REPRESENTATIONS OF FINITE GROUPS BY POSETS OF SMALL HEIGHT

GERGŐ GYENIZSE, PÉTER HAJNAL, AND LÁSZLÓ ZÁDORI

ABSTRACT. We characterize the finite groups as the automorphism groups of the finite height one posets with at most four orbits. We also prove that for each $n \geq 8$, the cyclic group $\mathbf{Z}_n$ is isomorphic to the automorphism group of a finite height one poset with at most two orbits. As a consequence, for each n, we determine the minimum size of the posets whose automorphism groups are isomorphic to $\mathbf{Z}_n$.

1. INTRODUCTION

Throughout the text, we use blackboard bold capitals to denote digraphs, in particular posets, and bold capitals to denote groups. We use common capital letters for base sets of digraphs, posets and groups as well. Let $\mathbf{G}$ be a group, and $\mathbb{D}$ a digraph. We say that $\mathbb{D}$ is a *representation* of $\mathbf{G}$ if $\mathbf{G}$ is isomorphic to the automorphism group of $\mathbb{D}$. A representation is called *transitive* if its automorphism group is transitive as a permutation group. A representation is *regular* if its automorphism group is regular as a permutation group. Let $\mathbf{Z}_n$ denote the additive group of integers modulo n. In [3], Babai proved that all finite groups with the exception of the groups $\mathbf{Z}_2^2$, $\mathbf{Z}_2^3$, $\mathbf{Z}_2^4$, $\mathbf{Z}_3^2$ and the quaternion group have regular representations.

In this paper, we study the poset representations of the finite groups. Notice that in a poset representation every orbit of the automorphism group is an antichain. So only the finite symmetric groups have transitive poset representations. Thus the remaining finite groups have at least two orbits in each of their poset representations. In this paper, we consider poset representations in which the number of orbits is at most 4. We also want to keep the height of the poset representations we study as small as possible, in most of the cases, just 1. Recall that a poset has *height n* if it has a subchain of $n+1$ elements and has no subchain of $n+2$ elements.

By Proposition 7.3 in [2], a regular digraph representation of a finite group yields a poset representation of height 2 with exactly three orbits. This with the above mentioned Babai result in [3] gives that all finite groups with at most five exceptions have height two poset representations with at most three orbits. In [4], for any finite group, Barmak gave a different poset representation of height 3 with just four orbits. In both cases, the group $\mathbf{G}$ represented acts regularly on each of the orbits, so each of the orbits has size $|G|$.

In this paper we prove that every finite group has a height one poset representation with at most four orbits. We also prove that with few exceptions, all finite

2010 *Mathematics Subject Classification.* 05E18, 06A11.

Key words and phrases. finite groups, digraphs, posets, automorphism group, cyclic group, optimal representation with posets.

cyclic groups have poset representations with at most two orbits, on each of which the automorphism group of the poset acts regularly. By the use of this result, for each n, we determine the minimum size of the posets that represent $\mathbf{Z}_n$. Our result is similar in flavour to that of Meriwether in [8] where for each n, he determined the minimum size of the graphs that represent $\mathbf{Z}_n$. Meriwether's theorem was later reproved by Arlinghaus, see Theorems 5.4 and 6.4 in [1]. In [1], Arlinghaus went further by extending Meriwether's theorem to finite Abelian groups.

Note Added in Proof: Our article was motivated by [4], an arXiv publication of Barmak. After submitting our results, we learned that Barreto had achieved similar results in a Spanish-language thesis in 2021 [6]. An English translation of part of this thesis [5] was published when our paper had already been submitted to Order. The translated part deals with the representation of cyclic groups, that is one of the topics of our paper. Further valuable results of the thesis are still only available in Spanish. Below we compare [5] with our results.

In Section 2 we prove that every finite group has a height one poset representation with at most four orbits. To verify the claim we need to represent the five exceptional groups that occur in Babai's classical result on regular digraph representations. Specifically, we also include a representation of the quaternion group. These claims are not presented in [5].

Section 3 and parts of section 4 deal with representations of cyclic groups. The proofs are the most natural: they give constructions for each group. [5] uses similar constructions. In the constructions, we want to keep the height of the representative poset, the number of orbits, and the degree of the Hasse diagram as small as possible (see our Theorem 3.3). [5] is more purposeful towards proving the main theorem, e. g., they state their version of our Theorem 3.3 only for prime powers.

