

Comparison theorems and convergence properties for functional differential equations with infinite delay

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Dedicated to Lajos Pintér on his 60th birthday

1. Introduction

In the general area of stability theory for functional differential equations, Lyapunov functions (Lyapunov—Razumikhin or — Krasovskii functions) often are employed instead of Lyapunov functionals [8, 12]. The derivative of such a function with respect to the equation under investigation is estimated from above on some appropriately chosen subset of the underlying solution (phase) space. The method requires a comparison theorem (or theorems) since the Lyapunov function in question usually is compared to a solution of a certain ordinary differential equation.

The technique of comparison theorems has been thoroughly investigated for functional differential equations with finite delay. (See, for example, [2, 6, 9].) For infinite delay cases DRIVER [1] obtained the first results, and his technique has been generalized in several directions and applied to examine various notions of stability. For instance, KATO [7] and ZHICHENG [13] have obtained results for general “admissible” phase spaces, while PARROTT [11] developed her work in terms of certain (exponentially weighted) C_γ spaces. In a recent paper of the authors [3], this method was applied for general C_γ spaces, but the comparison differential equation was only a trivial one.

In the present paper we examine the technique of comparison results from several points of view. In Section 2 we formulate general comparison theorems in terms of arbitrary real functions and then apply the theorems (in Section 3) to obtain various convergence results for these functions. Among the consequences of Section 3 there is a generalization of the main convergence result of [4] for semigroups on a special function space.

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As may be surmised from the title, one of our primary motivations has been to generate convergence theorems for solutions of functional differential equations with infinite delay. This is accomplished in Section 4 with the aid of the work in Sections 2 and 3. The main thrust in Section 4 is to compare convergence properties of certain functionals $W(=W(t, x_t))$ to corresponding properties of related Lyapunov functions $V(=V(t, x(t)))$.

The paper is concluded with several examples given in Section 5.

2. Comparison theorems

Let $\omega: R^+ \times R^+ \rightarrow R^+$ be a continuous function, $t_0, u_0 \in R^+$ and let $u(t)$ be the maximal solution of

$$(1) \quad \begin{cases} u'(t) = \omega(t, u(t)) & (t \geq t_0) \\ u(t_0) = u_0 \end{cases}$$

on an interval $[t_0, a)$ ($t_0 < a \leq \infty$). Let $f: R^+ \rightarrow R^+$ $g: R \rightarrow R^+$, and let g be continuous on $[t_0, \infty)$.

Theorem 1. *If for all $t \in [t_0, a)$ the inequalities*

$$(A_1) \quad g(t) \leq f(t),$$

$$(B_1) \quad f(t) \leq \max \left\{ \max_{-r \leq s \leq 0} g(t+s), f(t-r) \right\} \quad (r \in [0, t-t_0]),$$

are fulfilled and if for $t \in [t_0, a)$

$$(C_1) \quad 0 < g(t) = f(t)$$

implies

$$(D_1) \quad D^+g(t) \leq \omega(t, g(t)),$$

then $f(t_0) \leq u_0$ implies $f(t) \leq u(t)$ ($t \in [t_0, a)$).

Proof. First we remark that (A_1) , (B_1) imply

$$(2) \quad \liminf_{h \rightarrow 0^+} f(t-h) \leq f(t) \quad (t \in (t_0, a)),$$

$$(3) \quad \limsup_{h \rightarrow 0^+} f(t+h) \leq f(t) \quad (t \in [t_0, a)).$$

Let $\varepsilon > 0$ and define the function

$$F(t) = \max \left\{ \sup_{t_0 \leq s \leq t} f(s), \varepsilon \right\} \quad (t \geq t_0).$$

Clearly F is monotone nondecreasing. So, (2) and (3) imply F is continuous. Obviously

$$(4) \quad g(t) \cong f(t) \cong F(t) \quad (t \cong t_0)$$

and

$$(5) \quad \begin{aligned} F(t) &= \max \left\{ \sup_{t-r \cong s \cong t} f(s), F(t-r) \right\} \cong \\ &\cong \max \left\{ \sup_{t-r \cong s \cong t} \max_{t-r-s \cong u \cong 0} \{g(s+u), f(t-r)\}, F(t-r) \right\} \cong \\ &\cong \max \left\{ \max_{-r \cong s \cong 0} g(t+s), F(t-r) \right\} \quad (t \cong t_0, r \in [0, t-t_0]). \end{aligned}$$

If $g(t) < F(t)$, then by the continuity of g there is a $\delta > 0$ so that $\max_{0 \cong s \cong \delta} g(t+s) < F(t)$. Hence by using (5)

$$F(t+h) \cong \max \left\{ \max_{0 \cong s \cong \delta} g(t+s), F(t) \right\} \cong F(t)$$

whenever $0 < h \cong \delta$. So, $g(t) < F(t)$ implies $D^+F(t) \cong 0$.

