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Asymptotic stability for a differential-difference equation containing terms with and without a delay

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Dedicated to Professor Károly Tandori on his 70th birthday and to Professor László Leindler on his 60th birthday

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Abstract. The nonlinear scalar equation

$$x'(t) = b(t)f(x(t-T)) - c(t)g(x(t)) \quad (c(t) \geq 0)$$

is considered under the assumption $|f(x)| \leq \kappa |g(x)|$ $(|x| \leq \epsilon_0)$ with appropriate constants $\kappa, \epsilon_0 > 0$. Sufficient conditions are given for the asymptotic stability of the zero solution by Lyapunov's direct method with Lyapunov functionals. The effect of the dominating conditions

$$c(t) - \kappa|b(t+T)| \geq \mu \geq 0, \qquad c(t) - \kappa|b(t)| \geq \nu \geq 0$$

for all $t \geq 0$ with constant μ, ν is discussed by examples.

1. Introduction

Consider the equation

$$x'(t) = b(t)f(x(t-T)) - c(t)g(x(t)), \tag{1.1}$$

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where $b, c, f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous functions; $c(t) \geq 0$ for all t; g satisfies the usual sign condition $xg(x) > 0$, $(x \neq 0)$; $f(0) = 0$, and the positive constant T denotes the time lag.

Let C denote the space of the continuous functions $\varphi\colon [-T, 0] \to \mathbb{R}$ with the norm $\|\varphi\| := \max_{-T \leq s \leq 0} |\varphi(s)|$. If $x\colon [t_0 - T, t_*) \to \mathbb{R}$ $(-\infty < t_0 < t_* \leq \infty)$ is a continuous function and $t \in [t_0, t_*)$ then x_t denotes the element of C defined by $x_t(s) := x(t+s)$ $(-T \leq s \leq 0)$. It is well-known [9] that for any pair $(t_0, \varphi) \in \mathbb{R} \times C$ there exists a solution $x(\cdot) = x(\cdot; t_0, \varphi)\colon [t_0 - T, t_*) \to \mathbb{R}$ of (1.1) satisfying the initial condition $x_{t_0} = \varphi$. The zero solution is said to be *stable* if for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|\varphi\| < \delta$ implies $|x(t; t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$. If δ does not depend on t_0, then the stability is called *uniform.* The zero solution is said to be *asymptotically stable* if it is stable and, in addition, for every $t_0 \in \mathbb{R}$ there is a $\sigma = \sigma(t_0) > 0$ such that $\|\varphi\| < \sigma$ implies

$$\lim_{t \to \infty} x(t; t_0, \varphi) = 0. \tag{1.2}$$

The asymptotic stability is called *uniform* if the stability is uniform, σ can be independent of t_0, and limit (1.2) is uniform with respect to t_0 and φ $(t_0 \geq 0$, $\|\varphi\| < \sigma)$ [9], [23].

In this paper we are dealing with the asymptotic stability and uniform asymptotic stability of the zero solution of (1.1), which have been studied in numerous papers and books (see, e.g., [1–24] and the references therein). The first results in this direction concerned the corresponding linear equation

$$x'(t) = b(t)x(t - T) - c(t)x(t). \tag{1.3}$$

If $b(t)$ and $c(t)$ are constant (autonomous case), then the exact region of asymptotic stability independent of the size of the delay T is described on the parameter plane (b, c) by the inequality $|b| < c$. This can be interpreted by saying that the undelayed part dominates the delayed one. As it can be expected, the theorems for the case of varying coefficients $b(t)$, $c(t)$ (nonautonomous case) also contain conditions demanding that function c dominates function $|b|$ in some sense. However, the first results used also the boundedness of b and c. It was needed only by the techniques of the proofs and seemed to be unnatural since, e.g., the larger $c(t)$ the better from the point of view of asymptotic stability. Therefore, it is an old problem to guarantee asymptotic stability for the nonautonomous equation (1.3) allowing also unbounded coefficients b, c and so that the consequences of the nonautonomous results for the autonomous case approximate the region $|b| < c$ as much as possible.