In the constructions, we introduce a sequence of posets whose automorphism group is mostly cyclic. We also examine the automorphism groups of all the elements of the sequence, and in Corollary 3.4 obtain poset representations of additional groups, such as $\mathbf{PSL}(2,7)$ and $\mathbf{GL}(2,3)$ among other groups. In their proof they also introduce a sequence of posets parameterized by prime powers. The only poset that belongs to both sequences has parameter 8.

The main result of our paper is Theorem 4.5. This requires a lower bound after the constructions. Its essence is technical, elementary number theoretical analysis, following natural lines of thought. In this respect our article and [5] are very similar. We emphasize that we obtained Theorem 4.5 of the present paper independently of the results in Barreto's thesis [6].

To summarize, [6] obviously preceded our results, but is not available to the general public. The two English language papers ours and [5] have significant parts that are similar, but the attentive expert reader will find several statements, constructions and remarks which are in our paper only.

2. HEIGHT ONE POSET REPRESENTATIONS OF FINITE GROUPS

In this section, we construct a poset representation of height 1 for every finite group. We start with a theorem that renders a poset representation of height 1 to every finite group but the above mentioned five exceptional groups in Babai's regular digraph result of [3].

Theorem 2.1. *If a finite group* $\mathbf{G}$ *has a regular digraph representation, then* $\mathbf{G}$ *has a* $4|G|$*-element poset representation of height* 1 *with four orbits of size* $|G|$.

Proof. Let $\mathbb{D}$ be a digraph such that $\mathbf{Aut}(\mathbb{D})\cong \mathbf{G}$ and $\mathbf{Aut}(\mathbb{D})$ is a regular permutation group on D. By regularity, the out-degree of every vertex is the same in $\mathbb{D}$, say d. Clearly, the in-degree of every vertex also equals d. If $|D| \geq 3$, by taking complement, we may assume that $d \geq 2$. If $|D| \leq 2$, we may assume that $d = 0$. We also assume that $\mathbb{D}$ has the vertex set $\{a_0, \dots, a_n\}$. Let $\{b_0, \dots, b_n\}$ be another copy of this vertex set. We define the poset $\mathbb{P}$ of height 1 with minimal elements $\underline{a_0}, \dots, \underline{a_n}, \underline{b_0}, \dots, \underline{b_n}$, with maximal elements $\overline{a_0}, \dots, \overline{a_n}, \overline{b_0}, \dots, \overline{b_n}$ and covering pairs

$$\underline{a_i} < \overline{a_j} \text{ iff } a_i \to a_j \text{ in } D, \text{ and } \underline{a_i} < \overline{b_i},\ \underline{b_i} < \overline{a_i},\ \underline{b_i} < \overline{b_i} \text{ for all } 0 \leq i \leq n.$$

Let $\alpha \in \mathbf{Aut}(\mathbb{D})$. We conceive α as if it also acted on the elements b_i by $\alpha(b_i) = b_j$ iff $\alpha(a_i) = a_j$. Now it should be clear that the map β defined by

$$\beta(\underline{a_i}) = \underline{\alpha(a_i)},\ \beta(\underline{b_i}) = \underline{\alpha(b_i)},\ \beta(\overline{a_i}) = \overline{\alpha(a_i)},\ \beta(\overline{b_i}) = \overline{\alpha(b_i)}$$

is an element of $\mathbf{Aut}(\mathbb{P})$.

On the other hand, if $\beta \in \mathbf{Aut}(\mathbb{P})$, then β is a permutation restricted to the set of the $\overline{b_i}$ and to the set of the $\underline{b_i}$, respectively, because of $d \neq 2$. Since β permutes the covering pairs $\underline{b_i} < \overline{b_i}$ where $0 \leq i \leq n$, these two restrictions are the same if we ignore the underline and overline notation. The so obtained permutation of the b_i forces — as β permutes the pairs $\underline{a_i} < \overline{b_i}$, $0 \leq i \leq n$, and the pairs $\underline{b_i} < \overline{a_i}$, $0 \leq i \leq n$, respectively — that β is the same permutation restricted to the set of the $\underline{a_i}$ and to the set of the $\overline{a_i}$, respectively, if we ignore the underline and overline notation. The so obtained permutation of the a_i is clearly an automorphism of $\mathbb{D}$. So $\mathbf{Aut}(\mathbb{P})\cong\mathbf{Aut}(\mathbb{D})\cong \mathbf{G}$. □

Let $\mathbf{G}$ be any finite group that admits a digraph representation $\mathbb{D}$ such that both of the in-valency and the out-valency of every vertex in $\mathbb{D}$ are different from 1. We note that an analogue of the preceding proof yields a height one poset representation $\mathbb{P}$ of $\mathbf{G}$. Clearly, the number of orbits in $\mathbb{P}$ is four times that in $\mathbb{D}$.