Assume $g(t) = F(t)$ and $D^+F(t) > 0$. Then there exists a sequence $\{\delta_n\}$ such that $\delta_n > 0$, $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, $F(t+\delta_n) > F(t)$ and

$$D^+F(t) = \lim_{n \rightarrow \infty} \frac{F(t+\delta_n) - F(t)}{\delta_n}.$$

From (5) it follows that for any n there is a γ_n , $0 < \gamma_n \cong \delta_n$, such that

$$g(t+\gamma_n) \cong F(t+\delta_n).$$

Using (4) and (D₁) we have

$$\begin{aligned} D^+F(t) &= \lim_{n \rightarrow \infty} \frac{F(t+\delta_n) - F(t)}{\delta_n} \cong \limsup_{n \rightarrow \infty} \frac{g(t+\gamma_n) - g(t)}{\gamma_n} \cong \\ &\cong D^+g(t) \cong \omega(t, g(t)) = \omega(t, f(t)) = \omega(t, F(t)). \end{aligned}$$

Since ω is a nonnegative function, we obtain

$$D^+F(t) \cong \omega(t, F(t)) \quad (t \in [t_0, a]).$$

By using this inequality, the continuity of F , $F(t_0) \cong \max \{u(t_0), \varepsilon\}$ and a well-known differential inequality [9, vol. 1, pp. 15] we get

$$f(t) \cong F(t) \cong u_\varepsilon(t) \quad \text{on } [t_0, a_\varepsilon),$$

where $u_\varepsilon(t)$ is the maximal solution of

$$\begin{cases} u'_\varepsilon(t) = \omega(t, u_\varepsilon(t)) & (t \cong t_0) \\ u_\varepsilon(t_0) = \max \{u_0, \varepsilon\} \end{cases}$$

on $[t_0, a_\varepsilon)$. If $\varepsilon \rightarrow 0+$, then $a_\varepsilon \rightarrow a$ and $u_\varepsilon(t) \rightarrow u(t)$ uniformly on every compact interval of $[t_0, a)$. This completes the proof.

Corollary 1. Let (A_1) , (B_1) hold and suppose that (C_1) implies

$$(D_2) \quad D^+g(t) \leq 0.$$

Then $f(t)$ is a monotone non-increasing function on $[t_0, a)$.

Theorem 2. Suppose that $a = \infty$, (A_1) , (B_1) are satisfied and (C_1) implies (D_1) , moreover $\omega(t, u)$ is nondecreasing in u and the solutions of equation (1) are bounded on $[t_0, \infty)$ for every u_0 . Then $\lim_{t \rightarrow \infty} f(t)$ exists.

Proof. Since f is bounded below, it is enough to prove that $V^+f < \infty$, where V^+f denotes the positive variation of f on $[t_0, \infty)$. Let $\tilde{u}(t)$ be the maximal solution of (1) on $[t_0, \infty)$ with $\tilde{u}(t_0) = f(t_0)$. Theorem 1 implies $f(t) \leq \tilde{u}(t)$ for $t \geq t_0$. From $\omega(t, u) \geq 0$ and the boundedness of $\tilde{u}(t)$ it follows that $\tilde{u}' \in L^1([t_0, \infty))$. If $0 < f(t) = g(t)$, then

$$D^+g(t) \leq \omega(t, g(t)) = \omega(t, f(t)) \leq \omega(t, \tilde{u}(t)) = \tilde{u}'(t) \quad (t \geq t_0).$$

That is Theorem 1 is applicable with $\omega(t, u) = \tilde{u}'(t)$.

Obviously the maximal solution of

$$\begin{cases} u'(t) = \tilde{u}'(t), & t \geq t_1 \\ u(t_1) = f(t_1) \end{cases}$$

is $u(t) = f(t_1) + \int_{t_1}^t \tilde{u}'(s) ds = f(t_1) + \tilde{u}(t) - \tilde{u}(t_1)$. Replace t_0 by t_1 and apply Theorem 1 to get

$$f(t) \leq f(t_1) + \tilde{u}(t) - \tilde{u}(t_1) \quad \text{for all } t_0 \leq t_1 \leq t.$$

Using that $\tilde{u}(t)$ is nondecreasing on $[t_0, \infty)$, this inequality gives $V^+f < \infty$. This completes the proof.

Remark 1. Theorem 1 is an extension of Driver's result [1, Lemma 1]. He examined the case $f(t) = \sup_{\alpha \leq s \leq t} g(s)$, $-\infty \leq \alpha \leq t_0$ and g is continuous on $[\alpha, a)$.

Remark 2. Theorem 2 may be false if $\omega(t, u)$ is decreasing in u . For example, let

$$\omega(t, u) = \begin{cases} 3 - u & \text{if } u \leq 3 \\ 0 & \text{if } u > 3, \end{cases}$$

and put $f(t) = g(t) = \sin t$. Then all the assumptions of Theorem 2 are satisfied except the monotonicity condition on $\omega(t, u)$ and $\lim_{t \rightarrow \infty} f(t)$ does not exist.

Further on, we need a sharper version of Theorem 1. Namely, inequality (D_1) will be required only on a subset of the set of the points of $[t_0, a)$ where (C_1) is satisfied. In order to give this subset we introduce the following notation.

Let us suppose $a(t, r), p(t, r), h(t, r)$ are continuous functions on $[\tau, \infty) \times R^+$, where $\tau \geq 0$ is a constant, $p(t, r)$ is nondecreasing in r , $a(t, r) < r$ for all $r > 0, t \geq 0$. Suppose that $\tau \leq h(t, r), p(t, r) \leq t$ for all $r > 0, t \geq \tau$. Let $\sigma(t, r) = \sup \{s: p(s, r) \leq t\}$. It is not difficult to see that $\sigma(t, r)$ is nonincreasing in $r, \sigma(t, r) \leq t$ and if f is a locally bounded function on $[\tau, \infty), \sigma(\tau, r) < \infty$ for all $r > 0$, then there is $0 < u_0 (=u_0(f, \tau))$ such that $f(t) \leq u_0$ on $[\tau, \sigma(\tau, u_0)]$. For $r > 0, 0 \leq z \leq s \leq t$ define the function

$$g^*(z, s, t, r) = \begin{cases} D^+g(s) & \text{if } a(t, r) < g(v), f(v) \leq r \text{ for all } v \in [z, s] \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. *Suppose g is continuously differentiable on $[\tau, \infty), (A_1), (B_1)$ are satisfied on $[\tau, \infty)$ and that*

