_{\ell = \ell_1 + 1}^{\ell_2 - 1} \theta_\ell (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor) + (k_2 - 1 - \lfloor n t_{\ell_2 - 1} \rfloor) \theta_{\ell_2} \big)} \\
& + \sum_{\ell=1}^m \sum_{\lfloor n t_{\ell-1} \rfloor + 1 \leqslant k_1 < k_2 \leqslant \lfloor n t_\ell \rfloor} a^{k_2 - k_1} \big( \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell} - 1 \big)^2 \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} (k_2 - k_1 - 1) \theta_\ell}
\end{aligned}$$

for $a \in [0,1]$. The aim of the following discussion is to apply Lemma 4.5 with $z_n(a) := A_n(a)$, $n \in \mathbb{N}$, $a \in (0,1)$, $\varepsilon_n := N_n^{\frac{\beta}{1+\beta}}$, $n \in \mathbb{N}$, and

$$I := \psi_1^{-1} K_\beta \left| \sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}) \right|^{2(1+\beta)}.$$

Since $\beta \in (-1, 0)$, we have $\varepsilon_n \in (0,1)$ for $n \geqslant n_0$, where $n_0$ is sufficiently large, and $\lim_{n\to\infty} \varepsilon_n = 0$. First we check (4.1). Using (3.11), for any $a \in (0,1)$ we get

$$\begin{aligned}
|A_n(a)| \leqslant & \sum_{\ell=1}^m n^{-2} N_n^{-\frac{1}{1+\beta}} \frac{\theta_\ell^2}{2} (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor) \\
& + \sum_{1 \leqslant \ell_1 < \ell_2 \leqslant m} n^{-2} N_n^{-\frac{1}{1+\beta}} |\theta_{\ell_1}| |\theta_{\ell_2}| (\lfloor n t_{\ell_1} \rfloor - \lfloor n t_{\ell_1 - 1} \rfloor)(\lfloor n t_{\ell_2} \rfloor - \lfloor n t_{\ell_2 - 1} \rfloor) \\
& + \sum_{\ell=1}^m n^{-2} N_n^{-\frac{1}{1+\beta}} \frac{\theta_\ell^2}{2} (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor)(\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor - 1) \\
= & \frac{1}{2} n^{-2} N_n^{-\frac{1}{1+\beta}} \left( \sum_{\ell=1}^m |\theta_\ell| (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor) \right)^2 \leqslant \frac{1}{2} N_n^{-\frac{1}{1+\beta}} \left( \sum_{\ell=1}^m |\theta_\ell| (t_\ell - t_{\ell-1} + 1) \right)^2,
\end{aligned}$$

since $\frac{1}{n}(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor) \leqslant \frac{1}{n}(nt_\ell - nt_{\ell-1}+1) = t_\ell - t_{\ell-1} + \frac{1}{n} \leqslant t_\ell - t_{\ell-1}+1$. Consequently, since $\varepsilon_n^{-1} N_n = N_n^{\frac{1}{1+\beta}}$, we have

$$\sup_{n\geqslant n_0} \varepsilon_n^{-1} N_n \sup_{a\in(0,1-\varepsilon_n)} |A_n(a)| \leqslant \frac{1}{2}\biggl(\sum_{\ell=1}^m |\theta_\ell|(t_\ell - t_{\ell-1}+1)\biggr)^2 < \infty,$$

i.e., (4.1) is satisfied. Therefore, by Lemma 4.5, substituting $a = 1 - z^{-1} N_n^{-\frac{1}{1+\beta}}$ with $z > 0$, the statement of the theorem will follow from

$$\begin{aligned} &\limsup_{n\to\infty} N_n \int_{1-N_n^{\frac{\beta}{1+\beta}}}^1 \Bigl|1 - \mathrm{e}^{\frac{\lambda}{1-a} A_n(a)}\Bigr| (1-a)^\beta \,\mathrm{d}a \\ &= \limsup_{n\to\infty} \int_{N_n^{-1}}^\infty \Bigl|1 - \mathrm{e}^{\lambda z N_n^{\frac{1}{1+\beta}} A_n\left(1 - z^{-1} N_n^{-\frac{1}{1+\beta}}\right)}\Bigr| z^{-(2+\beta)} \,\mathrm{d}z < \infty \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \biggl| N_n \int_{1-N_n^{\frac{\beta}{1+\beta}}}^1 \Bigl(1 - \mathrm{e}^{\frac{\lambda}{1-a} A_n(a)}\Bigr) (1-a)^\beta \,\mathrm{d}a - I \biggr| \\ &= \lim_{n\to\infty} \biggl| \int_{N_n^{-1}}^\infty \Bigl(1 - \mathrm{e}^{\lambda z N_n^{\frac{1}{1+\beta}} A_n\left(1 - z^{-1} N_n^{-\frac{1}{1+\beta}}\right)}\Bigr) z^{-(2+\beta)} \,\mathrm{d}z - I \biggr| = 0 \end{aligned} \tag{4.3}$$

with

$$\begin{aligned} I &= \psi_1^{-1} K_\beta \biggl|\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1})\biggr|^{2(1+\beta)} = \left(\frac{\lambda}{2}\biggl|\sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1})\biggr|^2\right)^{1+\beta} \int_0^\infty (1-\mathrm{e}^{-z}) z^{-(2+\beta)} \,\mathrm{d}z \\ &= \int_0^\infty \left(1 - \mathrm{e}^{-\frac{\lambda z}{2}\left(\sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1})\right)^2}\right) z^{-(2+\beta)} \,\mathrm{d}z, \end{aligned}$$

where the first equality is justified by Li [31, formula (1.28)].

Next we check (4.2) and (4.3). By Taylor expansion,

$$\begin{aligned} \mathrm{e}^{\mathrm{i}n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell} - 1 &= \mathrm{i}n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell + n^{-2} N_n^{-\frac{1}{1+\beta}} \mathrm{O}(1) = n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \mathrm{O}(1), \\ \mathrm{e}^{\mathrm{i}n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell} - 1 - \mathrm{i}n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell &= -n^{-2} N_n^{-\frac{1}{1+\beta}} \frac{\theta_\ell^2}{2} + n^{-3} N_n^{-\frac{3}{2(1+\beta)}} \mathrm{O}(1) \\ &= n^{-2} N_n^{-\frac{1}{1+\beta}} \mathrm{O}(1) \end{aligned}$$

for all $\ell \in \{1, \ldots, m\}$, resulting

$$\begin{aligned} \lambda z N_n^{\frac{1}{1+\beta}} A_n\left(1 - z^{-1} N_n^{-\frac{1}{1+\beta}}\right) &= -\frac{\lambda z \bigl(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)\bigr)^2}{2n^2} + \frac{z\,\mathrm{O}(1)}{N_n^{\frac{1}{2(1+\beta)}}} + \frac{n\,\mathrm{O}(1)}{N_n^{\frac{1}{1+\beta}}} \\ &= -\frac{\lambda z \bigl(\sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1})\bigr)^2}{2} + z\,\mathrm{o}(1) + \frac{z\,\mathrm{O}(1)}{N_n^{\frac{1}{2(1+\beta)}}} + \frac{n\,\mathrm{O}(1)}{N_n^{\frac{1}{1+\beta}}} \end{aligned} \tag{4.4}$$

for $z > N_n^{-1}$. (We recall that for a sequence $(a_n)_{n\in\mathbb{N}}$ the notation $a_n = \mathrm{o}(1)$ means that $a_n \to 0$ as $n \to \infty$. ) Indeed, for $z > N_n^{-1}$, we also have $z > N_n^{-\frac{1}{1+\beta}}$, yielding that $1 - z^{-1} N_n^{-\frac{1}{1+\beta}} \in (0,1)$, and

$$
\begin{aligned}
& A_n \left(1 - z^{-1} N_n^{-\frac{1}{1+\beta}}\right) \\
& = \sum_{\ell=1}^{m} \big(\mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell} - 1 - \mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell\big) (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \\
& \quad + \sum_{1 \leqslant \ell_1 < \ell_2 \leqslant m} \sum_{k_1 = \lfloor nt_{\ell_1 - 1} \rfloor + 1}^{\lfloor nt_{\ell_1} \rfloor} \sum_{k_2 = \lfloor nt_{\ell_2 - 1} \rfloor + 1}^{\lfloor nt_{\ell_2} \rfloor} \left(1 - z^{-1} N_n^{-\frac{1}{1+\beta}}\right)^{k_2 - k_1} \big(\mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_{\ell_1}} - 1\big) \\
& \qquad \times \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \big((\lfloor nt_{\ell_1} \rfloor - k_1) \theta_{\ell_1} + \sum_{\ell = \ell_1 + 1}^{\ell_2 - 1} \theta_\ell (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) + (k_2 - 1 - \lfloor nt_{\ell_2 - 1} \rfloor) \theta_{\ell_2}\big)} \\
& \qquad \times \big(\mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_{\ell_2}} - 1\big) \\
& \quad + \sum_{\ell=1}^{m} \sum_{\lfloor nt_{\ell-1} \rfloor + 1 \leqslant k_1 < k_2 \leqslant \lfloor nt_\ell \rfloor} \left(1 - z^{-1} N_n^{-\frac{1}{1+\beta}}\right)^{k_2 - k_1} \big(\mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \theta_\ell} - 1\big)^2 \\
& \qquad \qquad \times \mathrm{e}^{\mathrm{i} n^{-1} N_n^{-\frac{1}{2(1+\beta)}} (k_2 - k_1 - 1) \theta_\ell} \\
& = \sum_{\ell=1}^{m} \Bigg(-\frac{\theta_\ell^2}{2 n^2 N_n^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{n^3 N_n^{\frac{3}{2(1+\beta)}}}\Bigg) (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \\
& \quad + \sum_{1 \leqslant \ell_1 < \ell_2 \leqslant m} \Bigg(1 + \frac{n\,\mathrm{O}(1)}{z N_n^{\frac{1}{1+\beta}}}\Bigg) \Bigg(\frac{\mathrm{i}\theta_{\ell_1}}{n N_n^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{n^2 N_n^{\frac{1}{1+\beta}}}\Bigg) \Bigg(1 + \frac{\mathrm{O}(1)}{N_n^{\frac{1}{2(1+\beta)}}}\Bigg) \\
& \qquad \times \Bigg(\frac{\mathrm{i}\theta_{\ell_2}}{n N_n^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{n^2 N_n^{\frac{1}{1+\beta}}}\Bigg) (\lfloor nt_{\ell_1} \rfloor - \lfloor nt_{\ell_1 - 1} \rfloor)(\lfloor nt_{\ell_2} \rfloor - \lfloor nt_{\ell_2 - 1} \rfloor) \\
& \quad + \frac{1}{2} \sum_{\ell=1}^{m} \Bigg(1 + \frac{n\,\mathrm{O}(1)}{z N_n^{\frac{1}{1+\beta}}}\Bigg) \Bigg(\frac{\mathrm{i}\theta_\ell}{n N_n^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{n^2 N_n^{\frac{1}{1+\beta}}}\Bigg)^2 \Bigg(1 + \frac{\mathrm{O}(1)}{N_n^{\frac{1}{2(1+\beta)}}}\Bigg) \\
& \qquad \times (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor)(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} - 1 \rfloor) \\
& = -\frac{\sum_{\ell=1}^{m} \theta_\ell^2 (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor)}{2 n^2 N_n^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{n^2 N_n^{\frac{3}{2(1+\beta)}}} \\
& \quad - \frac{\sum_{1 \leqslant \ell_1 < \ell_2 \leqslant m} \theta_{\ell_1} \theta_{\ell_2} (\lfloor nt_{\ell_1} \rfloor - \lfloor nt_{\ell_1 - 1} \rfloor)(\lfloor nt_{\ell_2} \rfloor - \lfloor nt_{\ell_2 - 1} \rfloor)}{n^2 N_n^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N_n^{\frac{3}{2(1+\beta)}}} + \frac{n\,\mathrm{O}(1)}{z N_n^{\frac{2}{1+\beta}}} \\
& \quad - \frac{\sum_{\ell=1}^{m} \theta_\ell^2 (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor)(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} - 1 \rfloor)}{2 n^2 N_n^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N_n^{\frac{3}{2(1+\beta)}}} + \frac{n\,\mathrm{O}(1)}{z N_n^{\frac{2}{1+\beta}}} \\
& = -\frac{\big(\sum_{\ell=1}^{m} \theta_\ell (\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor)\big)^2}{2 n^2 N_n^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N_n^{\frac{3}{2(1+\beta)}}} + \frac{n\,\mathrm{O}(1)}{z N_n^{\frac{2}{1+\beta}}},
\end{aligned}
$$

where we used the following facts:

- $$\mathrm{e}^{\mathrm{i}n^{-1}N_n^{-\frac{1}{2(1+\beta)}}\left((\lfloor nt_{\ell_1}\rfloor-k_1)\theta_{\ell_1}+\sum_{\ell=\ell_1+1}^{\ell_2-1}\theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)+(k_2-1-\lfloor nt_{\ell_2-1}\rfloor)\theta_{\ell_2}\right)} = \mathrm{e}^{\mathrm{i}N_n^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1)} = 1+N_n^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1) \tag{4.5}$$

- $$\mathrm{e}^{\mathrm{i}n^{-1}N_n^{-\frac{1}{2(1+\beta)}}(k_2-k_1+1)\theta_\ell} = \mathrm{e}^{\mathrm{i}N_n^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1)} = 1+N_n^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1), \tag{4.6}$$

  due to $\lfloor nt_{\ell-1}\rfloor+1 \leqslant k_1 < k_2 \leqslant \lfloor nt_\ell\rfloor$,

- $$\left(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\right)^{k_2-k_1} = 1+\frac{n\,\mathrm{O}(1)}{zN_n^{\frac{1}{1+\beta}}},$$

  following from an application of Bernoulli's inequality:

  $$\left|\left(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\right)^{k_2-k_1}-1\right| \leqslant \frac{k_2-k_1}{zN_n^{\frac{1}{1+\beta}}} \leqslant \frac{\lfloor nt_m\rfloor}{zN_n^{\frac{1}{1+\beta}}}.$$

By (4.4), for large enough $n$ and for any $z\in[1,\infty)$, we have

$$\begin{aligned}
&\lambda zN_n^{\frac{1}{1+\beta}}\operatorname{Re}A_n\big(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\big)\\
&= -\frac{\lambda z(\sum_{\ell=1}^m\theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor))^2}{2n^2}\left(1-\frac{\operatorname{Re}\mathrm{O}(1)}{N_n^{\frac{1}{2(1+\beta)}}}\right)+\frac{n\operatorname{Re}\mathrm{O}(1)}{N_n^{\frac{1}{1+\beta}}}\\
&\leqslant -\frac{\lambda z(\sum_{\ell=1}^m\theta_\ell(t_\ell-t_{\ell-1}))^2}{4}+\frac{n|\mathrm{O}(1)|}{N_n^{\frac{1}{1+\beta}}} \leqslant -\frac{\lambda(\sum_{\ell=1}^m\theta_\ell(t_\ell-t_{\ell-1}))^2}{4}+\frac{n|\mathrm{O}(1)|}{N_n^{\frac{1}{1+\beta}}} \leqslant 0,
\end{aligned}$$

since $N_n^{\frac{1}{2(1+\beta)}}\to\infty$ as $n\to\infty$, and $nN_n^{-\frac{1}{1+\beta}}\leqslant nN_n^{\frac{\beta}{1+\beta}}\to 0$ as $n\to\infty$, hence we obtain for large enough $n$,

$$\begin{aligned}
&\int_1^\infty\left|1-\mathrm{e}^{\lambda zN_n^{\frac{1}{1+\beta}}A_n(1-z^{-1}N_n^{-\frac{1}{1+\beta}})}\right|z^{-(\beta+2)}\,\mathrm{d}z\\
&\leqslant\int_1^\infty\left(1+\mathrm{e}^{\lambda zN_n^{\frac{1}{1+\beta}}\operatorname{Re}A_n(1-z^{-1}N_n^{-\frac{1}{1+\beta}})}\right)z^{-(\beta+2)}\,\mathrm{d}z\leqslant 2\int_1^\infty z^{-(\beta+2)}\,\mathrm{d}z<\infty.
\end{aligned}\tag{4.7}$$

Again by (4.4), for large enough $n$ and for any $z\in\big(N_n^{-1},1\big]$, we have

$$\begin{aligned}
&\left|\lambda zN_n^{\frac{1}{1+\beta}}A_n\Big(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\Big)\right| \leqslant \frac{\lambda z(\sum_{\ell=1}^m\theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor))^2}{2n^2}+\frac{z|\mathrm{O}(1)|}{N_n^{\frac{1}{2(1+\beta)}}}+\frac{n|\mathrm{O}(1)|}{N_n^{\frac{1}{1+\beta}}}\\
&\leqslant z\bigg(\frac{\lambda(\sum_{\ell=1}^m|\theta_\ell|(t_\ell-t_{\ell-1}+1))^2}{2}+\frac{|\mathrm{O}(1)|}{N_n^{\frac{1}{2(1+\beta)}}}+|\mathrm{O}(1)|\bigg)\leqslant z|\mathrm{O}(1)|\leqslant|\mathrm{O}(1)|,
\end{aligned}$$

where we used that $z\in\big(N_n^{-1},1\big]$ and $nN_n^{\frac{\beta}{1+\beta}}\to 0$ as $n\to\infty$ imply that

$$\frac{1}{z}\frac{n|\mathrm{O}(1)|}{N_n^{\frac{1}{1+\beta}}}\leqslant N_n\frac{n|\mathrm{O}(1)|}{N_n^{\frac{1}{1+\beta}}}=\frac{n|\mathrm{O}(1)|}{N_n^{-\frac{\beta}{1+\beta}}}=|\mathrm{O}(1)|. \tag{4.8}$$

Hence, using (3.12), we obtain for large enough $n$

$$\begin{aligned}
&\int_{N_n^{-1}}^1 \left|1-\mathrm{e}^{\lambda z N_n^{\frac{1}{1+\beta}} A_n\left(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\right)}\right| z^{-(2+\beta)}\,\mathrm{d}z \\
&\leqslant \int_{N_n^{-1}}^1 \left|\lambda z N_n^{\frac{1}{1+\beta}} A_n\big(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\big)\right| \mathrm{e}^{\left|\lambda z N_n^{\frac{1}{1+\beta}} A_n\left(1-z^{-1}N_n^{-\frac{1}{1+\beta}}\right)\right|} z^{-(2+\beta)}\,\mathrm{d}z \\
&\leqslant |\operatorname{O}(1)|\mathrm{e}^{|\operatorname{O}(1)|}\int_0^1 z^{-(1+\beta)}\,\mathrm{d}z < \infty,
\end{aligned}$$

which, together with (4.7), imply (4.2).

Now we turn to prove (4.3). By (3.10), we have

$$\begin{aligned}
&\left|\int_0^{N_n^{-1}}\Big(1-\mathrm{e}^{-\frac{\lambda z}{2}(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2}\Big) z^{-(2+\beta)}\,\mathrm{d}z\right| \\
&\leqslant \int_0^{N_n^{-1}} \frac{\lambda z(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2}{2} z^{-(2+\beta)}\,\mathrm{d}z \\
&= \frac{\lambda(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2}{2}\int_0^{N_n^{-1}} z^{-(1+\beta)}\,\mathrm{d}z = \frac{\lambda(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2}{2}\,\frac{N_n^{\beta}}{(-\beta)} \to 0
\end{aligned}$$

as $n\to\infty$, hence (4.3) reduces to checking that $\lim_{n\to\infty} I_n = 0$, where

$$I_n := \int_{N_n^{-1}}^\infty \Big[\mathrm{e}^{\lambda z N_n^{\frac{1}{1+\beta}} A_n(1-z^{-1}N_n^{-\frac{1}{1+\beta}})} - \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2}\Big] z^{-(2+\beta)}\,\mathrm{d}z.$$

Applying again (4.4), we obtain

$$|I_n| \leqslant \int_{N_n^{-1}}^\infty \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2}\left|\mathrm{e}^{z\operatorname{o}(1)+zN_n^{-\frac{1}{2(1+\beta)}}\operatorname{O}(1)+nN_n^{-\frac{1}{1+\beta}}\operatorname{O}(1)}-1\right| z^{-(2+\beta)}\,\mathrm{d}z.$$

Here, for $z\in(N_n^{-1},\infty)$, we have

$$\big|z\operatorname{o}(1)+zN_n^{-\frac{1}{2(1+\beta)}}\operatorname{O}(1)+nN_n^{-\frac{1}{1+\beta}}\operatorname{O}(1)\big| \leqslant z\big(\operatorname{o}(1)+N_n^{-\frac{1}{2(1+\beta)}}+nN_n^{\frac{\beta}{1+\beta}}\big)|\operatorname{O}(1)|,$$

and hence, by (3.12), we get

$$\begin{aligned}
&\left|\mathrm{e}^{z\operatorname{o}(1)+zN_n^{-\frac{1}{2(1+\beta)}}\operatorname{O}(1)+nN_n^{-\frac{1}{1+\beta}}\operatorname{O}(1)}-1\right| \\
&\leqslant \big|z\operatorname{o}(1)+zN_n^{-\frac{1}{2(1+\beta)}}\operatorname{O}(1)+nN_n^{-\frac{1}{1+\beta}}\operatorname{O}(1)\big|\,\mathrm{e}^{\left|z\operatorname{o}(1)+zN_n^{-\frac{1}{2(1+\beta)}}\operatorname{O}(1)+nN_n^{-\frac{1}{1+\beta}}\operatorname{O}(1)\right|} \\
&\leqslant z\big(\operatorname{o}(1)+N_n^{-\frac{1}{2(1+\beta)}}+nN_n^{\frac{\beta}{1+\beta}}\big)|\operatorname{O}(1)|\,\mathrm{e}^{z\left(\operatorname{o}(1)+N_n^{-\frac{1}{2(1+\beta)}}+nN_n^{\frac{\beta}{1+\beta}}\right)|\operatorname{O}(1)|}.
\end{aligned}$$