Very recently, applying their general Lyapunov type theorem to equation (1.3), T. A. Burton and G. Makay [6] proved the following

Theorem A. *Suppose there are constants* $c_1, c_2, c_3 > 0$ *with*
(a) $c(t) - |b(t+T)| \geq c_1$*;*
(b) *there is a sequence* $\{t_n\} \uparrow \infty$ *and* $K > 0$ *with* $t_{n+1} - t_n \leq K$ *and*

$$\int_{t_n - T}^{t_n} |b(s+T)|\, ds \leq c_2;$$

(c) $c(t) + |b(t)| \leq c_3(t+1)\ln(t+2)$.
Then the zero solution of (1.3) *is asymptotically stable.*

This paper is devoted to the study of the consequences of the dominating conditions of type (a) for (1.1). In order to obtain a better approximation of the region $c > |b|$, we start with the condition

$$(\mathrm{a}_0) \qquad c(t) - |b(t+T)| \geq 0 \quad \text{for} \quad t \geq 0.$$

To approach asymptotic stability, at first we study the conditions of the existence of finite limits of the solutions as $t \to \infty$. We will prove by an example that (a_0) is not sufficient for this property even if we suppose also that b is bounded on $\mathbb{R}_+$. However, if, in addition to (a_0), either $\int_0^\infty |b| < \infty$ or $c - |b|$ dominates $|b|$ in a certain integral sense, then the solutions tend to finite limits (see Theorem 2.1). If we assume also condition (b) then this dominating condition can be weakened (see Theorem 2.2). As a consequence, we get a stronger version of Theorem A; namely, we can replace condition (c) in Theorem A by

$$(\mathrm{c}') \qquad |b(t)| \leq c_3(t+1)\ln(t+2).$$

It means that our method does not require any growth condition on function c.

As we mentioned, we will show that (a_0) and the boundedness of b do not imply even the existence of finite limits of the solutions of (1.3). Theorem 2.5 says that (a) and the boundedness of b are sufficient for this property. However, another example will show that (a) and (b) are not sufficient for the asymptotic stability; in other words, condition (c) cannot be dropped from Theorem A. We conjecture but are not able to prove that (a) and the boundedness of $\int_{t-T}^{t} |b(s)|\, ds$ are not sufficient either.

Finally, we discuss the consequences of the dominating conditions

$$(\mathrm{A}_0) \qquad c(t) - |b(t)| \geq 0,$$

$$(\mathrm{A}_\varepsilon) \qquad c(t) - |b(t)| \geq \varepsilon \quad \text{for some } \varepsilon > 0$$

for the existence of the limits of the solutions of (1.3).

2. Sufficient conditions for the existence of limits and asymptotic stability

The method of the proofs of our theorems is based upon Lyapunov's direct method [9], [15] by the Lyapunov–Krasovskii functional

$$V(t,\varphi) := |\varphi(0)| + \int_{-T}^{0} |b(t+s+T)|\,|f(\varphi(s))|\,ds. \tag{2.1}$$

The derivative $V'(t,\varphi)$ of V with respect to (1.1) satisfies the inequality

$$V'(t,\varphi) \le |b(t+T)||f(\varphi(0))| - c(t)|g(\varphi(0))|. \tag{2.2}$$

Now we formulate our basic hypotheses expressing that the undelayed term dominates the delayed one in the nonlinear equation (1.1):
(H_1) there are numbers $\varepsilon_0 > 0$, $\kappa > 0$ such that $|x| \le \varepsilon_0$ implies $|f(x)| \le \kappa |g(x)|$;
(H_2) $c(t) - \kappa|b(t+T)| \ge 0$ for all $t \in \mathbb{R}_+$.
In the sequel we suppose that these two conditions are automatically satisfied.

Theorem 2.1. *Suppose that either $\int_0^\infty |b| < \infty$ or there is a continuous, strictly increasing function $W\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $W(0) = 0$, and such that*

$$\int_s^t [c(u) - \kappa|b(u+T)|]\,du \ge W\left(\int_s^t |b(u)|\,du\right) \tag{2.3}$$

for all $s \le t$.