$$(E_1) \quad \int_z^t g^*(z, s, t, r) ds < r - a(t, r)$$

for all $r > 0, t \geq \sigma(\tau, r) t > z \geq h(t, r)$. Moreover, if the inequalities

$$(C_2) \quad \begin{cases} 0 < g(t) = f(t), p(t, f(t)) \leq \tau, \\ a(t, f(t)) < g(v) \leq f(v) \leq f(t) \text{ for all } v \in [h(t, f(t)), t] \end{cases}$$

imply (D_1) , then

$$f(v) \leq u_0 \text{ for all } v \in [\tau, \sigma(\tau, u_0)]$$

implies

$$f(t) \leq u(t) \quad (t \in [\sigma(\tau, u_0), a]),$$

where $u(t)$ is the maximal solution of (1) on $[t_0, a)$ with $t_0 = \sigma(\tau, u_0)$.

Proof. Define $t_0 = \sigma(\tau, u_0)$ and for $t \geq t_0$

$$G(t) = \max(g(t), u_0), \quad F(t) = \sup_{t_0 \leq s \leq t} \max(f(s), u_0).$$

Then in the same way as in the proof of Theorem 1 we can see that

$$G(t) \leq F(t) \quad (t \geq t_0),$$

$$F(t) \leq \max \left\{ \max_{-r \leq s \leq 0} G(t+s), F(t-r) \right\} \quad (t \geq t_0, r \in [0, t-t_0]),$$

$G(t) < F(t)$ implies $D^+F(t) \leq 0$, and if $G(t) = F(t), D^+F(t) > 0$ then $D^+F(t) \leq D^+G(t)$. It is easy to see that in the case $t \geq t_0, G(t) = F(t), D^+F(t) > 0$ the following relations are true: $F(t) = f(t) = G(t) = g(t) \leq u_0, \frac{d}{dt}g(t) = D^+G(t)$. We want to show that in this case $D^+G(t) \leq \omega(t, G(t))$ is fulfilled, too. This would be sufficient to the completeness of the proof by using Theorem 1.

Since $F(t) = f(t)$ implies $f(v) \leq f(t)$ for all $v \in [h(t, f(t)), t]$, by the conditions

of Theorem 3 it is enough to prove that $a(t, f(t)) < g(v)$ for all $v \in [h(t, f(t)), t]$. Suppose the contrary, that is there exists a $z \in [h(t, f(t)), t]$ such that $a(t, f(t)) < g(v)$ for all $v \in (z, t]$, $a(t, f(t)) = g(z)$. Then $g^*(z, s, t, f(t)) = D^+g(s)$ for all $s \in (z, t)$. Therefore, by inequality (E₁) one gets

$$f(t) - a(t, f(t)) = g(t) - g(z) = \int_z^t g^*(z, s, t, f(t)) ds < f(t) - a(t, f(t)),$$

which is a contradiction, thereby completing the proof.

We can extend Theorem 2 in a similar way:

Theorem 4. *Suppose that $a = \infty$, (A₁), (B₁), (E₁) are satisfied and (C₂) implies (D₁), moreover $\omega(t, u)$ is nondecreasing in u and the solutions of equation (1) are bounded on $[t_0, \infty)$ for every u_0 . Then $\lim_{t \rightarrow \infty} f(t)$ exists.*

If we analyse the proof of Theorem 3 we can find that the differentiability property of function $g(t)$ is used only in relation $g(t) - g(z) = \int_z^t g^*(z, s, t, f(t)) ds$, where $z \in [h(t, f(t)), t]$. So, if $h(t, r) \equiv t$, then it is sufficient for g to be continuous. Therefore, a J. KATO and W. ZHICHENG type comparison theorem [7, 13] can be deduced from Theorem 1. We shall formulate it in the next

Corollary 2. *Assume $\tau \geq 0$, $g: [\tau, \infty) \rightarrow R^+$ is a continuous function and*

$$p(t, g(t)) \geq \tau, \quad 0 < g(t) = \max_{p(t, g(t)) \leq s \leq t} g(s)$$

imply

$$D^+g(t) \leq \omega(t, g(t)).$$

If there is $u_0 > 0$ such that $\sigma(\tau, u_0) < \infty$, $g(t) \leq u_0$ on $[\tau, \sigma(\tau, u_0)]$, then $g(t) \leq u(t)$ for all $t \in [\sigma(\tau, u_0), a)$, where $u(t)$ is the maximal solution of (1) on $[t_0, a)$ with $t_0 = \sigma(\tau, u_0)$.

Proof. Define $h(t, r) \equiv t$, and $f(t) = \max_{\tau \leq s \leq t} g(s)$ for $t \geq t_0$. If $p(t, f(t)) \geq \tau$, $0 < g(t) = f(t)$, then $g(t) = \max_{\tau \leq s \leq t} g(s)$, consequently $g(t) = \max_{p(t, f(t)) \leq s \leq t} g(s)$, therefore (D₁) is fulfilled, and the assertion follows from Theorem 3.

Z. MIKOLAJSKA [10] used a comparison result analogous with the special case $p(t, r) = t_0$. This case is stated in the following corollary. The proof is omitted because it is similar to that of Corollary 2.

Corollary 3. *Suppose $\tau \leq t_0$, $g: [\tau, \infty) \rightarrow R^+$ is continuously differentiable, (E₁) is satisfied for all $r > 0$, $t \geq t_0$, $t > z \geq h(t, r)$. If $h(t, r) \geq \tau$ for all $r > 0$, $t \geq t_0$, and if $t \geq t_0$,*

$$a(t, g(t)) < \min_{h(t, g(t)) \leq s \leq t} g(s) \leq \max_{h(t, g(t)) \leq s \leq t} g(s) = g(t)$$

imply (D₁), then $\max_{\tau \leq s \leq t_0} g(s) \leq u_0$ implies $g(t) \leq u(t)$ for all $t \geq t_0$.