Consequently, for large enough $n$,

$$\begin{aligned}
|I_n| &\leqslant \big(\operatorname{o}(1)+N_n^{-\frac{1}{2(1+\beta)}}+nN_n^{\frac{\beta}{1+\beta}}\big)|\operatorname{O}(1)| \\
&\qquad\times\int_{N_n^{-1}}^\infty \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2+z\left(\operatorname{o}(1)+N_n^{-\frac{1}{2(1+\beta)}}+nN_n^{\frac{\beta}{1+\beta}}\right)|\operatorname{O}(1)|} z^{-(1+\beta)}\,\mathrm{d}z \\
&\leqslant \big(\operatorname{o}(1)+N_n^{-\frac{1}{2(1+\beta)}}+nN_n^{\frac{\beta}{1+\beta}}\big)|\operatorname{O}(1)|\int_0^\infty \mathrm{e}^{-\frac{\lambda z}{4}(\sum_{\ell=1}^m \theta_\ell(t_\ell-t_{\ell-1}))^2} z^{-(1+\beta)}\,\mathrm{d}z,
\end{aligned}$$

that gets arbitrarily close to zero as $n$ approaches infinity, since the integral is finite due to the fact that

$$\frac{1}{\Gamma(-\beta)}\left(\frac{\lambda}{4}\left(\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1})\right)^2\right)^{-\beta}\mathrm{e}^{-\frac{\lambda z}{4}(\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1}))^2}\,z^{-(1+\beta)},\qquad z>0,$$

is the density function of a Gamma distributed random variable with parameters $-\beta$ and $\lambda(\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1}))^2/4$. This yields (4.3) completing the proof. □

**Proof of Theorem 4.2.** To prove this limit theorem we have to show that for any sequence $(N_n)_{n\in\mathbb{N}}$ of positive integers with $(\log N_n)^2 n^{-1}\to\infty$ we have

$$n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\,\widetilde{S}^{(N_n,n)}\xrightarrow{\mathcal{D}_{\mathrm{f}}}(W_{\lambda\psi_1}t)_{t\in\mathbb{R}_+}\qquad\text{as}\quad n\to\infty.$$

For this, by continuous mapping theorem, it is enough to verify that for any $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m\in\mathbb{R}_+$ with $0=:t_0<t_1<\ldots<t_m$, we have

$$\begin{aligned}&n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\sum_{j=1}^{N_n}\Bigg(\sum_{k=1}^{\lfloor nt_1\rfloor}\Big(X_k^{(j)}-\frac{\lambda}{1-\alpha^{(j)}}\Big),\sum_{k=\lfloor nt_1\rfloor+1}^{\lfloor nt_2\rfloor}\Big(X_k^{(j)}-\frac{\lambda}{1-\alpha^{(j)}}\Big),\ldots\\&\qquad\qquad\ldots,\sum_{k=\lfloor nt_{m-1}\rfloor+1}^{\lfloor nt_m\rfloor}\Big(X_k^{(j)}-\frac{\lambda}{1-\alpha^{(j)}}\Big)\Bigg)\\&\xrightarrow{\mathcal{D}}W_{\lambda\psi_1}(t_1,t_2-t_1,\ldots,t_m-t_{m-1})\qquad\text{as}\quad n\to\infty.\end{aligned}$$

So, by continuity theorem, we have to check that for any $m\in\mathbb{N}$, $t_0,t_1,\ldots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\ldots<t_m$ and $\theta_1,\ldots,\theta_m\in\mathbb{R}$ the convergence

$$\begin{aligned}&\mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}\sum_{\ell=1}^{m}\theta_\ell n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\sum_{j=1}^{N_n}\sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor}\Big(X_k^{(j)}-\frac{\lambda}{1-\alpha^{(j)}}\Big)\Bigg\}\Bigg)\\&\quad=\mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\sum_{j=1}^{N_n}\sum_{\ell=1}^{m}\theta_\ell\sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor}\Big(X_k^{(j)}-\frac{\lambda}{1-\alpha^{(j)}}\Big)\Bigg\}\Bigg)\\&\quad=\Bigg[\mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\sum_{\ell=1}^{m}\theta_\ell\sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor}\Big(X_k-\frac{\lambda}{1-\alpha}\Big)\Bigg\}\Bigg)\Bigg]^{N_n}\\&\quad\to\mathbb{E}\Big(\mathrm{e}^{\mathrm{i}\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1})W_{\lambda\psi_1}}\Big)=\mathrm{e}^{-\frac{\lambda\psi_1(\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1}))^2}{2}}\qquad\text{as}\quad n\to\infty\end{aligned}$$

holds. Note that it suffices to show

$$\begin{aligned}\Theta_n&:=N_n\Bigg[1-\mathbb{E}\Bigg(\exp\Bigg\{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\sum_{\ell=1}^{m}\theta_\ell\sum_{k=\lfloor nt_{\ell-1}\rfloor+1}^{\lfloor nt_\ell\rfloor}\Big(X_k-\frac{\lambda}{1-\alpha}\Big)\Bigg\}\Bigg)\Bigg]\\&\to\frac{\lambda\psi_1(\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1}))^2}{2}\qquad\text{as}\quad n\to\infty,\end{aligned}$$

since it implies that $(1-\Theta_n/N_n)^{N_n}\to\mathrm{e}^{-\frac{\lambda\psi_1(\sum_{\ell=1}^{m}\theta_\ell(t_\ell-t_{\ell-1}))^2}{2}}$ as $n\to\infty$, as desired. Let us recall that, as it was introduced before the proof of Theorem 3.8, the joint conditional generator function of $X_0,X_1,\ldots,X_k$ given $\alpha$ is denoted by $F_{0,\ldots,k}(z_0,\ldots,z_k\,|\,\alpha)$,

$z_0,\dots,z_k \in D = \{z\in\mathbb{C} : |z| \leqslant 1\}$. Then, by applying (2.21) to the left hand side, we get

$$\begin{aligned}
\Theta_n &= N_n\,\mathbb{E}\Bigg[1-\mathrm{e}^{-\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\frac{\lambda}{1-\alpha}\sum_{\ell=1}^m\theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)} \\
&\qquad\times F_{0,\dots,\lfloor nt_m\rfloor-1}\Big(\underbrace{\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_1},\dots,\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_1}}_{\lfloor nt_1\rfloor\text{ items}},\dots \\
&\qquad\qquad\dots,\underbrace{\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_m},\dots,\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_m}}_{\lfloor nt_m\rfloor-\lfloor nt_{m-1}\rfloor\text{ items}}\;\Big|\;\alpha\Big)\Bigg] \\
&= N_n\,\mathbb{E}\left[1-\mathrm{e}^{\frac{\lambda}{1-\alpha}B_n(\alpha)}\right] = N_n\int_0^1\left(1-\mathrm{e}^{\frac{\lambda}{1-a}B_n(a)}\right)\psi(a)\,\mathrm{d}a
\end{aligned}$$

with

$$\begin{aligned}
B_n(a) &:= \sum_{\ell=1}^m\Big(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell}-1-\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell\Big)(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor) \\
&+\sum_{1\leqslant\ell_1<\ell_2\leqslant m}\;\sum_{k_1=\lfloor nt_{\ell_1-1}\rfloor+1}^{\lfloor nt_{\ell_1}\rfloor}\;\sum_{k_2=\lfloor nt_{\ell_2-1}\rfloor+1}^{\lfloor nt_{\ell_2}\rfloor} a^{k_2-k_1}\big(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_{\ell_1}}-1\big) \\
&\qquad\times\big(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_{\ell_2}}-1\big) \\
&\qquad\times\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\big((\lfloor nt_{\ell_1}\rfloor-k_1)\theta_{\ell_1}+\sum_{\ell=\ell_1+1}^{\ell_2-1}\theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)+(k_2-1-\lfloor nt_{\ell_2-1}\rfloor)\theta_{\ell_2}\big)} \\
&+\sum_{\ell=1}^m\;\sum_{\lfloor nt_{\ell-1}\rfloor+1\leqslant k_1<k_2\leqslant\lfloor nt_\ell\rfloor} a^{k_2-k_1}\big(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell}-1\big)^2\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}(k_2-k_1-1)\theta_\ell}
\end{aligned}$$

for $a\in[0,1]$. The aim of the following discussion is to apply Lemma 4.5 with $z_n(a) := B_n(a)$, $n\in\mathbb{N}$, $a\in(0,1)$, $\varepsilon_n := (\log N_n)^{-1}$, $n\in\mathbb{N}$, and $I := \frac{\lambda}{2}(\sum_{\ell=1}^m\theta_\ell(t_\ell-t_{\ell-1}))^2$. Note that $\varepsilon_n\in(0,1)$ for $n\geqslant n_0$, where $n_0$ is sufficiently large, and $\lim_{n\to\infty}\varepsilon_n=0$. First we check (4.1). Using (3.11), for any $a\in(0,1)$ we get

$$\begin{aligned}
|B_n(a)| &\leqslant \sum_{\ell=1}^m n^{-2}(N_n\log N_n)^{-1}\frac{\theta_\ell^2}{2}(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor) \\
&\quad+\sum_{1\leqslant\ell_1<\ell_2\leqslant m} n^{-2}(N_n\log N_n)^{-1}|\theta_{\ell_1}||\theta_{\ell_2}|(\lfloor nt_{\ell_1}\rfloor-\lfloor nt_{\ell_1-1}\rfloor)(\lfloor nt_{\ell_2}\rfloor-\lfloor nt_{\ell_2-1}\rfloor) \\
&\quad+\sum_{\ell=1}^m n^{-2}(N_n\log N_n)^{-1}\frac{\theta_\ell^2}{2}(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor-1) \\
&= \frac12 n^{-2}(N_n\log N_n)^{-1}\left(\sum_{\ell=1}^m|\theta_\ell|(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)\right)^2 \\
&\leqslant \frac12(N_n\log N_n)^{-1}\left(\sum_{\ell=1}^m|\theta_\ell|(t_\ell-t_{\ell-1}+1)\right)^2,
\end{aligned}$$

since $\frac{1}{n}(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor) \leqslant \frac{1}{n}(nt_\ell - nt_{\ell-1}+1) = t_\ell - t_{\ell-1} + \frac{1}{n} \leqslant t_\ell - t_{\ell-1}+1$. Consequently, since $\varepsilon_n = (\log N_n)^{-1}$, we have

$$\sup_{n\geqslant n_0} \varepsilon_n^{-1} N_n \sup_{a\in(0,1-\varepsilon_n)} |B_n(a)| \leqslant \frac{1}{2}\bigg(\sum_{\ell=1}^m |\theta_\ell|(t_\ell - t_{\ell-1}+1)\bigg)^2 < \infty,$$

i.e., (4.1) is satisfied. Therefore, by Lemma 4.5, substituting $a = 1 - zN_n^{-1}$ with $z > 0$, the statement of the theorem will follow from

$$\begin{aligned} &\limsup_{n\to\infty} N_n \int_{1-(\log N_n)^{-1}}^1 \left|1 - \mathrm{e}^{\frac{\lambda}{1-a}B_n(a)}\right| \mathrm{d}a \\ &= \limsup_{n\to\infty} \int_0^{\frac{N_n}{\log N_n}} \left|1 - \mathrm{e}^{\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right)}\right| \mathrm{d}z < \infty \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \left| N_n \int_{1-(\log N_n)^{-1}}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a}B_n(a)}\right) \mathrm{d}a - I\right| \\ &= \lim_{n\to\infty} \left| \int_0^{\frac{N_n}{\log N_n}} \left(1 - \mathrm{e}^{\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right)}\right) \mathrm{d}z - I\right| = 0 \end{aligned} \tag{4.10}$$

with $I = \frac{\lambda}{2}(\sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1}))^2$.