Then each solution of (1.1) *starting from a small neighborhood of the origin has a finite limit as $t \to \infty$.*

If, in addition, the condition

$$\int_0^\infty [c(t) - \kappa|b(t+T)|]\,dt = \infty \tag{2.4}$$

holds, then the zero solution of (1.1) *is asymptotically stable.*

Proof. By hypotheses (H_1), (H_2) and estimate (2.2) the derivative of functional (2.1) with respect to equation (1.1) satisfies the inequality

$$V'(t,\varphi) \le -[c(t) - \kappa|b(t+T)|] \mid g(\varphi(0)) \mid \le 0 \tag{2.5}$$

whenever $|\varphi(0)| \le \varepsilon_0$. By the basic theorem on the stability for FDE's (see, e.g., [3, Th. 8.1.6]) the zero solution of (1.1) is stable. Take $\delta(\varepsilon, t_0)$ from the definition of stability and introduce the notation $\sigma(t_0) := \delta(\varepsilon_0, t_0) > 0$. Consider an arbitrary $(t_0, \varphi) \in \mathbb{R} \times C$. First we show that condition (2.3) implies the existence of the finite limit $\lim_{t\to\infty} x(t; t_0, \varphi)$.

For the sake of brevity let us use the notation $x(t) := x(t; t_0, \varphi)$. If the limit does not exist, then there are λ_1, λ_2 $(0 < \lambda_1 < \lambda_2 < \varepsilon_0)$ and sequences $\{t_k'\}, \{t_k''\}$ such that

(2.6)
$$t_k' < t_k'' < t_{k+1}', \quad |x(t_k')| = \lambda_1, \quad |x(t_k'')| = \lambda_2, \quad \lambda_1 \le |x(t)| \le \lambda_2 \quad (t_k' \le t \le t_k'')$$

for all $k = 1, 2, \ldots$. Let $[a]_+$ denote the *positive part* of the real number a; i.e., $[a]_+ := \max\{a,\, 0\}$. Then by equation (1.1) there is a constant $\alpha > 0$ such that

$$[|x(t)|']_+ \le |b(t) f(x(t-T))| \le \alpha |b(t)|, \tag{2.7}$$

provided that $x(t) \ne 0$.

If $\int_0^\infty |b| < \infty$, then (2.7) contradicts (2.6). Supposing $\int_0^\infty |b| = \infty$ and introducing the notation

$$\Delta(t,\varepsilon) := \inf\{\tau > 0 : \int_{t-\tau}^{t} |b(r)|\, dr \ge \varepsilon\} \quad (t \in \mathbb{R},\ \varepsilon > 0), \tag{2.8}$$

we define the sequence $t_k^* := t_k'' - \Delta\Big(t_k''; (\lambda_2 - \lambda_1)/2\alpha\Big)$. Condition (2.3) and formulae (2.5)–(2.7) imply the existence of a constant $\beta > 0$ such that

$$\begin{aligned} V(t_K'', x_{t_K''}) - V(t_0, \varphi) &\le \sum_{k=1}^{K} \int_{t_k'}^{t_k''} V'(t, x_t)\, dt \le -\beta \sum_{k=1}^{K} \int_{t_k^*}^{t''_k} [c(t) - \kappa|b(t+T)|]\, dt \\ &\le -\beta \sum_{k=1}^{K} W\Big(\int_{t_k^*}^{t_k''} |b(t)|\, dt\Big) \le -\beta K W\Big(\frac{\lambda_2 - \lambda_1}{2\alpha}\Big) \to -\infty \end{aligned}$$

as $K \to \infty$, which is a contradiction. Therefore, x has a finite limit.

If condition (2.4) is also satisfied, then estimate (2.5) implies that $\lim_{t\to\infty} x(t)$ has to be equal to zero; i.e., the zero solution is asymptotically stable.