3. Convergence properties of real functions

In the previous chapter sufficient conditions on functions f and g were given to guarantee the existence of the limit of f as $t \rightarrow \infty$. Now, we show that it is possible to modify condition (B_1) such that the existence of $\lim_{t \rightarrow \infty} f(t)$ implies that of $\lim_{t \rightarrow \infty} g(t)$.

Lemma 1. *Suppose (A_1) for $t \geq t_0$ and that there exists a function $h: R^+ \times R^+ \rightarrow R^+$ such that*

$$(F_1) \quad \lim_{t \rightarrow \infty} h(t-r, t) = 0 \quad (r > 0),$$

$$(B_2) \quad f(t) \leq \max_{-r \leq s \leq 0} g(t+s) + h(r, t) \quad (t \geq t_0, r \in [0, t-t_0]).$$

Then $\limsup_{t \rightarrow \infty} g(t) = \limsup_{t \rightarrow \infty} f(t)$.

Proof. (A_1) implies $\limsup_{t \rightarrow \infty} g(t) \leq \limsup_{t \rightarrow \infty} f(t)$. On the other hand, if $c = \limsup_{t \rightarrow \infty} g(t) < \infty$, then for all $\varepsilon > 0$ there is a $T = T(\varepsilon) \geq t_0$ such that $g(t) \leq c + \varepsilon$ for $t \geq T$. By (B_2) we have $f(t) \leq c + \varepsilon + h(t-T, t)$ for all $t \geq T$. Using (F_1) , we obtain $\limsup_{t \rightarrow \infty} f(t) \leq c + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the theorem is proved.

Theorem 5. *Suppose g is uniformly continuous on $[t_0, \infty)$, (A_1) is satisfied for $t \geq t_0$ and there exist functions $h, k_1, k_2: R^+ \times R^+ \rightarrow R^+$ such that (F_1) is fulfilled, $k_1(r, u), k_2(r, u)$ are monotone nondecreasing and continuous in u for all $r \in R^+$,*

$$k_1(0, u) = \lim_{r \rightarrow 0^+} k_1(r, u) = u \quad (u > 0),$$

$$k_2(r, u) < u \text{ for all } r, u > 0, \quad k_2(0, u) \leq u \text{ and}$$

$$(B_3) \quad f(t) \leq \max \left\{ k_1(r, \max_{-r \leq s \leq 0} g(t+s)), k_2(r, \max_{-\tau \leq s \leq -r} g(t+s)) \right\} + h(\tau, t) \\ (t \geq t_0, \tau \in [0, t-t_0], r \in [0, \tau]).$$

Then $\lim_{t \rightarrow \infty} g(t) = c$ if and only if $\lim_{t \rightarrow \infty} f(t) = c$.

Proof. If $\lim_{t \rightarrow \infty} g(t) = c$, then according to $(A_1), (B_3)$ with $r=0$ and Lemma 1

$$c = \liminf_{t \rightarrow \infty} g(t) \leq \liminf_{t \rightarrow \infty} f(t) \leq \limsup_{t \rightarrow \infty} f(t) = \limsup_{t \rightarrow \infty} g(t) = c,$$

i.e. $\lim_{t \rightarrow \infty} f(t) = c$.

Now, assume $\lim_{t \rightarrow \infty} f(t) = c$. It is enough to prove that $\liminf_{t \rightarrow \infty} g(t) \geq c$. Suppose the contrary, i.e. $\liminf_{t \rightarrow \infty} g(t) < c$. Let $c_1 \in (\liminf_{t \rightarrow \infty} g(t), c)$. From the uniform continuity of g there is a $\delta > 0$ such that $t_1, t_2 \geq t_0, |t_2 - t_1| < \delta$ imply $|g(t_1) - g(t_2)| <$

$< (c - c_1)/4$. Define a sequence $\{t_n\}$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $g(t_n) \equiv c_1$ for $n = 1, 2, \dots$. Then

$$\begin{aligned} \max_{-\delta \leq s \leq 0} g(t_n + s) &\equiv \max_{-\delta \leq s \leq 0} (g(t_n + s) - g(t_n)) + g(t_n) \equiv \\ &\equiv \frac{c - c_1}{4} + c_1 = \frac{c + 3c_1}{4}. \end{aligned}$$

Let $r \in (0, \delta)$ be chosen such that $k_1(r, (c + 3c_1)/4) \equiv (c + c_1)/2$. Choose $\varepsilon > 0$, $T = T(\varepsilon) \equiv t_0$ such that $k_2(r, c + \varepsilon) < c$ and $g(t) \equiv c + \varepsilon$ for $t \equiv T$. From (B₃) we obtain

$$\begin{aligned} f(t_n) &\equiv \max \{k_1(r, \max_{-\delta \leq s \leq 0} g(t_n + s)), k_2(r, \max_{T - t_n \leq s \leq -r} g(t_n + s))\} + \\ &+ h(t_n - T, t_n) \equiv \max \left\{ \frac{c + c_1}{2}, k_2(r, c + \varepsilon) \right\} + h(t_n - T, t_n) \end{aligned}$$

for $t_n \equiv T$. Using $\lim_{t \rightarrow \infty} h(t - T, t) = 0$ we get the contradiction

$$c = \limsup_{n \rightarrow \infty} f(t_n) \equiv \max \{(c + c_1)/2, k_2(r, c + \varepsilon)\} < c.$$

This completes the proof.