Next we check (4.9) and (4.10). We get

$$\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right) = -\frac{\lambda\big(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)\big)^2}{2zn^2 \log N_n} + \frac{\mathrm{O}(1)}{zN_n^{\frac{1}{2}}(\log N_n)^{\frac{3}{2}}} + \frac{n\,\mathrm{O}(1)}{N_n \log N_n} \tag{4.11}$$

for $z < N_n$. Indeed, $z < N_n$ yields that $1 - z{N_n}^{-1} \in (0,1)$, and

$$\begin{aligned} &B_n\left(1 - z{N_n}^{-1}\right) \\ &= \sum_{\ell=1}^m \big(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\theta_\ell} - 1 - \mathrm{i}n^{-1}(N_n \log N_n)^{-\frac{1}{2}}\theta_\ell\big)(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor) \\ &\quad + \sum_{1\leqslant \ell_1<\ell_2\leqslant m} \sum_{k_1=\lfloor nt_{\ell_1-1}\rfloor+1}^{\lfloor nt_{\ell_1}\rfloor} \sum_{k_2=\lfloor nt_{\ell_2-1}\rfloor+1}^{\lfloor nt_{\ell_2}\rfloor} \left(1 - z{N_n}^{-1}\right)^{k_2-k_1} \left(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\theta_{\ell_1}} - 1\right) \\ &\qquad\quad \times \mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\left((\lfloor nt_{\ell_1}\rfloor - k_1)\theta_{\ell_1} + \sum_{\ell=\ell_1+1}^{\ell_2-1}\theta_\ell(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor) + (k_2 - 1 - \lfloor nt_{\ell_2-1}\rfloor)\theta_{\ell_2}\right)} \\ &\qquad\quad \times \left(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\theta_{\ell_2}} - 1\right) \\ &\quad + \sum_{\ell=1}^m \sum_{\lfloor nt_{\ell-1}\rfloor+1\leqslant k_1<k_2\leqslant\lfloor nt_\ell\rfloor} \left(1 - z{N_n}^{-1}\right)^{k_2-k_1} \left(\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}\theta_\ell} - 1\right)^2 \\ &\qquad\qquad\qquad \times \mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac{1}{2}}(k_2-k_1-1)\theta_\ell} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{m}\Big(-\frac{\theta_\ell^2}{2n^2 N_n \log N_n} + \frac{\mathrm{O}(1)}{n^3(N_n\log N_n)^{\frac32}}\Big)(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)\\
&\quad + \sum_{1\leqslant \ell_1<\ell_2\leqslant m}\left(1+\frac{n\,z\,\mathrm{O}(1)}{N_n}\right)\left(\frac{\mathrm{i}\theta_{\ell_1}}{n(N_n\log N_n)^{\frac12}} + \frac{\mathrm{O}(1)}{n^2 N_n\log N_n}\right)\left(1+\frac{\mathrm{O}(1)}{(N_n\log N_n)^{\frac12}}\right)\\
&\qquad\qquad \times\left(\frac{\mathrm{i}\theta_{\ell_2}}{n(N_n\log N_n)^{\frac12}} + \frac{\mathrm{O}(1)}{n^2 N_n\log N_n}\right)(\lfloor nt_{\ell_1}\rfloor - \lfloor nt_{\ell_1-1}\rfloor)(\lfloor nt_{\ell_2}\rfloor - \lfloor nt_{\ell_2-1}\rfloor)\\
&\quad + \frac12\sum_{\ell=1}^{m}\left(1+\frac{n\,z\,\mathrm{O}(1)}{N_n}\right)\left(\frac{\mathrm{i}\theta_{\ell}}{n(N_n\log N_n)^{\frac12}} + \frac{\mathrm{O}(1)}{n^2 N_n\log N_n}\right)^2\left(1+\frac{\mathrm{O}(1)}{(N_n\log N_n)^{\frac12}}\right)\\
&\qquad \times(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}-1\rfloor)\\
&= -\frac{\sum_{\ell=1}^m \theta_\ell^2(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)}{2n^2N_n\log N_n} + \frac{\mathrm{O}(1)}{n^2(N_n\log N_n)^{\frac32}}\\
&\quad - \frac{\sum_{1\leqslant\ell_1<\ell_2\leqslant m}\theta_{\ell_1}\theta_{\ell_2}(\lfloor nt_{\ell_1}\rfloor - \lfloor nt_{\ell_1-1}\rfloor)(\lfloor nt_{\ell_2}\rfloor - \lfloor nt_{\ell_2-1}\rfloor)}{n^2N_n\log N_n} + \frac{\mathrm{O}(1)}{(N_n\log N_n)^{\frac32}}\\
&\quad + \frac{n\,z\,\mathrm{O}(1)}{N_n^2\log N_n} - \frac{\sum_{\ell=1}^m \theta_\ell^2(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}-1\rfloor)}{2n^2N_n\log N_n}\\
&\quad + \frac{\mathrm{O}(1)}{(N_n\log N_n)^{\frac32}} + \frac{n\,z\,\mathrm{O}(1)}{N_n^2\log N_n}\\
&= -\frac{\big(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)\big)^2}{2n^2N_n\log N_n} + \frac{\mathrm{O}(1)}{(N_n\log N_n)^{\frac32}} + \frac{n\,z\,\mathrm{O}(1)}{N_n^2\log N_n},
\end{aligned}
$$

where we used the corresponding versions of (4.5) and (4.6) after replacing $N_n^{-\frac{1}{2(1+\beta)}}$ by $(N_n\log N_n)^{-\frac12}$, the Taylor expansions

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell} - 1 &= \mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell + n^{-2}(N_n\log N_n)^{-1}\,\mathrm{O}(1)\\
&= n^{-1}(N_n\log N_n)^{-\frac12}\,\mathrm{O}(1),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell} - 1 - \mathrm{i}n^{-1}(N_n\log N_n)^{-\frac12}\theta_\ell\\
&\qquad = -n^{-2}(N_n\log N_n)^{-1}\frac{\theta_\ell^2}{2} + n^{-3}(N_n\log N_n)^{-\frac32}\,\mathrm{O}(1) = n^{-2}(N_n\log N_n)^{-1}\,\mathrm{O}(1)
\end{aligned}
$$

for all $\ell\in\{1,\ldots,m\}$, and that

$$
\big(1 - zN_n{}^{-1}\big)^{k_2-k_1} = 1 + \frac{n\,z\,\mathrm{O}(1)}{N_n}
$$

following from Bernoulli's inequality. By (4.11), for large enough $n$ and for any $z\in(0,N_n)$, we have

$$
\begin{aligned}
&\lambda\frac{N_n}{z}\operatorname{Re}B_n\big(1 - zN_n{}^{-1}\big)\\
&\qquad = -\frac{\lambda\big(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)\big)^2}{2zn^2\log N_n}\left(1 - \frac{\operatorname{Re}\mathrm{O}(1)}{(N_n\log N_n)^{\frac12}}\right) + \frac{n\operatorname{Re}\mathrm{O}(1)}{N_n\log N_n}\\
&\qquad \leqslant -\frac{\lambda\big(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor - \lfloor nt_{\ell-1}\rfloor)\big)^2}{4zn^2\log N_n} + \frac{n|\,\mathrm{O}(1)|}{N_n\log N_n},
\end{aligned}
$$

hence we obtain that

$$
\begin{aligned}
&\int_0^{(\log N_n)^{-1}} \left|1-\mathrm{e}^{\lambda\frac{N_n}{z}B_n\left(1-zN_n{}^{-1}\right)}\right| \mathrm{d}z \leqslant \int_0^{(\log N_n)^{-1}} \left(1+\mathrm{e}^{\lambda\frac{N_n}{z}\operatorname{Re}B_n\left(1-zN_n{}^{-1}\right)}\right) \mathrm{d}z \\
&\leqslant (\log N_n)^{-1}\left(1+\exp\left\{-\frac{\lambda\left(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)\right)^2}{4zn^2\log N_n}+\frac{n|\,\mathrm{O}(1)|}{N_n\log N_n}\right\}\right)\to 0
\end{aligned}
\tag{4.12}
$$

as $n\to\infty$, since

$$
\lim_{n\to\infty}\frac{\left(\sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell\rfloor-\lfloor nt_{\ell-1}\rfloor)\right)^2}{n^2\log N_n}=0,
$$

and, due to the assumption $(\log N_n)^2 n^{-1}\to\infty$ as $n\to\infty$, we have

$$
\frac{n}{N_n\log N_n}=\frac{n}{(\log N_n)^2}\frac{\log N_n}{N_n}\to 0 \qquad \text{as } n\to\infty.
$$

Note that for every $z\in((\log N_n)^{-1}, N_n(\log N_n)^{-1})$ we have

$$
\begin{aligned}
&\left|\lambda\frac{N_n}{z}B_n\left(1-zN_n{}^{-1}\right)\right| \leqslant \frac{\lambda\left(\sum_{\ell=1}^m|\theta_\ell|(t_\ell-t_{\ell-1}+1)\right)^2}{2z\log N_n}+\frac{|\,\mathrm{O}(1)|}{zN_n^{\frac12}(\log N_n)^{\frac32}}+\frac{n\,|\,\mathrm{O}(1)|}{N_n\log N_n} \\
&\leqslant \frac{\lambda\left(\sum_{\ell=1}^m|\theta_\ell|(t_\ell-t_{\ell-1}+1)\right)^2}{2}+\frac{|\,\mathrm{O}(1)|}{N_n^{\frac12}(\log N_n)^{\frac12}}+\frac{n\,|\,\mathrm{O}(1)|}{N_n\log N_n}=|\,\mathrm{O}(1)|,
\end{aligned}
\tag{4.13}
$$

since $n(N_n\log N_n)^{-1}\to 0$ as $n\to\infty$, as we have seen before.