The theorem is proved. ■

Theorem 2.1 does not contain any boundedness or growth condition on function $|b|$. This was made possible by condition (2.3) controlling the behaviour of this function. If we have some growth information of coefficient b (see (b) in Theorem A) then condition (2.3) can be weakened.

Theorem 2.2. *Suppose that the following conditions are satisfied:*

(i) *there are an increasing sequence* $\{t_i\}$ *and a number* B *such that*

$$\lim_{t\to\infty} t_i = \infty, \quad \int_{t_i-T}^{t_i} |b(s)|\,ds \le B \quad (i = 1, 2, \ldots);$$

(ii) *for any* $\varepsilon > 0$ *and* $\tilde{t}_i \in [t_i - T, t_i]$ *we have*

$$\sum_{i=1}^{\infty} \int_{\max\{t_{i-1};\, \tilde{t}_i - \Delta(\tilde{t}_i,\varepsilon)\}}^{\tilde{t}_i} [c(t) - \kappa|b(t+T)|]\,dt = \infty,$$

where $\Delta(t,\varepsilon)$ *is defined by* (2.8).

Then the zero solution of (1.1) *is asymptotically stable.*

Proof. As was shown in the proof of Theorem 2.1, the zero solution is stable. Define $\sigma(t_0) > 0$ in the same way and consider an arbitrary solution $x(t) = x(t; t_0, \varphi)$ with $\|\varphi\| < \sigma(t_0)$. It is enough to prove that $V(t, x_t) \to 0$ as $t \to \infty$.

Suppose that the limit of V is greater than zero. Then there is a $\mu > 0$ such that $V(t, x_t) \ge \mu$ for all $t \ge 0$. From condition (i) it follows that

$$\mu \le V(t_i, x_{t_i}) \le \|x_{t_i}\| + B \max_{t_i - T \le s \le t_i} |f(x(s))|,$$

which implies the existence of a $\nu > 0$ with $\|x_{t_i}\| \ge \nu$ $(i = 1, 2, \ldots)$. This means that for every i there is a $\tilde{t}_i \in [t_i - T, t_i]$ with $|x(\tilde{t}_i)| = \nu$. Similarly to the proof of Theorem 2.1 we obtain

$$V(t_k, x_{t_k}) - V(t_0, \varphi) \le -\beta \sum_{i=1}^{k} \int_{\max\{t_{i-1};\, \tilde{t}_i - \Delta(\tilde{t}_i, \nu/2\alpha)\}}^{\tilde{t}_i} [c(t) - \kappa|b(t+T)|]\,dt, \tag{2.9}$$

whence, by condition (ii), we have $V(t, x_t) \to -\infty$ as $t \to \infty$, which is a contradiction.

Theorem 2.2 is proved. ■

If $|b(t)| \le \lambda(t)$ for all t and λ is nondecreasing then

$$\int_{t-\tau}^{t} |b(r)|\,dr \le \int_{t-\tau}^{t} \lambda(r)\,dr \le \lambda(t)\tau, \qquad (\tau > 0);$$

therefore, $\Delta(t,\varepsilon) \ge \varepsilon/\lambda(t)$ and we obtain the following

Corollary 2.3. *Suppose that there is a continuous nondecreasing* $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $|b(t)| \le \lambda(t)$ *for all* $t \in \mathbb{R}_+$*. If condition* (i) *in Theorem* 2.2 *is satisfied, and*

(ii) *for any* $\varepsilon > 0$ *and* $\tilde{t}_i \in [t_i - T, t_i]$ *we have*

$$\sum_{i=1}^{\infty} \int_{\max\{t_{i-1};\, \tilde{t}_i - \varepsilon/\lambda(\tilde{t}_i)\}}^{\tilde{t}_i} [c(t) - \kappa |b(t+T)|]\, dt = \infty,$$

then the zero solution of (1.1) *is asymptotically stable.*

For example, if $c(t) - \kappa|b(t+T)| \ge c_1 > 0$ and $t_{i+1} - t_i \le K$ hold for all $t \ge 0,\ \ i = 1, 2, \dots$ with appropriate constants c_1, K, then $\int_0^\infty 1/\lambda = \infty$ is sufficient for (ii). This means that Theorem A of Burton and Makay is a consequence of Corollary 2.3. What is more, function c can be omitted from condition (c) in Theorem A.

