

On the fundamental solution of a linear delay differential equation

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Dedicated to the 65th birthday of Professor István Györi

Abstract. We consider the integral $\Psi(\alpha) = \int_0^\infty |X(t, \alpha)| dt$ for the fundamental solution $X(\cdot, \alpha)$ of the linear delay differential equation $\dot{x}(t) = -\alpha x(t-1)$. It is shown that $(\pi/2 - \alpha)\Psi(\alpha) \rightarrow 4\sqrt{4 + \pi^2}/\pi^2$ as $\alpha \rightarrow \pi/2^-$.

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1. Introduction

The fundamental solution appears naturally in the variation of constants formula for linear inhomogeneous delay differential equations. Explicit estimations on the integral of its absolute value are important in perturbation results. This is a nontrivial task even for the simplest linear delay differential equations.

Let $\alpha > 0$, and consider the delay differential equation

$$\dot{x}(t) = -\alpha x(t-1) \tag{1.1}$$

with parameter $\alpha > 0$. Define $X : [-1, \infty) \rightarrow \mathbb{R}$ so that

$$X(t) = \begin{cases} 0 & \text{if } -1 \leq t < 0 \\ 1 & \text{if } t = 0, \end{cases}$$

and for all $t > 0$ the equation

$$X(t) = 1 - \alpha \int_{-1}^{t-1} X(s) ds$$

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holds. Clearly, X is well defined by the method of steps, $X|_{[0,\infty)}$ is continuous, $X|_{(1,\infty)}$ is differentiable, and X satisfies (1.1) for all $t \in (0,1) \cup (1,\infty)$. X is called the fundamental solution of (1.1). We write $X(\cdot, \alpha)$ if we want to emphasize the dependence on the parameter $\alpha > 0$. Define the integral

$$\Psi(\alpha) = \int_0^\infty |X(t, \alpha)| dt.$$

For $\alpha > 1/e$, the zeros of the characteristic function $\mathbb{C} \ni z \mapsto z + \alpha e^{-z} \in \mathbb{C}$ associated with (1.1) are simple, and appear in complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_{j=0}^\infty$ with

$$\begin{aligned} \operatorname{Re} \lambda_0 > \operatorname{Re} \lambda_1 > \dots > \operatorname{Re} \lambda_{j-1} > \operatorname{Re} \lambda_j \rightarrow -\infty \quad (j \rightarrow \infty), \\ \operatorname{Im} \lambda_j \in (2j\pi, (2j+1)\pi) \quad \text{for all } j \in \{0, 1, \dots\}. \end{aligned}$$

If $0 < \alpha \leq 1/e$ then two real zeros λ_0, λ_0^* appear instead of $\lambda_0, \overline{\lambda_0}$ with $0 > \lambda_0 \geq \lambda_0^* > \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2 > \dots$

It is well known that $\Psi(\alpha) < \infty$, the zero solution of (1.1) is asymptotically stable, $\alpha < \pi/2$ and $\operatorname{Re} \lambda_0 < 0$ are equivalent statements. Explicit estimations for $\Psi(\alpha)$, in particular upper bounds are important for stability results of perturbations of (1.1), see e.g. [2,3,4,5].

The following upper bound was obtained by Györi [2] for $0 < \alpha < \pi/2$:

$$\Psi(\alpha) \leq \frac{1}{\alpha} \left[1 + \left(\frac{\operatorname{Im} \lambda_0(\alpha)}{\operatorname{Re} \lambda_0(\alpha)} \right)^2 \right]. \quad (1.2)$$

This inequality is sharp if $0 < \alpha \leq 1/e$.

Györi and Hartung [3] proved several numerically computable upper bounds. Moreover, they obtained

$$\left(\frac{\pi}{2} - \alpha \right) \Psi(\alpha) \geq \frac{1}{\alpha} \quad \left(0 < \alpha < \frac{\pi}{2} \right). \quad (1.3)$$

In this note we investigate $\Psi(\alpha)$ near the critical value $\pi/2$. Our main result is that

$$\lim_{\alpha \rightarrow \pi/2-} \left(\frac{\pi}{2} - \alpha \right) \Psi(\alpha) = \frac{4\sqrt{4+\pi^2}}{\pi^2} \approx 2.371. \quad (1.4)$$

Remark that (1.3) implies $\liminf_{\alpha \rightarrow \pi/2-} (\pi/2 - \alpha) \Psi(\alpha) \geq 2/\pi \approx 0.637$, while from (1.2) $\limsup_{\alpha \rightarrow \pi/2-} (\pi/2 - \alpha) \Psi(\alpha) \leq \infty$ follows.