4. Applications for functional differential equations

Let X be a Banach space with the norm $\| \cdot \|_X$ and let B be a space of functions mapping R^- into X with a semi-norm $\| \cdot \|_B$. For a function $x: (-\infty, a) \rightarrow X$ and for $t \in (-\infty, a)$ define x_t as a function from R^- into X by $x_t(s) = x(t + s)$, $s \in R^-$. For $\tau \in R^+$ define B_τ as the set of $\varphi \in B$ such that $\varphi_t \in B$ for each $t \in [-\tau, 0]$ and $\varphi(s)$ is continuous on $[-\tau, 0]$. Let $D \subset B$ and let $f: R^+ \times D \rightarrow X$ be a given function. Consider the functional differential equation

$$(6) \quad \dot{x}(t) = f(t, x_t).$$

A solution of equation (6) on $[t_0, a)$, $t_0 < a \equiv \infty$ is a function $x: (-\infty, a) \rightarrow X$ such that $x_t \in D$ for $t \in [t_0, a)$, $x(t)$ is continuous on $[t_0, a)$, differentiable on (t_0, a) and $\dot{x}(t) = f(t, x_t)$ on (t_0, a) .

Let $V: R \times X \rightarrow R^+$ be a locally Lipschitzian function.

Suppose that there exists a function $W: R^+ \times D \rightarrow R^+$ such that

$$(AV) \quad V(t, \varphi(0)) \equiv W(t, \varphi) \quad (t \in R^+, \varphi \in D)$$

and

$$\begin{aligned} (BV_1) \quad W(t, \varphi) &\equiv \max \left\{ \max_{-r \leq s \leq 0} V(t + s, \varphi(s)), W(t - r, \varphi_{-r}) \right\} \\ &(t \in R^+, r \in [0, t], \varphi \in B_r). \end{aligned}$$

If $x(t)$ is a solution of (6), then $g(t)=V(t, x(t))$ and $f(t)=W(t, x_t)$ satisfy conditions (A₁) and (B₁). So, we may apply Theorem 1, when the derivative of $V(t, x(t))$ has an appropriate estimate on the set $V(t, \varphi(0))=W(t, \varphi)$.

If $W(t, \varphi) = \sup_{-\tau \leq s \leq 0} V(t+s, \varphi(s))$, $\tau \in R^+$, then we get a RAZUMIKHIN type comparison result [6, 12]. One may put

$$(7) \quad W(t, \varphi) = \sup_{s \in R^-} V(t+s, \varphi(s))$$

or

$$(8) \quad W(t, \varphi) = \sup_{s \in R^-} l(s, V(t+s, \varphi(s))),$$

where $l: R^- \times R^+ \rightarrow R^+$ is a continuous function such that $l(s_1, v_1) < l(s_2, v_2) < v_2$ for all $s_1 < s_2 < 0$, $0 \leq v_1 < v_2$ and supposing that the supremums on the right-hand side of (7) and (8) exist for all $\varphi \in D$. If $l(s, v) = e^{\gamma s} v$ for a $\gamma > 0$, then we obtain the case examined by M. PARROTT [11].

Let $k: R^- \rightarrow R^+$ be a measurable function such that $k(s_0) = 0$ implies $k(s) = 0$ for all $s \leq s_0$, for each $r \geq 0$

$$(9) \quad \operatorname{ess\,sup}_{s \in R^-, k(s) > 0} \frac{k(s-r)}{k(s)} + \int_{-r}^0 k(s) ds \leq 1$$

holds and $\int_{-\infty}^0 k(s)V(t+s, \varphi(s)) ds$ exists for all $t \geq 0$, $\varphi \in D$. Then one can choose

$$(10) \quad W(t, \varphi) = \max \left\{ V(t, \varphi(0)), \int_{-\infty}^0 k(s)V(t+s, \varphi(s)) ds \right\}.$$

We remark if k is continuous then (9) implies $k(s) \leq Me^{\gamma s}$ for all $s \in (-\infty, 0]$ where $M, \gamma > 0$. On the other hand, (9) is true if $k(s) = Me^{\gamma s}$ such that $\gamma \geq M > 0$.

Our comparison results are useful to prove stability, uniqueness and continuous dependence of the solutions (see e.g. [1]). In this paper we deal with the convergence properties of solutions as $t \rightarrow \infty$. From Theorems 2 and 4 we get the following results. The derivative of V with respect to (6) is defined by

$$\dot{V}(t, \varphi) = \limsup_{h \rightarrow 0^+} (V(t+h, \varphi(0) + hf(t, \varphi)) - V(t, \varphi(0))) h^{-1}.$$

Corollary 4. Suppose (AV), (BV₁) and

$$(DV) \quad \dot{V}(t, \varphi) \leq \omega(t, V(t, \varphi(0)))$$

whenever

$$(CV_1) \quad 0 < V(t, \varphi(0)) = W(t, \varphi)$$

for $t \in R^+$, $\varphi \in D$, where $\omega: R^+ \times R^+ \rightarrow R^+$ is continuous, nondecreasing in its second variable and the solutions of the equation $\dot{u}(t) = \omega(t, u(t))$ are defined and bounded

on R^+ . Then for each solution $x(t)$ of (6) defined on $[0, \infty)$ the limit $\lim_{t \rightarrow \infty} W(t, x_t)$ exists.

