

# Corrigendum to “Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback”

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The purpose of this note is to correct a mistake left in our previous paper [2]. The paper concerns the scalar equation

$$(1) \quad \dot{x}(t) = -\mu x(t) + f(x(t-1))$$

with  $\mu = 1$  and a special strictly increasing, continuously differentiable  $f$ . The natural phase space for Eq. (1) is  $C = C([-1, 0], \mathbb{R})$  equipped with the supremum norm. For any  $\varphi \in C$ , there is a unique solution  $x^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  of (1). For each  $t \geq 0$ , the segment  $x_t^\varphi \in C$  is defined by  $x_t^\varphi(s) = x^\varphi(t+s)$ ,  $-1 \leq s \leq 0$ . Let  $\Phi$  denote the semiflow induced by E.q. (1):

$$\Phi : [-1, \infty) \times C \ni (t, \varphi) \mapsto x_t^\varphi \in C.$$

Theorem 1.1 of paper [2] gives a periodic solution  $p : \mathbb{R} \rightarrow \mathbb{R}$  of E.q. (1) with  $p(-1) = 0$  and  $\dot{p}(-1) \neq 0$ . The proof of Theorem 1.2 in Section 8 then applies a Poincaré return map defined on a neighborhood of  $p_0$  in  $H$ , where  $H = \{\varphi : \varphi(-1) = 0\}$  is a hyperplane transversal to the periodic orbit  $\mathcal{O}_p = \{p_t : t \in \mathbb{R}\}$ . As we shall see, this hyperplane was not selected appropriately.

We evoke results from Floquet theory before pointing at the error and showing its correction.

## 1. FLOQUET THEORY

Let  $\omega \in (1, 2)$  denote the minimal period of  $p$ . Consider the period map  $Q = \Phi(\omega, \cdot)$  with fixed point  $p_0$ . Consider its derivative  $M = D_2\Phi(\omega, p_0)$  at  $p_0$ . Then  $M\varphi = u_\omega^\varphi$  for all  $\varphi \in C$ , where  $u^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  is the solution of the variational equation

$$(2) \quad \dot{u}(t) = -u(t) + f'(p(t-1))u(t-1)$$

with  $u_0^\varphi = \varphi$ .  $M$  is called the monodromy operator.  $M$  is a compact operator, 0 belongs to its spectrum  $\sigma = \sigma(M)$ , and eigenvalues of finite multiplicity – the so called Floquet multipliers – form  $\sigma(M) \setminus \{0\}$ .

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As  $\dot{p}$  is a nonzero solution of the variational equation, 1 is a Floquet multiplier with eigenfunction  $\dot{p}_0$ . The paper [2] proves that  $\mathcal{O}_p$  is hyperbolic, which means that the generalized eigenspace of  $M$  corresponding to the eigenvalue 1 is one-dimensional, furthermore there are no Floquet multipliers on the unit circle besides 1.

If  $\varphi$  is a nonzero element of the phase space  $C = C([-1, 0], \mathbb{R})$ , let  $V(\varphi)$  denote the number of sign changes of  $\varphi$  if it is even or  $\infty$ , otherwise let  $V(\varphi)$  be the number of sign changes plus one. This is the so-called discrete Lyapunov functional of Mallet-Paret and Sell [4].

By Section 4 of [2],  $\mathcal{O}_p$  has two real and simple Floquet multipliers  $\lambda_1$  and  $\lambda_2$  outside the unit circle with  $\lambda_1 > \lambda_2 > 1$ . Regarding the associated eigenspaces, we have the following information from [4] and from Appendix VII of [3]. The eigenvector  $u_1$  of  $M$  corresponding to  $\lambda_1$  is strictly positive. The realified generalized eigenspace  $C_{<\lambda_1}$  associated with the spectral set  $\{z \in \sigma : |z| < \lambda_1\}$  satisfies

$$(3) \quad C_{<\lambda_1} \cap V^{-1}(0) = \emptyset.$$

Let  $C_{\leq\rho}$ ,  $\rho > 0$ , denote the realified generalized eigenspace of  $M$  associated with the spectral set  $\{z \in \sigma : |z| \leq \rho\}$ . The set

$$\{\rho \in (0, \infty) : \sigma(M) \cap \rho S_{\mathbb{C}}^1 \neq \emptyset, C_{\leq\rho} \cap V^{-1}(\{0, 2\}) = \emptyset\}$$

is nonempty and has a maximum  $r_M$ . Then

$$(4) \quad C_{\leq r_M} \cap V^{-1}(\{0, 2\}) = \emptyset, \quad C_{r_M <} \setminus \{\hat{0}\} \subset V^{-1}(\{0, 2\}) \text{ and } \dim C_{r_M <} \leq 3,$$

where  $C_{r_M <}$  is the realified generalized eigenspace of  $M$  associated with the nonempty spectral set  $\{z \in \sigma : |z| > r_M\}$ . It follows from the construction of  $p$  in [2] that  $V(\dot{p}_0) = 2$ . Hence  $r_M < 1$  in our case, and  $V(u_2) = 2$  for the eigenvector  $u_2$  of  $M$  corresponding to  $\lambda_2$ .