Hence, using (3.12), we obtain for large enough $n$

$$
\begin{aligned}
&\int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \left|1-\mathrm{e}^{\lambda\frac{N_n}{z}B_n\left(1-zN_n{}^{-1}\right)}\right| \mathrm{d}z \\
&\leqslant \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \left|\lambda\frac{N_n}{z}B_n\left(1-zN_n{}^{-1}\right)\right| \mathrm{e}^{\left|\lambda\frac{N_n}{z}B_n\left(1-zN_n{}^{-1}\right)\right|}\mathrm{d}z \\
&\leqslant \mathrm{e}^{|\,\mathrm{O}(1)|}\int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}}\left[\frac{\lambda\left(\sum_{\ell=1}^m|\theta_\ell|(t_\ell-t_{\ell-1}+1)\right)^2}{2z\log N_n}+\frac{|\,\mathrm{O}(1)|}{zN_n^{\frac12}(\log N_n)^{\frac32}}+\frac{n\,|\,\mathrm{O}(1)|}{N_n\log N_n}\right]\mathrm{d}z \\
&<\infty,
\end{aligned}
$$

since for every $N_n\in\mathbb{N}$, we have

$$
\frac{1}{\log N_n}\int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}}\frac{1}{z}\,\mathrm{d}z=1, \tag{4.14}
$$

and

$$
\frac{n}{N_n\log N_n}\int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}}1\,\mathrm{d}z=\frac{n(N_n-1)}{N_n(\log N_n)^2}=\frac{n}{(\log N_n)^2}\left(1-\frac{1}{N_n}\right)\to 0 \tag{4.15}
$$

as $n\to\infty$ due to the assumption $n^{-1}(\log N_n)^2\to\infty$ as $n\to\infty$. Together with (4.12), this implies (4.9).

Now we turn to prove (4.10). By (4.12), the convergence (4.10) reduces to showing that

$$\left| \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \left(1 - \mathrm{e}^{\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right)}\right) \mathrm{d}z - \frac{\lambda(\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}))^2}{2} \right| \to 0$$

as $n \to \infty$. Using (4.14), it is enough to check that

$$\left| \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \left( \mathrm{e}^{\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right)} - 1 + \frac{\lambda(\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}))^2}{2z \log N_n} \right) \mathrm{d}z \right| \to 0$$

as $n \to \infty$. By applying (3.13), (4.11) and (4.13), we have

$$\begin{aligned}
&\left| \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \left( \mathrm{e}^{\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right)} - 1 + \frac{\lambda(\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}))^2}{2z \log N_n} \right) \mathrm{d}z \right| \\
&\leqslant \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \Bigg( \frac{1}{2} \left| \lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right) \right|^2 \mathrm{e}^{\left|\lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right)\right|} \\
&\qquad\qquad + \left| \lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right) + \frac{\lambda(\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}))^2}{2z \log N_n} \right| \Bigg) \mathrm{d}z \\
&\leqslant \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \Bigg( \frac{1}{2} \left( \frac{\lambda(\sum_{\ell=1}^m |\theta_\ell| (t_\ell - t_{\ell-1} + 1))^2}{2z \log N_n} + \frac{|\operatorname{O}(1)|}{z N_n^{\frac{1}{2}} (\log N_n)^{\frac{3}{2}}} + \frac{n|\operatorname{O}(1)|}{N_n \log N_n} \right)^2 \\
&\qquad\qquad \times \mathrm{e}^{|\operatorname{O}(1)|} \\
&\qquad\qquad + \frac{|\operatorname{O}(1)|}{z N_n^{\frac{1}{2}} (\log N_n)^{\frac{3}{2}}} + \frac{n|\operatorname{O}(1)|}{N_n \log N_n} \\
&\qquad\qquad + \frac{\lambda}{2z \log N_n} \left| \Big( \sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}) \Big)^2 - n^{-2} \Big( \sum_{\ell=1}^m \theta_\ell (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor) \Big)^2 \right| \Bigg) \mathrm{d}z, \\
&\leqslant \int_{(\log N_n)^{-1}}^{N_n(\log N_n)^{-1}} \Bigg( \frac{3}{2} \left( \frac{\lambda^2(\sum_{\ell=1}^m |\theta_\ell| (t_\ell - t_{\ell-1} + 1))^4}{4z^2 (\log N_n)^2} + \frac{|\operatorname{O}(1)|}{z^2 N_n (\log N_n)^3} + \frac{n^2|\operatorname{O}(1)|}{N_n^2 (\log N_n)^2} \right) \\
&\qquad\qquad \times \mathrm{e}^{|\operatorname{O}(1)|} \\
&\qquad\qquad + \frac{|\operatorname{O}(1)|}{z N_n^{\frac{1}{2}} (\log N_n)^{\frac{3}{2}}} + \frac{n|\operatorname{O}(1)|}{N_n \log N_n} \\
&\qquad\qquad + \frac{\lambda}{2z \log N_n} \left| \Big( \sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}) \Big)^2 - n^{-2} \Big( \sum_{\ell=1}^m \theta_\ell (\lfloor n t_\ell \rfloor - \lfloor n t_{\ell-1} \rfloor) \Big)^2 \right| \Bigg) \mathrm{d}z.
\end{aligned}$$

Indeed, the last but one inequality follows from

$$\left| \lambda \frac{N_n}{z} B_n\left(1 - z{N_n}^{-1}\right) + \frac{\lambda(\sum_{\ell=1}^m \theta_\ell (t_\ell - t_{\ell-1}))^2}{2z \log N_n} \right|$$

$$
\begin{aligned}
&\leqslant \left| \lambda \frac{N_n}{z} B_n \left(1 - z{N_n}^{-1}\right) + \frac{\lambda \Big( \sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \Big)^2}{2zn^2 \log N_n} \right| \\
&\quad + \left| - \frac{\lambda \Big( \sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \Big)^2}{2zn^2 \log N_n} + \frac{\lambda (\sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1}))^2}{2z \log N_n} \right| \\
&\leqslant \frac{|\operatorname{O}(1)|}{z N_n^{\frac{1}{2}} (\log N_n)^{\frac{3}{2}}} + \frac{n |\operatorname{O}(1)|}{N_n \log N_n} \\
&\quad + \frac{\lambda}{2z \log N_n} \left| \Big( \sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1}) \Big)^2 - n^{-2} \Big( \sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \Big)^2 \right|,
\end{aligned}
$$

and the last inequality from $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, $a, b, c \in \mathbb{R}$. Consequently,

$$
\left| \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} \left( \mathrm{e}^{\lambda \frac{N_n}{z} B_n \left(1 - z{N_n}^{-1}\right)} - 1 + \frac{\lambda (\sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1}))^2}{2z \log N_n} \right) \mathrm{d}z \right| \to 0 \qquad \text{as } n \to \infty.
$$

Indeed,

$$
\begin{aligned}
\frac{1}{(\log N_n)^2} \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} \frac{1}{z^2} \, \mathrm{d}z &= \frac{1}{(\log N_n)^2} \left( \log N_n - \frac{\log N_n}{N_n} \right) \\
&= \frac{1}{\log N_n} - \frac{1}{N_n \log N_n} \to 0
\end{aligned}
$$

as $n \to \infty$, and hence

$$
\frac{1}{N_n (\log N_n)^3} \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} \frac{1}{z^2} \, \mathrm{d}z \to 0 \qquad \text{as } n \to \infty.
$$

Further, using the assumption $(\log N_n)^2 n^{-1} \to \infty$ as $n \to \infty$, we have

$$
\begin{aligned}
\frac{n^2}{N_n^2 (\log N_n)^2} \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} 1 \, \mathrm{d}z &= \frac{n^2}{N_n^2 (\log N_n)^2} (\log N_n)^{-1} (N_n - 1) \\
&= \frac{n^2}{N_n (\log N_n)^3} \left( 1 - \frac{1}{N_n} \right) \\
&= \left( \frac{n}{(\log N_n)^2} \right)^2 \frac{\log N_n}{N_n} \left( 1 - \frac{1}{N_n} \right) \to 0 \qquad \text{as } n \to \infty.
\end{aligned}
$$

Moreover, (4.14) yields that

$$
\frac{1}{N_n^{\frac{1}{2}} (\log N_n)^{\frac{3}{2}}} \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} \frac{1}{z} \, \mathrm{d}z \to 0 \qquad \text{as } n \to \infty,
$$

and

$$
\begin{aligned}
&\left( \frac{1}{\log N_n} \int_{(\log N_n)^{-1}}^{N_n (\log N_n)^{-1}} \frac{1}{z} \, \mathrm{d}z \right) \left| \Big( \sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1}) \Big)^2 - n^{-2} \Big( \sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \Big)^2 \right| \\
&= \left| \Big( \sum_{\ell=1}^m \theta_\ell(t_\ell - t_{\ell-1}) \Big)^2 - n^{-2} \Big( \sum_{\ell=1}^m \theta_\ell(\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor) \Big)^2 \right| \to 0 \qquad \text{as } n \to \infty.
\end{aligned}
$$

This together with (4.15) yields (4.10), completing the proof. $\square$

**Proof of Lemma 4.5** For all $a \in (0,1)$ and for sufficiently large $n \in \mathbb{N}$, we have $1 - \varepsilon_n > a$, hence, by (4.1),

$$N_n|z_n(a)| \leqslant \varepsilon_n \varepsilon_n^{-1} N_n \sup_{b\in(0,1-\varepsilon_n)} |z_n(b)| \to 0 \qquad \text{as } n \to \infty, \tag{4.16}$$

thus we conclude $\lim_{n\to\infty} N_n|z_n(a)| = 0$. By applying (3.12) and using (4.16), for any $n \in \mathbb{N}$ and $a \in (0,1)$, we get

$$\left|N_n\left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right)\right| \leqslant N_n\left|\frac{\lambda}{1-a} z_n(a)\right| \mathrm{e}^{\left|\frac{\lambda}{1-a} z_n(a)\right|} \to 0 \qquad \text{as } n \to \infty. \tag{4.17}$$

If $n \geqslant n_0$ and $a \in (0, 1 - \varepsilon_n)$, then $\frac{1}{1-a} < \varepsilon_n^{-1}$ and

$$\left|N_n\left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right)\right| \leqslant \lambda \sup_{n\geqslant n_0} \varepsilon_n^{-1} N_n \sup_{a\in(0,1-\varepsilon_n)} |z_n(a)|\, \mathrm{e}^{\lambda \sup_{n\geqslant n_0} \varepsilon_n^{-1} \sup_{a\in(0,1-\varepsilon_n)} |z_n(a)|} =: C,$$

where $C \in \mathbb{R}_+$ (due to (4.1)). Since $\int_0^1 \psi(a)(1-a)^\beta \,\mathrm{d}a = 1$, we have

$$\begin{aligned}
&\left|N_n \int_0^{1-\varepsilon_n} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \psi(a)(1-a)^\beta \,\mathrm{d}a\right| \\
&\qquad = \left|\int_0^1 N_n\left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \mathbf{1}_{(0,1-\varepsilon_n)}(a)\psi(a)(1-a)^\beta \,\mathrm{d}a\right| \\
&\qquad \leqslant \int_0^1 C\psi(a)(1-a)^\beta \,\mathrm{d}a < \infty
\end{aligned}$$

for $n \geqslant n_0$. Therefore, $(0,1) \ni a \mapsto C\psi(a)(1-a)^\beta$ serves as a dominating integrable function. Thus, by the dominated convergence theorem, the pointwise convergence in (4.17) results

$$\lim_{n\to\infty} N_n \int_0^{1-\varepsilon_n} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \psi(a)(1-a)^\beta \,\mathrm{d}a = 0. \tag{4.18}$$