Corollary 2.4. *Suppose that the following conditions are satisfied:*

(i) *there are a sequence* $\{t_i\}$ *and a constant* $\Gamma > T$ *such that* $t_{i+1} \ge t_i + \Gamma$ *for all* $i = 1, 2, \dots,$ *and* $\int_0^t |b(s)|\, ds$ *is uniformly continuous on the set* $\bigcup\limits_{i=1}^{\infty} [t_i - \Gamma, t_i]$;

(ii) *for any* $\delta > 0$ *and* $\tilde{t}_i \in [t_i - T, t_i]$ *we have*

$$\sum_{i=1}^{\infty} \int\limits_{\tilde{t}_i - \delta}^{\tilde{t}_i} [c(t) - \kappa |b(t+T)|]\, dt = \infty.$$

Then the zero solution of (1.1) *is asymptotically stable.*

Proof. Obviously, (i) implies condition (i) in Theorem 2.2. Moreover, for every $\xi > 0$ there is a $\rho = \rho(\xi) > 0$ such that $t_i - \Gamma \le s \le t \le t_i$, $t - s < \rho$ imply $\int_s^t |b(r)|\, dr < \xi$. This means that $\Delta(t, \xi) \ge \rho(\xi)$ for all $t \in [t_i - T, t_i]$, and condition (ii) implies condition (ii) in Theorem 2.2 Corollary 2.4 is proved. ■

Theorem 2.5. *Suppose that the following conditions are satisfied:*

(i) *there are a sequence* $\{t_i\}$ *and positive constants* $\Gamma > T$, K *such that* $t_i + \Gamma \le t_{i+1} \le t_i + K$ *for all* $i = 1, 2, \dots,$ *and* $\int_0^t |b(s)|\, ds$ *is uniformly continuous on the set* $\cup_{i=1}^{\infty} [t_i - \Gamma, t_i]$;

(ii) *for every* $\beta > 0$ *there is a* $\gamma = \gamma(\beta) > 0$ *such that* $\int_{t-\beta}^t [c(s) - \kappa |b(s+T)|]\, ds \ge \gamma$ *for all* $t \in \cup_{i=1}^{\infty} [t_i - T, t_i]$;

(iii) $\int_{t-T}^t |b(s)|\, ds$ *is bounded on* $\mathbb{R}_+$.

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. By condition (iii), for the Lyapunov functional (2.1) we have $V(t,\varphi) \le W(\|\varphi\|)$ with an appropriate continuous increasing function W vanishing at zero. As is proved in [9, Theorem 5.2.1], the zero solution of (1.1) is uniformly stable; namely, for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $\|\varphi\| < \beta$ implies $|x(t;t_0,\varphi| < \varepsilon$ for all t_0, t $(t_0 \le t)$. Denote $\sigma := \delta(\varepsilon_o)$.

For the uniform asymptotic stability it is enough to show that for every $\eta > 0$ there is a $S(\eta)$ such that $\|x_{t_0+S(\eta)}(\cdot;t_0,\varphi)\| < \delta(\eta)$ for all t_0, φ $(t_0 \ge 0, \|\varphi\| < \sigma)$.

Let j denote the smallest $i \ge 1$ with $t_i \ge t_0$. Suppose that $\|x_{t_i}\| \ge \delta(\eta)$ for $i = j, j+1, ..., J$. By the uniform continuity of $\int_0^t |b|$ (see the proof of Corollary 2.4), the inequality analogous to (2.9) reads as follows:

$$\begin{aligned} V(t_J, x_{t_J}) - V(t_0,\varphi) &\le -\beta \sum_{i=j}^{J} \int_{\tilde{t}_i - \Delta(\tilde{t}_i, \delta(\eta)/2\alpha)}^{\tilde{t}_i} [c(t) - \kappa|b(t+T)|]\, dt \\ &\le -\beta \sum_{i=j}^{J} \int_{\tilde{t}_i - \rho(\delta(\eta)/2\alpha)}^{\tilde{t}_i} [c(t) - \kappa|b(t+T)|]\, dt \\ &\le -\beta(J-j)\gamma(\rho(\delta(\eta)/2\alpha)). \end{aligned} \tag{2.10}$$