Corollary 5. Let $a(t, r), p(t, r), h(t, r)$ be the same functions as in Theorem 3 and for $r > 0, 0 \leq z \leq s \leq t$ define

$$g^*(z, s, t, r) = \sup\{\dot{V}(s, \varphi) : a(t, r) < V(v, \varphi(v-s)), \\ W(v, \varphi_{v-s}) \leq r \text{ for all } v \in [z, s]\}.$$

Suppose $V(t, x)$ has continuous partial derivatives, (AV), (BV₁), (E₁) are fulfilled and (DV) is true whenever $(CV_1), p(t, V(t, \varphi(0))) > 0$ and for all $z \in [h(t, V(t, \varphi(0))), t]$ the inequality

$$a(t, V(t, \varphi(0)) < V(z, \varphi(z-t)) \leq W(z, \varphi_{z-t}) \leq W(t, \varphi)$$

is satisfied. Then for each solution $x(t)$ of (6) that is defined on $[0, \infty)$, the limit $\lim_{t \rightarrow \infty} W(t, x_t)$ exists.

Generally, the existence of the limit $\lim_{t \rightarrow \infty} W(t, x_t)$ gives little information about the asymptotic behavior of solutions. For example, if $W(t, \varphi) = \sup_{s \in R^-} V(t+s, \varphi(s))$, then the existence of $\lim_{t \rightarrow \infty} W(t, x_t)$ means the boundedness of $V(t, x(t))$ on $[t_0, \infty)$ only. Using Theorem 5 we may obtain conditions for $W(t, \varphi)$ to guarantee the existence of $\lim_{t \rightarrow \infty} V(t, x(t))$, which gives much more information about $x(t)$.

Corollary 6. Suppose that all conditions of Corollary 4 (or 5) are satisfied and there exist functions $k_1, k_2: R^+ \times R^+ \rightarrow R^+$ and $h: R^+ \times R^+ \times D \rightarrow R^+$ such that $k_1(r, u), k_2(r, u)$ are monotone nondecreasing and continuous in u for all $r \in R^+$, $\lim_{r \rightarrow 0^+} k_1(r, u) = u$ for all $u > 0, k_2(r, u) < u$ for all $r, u > 0, k_2(0, u) \leq u$ for all $u \geq 0, h(t-r, t, \varphi) \rightarrow 0$ as $t \rightarrow \infty$ for all $r > 0, \varphi \in D$, moreover

$$(BV_2) \quad W(t, \varphi) \leq \max\{k_1(r, \max_{-r \leq s \leq 0} V(t+s, \varphi(s))), \\ k_2(r, \max_{-\tau \leq s \leq -r} V(t+s, \varphi(s)))\} + h(\tau, t, \varphi_{-\tau})$$

for all $t \in R^+, \tau \in [0, t], r \in [0, \tau], \varphi \in B_{-\tau} \cap D$. Then $\lim_{t \rightarrow \infty} V(t, x(t))$ exists for every solution $x(t)$ of (7) which is defined on $[0, \infty)$ and for which $V(t, x(t))$ is uniformly continuous on $[0, \infty)$.

If $W(t, \varphi)$ is defined by (8), where $l(s, v) \rightarrow 0$ as $s \rightarrow -\infty$ for every $v > 0, V(t+s, \varphi(s))$ is bounded on R^- , then (BV₂) is true with $k_1(r, u) = u, k_2(r, u) = l(-r, u)$ and $h(r, t, \varphi) = \sup_{s \leq -r} l(s, V(t+s, \varphi(s)))$. If $W(t, \varphi)$ is defined by (10),

$\int_{-\infty}^0 k(s)V(t+s, \varphi(s)) ds < \infty$ for all $\varphi \in D$ and $t \in R^+, k(s)$ is nondecreasing,

$\int_{-\infty}^0 k(s)ds=1$, then (BV_2) is true with

$$k_1(r, u) = u \left(1 + \left(\int_{-r}^0 k(s) ds \right)^2 - \int_{-r}^0 k(s) ds \right)^{-1},$$

$$k_2(r, u) = u \left(1 + \left(\int_{-r}^0 k(s) ds \right)^2 \right)^{-1},$$

$$h(\tau, t, \varphi) = \int_{-\infty}^0 k(s-\tau)V(t+s-\tau, \varphi(s)) ds.$$

We get an important special case if

$$(11) \quad D = B, \quad V(t, x) = \|x\|_X, \quad W(t, \varphi) = \|\varphi\|_B.$$

Then (AV) , (BV_1) and (BV_2) are axioms for these norms as it is used generally in functional differential equations with infinite delay.

These axioms resemble axioms of admissible phase spaces in which the estimation

$$(12) \quad \mu \|\varphi(0)\|_X \cong \|\varphi\|_B \cong K(r) \sup_{-r \leq s \leq 0} \|\varphi(s)\|_X + M(r)\|\varphi_{-r}\|_B$$

is true with $\mu > 0$ and some continuous functions $K, M: R^+ \rightarrow R^+$ [7]. If $\mu=1$ and $K(r)+M(r) \leq 1$ then (12) implies (AV) and (BV_1) in the case (11). So (AV) and (BV_1) are true in special admissible phase spaces. In case (11) property (BV_2) cannot be compared to (12).