Moreover, for all $n \geqslant n_0$, we have

$$\begin{aligned}
&\left|N_n \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \psi(a)(1-a)^\beta \,\mathrm{d}a - \psi_1 I\right| \\
&\qquad \leqslant \left|N_n \int_0^{1-\varepsilon_n} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) \psi(a)(1-a)^\beta \,\mathrm{d}a\right| \\
&\qquad\quad + \left|N_n \int_{1-\varepsilon_n}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) (\psi(a) - \psi_1)(1-a)^\beta \,\mathrm{d}a\right| \\
&\qquad\quad + \psi_1 \left|N_n \int_{1-\varepsilon_n}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) (1-a)^\beta \,\mathrm{d}a - I\right|,
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
&\left|N_n \int_{1-\varepsilon_n}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right) (\psi(a) - \psi_1)(1-a)^\beta \,\mathrm{d}a\right| \\
&\leqslant \left(\sup_{a\in[1-\varepsilon_n,1)} |\psi(a) - \psi_1|\right) N_n \int_{1-\varepsilon_n}^1 \left|1 - \mathrm{e}^{\frac{\lambda}{1-a} z_n(a)}\right| (1-a)^\beta \,\mathrm{d}a,
\end{aligned}$$

with $\sup_{a\in[1-\varepsilon_n,1)} |\psi(a) - \psi_1| \to 0$ as $n \to \infty$, by the assumption $\lim_{x\uparrow 1} \psi(x) = \psi_1$. Taking $\limsup_{n\to\infty}$ of both sides of (4.19), by (4.18) and the assumptions of the lemma, we obtain the statement. $\square$

# Appendix A

# Abridged results of Chapters 3 and 4

Here we summarize some of the results of Chapters 3 and 4 related to the aggregates of the randomized INAR(1) processes (defined in Chapter 3) with the specific mixing distribution introduced in (3.6). We summarize so one can compare the results of the many theorems presented in the thesis. The tables contain the scaling factors of the limit theorems, the limit processes, and the numbers of the corresponding theorems. Note that in some cases the limit processes are simplified in the sense that some constants are only represented with a general $c \in \mathbb{R}$ notation. For the definitions of the limit processes, see the referred theorems.

### Limit theorems for $\widetilde{S}^{(N,n)}$, centered by the conditional expectations.

The following three tables contain the limit theorems concerning the aggregates centered with the conditional expectations: $\widetilde{S}^{(N,n)}$, defined in (3.7).

Table of results when first $n \to \infty$, then $N \to \infty$:

| $\beta$ | $(-1,1)$ | $1$ | $(1,\infty)$ |
|---|---|---|---|
| Scaling | $N^{\frac{1}{1+\beta}} n^{\frac{1}{2}}$ | $(nN \log N)^{\frac{1}{2}}$ | $(nN)^{\frac{1}{2}}$ |
| Limit | $\sqrt{Y_{(1+\beta)/2}}\, B$ | $\sqrt{\lambda\psi_1} B$ | $\sigma B$ |
| Theorem | 3.12 | 3.13 | 3.14 |

Let us point out that here and in the forthcoming tables the scaling becomes heavier as the parameter $\beta$ decreases. This is related to the fact that by Remark 3.6 the finiteness of the first and second moments of the stationary distribution depends on this parameter.

Table of results when first $N \to \infty$, then $n \to \infty$:

| $\beta$ | $(-1,0)$ | $0$ | $(0,1)$ | $1$ | $(1,\infty)$ |
|---|---|---|---|---|---|
| Scaling | $nN^{\frac{1}{2(1+\beta)}}$ | $n(N \log N)^{\frac{1}{2}}$ | $n^{1-\frac{\beta}{2}} N^{\frac{1}{2}}$ | $(n(\log n)N)^{\frac{1}{2}}$ | $(nN)^{\frac{1}{2}}$ |
| Limit | $(V_{2(1+\beta)} t)_{t\in\mathbb{R}_+}$ | $(W_{\lambda\psi_1} t)_{t\in\mathbb{R}_+}$ | $c\mathcal{B}_{1-\frac{\beta}{2}}$ | $\sqrt{2\lambda\psi_1} B$ | $\sigma B$ |
| Theorem | 3.8 | 3.9 | 3.10 | 3.11 | 3.14 |

Table of results when $n$ and $N$ increase to infinity simultaneously:

| $\beta$ | $(-1,0)$ | $0$ | $(0,\infty)$ |
|---|---|---|---|
| Rate | $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$ | $(\log N_n)^2 n^{-1} \to \infty$ | Future plan |
| Scaling | $n N_n^{\frac{1}{2(1+\beta)}}$ | $n(N_n \log N_n)^{\frac{1}{2}}$ | Future plan |
| Limit | $(V_{2(1+\beta)} t)_{t\in\mathbb{R}_+}$ | $(W_{\lambda\psi_1} t)_{t\in\mathbb{R}_+}$ | Future plan |
| Theorem | 4.1 | 4.2 | |

**Limit theorems for $\boldsymbol{S}^{(N,n)}$, centered by the expectations.**

The following table contains the limit theorems concerning the aggregates centered with the expectations: $\boldsymbol{S}^{(N,n)}$, defined in (2.10). Note that such results exist only when the expectation is finite, which, by Remark 3.6, occurs when $\beta > 0$. As we have no simultaneous limit theorems yet for $\beta > 0$, we only summarize the iterated limit theorems for these aggregates.

Table of results when first $n \to \infty$, then $N \to \infty$, or vice versa:

| $\beta$ | $(0,1)$ | $1$ | $(1,\infty)$ |
|---|---|---|---|
| Scaling | $nN^{\frac{1}{1+\beta}}$ | $na_N$, where $\sqrt{N}/a_N = \mathrm{o}(1)$ | $nN^{\frac{1}{2}}$ |
| Limit | $\big(Z_{1+\beta}\, t\big)_{t\in\mathbb{R}_+}$ | $(W\, t)_{t\in\mathbb{R}_+}$ | $(W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}\, t)_{t\in\mathbb{R}_+}$ |
| Theorem | 3.16 | 3.17 | 3.18 |

# Appendix B

# Notations

| Notation | Formula, theorem, proposition. | Page number |
|---|---|---|
| $\boldsymbol{Y}$, $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$ | (2.1) | 9 |
| $\boldsymbol{m}_{\varepsilon}$, $\boldsymbol{M}_{\boldsymbol{\xi}}$ | | 9 |
| $\boldsymbol{v}$ | (2.2) | 9 |
| $\boldsymbol{V}$ | (2.5) | 10 |
| $\boldsymbol{S}^{(N)}$ | (2.7) | 12 |
| $\boldsymbol{\mathcal{Y}}$ | Proposition 2.6 | 12 |
| $\boldsymbol{B}$ | Proposition 2.7 | 12 |
| $\boldsymbol{S}^{(N,n)}$ | (2.10) | 13 |
| $Z$ | (2.14) | 14 |
| $F_k(\cdot)$, $G(\cdot)$, $H(\cdot)$, $D$ | (2.16) | 16 |
| $\widetilde{F}(\cdot)$ | (2.18) | 17 |
| $F_{0,\dots,k}(\cdot)$ | (2.21) | 18 |
| $\widetilde{S}^{(N)}$ | (2.23), (3.4) | 18, 35 |
| $\widetilde{\mathcal{Y}}$ | Proposition 3.1 | 35 |
| $\mathcal{Y}$ | Proposition 3.3 | 36 |
| $\psi(\cdot)$, $\psi_1$ | (3.6) | 37 |
| $\widetilde{S}^{(N,n)}$ | (3.7) | 37 |
| $\mathcal{B}_{1-\frac{\beta}{2}}$ | (3.8) | 38 |
| $V_{2(1+\beta)}$ | Theorem 3.8 | 38 |
| $\mathcal{Y}_{1+\beta}$, $Y_{(1+\beta)/2}$ | Theorem 3.12 | 39 |
| $Z_{1+\beta}$ | Theorem 3.16 | 40 |
| $\mathrm{o}(1)$ | Theorem 3.17 | 40 |
| $\widehat{S}^{(N,n)}$ | (2.26), (3.9) | 20, 41 |
| $\mathrm{O}(1)$, $\lvert\,\mathrm{O}(1)\rvert$ | | 45 |
| $F_0(\cdot\,\vert\,\alpha)$ | (3.17) | 45 |

Table B.1: Notations

# Appendix C

# Summary

This thesis is about the limit behavior of the temporal and contemporaneous aggregates of certain branching processes. Aggregates, also known as partial sums, are well-known to be very important in stochastics. All of the chapters of this current work deal with this question, either for different processes, different manners regarding the convergence (iterated, simultaneous), or different centralizations.

Chapter 1 contains an introduction where we explain the goal of the thesis, the historical background of the studied topic, along with an overview of this work. In the following paragraph we specify the scheme of aggregation that we use.

The aggregation problem is concerned with the relationship between individual (micro) and aggregate (macro) behavior. In general, we consider independent copies of a stationary branching process, we denote these by $(X_k^{(j)})_{k\in\mathbb{N}}$, $j \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, \dots\}$. We are interested in the limit behavior of the aggregate process $\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$, as both $n$, the time parameter, and $N$, the number of copies tend to infinity in some manner. If we take the limits in an iterated manner, i.e., first $n$ tends to infinity and then $N$ tends to infinity, or vice versa, then the resulting limit theorem is called an iterated one. If both converge to infinity at the same time, then it is called a simultaneous limit theorem. To achieve such limit theorems, we also consider the simple aggregates, $\sum_{k=1}^{n} X_k^{(j)}$, which is called temporal (or time-aggregated), and $\sum_{j=1}^{N} X_k^{(j)}$, which is called contemporaneous (or space-aggregated).

Let us recall the most important antecedents of the work presented in this thesis. The scheme of contemporaneous aggregation of random coefficient autoregressive processes of order 1 (AR(1)) was firstly proposed by the Nobel prize winner Clive W. J. Granger [18] in order to obtain the long memory phenomena in aggregated time series. In a series of papers, Donatas Surgailis and his co-authors studied the aggregation of random coefficient AR(1) processes, where $(X_k^{(j)})_{k\in\mathbb{Z}_+:=\{0,1,\dots\}}$, $j \in \mathbb{N}$, are independent copies of a stationary random coefficient AR(1) process

$$X_k = aX_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

with standardized independent and identically distributed (i.i.d.) innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ and a random coefficient $a$ with values in $(0, 1)$, being independent of $(\varepsilon_k)_{k\in\mathbb{N}}$ and admitting a probability density function of the form

$$\psi(x)(1 - x)^{\beta}, \qquad x \in (0, 1),$$

where $\beta \in (-1, \infty)$ and $\psi$ is an integrable function on $(0,1)$ with $\lim_{x\uparrow 1} \psi(x) := \psi_1 \in (0, \infty)$. In the paper Pilipauskaitė and Surgailis [38], both iterated and simultaneous limit theorems were presented concerning the limit behavior of the aggregate process $\left(\sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$.