## 2. POINCARÉ RETURN MAPS

Choose  $X$  to be a hyperplane with codimension 1 so that  $\dot{p}_0 \notin X$ . An application of the implicit function theorem yields a convex bounded open neighborhood  $N$  of  $p_0$  in  $C$ ,  $\varepsilon \in (0, \omega)$  and a  $C^1$ -map  $\gamma : N \rightarrow (\omega - \varepsilon, \omega + \varepsilon)$  with  $\gamma(p_0) = \omega$  so that for each  $(t, \varphi) \in (\omega - \varepsilon, \omega + \varepsilon) \times N$ , the segment  $x_t^\varphi$  belongs to  $p_0 + X$  if and only if  $t = \gamma(\varphi)$  (see [1], Appendix I in [3]). The Poincaré return map  $P_X$  is defined by

$$P_X : N \cap (p_0 + X) \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in p_0 + X.$$

Then  $P_X$  is continuously differentiable with fixed point  $p_0$ .

Let  $\sigma(P_X)$  and  $\sigma(M)$  denote the spectra of  $DP_X(p_0) : X \rightarrow X$  and the monodromy operator  $M$ , respectively. We obtain the following result from Theorem XIV.4.5 in [1].

**Lemma.**

(i)  $\sigma(P_X) \setminus \{0, 1\} = \sigma(M) \setminus \{0, 1\}$ , and for every  $\lambda \in \sigma(M) \setminus \{0, 1\}$ , the projection along  $\mathbb{R}\dot{p}_0$  onto  $X$  defines an isomorphism from the realified generalized eigenspace of  $\lambda$  and  $M$  onto the realified generalized eigenspace of  $\lambda$  and  $DP_X(p_0)$ .

(ii)  $1 \notin \sigma(P_X)$ .

In Section 8 of [2] we selected the hyperplane  $H = \{\varphi : \varphi(-1) = 0\}$  and the associated Poincaré map  $P = P_H$ . It follows from the above proposition that  $DP(p_0)$  has exactly two real eigenvalues  $\lambda_1 > \lambda_2 > 1$  outside the unit circle. Let  $v_1$  and  $v_2$  denote the eigenvectors of  $DP(p_0)$  corresponding to  $\lambda_1$  and  $\lambda_2$ , respectively. Section 8 of [2] used the statement that  $V(v_1) = 0$  and  $V(v_2) = 2$ . This is not necessarily true. The mistake can be corrected by selecting a different hyperplane.

Let  $C_s$  and  $C_u$  be the closed subspaces of  $C$  chosen so that  $C = C_s \oplus \mathbb{R}p_0 \oplus C_u$ ,  $C_s$  and  $C_u$  are invariant under  $M$ , and the spectra  $\sigma_s(M)$  and  $\sigma_u(M)$  of the induced maps  $C_s \ni x \mapsto Mx \in C_s$  and  $C_u \ni x \mapsto Mx \in C_u$  are contained in  $\{\mu \in \mathbb{C} : |\mu| < 1\}$  and  $\{\mu \in \mathbb{C} : |\mu| > 1\}$ , respectively. As  $\mathcal{O}_p$  has two real and simple Floquet multipliers  $\lambda_1$  and  $\lambda_2$  outside the unit circle with eigenvectors  $u_1$  and  $u_2$ , we have  $C_u = \{c_1 u_1 + c_2 u_2\}$ .

Set  $Y = C_s \oplus C_u$ . Then  $Y$  is a hyperplane in  $C$ ,  $\dot{p}_0 \notin Y$  and  $C = Y \oplus \mathbb{R}\dot{p}_0$ .

The special choice of  $Y$  and Lemma imply that  $\lambda_i$  and  $u_i$  is an eigenvalue-eigenvector pair of  $DP_Y(p_0)$  for both  $i \in \{1, 2\}$ . In addition,  $C_s$  and  $C_u$  are invariant under  $DP_Y(p_0)$ , and the spectra  $\sigma_s(P_Y)$  and  $\sigma_u(P_Y)$  of the induced maps  $C_s \ni x \mapsto DP_Y(p_0)x \in C_s$  and  $C_u \ni x \mapsto DP_Y(p_0)x \in C_u$  are contained in  $\{\mu \in \mathbb{C} : |\mu| < 1\}$  and  $\{\mu \in \mathbb{C} : |\mu| > 1\}$ , respectively. Summing up,  $DP_Y(p_0)$  has exactly two real and simple eigenvalues  $\lambda_1 > \lambda_2 > 1$  outside the unit circle, and for the corresponding eigenvectors  $u_1$  and  $u_2$ , we have the desired properties  $V(u_1) = 0$  and  $V(u_2) = 2$ .

In accordance,  $H$  and  $P = P_H$  should be changed to  $Y$  and  $P_Y$  in Section 8 of [2]. Then the proof of Theorem 1.2. (found in Section 8 of [2]) becomes correct without any further change.

## REFERENCES

- [1] Diekmann, O., van Gils, S. A., Verduyn Lunel, S. M., and Walther, H.-O. (1995). Delay equations. Functional, complex, and nonlinear analysis, Springer-Verlag, New York.
- [2] Krisztin, T. and Vas, G. (2011). Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback. J. Dynam. Differential Equations 23, no. 4, 727–790.
- [3] Krisztin T., Walter H.-O., and Wu J. (1999). Shape, smoothness and invariant stratification of an attracting set for delayed monotone positive feedback, Amer. Math. Soc., Providence, RI.
- [4] Mallet-Paret, J., and Sell G. R. (1996). Systems of differential delay equations: Floquet multipliers and discrete Lyapunov Functions. J. Differential Equations 125, 385–440.