In case of several phase spaces used in theory of functional differential equations with infinite delay we may define a norm such that (AV) , (BV_1) and (BV_2) are fulfilled. So, in the special case (11), if

a) $B=BC$ is the space of bounded continuous functions on $(-\infty, 0]$ into X with norm

$$\|\varphi\|_{BC} = \sup_{s \in R^-} \|\varphi(s)\|_X$$

then (AV) and (BV_1) are fulfilled but (BV_2) is not satisfied. If we put

$$\|\varphi\|_{BC} = \sup_{s \in R^-} p(s)\|\varphi(s)\|_X,$$

where $p: R^- \rightarrow R^+$, $p(s_1) < p(s_2) < 1$ for all $s_1 < s_2 < 0$, $p(0)=1$ and $\lim_{s \rightarrow -\infty} p(s)=0$, then (AV) , (BV_1) and (BV_2) are fulfilled.

b) $B=C_\gamma$ ($\gamma \in R^+$) is the space of continuous functions φ on $(-\infty, 0]$ such that $\lim_{s \rightarrow -\infty} e^{\gamma s} \|\varphi(s)\|_X$ exists and

$$\|\varphi\|_{C_\gamma} = \sup_{s \in R^-} e^{\gamma s} \|\varphi(s)\|_X,$$

then for $\gamma > 0$ (AV), (BV_1) and (BV_2) are fulfilled. For $\gamma = 0$ (AV), (BV_1) hold, but (BV_2) does not.

c) $B = L_k^p$, $p \geq 1$ is the space of measurable functions on R^- such that

$$\int_{-\infty}^0 k(s) \|\varphi(s)\|^p ds < \infty,$$

where $k: R^- \rightarrow R^+$ is measurable, $\int_{-\infty}^0 k(s) ds = 1$ and $\int_{-r}^0 k(s) ds > 0$ for all $r > 0$ then (AV) and (BV_2) are true with the norm

$$\|\varphi\|_{L_k^p} = \max \left(\|\varphi(0)\|_X, \left(\int_{-\infty}^0 k(s) \|\varphi(s)\|_X^p ds \right)^{1/p} \right).$$

If (9) is valid for all $r > 0$, then (BV_1) is fulfilled, too.

5. Examples

1. Consider the equation

$$(13) \quad \dot{x}(t) = H \left(t, x(t) - \int_{-\infty}^0 k(s)x(t+s) ds \right).$$

Here $H: R^+ \times R \rightarrow R$ is continuous, $H(t, u)u \leq 0$ for all $t \in R^+$, $u \in R$; $\sup_{t \in R^+, u \in K} |H(t, u)| < \infty$ for every compact set $K \subset R$; $k: R^- \rightarrow R^+$ is nondecreasing, measurable, $\int_{-\infty}^0 k(s) ds = 1$. So, for each constant c , $x(t) \equiv c$ is a solution of equation (13). Let us choose L_k^1 as a phase space for (13). Then the existence and continuity of a solution through every $\varphi \in L_k^1$ is insured, further, if a solution $x(t)$ is bounded, then it can be continued as $t \rightarrow \infty$.

Assertion. If (9) is fulfilled then every noncontinuable solution of (13) has a finite limit as $t \rightarrow \infty$.

In order to prove this assertion, we define the following functions for $t \in R^+$, $\varphi \in L_k^1$. $V(t, \varphi(0)) = |\varphi(0)|$,

$$W(t, \varphi) = \max \left(|\varphi(0)|, \int_{-\infty}^0 k(s) |\varphi(s)| ds \right).$$

If $x(t)$ is a noncontinuable solution of (13) on $[t_0, a)$ through φ , then $g(t) = V(t, x(t))$ and $W(t, x_t)$ satisfy the assumptions of Corollary 1 with $\omega(t, u) \equiv 0$. So, we have

$$|x(t)| \leq \max \left(|x(0)|, \int_{-\infty}^0 k(s) |x(t+s)| ds \right)$$

for $t \in [t_0, a)$. Consequently, $x(t), \dot{x}(t)$ are bounded, and $a = \infty$, therefore we may apply Corollaries 4 and 6 with $V(t, \varphi(0)), W(t, \varphi)$, which implies the assertion.

Assertion. If $k(s)$ is differentiable and $k'(s) \cong k(0)k(s)$ for $s \in \mathbb{R}^-$, then every bounded solution of equation (13) has a finite limit as $t \rightarrow \infty$.

Indeed, let $x(t)$ be a bounded solution of (13) on $[t_0, \infty)$ and put $g(t) = V(t, x(t)), f(t) = W(t, x_t)$ where V and W are defined above.

We want to estimate the derivative $D^+f(t)$. We have three cases:

$$a) \quad |x(t)| \cong \int_{-\infty}^0 k(s)|x(t+s)| ds.$$

Then $f(t) = g(t)$ and (13) implies $\frac{d}{dt}|x(t)| \cong 0$.

$$b) \quad |x(t)| < \int_{-\infty}^0 k(s)|x(t+s)| ds.$$

In this case

$$f(t) = \int_{-\infty}^0 k(s)|x(t+s)| ds = \int_{-\infty}^t k(s-t)|x(s)| ds,$$

so using the inequality

$$\begin{aligned} (14) \quad \frac{d}{dt} \int_{-\infty}^0 k(s)|x(t+s)| ds &= k(0)|x(t)| - \int_{-\infty}^t k'(s-t)|x(s)| ds \cong \\ &\cong k(0)|x(t)| - \int_{-\infty}^0 k'(s)|x(t+s)| ds \cong \\ &\cong k(0) \left(|x(t)| - \int_{-\infty}^0 k(s)|x(t+s)| ds \right) \end{aligned}$$

we get $\frac{d}{dt} f(t) \cong 0$.

$$c) \quad |x(t)| = \int_{-\infty}^0 k(s)|x(t+s)| ds.$$

Then using the case a) and inequality (14) we have

$$\begin{aligned} D^+f(t) &\cong D^+|x(t)| + D^+ \int_{-\infty}^0 k(s)|x(t+s)| ds \cong \\ &\cong D^+|x(t)| + k(0) \left(|x(t)| - \int_{-\infty}^0 k(s)|x(t+s)| ds \right) \cong 0. \end{aligned}$$

Therefore $D^+f(t) \leq 0$ for all $t \in [t_0, \infty)$, so $\lim_{t \rightarrow \infty} f(t)$ exists. Consequently, Theorem 5 implies our assertion.