The applied technique is suitable to obtain new explicit bounds for $\Psi(\alpha)$ for parameter values near $\pi/2$, see [6]. It can be used for equations with several delays as well.

Our estimation is based on the well known representation of the fundamental solution

$$X(t, \alpha) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{zt}}{z + \alpha e^{-z}} dz + \sum_{z=\lambda_0, \overline{\lambda_0}} \operatorname{Res} \frac{e^{zt}}{z + \alpha e^{-z}} \quad (1.5)$$

which is valid for $t > 0$ and a $\gamma \in \mathbb{R}$ with $\operatorname{Re} \lambda_1 < \gamma < \operatorname{Re} \lambda_0$. Setting $\mu = \mu(\alpha) = \operatorname{Re} \lambda_0(\alpha)$ and $\nu = \nu(\alpha) = \operatorname{Im} \lambda_0(\alpha)$, we have

$$\sum_{z=\lambda_0, \bar{\lambda}_0} \operatorname{Res} \frac{e^{zt}}{z + \alpha e^{-z}} = \frac{e^{\lambda_0 t}}{1 + \lambda_0} + \frac{e^{\bar{\lambda}_0 t}}{1 + \bar{\lambda}_0} = \frac{2e^{\mu t}}{(1 + \mu)^2 + \nu^2} [(1 + \mu) \cos \nu t + \nu \sin \nu t].$$

Define the functions $p(\alpha), q(\alpha) : [-1, \infty) \rightarrow \mathbb{R}$ by

$$p(t, \alpha) = \frac{2e^{\mu t}}{(1 + \mu)^2 + \nu^2} [(1 + \mu) \cos \nu t + \nu \sin \nu t]$$

and

$$q(t, \alpha) = X(t, \alpha) - p(t, \alpha).$$

We need information on $\mu(\alpha)$ and $\nu(\alpha)$.

Proposition 1.1. *If $3/2 \leq \alpha < \pi/2$ then*

$$\operatorname{Re} \lambda_1(\alpha) < -1, \quad -\frac{1}{2} < \mu(\alpha) < 0, \quad 0 < \nu(\alpha) < \frac{\pi}{2}, \quad |\lambda_0| > 1.$$

Moreover,

$$\lim_{\alpha \rightarrow \pi/2^-} \nu(\alpha) = \frac{\pi}{2}, \quad \lim_{\alpha \rightarrow \pi/2^-} \frac{-\mu(\alpha)}{\frac{\pi}{2} - \alpha} = \frac{2\pi}{4 + \pi^2}.$$

In the next step we estimate the integral of $\int_0^\infty |q(t, \alpha)| dt$. It is important that there is a uniform upper bound.

Proposition 1.2. *If $3/2 \leq \alpha < \pi/2$ then*

$$\int_0^\infty |q(t, \alpha)| dt \leq 3e^2.$$

The asymptotic behavior of $\int_0^\infty |p(t, \alpha)| dt$ can be obtained in an elementary way.

Proposition 1.3.

$$\lim_{\alpha \rightarrow \pi/2^-} \left(\frac{\pi}{2} - \alpha \right) \int_0^\infty |p(t, \alpha)| dt = \frac{4\sqrt{4 + \pi^2}}{\pi^2}.$$

The above propositions imply our result.

Theorem 1.1.

$$\lim_{\alpha \rightarrow \pi/2^-} \left(\frac{\pi}{2} - \alpha \right) \Psi(\alpha) = \frac{4\sqrt{4 + \pi^2}}{\pi^2}.$$

2. Proofs

Proof of Proposition 1.1. We show only the last statement. All the others are found in the basic monographs [1,5] or can be easily obtained from them. [6] also contains a detailed proof.