On the other hand, we have

$$V(t,x_t) - V(t_0,\varphi) \ge -|\varphi(0)| - \int_{t_0-T}^{t_0} |b(s+T)|\, ds \max_{|x|\le\varepsilon_0} |f(x)|.$$

By condition (iii) the right-hand side has a lower bound independent of t_0 and φ. Therefore, from estimate (2.10) it follows that $J-j$ has an upper bound depending only on η, which proves the existence of $S(\eta)$.

The theorem is proved. ■

3. Remarks, examples, open problems

1. In our first example we show that condition

$$c(t) - |b(t+T)| \ge 0 \qquad (t \ge 0) \tag{a$_0$}$$

and the boundedness of b do not guarantee the asymptotic constancy of the solutions of (1.3). The example is of the form

$$x'(t) = -a(t)x(t) + a(t-1)x(t-1) \tag{3.1}$$

where a is nonnegative, continuous, ω-periodic with $\omega \neq 1$ and not 1-periodic. Thus, (3.1) is a particular case of (1.3) with $c(t) = a(t)$, $b(t) = a(t-1)$, $T = 1$. Clearly, (a_0) and the boundedness of b are satisfied.

From the results [1, Theorem 2 and Proposition 8] it follows that each solution $x(\cdot; t_0, \phi)$ of (3.1) tends to the unique ω-periodic solution y of (3.1) for which

$$y(t) + \int_{t-1}^{t} a(s)y(s)\,ds \equiv x(t_0) + \int_{t_0-1}^{t_0} a(s)x(s)\,ds$$

holds. Since a is not 1-periodic, y cannot be a constant function. Therefore, $\lim_{t\to\infty} x(t; t_0, \phi)$ does not exist.

A more direct example can be given as follows. Let a be a 2-periodic function defined by

$$a(t) = \begin{cases} 4 - \frac{\pi}{2}t, & \text{if } -1 \le t \le 0; \\ \frac{-\pi\cos\pi t + (2-\sin\pi t)(4+\pi/2-\pi t/2)}{2+\sin\pi t}, & \text{if } 0 \le t \le 1. \end{cases}$$

Then a straightforward calculation gives that $x(t) = 2 + \sin\pi t$ is a solution of (3.1).

2. The next example shows that conditions (a) and (b) in Theorem A are not sufficient to guarantee the asymptotic constancy of all solutions of (1.3). The functions b and c of this example will be piecewise constants and equation (1.3) will hold only almost everywhere. By using the discontinuous example, it can be easily modified to get an example with continuous b and c.

In the example we will have $b(t) \ge 0$, $t \ge 0$, and $\phi(s) \ge 0$, $-1 \le s \le 0$. These imply $x(t; 0, \phi) \ge 0$, $t \ge 0$. Let $\varepsilon := c_1 > 0$ be given. Assume that $T = 1$. First choose a strictly decreasing sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \alpha_n > 0$. We can find another sequence $\{\delta_n\}_{n=1}^{\infty}$ with $0 < \delta_n < \frac{1}{2}$ and $\alpha_{n-1} > (1 + \varepsilon\delta_n)\alpha_n$, $n = 1, 2, \ldots$.