2. These results may be extended to the equation

$$(15) \quad \dot{x}(t) = H(t, x(t), h(t, x_t)),$$

where

$$H: R^+ \times R^n \times R^n \rightarrow R^n, \quad h: R^+ \times L_k^1 \rightarrow R^n,$$

$$\|h(t, \varphi)\| \leq \int_{-\infty}^0 k(s) \|\varphi(s)\| ds, \quad \sup_{u, v \in K, t \in R^+} \|H(t, u, v)\| < \infty$$

for every compact set $K \subset R^n$ and

$$\sup_{\|v\| \leq \|u\|} (H(t, u, v), u) \leq p(t) \|u\|^2$$

where (\cdot, \cdot) means the inner product in R^n , and $p: R^+ \rightarrow R^+$, $\int_0^\infty p(s) ds < \infty$.

We may put $V(t, x) = \|x\| = (x, x)^{1/2}$ and

$$W(t, \varphi) = \max \left\{ \|\varphi(0)\|, \int_{-\infty}^0 k(s) \|\varphi(s)\| ds \right\},$$

and we assert that $\lim_{t \rightarrow \infty} \|x(t)\|$ exists for every solution of (15), if k satisfies the same properties as in Example 1.

3. Let us examine the equation

$$(16) \quad \dot{x}(t) = -p(t)x(t) + q(t)x(t - \varrho(t)).$$

Let $p, q, \varrho: R^+ \rightarrow R$ be continuous, bounded functions, $\varrho(t) \geq 0$ for $t \in R^+$. Choose BC as a phase space for (15).

Put $V(t, x) = x^2$, $W(t, \varphi) = \sup_{s \in R^-} e^{2\gamma s} |\varphi(s)|^2$, where $\gamma > 0$ is a constant. Then

$$\dot{V}(t, \varphi) = -2p(t)\varphi^2(0) + 2q(t)\varphi(0)\varphi(-\varrho(t)),$$

therefore, if $W(t, \varphi) = V(t, \varphi(0))$, i.e.

$$e^{-2\gamma\varrho(t)} |\varphi(-\varrho(t))|^2 \leq \varphi^2(0), \quad \text{and} \quad |q(t)| e^{\gamma\varrho(t)} \leq p^+(t),$$

then

$$\dot{V}(t, \varphi) \leq -2p(t)\varphi^2(0) + 2|q(t)| e^{\gamma\varrho(t)} \varphi^2(0) \leq 2p^-(t)V(t, \varphi(0)),$$

where and in the sequel, for any $a \in R$, a^+, a^- are defined by $a^+ = \max\{0, a\}$, $a^- = \max\{0, -a\}$, respectively. Similarly to Example 1, the existence of solutions for all large t and their boundedness together with the derivative can be proved. Therefore, Corollary 6 gives:

Assertion. If $p^- \in L^1$ and there exists $\gamma > 0$ such that $|q(t)| e^{\gamma \varrho(t)} \leq p^+(t)$ for all $t \in R^+$, then $x(t) \rightarrow \text{constant}$ as $t \rightarrow \infty$ for every solution of (16).

4. Consider the equation

$$(17) \quad \dot{x}(t) = q(t)x(t - \varrho(t)),$$

where $q, \varrho: R^+ \rightarrow R$ are continuous, q is bounded, $\varrho(t) \geq 0$ for $t \in R^+$, and there exists a $T > 0$ such that $t - \varrho(t) \geq 0$ for all $t \geq T$. Choose BC as a phase space.

Assertion. Suppose that there exists a strictly increasing continuous function $g(s)$ on R^- such that $\lim_{s \rightarrow -\infty} g(s) = 0$,

$$\int_{t-\varrho(t)}^t |q(t)|/g(-\varrho(s)) ds < 1$$

for all $t \geq T$ and

$$\int_T^\infty q^+(t)/g(-\varrho(t)) dt < \infty.$$

Then for every solution $x(t)$ of equation (17) the limit $\lim_{t \rightarrow \infty} x(t)$ exists.

Put $V(t, x) = |x|$, $W(t, \varphi) = \sup_{s \in R^-} g(s)|\varphi(s)|$, $p(t, r) \equiv 0$, $h(t, r) = (t - \varrho(t))^+$,

$$a(t, r) = \frac{r}{2} \left(1 - \int_{t-\varrho(t)}^t |q(s)|/g(-\varrho(s)) ds \right).$$

Then

$$\dot{V}(s, \varphi) = q(s)\varphi(-\varrho(s)) \operatorname{sgn} \varphi(0)$$

for all $\varphi \in BC$, so we have

$$q^*(z, s, t, r) \leq r|q(z)|/g(-\varrho(z)),$$

therefore (E_1) is fulfilled for $t \geq T$. If $t \geq T$, $0 < |\varphi(0)| = \sup g(s)|\varphi(s)|$ and

$$a(t, |\varphi(0)|) < |\varphi(z)| \leq \sup_{s \in R^-} g(s+z)|\varphi(s+z)| \leq \sup_{s \in R^-} g(s)|\varphi(s)|$$

for all $z \in [-(t - \varrho(t))^+, 0]$, then $\operatorname{sgn} \varphi(0) = \operatorname{sgn} \varphi(-\varrho(t))$ and therefore

$$\dot{V}(t, \varphi) \leq q^+(t)V(t, \varphi(0))/g(-\varrho(t)).$$

The boundedness of solutions and their derivatives can be proved similarly to Example 1. So, we can apply Corollary 6.

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