Our aim is to provide such results when branching processes take the place of the random coefficient AR(1) model explained before. These processes are widely applicable as they can model integer-valued phenomena, such as migration and the spreading of contagious diseases like COVID-19. These possible applications are more thoroughly detailed in Chapter 2, which is devoted to the investigation of the aggregates of multitype Galton–Watson processes with immigration. The $p$-dimensional process $(\boldsymbol{Y}_k = [Y_{k,1}, \ldots, Y_{k,p}]^\top)_{k\in\mathbb{Z}_+}$, where $p \in \mathbb{N}$, is a $p$-type Galton–Watson branching process with immigration if

$$\boldsymbol{Y}_k = \sum_{\ell=1}^{Y_{k-1,1}} \begin{bmatrix} \xi_{k,\ell}^{(1,1)} \\ \vdots \\ \xi_{k,\ell}^{(1,p)} \end{bmatrix} + \cdots + \sum_{\ell=1}^{Y_{k-1,p}} \begin{bmatrix} \xi_{k,\ell}^{(p,1)} \\ \vdots \\ \xi_{k,\ell}^{(p,p)} \end{bmatrix} + \begin{bmatrix} \varepsilon_k^{(1)} \\ \vdots \\ \varepsilon_k^{(p)} \end{bmatrix} = \sum_{i=1}^{p} \sum_{\ell=1}^{Y_{k-1,i}} \boldsymbol{\xi}_{k,\ell}^{(i)} + \boldsymbol{\varepsilon}_k$$

for every $k \in \mathbb{N}$, where we define $\sum_{\ell=1}^{0} := \boldsymbol{0}$, and $\left\{\boldsymbol{Y}_0, \boldsymbol{\xi}_{k,\ell}^{(i)}, \boldsymbol{\varepsilon}_k : k, \ell \in \mathbb{N}, i \in \{1, \ldots, p\}\right\}$ are independent $\mathbb{Z}_+^p$-valued random vectors. Moreover, for all $i \in \{1, \ldots, p\}$, $\{\boldsymbol{\xi}_{k,\ell}^{(i)} : k, \ell \in \mathbb{N}\}$ and $\{\boldsymbol{\varepsilon}_k : k \in \mathbb{N}\}$ consist of identically distributed random vectors, respectively. By choosing $\boldsymbol{Y}_0$ as the unique stationary distribution, which is shown to exist under some general conditions, the considered process is strictly stationary. Then we take independent copies, $(\boldsymbol{Y}_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, of this process. For each $N, n \in \mathbb{N}$, we consider the stochastic process $\boldsymbol{S}^{(N,n)} = (\boldsymbol{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$\boldsymbol{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} (\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})), \qquad t \in [0, \infty).$$

We show that the finite dimensional distributions of the aggregate process scaled by the usual scaling sequence $\sqrt{nN}$ converge to those of a zero mean $p$-dimensional (non-standard, with a given covariance structure) Brownian motion if

- all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1, \ldots, p\}$, and $\boldsymbol{\varepsilon}$ have finite second moments and first $N$, then $n$ converges to infinity;
- all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1, \ldots, p\}$, and $\boldsymbol{\varepsilon}$ have finite third moments and first $n$, then $N$ converges to infinity, or they converge to infinity together at any rate.

The proofs of these theorems rely on the multidimensional central limit theorem and the functional martingale central limit theorem.

Among others, we also discuss these results in the special case of integer-valued autoregressive processes of order 1 (INAR(1)) with Poisson innovations. These are one-dimensional Galton–Watson processes with immigration where the offsprings have Bernoulli distribution with parameter $\alpha \in (0, 1)$ and the immigrations have Poisson distribution. The proofs of Chapter 2 are based on the paper Barczy et al. [3].

In Chapters 3 and 4 we consider a certain randomized INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$ with randomized thinning parameter $\alpha$, given formally by the recursive equation

$$X_k = \sum_{\ell=1}^{X_{k-1}} \xi_{k,\ell} + \varepsilon_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

where $\circ$ is the so-called Steutel and van Harn thinning operator, $\alpha$ is a random variable with values in $(0,1)$, and $X_0$ is some appropriate random variable. This means that, conditionally on $\alpha$, the process $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with thinning parameter $\alpha$, i.e., conditionally on $\alpha$, the offsprings, $(\xi_{k,\ell})_{k,\ell\in\mathbb{N}}$, have Bernoulli distribution with parameter $\alpha$. Conditionally on $\alpha$, the i.i.d. innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ have a Poisson distribution with parameter $\lambda \in (0,\infty)$, and the conditional distribution of the initial value $X_0$ given $\alpha$ is the unique stationary distribution, namely, a Poisson distribution with parameter $\lambda/(1-\alpha)$. In Chapter 3 we provide a rigorous construction of this process. For the desired iterated and simultaneous limit theorems we assume that the random parameter $\alpha$ admits a mixing distribution having a probability density of the form

$$\psi(x)(1-x)^{\beta}, \qquad x \in (0,1),$$

where $\psi$ is a function on $(0,1)$ having a limit $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$. Note that necessarily $\beta \in (-1,\infty)$ (otherwise $\int_0^1 \psi(x)(1-x)^{\beta}\,\mathrm{d}x = \infty$), and the function $(0,1) \ni x \mapsto \psi(x)$ is integrable on $(0,1)$. The Beta distribution is a special case of this form. Certain $\circ$ operators, where the summands are random parameter Bernoulli distributions with a parameter having Beta distribution, appear in catastrophe models. Moreover, Clive W. J. Granger used the square root of a Beta distribution as a mixing distribution for random coefficient AR(1) processes.

Chapter 3 contains an exhaustive list of iterated limit theorems related to the aggregates in multiple manners. For every $N, n \in \mathbb{N}$, we consider three different aggregate processes regarding the centralization:

$$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)} | \alpha^{(j)}))\right)_{t\in[0,\infty)},$$

where we center with the conditional expectation with respect to the random parameter belonging to the corresponding process,

$$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)}))\right)_{t\in[0,\infty)},$$

where we center with the expectation (note that this only exists for $\beta > 0$), and

$$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}\left(X_k^{(j)} - \frac{\sum_{\ell=1}^{n} X_\ell^{(j)}}{n}\right)\right)_{t\in[0,\infty)},$$

where we center with the empirical mean of the first $n$ observations for the corresponding process in order to provide a well-applicable, observable alternative. As there are two different approaches to iterated limit theorems ($n \to \infty$ and then $N \to \infty$ or vice versa), and we have different limit theorems for different ranges of the parameter $\beta$, this chapter contains many limit theorems. The proofs rely heavily on the multidimensional central limit theorem and a lemma that was developed for this research, which helps us prove the convergence of characteristic functions. We also use that in case of zero mean Gaussian processes, to show their convergence in distribution, the convergence of their covariance functions has to be shown. Furthermore, both Theorem 4.3 of Beran et al. [7], which is about convergence of partial sums of a Hermite function of a stationary sequence of standard normal random variables, and Theorem 7.1 of Resnick [46], which is about weak convergence of partial sum processes for a triangular array towards a Lévy process, are used once. The limit theorems corresponding to the process centered by the empirical mean follow from those of the process centered by the conditional expectation by Slutsky's lemma. In these cases, by the nature of this process, we get bridge-type limit processes. Let us point out that the the scaling of the processes becomes heavier as the parameter $\beta$ decreases, since the finiteness of the first and second moments of the stationary distribution of the randomized INAR(1) process depends on this parameter. Also, it is interesting that in most of the cases the two different orders of iteration result in significantly different limit theorems as the scaling factors and limit processes differ. In Chapter 3 the proofs are based on the papers Nedényi and Pap [35] and Barczy et al. [6].

In Chapter 4 two simultaneous limit theorems are presented. Contrary to the iterated limit theorems, the list of the simultaneous ones is not complete, some cases remain for future work. The proofs of the achieved limit theorems are based on our lemma developed for this research, which is a counterpart of the one applied for the iterated limit theorems. We plan to handle the remaining cases as future work which will require a different proof technique. The proofs of Chapter 4 are based on the paper Barczy et al. [4] that has been submitted to a journal.

# Appendix D

# Összefoglaló

Ebben a dolgozatban bizonyos elágazó folyamatok (centrált és skálázott) térbeli és időbeli aggregáltjainak határeloszlásaival foglalkozunk. Jól ismert, hogy az aggregáltaknak, más néven részletösszegeknek különösen fontos szerepe van a sztochasztikában. A dolgozat mindegyik fejezete ezzel a kérdéssel foglalkozik, különböző folyamatok, konvergencia-típus (iterált, vagy szimultán), illetve centrálás esetében.

Az első fejezet (Chapter 1) az értekezés bevezetője, ahol felvázoljuk a dolgozat célját, a kutatott téma előzményeit, valamint a disszertáció felépítését. A következő bekezdésben részletesebben bemutatjuk az általunk alkalmazott aggregációt.

Az aggregáció célja, hogy kapcsolatot teremtsen az egyéni (mikro) és az összesített (makro) viselkedés között. Mi minden esetben egy stacionárius elágazó folyamat független kópiáiból fogunk kiindulni, jelölje ezeket $(X_k^{(j)})_{k\in\mathbb{N}}$, $j \in \mathbb{N}$, ahol $\mathbb{N} := \{1, 2, \ldots\}$. Azt vizsgáljuk, hogyan viselkedik a $\left(\sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt\rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$ aggregált folyamat, amint $n$, az idő paramétere, valamint $N$, a kópiák száma valamilyen módon végtelenhez tart. Amennyiben iterált módon tekintjük a határérték-képzést, azaz először $n$ tart végtelenhez, majd $N$, vagy fordítva, akkor a kapott határeloszlás-tételt iteráltnak nevezzük. Amennyiben a két paraméter egyszerre konvergál végtelenhez, akkor a tételt szimultánnak nevezzük. Ahhoz, hogy ilyen határeloszlás-tételeket lássunk be, vizsgáljuk a következő egyszeres aggregáltakat is: $\sum_{k=1}^{n} X_k^{(j)}$, melyet időbeli, illetve $\sum_{j=1}^{N} X_k^{(j)}$, melyet térbeli aggregáltnak nevezünk.

A továbbiakban felelevenítjük az értekezés legfontosabb előzményeit. Az elsőrendű autoregressziós (AR(1)) folyamatok térbeli aggregációját először a Nobel-díjas Clive W. J. Granger [18] vizsgálta azzal a céllal, hogy hosszú memóriát váltson ki aggregált idősorok esetében. Donatas Surgailis és szerzőtársai cikksorozatukban véletlen együtthatójú AR(1) folyamatok aggregációját tanulmányozták, ahol $(X_k^{(j)})_{k\in\mathbb{Z}_+:=\{0,1,\ldots\}}$, $j \in \mathbb{N}$, a következő stacionárius véletlen együtthatójú AR(1) folyamat független kópiái:

$$X_k = aX_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

standardizált, független és azonos eloszlású $(\varepsilon_k)_{k\in\mathbb{N}}$ bevándorlással, ahol a $(0, 1)$ értékkészletű $a$ véletlen együttható független az $(\varepsilon_k)_{k\in\mathbb{N}}$ sorozattól, és

$$\psi(x)(1 - x)^{\beta}, \qquad x \in (0, 1),$$

a sűrűségfüggvénye, ahol $\beta \in (-1, \infty)$ és $\psi$ olyan integrálható függvény a $(0, 1)$ intervallumon, melyre $\lim_{x\uparrow 1} \psi(x) := \psi_1 \in (0, \infty)$. Pilipauskaitė és Surgailis [38] cikkükben a

$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$ aggregált folyamathoz tartozó iterált és szimultán határeloszlás-tételeket adtak meg.