For the zero $\lambda_0 = \mu + i\nu$ of the characteristic function $z + \alpha e^{-z}$ the equations

$$\mu + \alpha e^{-\mu} \cos \nu = 0, \quad \nu - \alpha e^{-\mu} \sin \nu = 0 \quad (2.1)$$

hold. It is well known that $\nu(\alpha) \in (0, \pi/2]$ for $1/e < \alpha \leq \pi/2$, and $\nu(\pi/2) = \pi/2$, see e.g. [1, Chapter XI]. Define the functions g, h from $(0, \pi/2]$ into \mathbb{R} by

$$g(s) = -s \frac{\cos s}{\sin s}, \quad h(s) = \frac{s}{\sin s} e^{g(s)}.$$

Then the solutions of (6) with $\nu \in (0, \pi/2]$ satisfy

$$\mu = g(\nu) \quad \text{and} \quad h(\nu) = \alpha.$$

Both g and h are smooth functions. Elementary calculations give

$$g'(s) = \frac{s - \sin s \cos s}{\sin^2 s} > 0 \quad \left(0 < s \leq \frac{\pi}{2}\right),$$

$$g''(s) = 2 \frac{\sin s - s \cos s}{\sin^3 s} > 0 \quad \left(0 < s \leq \frac{\pi}{2}\right).$$

Thus both g and g' are strictly increasing on $(0, \pi/2]$.

For the derivative of h we have

$$\begin{aligned} h'(s) &= e^{g(s)} \frac{s^2 + \sin^2 s - 2s \sin s \cos s}{\sin^3 s} \\ &> e^{g(s)} \frac{(s - \sin s)^2}{\sin^3 s} > 0 \quad \left(0 < s \leq \frac{\pi}{2}\right), \end{aligned}$$

that is, h is strictly increasing.

It is easy to check that

$$h(1.54) < \frac{3}{2}, \quad h\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

It follows that, for $\frac{3}{2} \leq \alpha \leq \frac{\pi}{2}$, the equation $h(\nu) = \alpha$ has a unique solution $\nu(\alpha)$ such that $1.54 < \nu(\alpha) \leq \frac{\pi}{2}$, $\nu(\pi/2) = \pi/2$, and $\nu(\alpha)$ is differentiable with

$$\nu'(\alpha) = \frac{1}{h'(\nu(\alpha))}.$$

This fact and the relation $\mu = g(\nu)$ combined yield for $\frac{3}{2} \leq \alpha < \frac{\pi}{2}$ that

$$\begin{aligned} \mu(\alpha) &= g(\nu(\alpha)) = g(\nu(\alpha)) - g(\nu(\pi/2)) \\ &= g'(\nu(\hat{\alpha})) \nu'(\hat{\alpha}) \left(\alpha - \frac{\pi}{2}\right) \\ &= \frac{g'(\nu(\hat{\alpha}))}{h'(\nu(\hat{\alpha}))} \left(\alpha - \frac{\pi}{2}\right) \end{aligned}$$

for some $\hat{\alpha} \in (3/2, \pi/2)$. If $\alpha \rightarrow \pi/2^-$ then $\nu(\hat{\alpha}) \rightarrow \pi/2$. Therefore

$$\frac{-\mu(\alpha)}{\frac{\pi}{2} - \alpha} = \frac{g'(\nu(\hat{\alpha}))}{h'(\nu(\hat{\alpha}))} \rightarrow \frac{g'(\pi/2)}{h'(\pi/2)} = \frac{2\pi}{4 + \pi^2}.$$

□

Proof of Proposition 1.2. By Proposition 1.1, for all $3/2 \leq \alpha < \pi/2$ we can choose $\gamma = -1$ in formula (1.5). Then, for $t > 0$

$$q(t, \alpha) = \frac{e^{-t}}{2\pi} Q(t, \alpha)$$

with

$$Q(t, \alpha) = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{i\tau t}}{-1 + i\tau + \alpha e e^{-i\tau}} d\tau.$$