Let $\phi \in C([-1, 0], R_+)$ such that $\phi(0) > \alpha_0$. Then $x(t) := x(t; \phi, 0) \ge 0$ for $t \ge 0$ whenever $b(t) \ge 0$. Define $\tau_0 = 0$ and $c(0) = \varepsilon$, $b(0) = 0$. Let $k \ge 0$ and suppose that $\tau_0, \tau_1, \ldots, \tau_k$ and $c(t)$, $b(t)$ on $[0, \tau_k]$ are given such that $x(\tau_k) > \alpha_k$. Let

$$c(t) = d_k,\ b(t) = 0 \qquad (t \in (\tau_k, \tau_k + 1]),$$

where $d_k > 0$ is so large that

$$(d_k - \varepsilon) \int_{\tau_k}^{\tau_k+\delta_{k+1}} x(s)\,ds > \alpha_k.$$

There is such a d_k because $x(t) = x(\tau_k)e^{-d_k(t-\tau_k)}$ for $t \in [\tau_k, \tau_k + 1]$ and thus

$$(d_k - \varepsilon) \int_{\tau_k}^{\tau_k+\delta_{k+1}} x(s)\, ds = \frac{d_k - \varepsilon}{d_k} x(\tau_k)(1 - e^{-d_k\delta_{k+1}}) \to x(\tau_k) > \alpha_k$$

as $d_k \to \infty$.

Choose $\tau_{k+1} \in (\tau_k + 1, \tau_k + 1 + \delta_{k+1}]$ such that if

$$b(t) = d_k - \varepsilon,\ c(t) = \varepsilon \qquad (t \in (\tau_k + 1, \tau_{k+1}])$$

then $x(\tau_{k+1}) > \alpha_{k+1}$. If there were not a τ_{k+1} with this property, then from $b(t) = d_k - \varepsilon$, $c(t) = \varepsilon$ on $(\tau_k + 1, \tau_k + 1 + \delta_{k+1}]$ it would follow that

$$\begin{aligned} x(\tau_k + 1 + \delta_{k+1}) = x(\tau_k + 1) + (d_k - \varepsilon) \int_{\tau_k}^{\tau_k+\delta_{k+1}} x(s)\, ds \\ - \varepsilon \int_{\tau_k+1}^{\tau_k+1+\delta_{k+1}} x(s)\, ds > \alpha_k - \varepsilon\delta_{k+1}\alpha_{k+1} > \alpha_{k+1}, \end{aligned}$$

a contradiction.

Therefore, by induction, $b(t)$, $c(t)$ can be defined on $[0, \infty)$ such that

$$(3.2) \qquad c(t) - |b(t + 1)| \geq \varepsilon \qquad (t \geq 0)$$

and $\limsup_{t\to\infty} x(t) \geq \lim_{n\to\infty} \alpha_n > 0$. Then $\lim_{t\to\infty} x(t)$ cannot exist since from (2.5) and condition (3.2) we have $\int_0^\infty |x(t)|\, dt < \infty$.

Conditions (a) and (b) are satisfied for this example with $c_1 = \varepsilon$, $T = 1$, $t_k = \tau_k$, $K = 2$.

3. Let us remark that if (a_0) is replaced by

$$(\mathrm{A}_0) \qquad c(t) - |b(t)| \geq 0 \qquad (t \geq 0)$$

and $\int_t^{t+T} |b(s)| ds$ is bounded, then it follows from [17, Theorem 3] that all solutions of (1.3) have a finite limit (not necessarily zero) as $t \to \infty$. Several papers used conditions of the type (A_0) to study the asymptotic behavior of solutions of (1.1) instead of (a_0) or (a), that is the functions b and c were compared at the same times (see e.g. [7], [9], [11], [16], [17], [19] and references therein).

4. Now we present an example to show that if

$$(\mathrm{A}_\varepsilon) \qquad c(t) - |b(t)| \geq \varepsilon \quad \text{for some } \varepsilon > 0$$

then the limit $\lim_{t\to\infty} x(t)$ does not always exist for the solutions of (1.3). Using methods of [17] it can be proved that if $\int_t^{t+T} |b(s)|\, ds$ is bounded, then asymptotic stability follows from condition (A_ε).