Célunk, hogy hasonló eredményeket írjunk le abban az esetben, amikor a véletlen együtthatójú AR(1) modellek helyét elágazó folyamatok veszik át. Ezen folyamatok széles körben alkalmazhatóak egészértékű jelenségek, mint a migráció, vagy fertőző betegségek (például COVID-19) modellezése. Részletesebben leírjuk ezt a két lehetséges alkalmazást a dolgozat második fejezetében (Chapter 2), melyben többtípusos bevándorlásos Galton–Watson folyamatok aggregáltjaival foglalkozunk. A $p$-dimenziós ( $p\in\mathbb{N}$ ) folyamat, $(\boldsymbol{Y}_k = [Y_{k,1},\ldots,Y_{k,p}]^\top)_{k\in\mathbb{Z}_+}$ bevándorlásos $p$-típusos Galton–Watson elágazó folyamat, amennyiben

$$\boldsymbol{Y}_k = \sum_{\ell=1}^{Y_{k-1,1}} \begin{bmatrix} \xi_{k,\ell}^{(1,1)} \\ \vdots \\ \xi_{k,\ell}^{(1,p)} \end{bmatrix} + \cdots + \sum_{\ell=1}^{Y_{k-1,p}} \begin{bmatrix} \xi_{k,\ell}^{(p,1)} \\ \vdots \\ \xi_{k,\ell}^{(p,p)} \end{bmatrix} + \begin{bmatrix} \varepsilon_k^{(1)} \\ \vdots \\ \varepsilon_k^{(p)} \end{bmatrix} = \sum_{i=1}^{p}\sum_{\ell=1}^{Y_{k-1,i}} \boldsymbol{\xi}_{k,\ell}^{(i)} + \boldsymbol{\varepsilon}_k$$

minden $k\in\mathbb{N}$ esetén, ahol $\sum_{\ell=1}^{0} := \boldsymbol{0}$, és $\left\{\boldsymbol{Y}_0, \boldsymbol{\xi}_{k,\ell}^{(i)}, \boldsymbol{\varepsilon}_k : k,\ell\in\mathbb{N}, i\in\{1,\ldots,p\}\right\}$ független $\mathbb{Z}_+^p$-értékű véletlen vektorok. Továbbá minden $i\in\{1,\ldots,p\}$ esetén mind $\{\boldsymbol{\xi}_{k,\ell}^{(i)} : k,\ell\in\mathbb{N}\}$, mind $\{\boldsymbol{\varepsilon}_k : k\in\mathbb{N}\}$ azonos eloszlású vektorokból áll. Amennyiben $\boldsymbol{Y}_0$-t úgy választjuk meg, hogy eloszlása az egyértelmű stacionárius eloszlás legyen (melyről megmutatjuk, hogy általános feltételek mellett létezik), akkor a tekintett folyamat erősen stacionárius. Ezen folyamat $(\boldsymbol{Y}_k^{(j)})_{k\in\mathbb{Z}_+}$, $j\in\mathbb{N}$, független kópiáit tekintjük. Minden $N,n\in\mathbb{N}$, esetén legyen $\boldsymbol{S}^{(N,n)} = (\boldsymbol{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$, ahol

$$\boldsymbol{S}_t^{(N,n)} := \sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})), \qquad t\in[0,\infty).$$

Megmutatjuk, hogy az előbbi aggregált folyamat véges dimenziós eloszlásai a szokásos $\sqrt{nN}$ sorozattal skálázva konvergálnak egy 0 várható értékű $p$-dimenziós (nem standard, adott kovariancia-struktúrájú) Brown-mozgás megfelelő véges dimenziós eloszlásaihoz, amennyiben

- a $\boldsymbol{\xi}^{(i)}$, $i\in\{1,\ldots,p\}$, és $\boldsymbol{\varepsilon}$ vektorok második momentuma véges és először $N$, majd $n$ konvergál végtelenhez;
- a $\boldsymbol{\xi}^{(i)}$, $i\in\{1,\ldots,p\}$, és $\boldsymbol{\varepsilon}$ vektorok harmadik momentuma véges és először $n$, majd $N$ konvergál végtelenhez, vagy együtt tartanak végtelenhez, bármilyen rátával.

Ezen tételek bizonyításának alapja a többdimenziós centrális határeloszlás-tétel és a funkcionális martingál centrális határeloszlás-tétel.

Több speciális esetben is tárgyaljuk az eredményeket, köztük Poisson bevándorlású, elsőrendű egészértékű autoregressziós (INAR(1)) folyamatokra. Ezek olyan bevándorlásos egydimenziós Galton–Watson folyamatok, melyek esetében az utódeloszlások Bernoulli eloszlásúak $\alpha\in(0,1)$ paraméterrel, a bevándorlások pedig Poisson eloszlásúak. A második fejezet (Chapter 2) bizonyításainak alapja a Barczy et al. [3] cikk.

A harmadik és negyedik fejezetben (Chapter 3 és 4) az $(X_k)_{k\in\mathbb{Z}_+}$ véletlenített, $\alpha$ ritkítási paraméterrel rendelkező, véletlenített INAR(1) folyamatokat vizsgáljuk, melyekre

$$X_k = \sum_{\ell=1}^{X_{k-1}} \xi_{k,\ell} + \varepsilon_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

ahol $\circ$ az úgynevezett Steutel and van Harn ritkítási operátor, $\alpha$ egy valószínűségi változó $(0,1)$-beli értékekkel, $X_0$ pedig egy megfelelő valószínűségi változó. Ez azt jelenti, hogy az $\alpha$ változóra feltételesen az $(X_k)_{k\in\mathbb{Z}_+}$ folyamat egy $\alpha$ ritkítási paraméterű INAR(1) folyamat, azaz $\alpha$-ra feltételesen a $(\xi_{k,\ell})_{k,\ell\in\mathbb{N}}$, utódeloszlások Bernoulli eloszlásúak $\alpha$ paraméterrel. Szintén $\alpha$-ra feltételesen a független $(\varepsilon_k)_{k\in\mathbb{N}}$ bevándorlások Poisson eloszlásúak $\lambda \in (0,\infty)$ paraméterrel, az $X_0$ kezdeti érték feltételes eloszlása pedig az egyértelmű stacionárius eloszlás, mely egy Poisson eloszlás $\lambda/(1-\alpha)$ paraméterrel. A harmadik fejezetben (Chapter 3) precízen belátjuk, hogy ilyen folyamat létezik. A célként kitűzött iterált és szimultán határeloszlás-tételekhez feltesszük, hogy az $\alpha$ véletlen együttható abszolút folytonos

$$\psi(x)(1-x)^\beta, \qquad x \in (0,1),$$

sűrűségfüggvénnyel, ahol $\psi$ olyan függvény a $(0,1)$ intervallumon, melyre $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$. Jegyezzük meg, hogy $\beta \in (-1,\infty)$ (másképp $\int_0^1 \psi(x)(1-x)^\beta \, \mathrm{d}x = \infty$), és a $(0,1) \ni x \mapsto \psi(x)$ függvény $(0,1)$-en integrálható. A Béta-eloszlás speciális esete ennek az alaknak. Katasztrófa modellek esetén megjelennek azok a $\circ$ operátorok, ahol az összeadandók véletlen együtthatójú Bernoulli eloszlású véletlen változók Béta-eloszlású paraméterrel. Továbbá Clive W. J. Granger Béta-eloszlás négyzetgyökét alkalmazta véletlenített AR(1) folyamatok paraméterének véletlenítésére.

A harmadik fejezetben (Chapter 3) az aggregáltakhoz tartozó, többféle módon tekintett iterált határeloszlástételeket prezentálunk. Minden $N, n \in \mathbb{N}$ esetén három, a centrálás tekintetében különböző aggregált folyamatot tekintünk:

$$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)}|\alpha^{(j)}))\right)_{t\in[0,\infty)},$$

ahol az adott folyamathoz tartozó véletlenített paraméterre vett feltételes várható értékkel centrálunk,

$$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)}))\right)_{t\in[0,\infty)},$$

ahol a várható értékkel centrálunk (mely csak $\beta > 0$ esetén létezik), valamint

$$\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}\left(X_k^{(j)} - \frac{\sum_{\ell=1}^{n} X_\ell^{(j)}}{n}\right)\right)_{t\in[0,\infty)},$$

ahol az első $n$ megfigyelés átlagával centrálunk, hogy egy jól alkalmazható, megfigyelhető alternatívát kapjunk. Mivel az iterált határeloszlás-tételeknek két típusa van ($n \to \infty$, majd $N \to \infty$, vagy fordítva), és különböző határeloszlás-tételeket kapunk $\beta$ különböző értékeire, így számos tétel szerepel a fejezetben. Ezek listája teljes. A bizonyítások alapja a többdimenziós centrális határeloszlás-tétel és egy általunk kidolgozott lemma, mely a karakterisztikus függvények konvergenciájának ellenőrzésében nyújt segítséget. Azt is alkalmazzuk, hogy 0 várható értékű Gauss-folyamatok eloszlásbeli konvergenciájának ellenőrzésekor elegendő a kovariancia-függvények konvergenciáját belátni. Továbbá egy-egy tétel bizonyítása során felhasználjuk Beran et al. [7] 4.3. Tételét, mely standard normális eloszlású véletlen változók stacionárius sorozatának Hermite-függvényének részletösszeg-sorozatának konvergenciájáról szól, valamint Resnick [46] 7.1. Tételét háromszögrendszerek részletösszeg-sorozatának Lévy-folyamatokhoz való gyenge konvergenciájáról. A mintaátlaggal centrált aggregált folyamathoz tartozó tételek Slutsky lemmájának segítségével vezethetőek le a feltételes várható értékkel centrált folyamathoz tartozó tételekből. Ezek esetében az aggregált folyamat szerkezete miatt híd típusú folyamatokat kapunk határeloszlásként. Megjegyezzük, hogy annál inkább súlyozni kell a folyamatokat, minél kisebb a $\beta$ paraméter. Ennek oka, hogy a véletlenített INAR(1) folyamat stacionárius eloszlásának első és második momentumának létezése ezen paraméter értékétől függ. További érdekesség, hogy a különböző sorrendű iterálás során igen különbözőek a kapott tételek is: sok esetben mind a skálázás, mind a határfolyamat más. A harmadik fejezet (Chapter 3) bizonyításainak alapja a Nedényi és Pap [35], valamint a Barczy et al. [6] cikk.

A negyedik fejezetben (Chapter 4) két szimultán határeloszlás-tétel szerepel. Az iterált határeloszlás-tételekkel szemben ezek listája nem teljes, a kimaradt eseteket jövőbeli kutatásunk során szeretnénk kezelni. A két belátott tétel bizonyításának alapja az a lemma, melyet az iterált tételeknél felhasznált lemma mintájára dolgoztunk ki. A jövőben más bizonyítási technikával szeretnénk kezelni a fennmaradó eseteket. A negyedik fejezet (Chapter 4) bizonyításainak alapja a Barczy et al. [4] cikk, mely benyújtásra került egy folyóirathoz.

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