Integration by parts gives

$$\begin{aligned} \int_{-T}^T \frac{e^{i\tau t}}{-1 + i\tau + \alpha e e^{-i\tau}} d\tau &= \left[\frac{e^{i\tau t}}{it(-1 + i\tau + \alpha e e^{-i\tau})} \right]_{\tau=-T}^{\tau=T} \\ &\quad - \int_{-T}^T \frac{e^{i\tau t}(1 - \alpha e e^{-i\tau})}{t(-1 + i\tau + \alpha e e^{-i\tau})^2} d\tau. \end{aligned}$$

Clearly

$$\frac{e^{i\tau t}}{it(-1 + i\tau + \alpha e e^{-i\tau})} \rightarrow 0 \quad \text{as } |\tau| \rightarrow \infty.$$

Hence, for $t \geq 2$

$$\begin{aligned} |Q(t, \alpha)| &\leq \left| \lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{i\tau t}(1 - \alpha e e^{-i\tau})}{t(-1 + i\tau + \alpha e e^{-i\tau})^2} d\tau \right| \\ &\leq \frac{1 + \pi e/2}{2} 2 \int_0^\infty \frac{d\tau}{|-1 + i\tau + \alpha e e^{-i\tau}|^2}. \end{aligned}$$

If $\tau \geq 2\pi$ then

$$|-1 + i\tau + \alpha e e^{-i\tau}| \geq |-1 + i\tau| - \frac{\pi e}{2} = \sqrt{1 + \tau^2} - \frac{\pi e}{2} > \frac{1}{4} \sqrt{1 + \tau^2}.$$

For $\tau \in [0, 2\pi]$ we have

$$\begin{aligned} |-1 + i\tau + \alpha e e^{-i\tau}|^2 &= 1 + \tau^2 + \alpha^2 e^2 - 2\alpha e (\tau \sin \tau + \cos \tau) \\ &\geq 1 + \tau^2 + \alpha^2 e^2 - 2\alpha e \max_{0 \leq \tau \leq 2\pi} (\tau \sin \tau + \cos \tau) \\ &= 1 + \tau^2 + \alpha^2 e^2 - 2\alpha e \frac{\pi}{2} \\ &> 1 + \tau^2. \end{aligned}$$

Therefore, for $t \geq 2$

$$|Q(t, \alpha)| \leq \left(1 + \frac{\pi e}{2}\right) \int_0^\infty \frac{16}{1 + \tau^2} d\tau = \left(1 + \frac{\pi e}{2}\right) 8\pi,$$

and

$$|q(t, \alpha)| \leq (4 + 2\pi e)e^{-t}.$$

For $0 \leq t \leq 2$ we use the inequality $|q(t, \alpha)| \leq |X(t, \alpha)| + |p(t, \alpha)|$. Clearly $|X(t, \alpha)| \leq 1$, $0 \leq t \leq 2$, since $X(t, \alpha) = 1$, $0 \leq t \leq 1$, and $X(t, \alpha) = -\alpha(t - 1) + 1$, $1 \leq t \leq 2$.

By Proposition 1.1 and the Schwarz inequality

$$|p(t, \alpha)| \leq \frac{2}{(1 + \mu)^2 + \nu^2} \sqrt{(1 + \mu)^2 + \nu^2} = \frac{2}{\sqrt{(1 + \mu)^2 + \nu^2}} \leq 2.$$

Thus

$$|q(t, \alpha)| \leq 3 \quad (0 \leq t \leq 2).$$

It follows that

$$|q(t, \alpha)| \leq e^{-t} \max\{3e^2, 4 + 2\pi e\} = 3e^2 e^{-t} \quad (t \geq 0).$$

Consequently

$$\int_0^\infty |q(t, \alpha)| dt \leq 3e^2.$$

□

Proof of Proposition 1.3. We have

$$p(t, \alpha) = \frac{2}{\sqrt{(1 + \mu)^2 + \nu^2}} e^{\mu t} \sin(\nu t + \omega)$$

with $\omega = \tan^{-1} \frac{1 + \mu}{\nu} \in (0, \pi/2)$. Setting $t_j = (j\pi - \omega)/\nu$ for $j \in \{1, 2, \dots\}$, $\sin(\nu t + \omega) < 0$ for some $t \geq 0$ if and only if $t \in (t_1, t_2) \cup (t_3, t_4) \cup (t_5, t_6) \cup \dots$. Therefore