Let $\varepsilon > 0$ be fixed. Let us choose a sequence $\{d_n\}_{n=1}^\infty$ such that

$$d_n \geq \varepsilon,\ (1 - \frac{\varepsilon}{d_n})(1 - e^{-d_n 2^{-n-1}})(1 + \frac{1}{2^{n-1}}) > 1 + \frac{1}{2^n},\ n = 1, 2, \ldots .$$

Define the continuous functions b and c on $[0,\infty)$ such that

$$b(t) = \begin{cases} d_n - \varepsilon, & \text{if } t \in [n - \frac{1}{2^n}, n] \quad (n = 1, 2, \ldots); \\ 0, & \text{if } t \in [n - 1 + \frac{1}{8}, n - 1 + \frac{3}{8}] \quad (n = 1, 2, \ldots); \\ \text{arbitrary } \geq 0 & \text{otherwise,} \end{cases}$$

$$c(t) = \begin{cases} d_n, & \text{if } t \in [n - \frac{1}{2^n}, n] \quad (n = 1, 2, \ldots); \\ \max\{8, \varepsilon\}, & \text{if } t \in [n - 1 + \frac{1}{8}, n - 1 + \frac{3}{8}] \quad (n = 1, 2, \ldots); \\ \text{arbitrary } \geq b(t) + \varepsilon & \text{otherwise.} \end{cases}$$

Let $\phi(s) = 2$ for $s \in [-1, 0]$ and consider the solution $x(t) := x(t; 0, \phi)$ of

$$x'(t) = -c(t)x(t) + b(t)x(t-1).$$

Then it is not difficult to see that $0 < x(t) \leq 2$ on $[0, \infty)$.

If $t \in [n - 1 + 1/8, n - 1 + 3/8]$ then

$$x'(t) = -c(t)x(t) = -\max\{8, \varepsilon\}x(t)$$

and thus

$$x(t) \leq e^{-8(t-(n-1+1/8))}x(n-1+1/8) \leq 2e^{-8(t-(n-1+1/8))},$$

from which $x(n-1+3/8) \leq 2e^{-2} < 1/2$, $n = 1, 2, \ldots$ follows. So, $\liminf_{t\to\infty} x(t) \leq 1/2$.

Now assume that $n \geq 1$ and $x(t) \geq 1 + \frac{1}{2^{n-1}}$ if $n - 1 - \frac{1}{2^n} \leq t \leq n - 1$. This holds for $n = 1$ because $x(t) = 2$ for $t \in [-1, 0]$. Then from the definition of b and c we obtain

$$x'(t) \geq -d_n x(t) + (d_n - \varepsilon)(1 + \frac{1}{2^{n-1}}) \qquad (n - \frac{1}{2^n} \leq t \leq n).$$

Hence

$$
\begin{aligned}
x(t) \geq{} & e^{-d_n(t-(n-1/2^n))} x(n - \frac{1}{2^n}) \\
& + e^{-d_n(t-(n-1/2^n))} \int_{n-1/2^n}^{t} e^{d_n(s-(n-1/2^n))} (d_n - \varepsilon)(1 + \frac{1}{2^{n-1}})\, ds \\
\geq{} & \frac{1 - e^{-d_n(t-(n-1/2^n))}}{d_n} (d_n - \varepsilon)(1 + \frac{1}{2^{n-1}}).
\end{aligned}
$$

If $n - 1/2^{n+1} \leq t \leq n$, then

$$
x(t) \geq (1 - \frac{\varepsilon}{d_n})(1 - e^{-d_n 2^{-n-1}})(1 + \frac{1}{2^{n-1}}) > 1 + \frac{1}{2^n}.
$$

Therefore, by induction, $\limsup_{t\to\infty} x(t) \geq 1$ follows and this means that $\lim_{t\to\infty} x(t)$ does not exist.

5. Finally, after analysing consequences of the different combinations of (A_0) or (A_ε) with the boundedness type conditions on b, the following problems have remained open:

(i) Does (A_0) (or only (A_ε)) imply the existence of the limits of the solutions of (1.3) provided that $\int_{t_i-T}^{t_i} |b(s)|\, ds$ is bounded for a sequence $\{t_i\} \uparrow \infty$ $(i \to \infty)$?

(ii) Does (A_0) (or only (A_ε)) imply the existence of the limits of the solutions of (1.3) provided (b) in Theorem A holds?

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