$$\begin{aligned} \int_{t_1}^\infty |p(t, \alpha)| dt &= \sum_{j=1}^\infty (-1)^j \int_{t_j}^{t_{j+1}} p(t, \alpha) dt \\ &= \frac{2}{\nu \sqrt{(1 + \mu)^2 + \nu^2}} \sum_{j=1}^\infty e^{\mu t_j} \int_0^\pi e^{\mu s/\nu} \sin s ds \\ &= \frac{2e^{-\mu\omega/\nu}}{\nu \sqrt{(1 + \mu)^2 + \nu^2}} \sum_{j=1}^\infty e^{j\mu\pi/\nu} \left[\frac{1}{1 + (\mu/\nu)^2} e^{\mu s/\nu} \left(\frac{\mu}{\nu} \sin s - \cos s \right) \right]_{s=0}^{s=\pi} \\ &= \frac{2e^{-\mu\omega/\nu} (e^{\mu\pi/\nu} + 1)}{\nu \sqrt{(1 + \mu)^2 + \nu^2} (1 + \mu^2/\nu^2)} \sum_{j=1}^\infty e^{j\mu\pi/\nu} \\ &= \frac{2e^{-\mu\omega/\nu} (e^{\mu\pi/\nu} + 1)}{\nu \sqrt{(1 + \mu)^2 + \nu^2} (1 + \mu^2/\nu^2)} \frac{e^{\mu\pi/\nu}}{1 - e^{\mu\pi/\nu}}. \end{aligned}$$

Combining the above equality multiplied by $\pi/2 - \alpha$, Proposition 1.1 and $\lim_{x \rightarrow 0} (e^x - 1)/x = 1$, it follows that

$$\left(\frac{\pi}{2} - \alpha\right) \int_{t_1}^{\infty} |p(t, \alpha)| dt \rightarrow \frac{4\sqrt{4 + \pi^2}}{\pi^2}$$

as $\alpha \rightarrow \pi/2^-$. It is obvious that

$$\left(\frac{\pi}{2} - \alpha\right) \int_0^{t_1} |p(t, \alpha)| dt \rightarrow 0$$

as $\alpha \rightarrow \pi/2^-$. This completes the proof. \square

Proof of Theorem 1.1. From $X(t, \alpha) = p(t, \alpha) + q(t, \alpha)$ one obtains

$$\int_0^{\infty} |p(t, \alpha)| dt - \int_0^{\infty} |q(t, \alpha)| dt \leq \int_0^{\infty} |X(t, \alpha)| dt \leq \int_0^{\infty} |p(t, \alpha)| dt + \int_0^{\infty} |q(t, \alpha)| dt.$$

From Proposition 1.2

$$\left(\frac{\pi}{2} - \alpha\right) \int_0^{\infty} |q(t, \alpha)| dt \rightarrow 0 \quad (\alpha \rightarrow \frac{\pi}{2}^-)$$

follows. Therefore

$$\lim_{\alpha \rightarrow \pi/2^-} \left(\frac{\pi}{2} - \alpha\right) \int_0^{\infty} |X(t, \alpha)| dt = \lim_{\alpha \rightarrow \pi/2^-} \left(\frac{\pi}{2} - \alpha\right) \int_0^{\infty} |p(t, \alpha)| dt,$$

and by Proposition 1.3 the proof is complete. \square

References

- [1] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel and H.-O. Walthert, Delay Equations, Functional-, Complex-, and Nonlinear Analysis, Springer-Verlag, New York, 1995.
- [2] I. Gyóri, Global attractivity in a perturbed linear delay differential equation, *Applicable Analysis* 34 (1989), 167–181.
- [3] I. Gyóri and F. Hartung, Stability in delay perturbed differential and difference equations, *Fields Institute Communications* 29 (2001), 181–194.
- [4] I. Gyóri and F. Hartung, Fundamental solution and asymptotic stability of linear delay equations, *Dyn. Contin. Discrete Impuls. Syst.* 13 (2006), 261–288.
- [5] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.
- [6] T. Krisztin, Upper bounds for the integral of the fundamental solution of a linear delay differential equation, preprint.