For all $n \geq 2$, we define the $2n$-element crowns $\mathbb{C}_n$ that are the height one posets as follows. The minimal and maximal elements of $\mathbb{C}_n$ are $\underline{i}$ and $\overline{i}$, respectively, where $0 \leq i \leq n-1$, and $\underline{i}$ has exactly two upper covers $\overline{i}$ and $\overline{i+1}$ in $\mathbb{C}_n$ where $i+1$ is meant modulo n. It is easy to check that $\mathbf{Aut}(\mathbb{C}_2)\cong \mathbf{Z}_2^2$ and for any $n \geq 3$, $\mathbf{Aut}(\mathbb{C}_n)\cong \mathbf{D}_n$ where $\mathbf{D}_n$ is the $2n$-element dihedral group. When $n \geq 3$, as usual, we call the powers of the automorphism $(\underline{0} \dots \underline{n-1})(\overline{0} \dots \overline{n-1})$ *rotations* and the remaining elements of $\mathbf{Aut}(\mathbb{C}_n)$ *reflections*.

Corollary 2.2. *Every finite group* $\mathbf{G}$ *has a height one poset representation with at most four orbits.*

Proof. It suffices to verify the claim for the five groups that occur as exceptions in Babai's regular digraph representation result in [3]. First we define some poset representations of $\mathbf{Z}_2^2$, $\mathbf{Z}_2^3$, and $\mathbf{Z}_2^4$, see Figure 1. As we noted after the definition of crowns, a 4-element crown is a height one poset representation of $\mathbf{Z}_2^2$. The disjoint sum of a 4-element crown and a 2-element antichain is a height one poset representation of $\mathbf{Z}_2^3$. The poset obtained from the disjoint sum of two 4-element crowns by adding all pairs whose first components are minimal in the first crown and whose second components are maximal in the second crown is a height one poset representation of $\mathbf{Z}_2^4$. Obviously, in each of these representations of $\mathbf{Z}_2^2$, $\mathbf{Z}_2^3$, and $\mathbf{Z}_2^4$, the number of orbits is at most four.

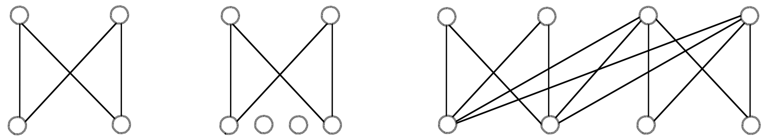

FIGURE 1. Poset representations of $\mathbf{Z}_2^2$, $\mathbf{Z}_2^3$, and $\mathbf{Z}_2^4$

Next we exhibit a poset representation of the group $\mathbf{Z}_3^2$. We assume that $\mathbf{Z}_3^2$ is defined on the set $H = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}$ with the usual componentwise modulo 3 addition. Let

$$\underline{H} = \{\underline{00}, \underline{01}, \underline{02}, \underline{10}, \underline{11}, \underline{12}, \underline{20}, \underline{21}, \underline{22}\}.$$

The set $\overline{H}$ is defined analogously. These definitions extend to the subsets of H in the natural way.

Let

$$A = \{10, 01, 22\},\ B = \{00, 12, 21\},\ C = \{02, 20, 11\} \text{ and}$$
$$D = \{00, 11, 22\},\ E = \{21, 02, 10\},\ F = \{12, 20, 01\}.$$

We note that the elements of B form a subgroup of $\mathbf{Z}_3^2$ with corresponding cosets A, B and C. Similarly, the elements of D constitute a subgroup of $\mathbf{Z}_3^2$ with related cosets D, E and F.

We define a poset $\mathbb{P}$ on the set

$$P = (\underline{H} \cup \{A, B, C\}) \cup (\{D, E, F\} \cup \overline{H}).$$

The covering relation of $\mathbb{P}$ is the union of the following sets

$$\{(\underline{h}, \overline{g}) :\ h \in H,\ g \in \{h + 10, h + 01, h + 02\}\},$$
$$\{A\} \times \overline{A},\ \{B\} \times \overline{B},\ \{C\} \times \overline{C} \text{ and } \underline{D} \times \{D\},\ \underline{E} \times \{E\},\ \underline{F} \times \{F\},$$

see Figure 2.

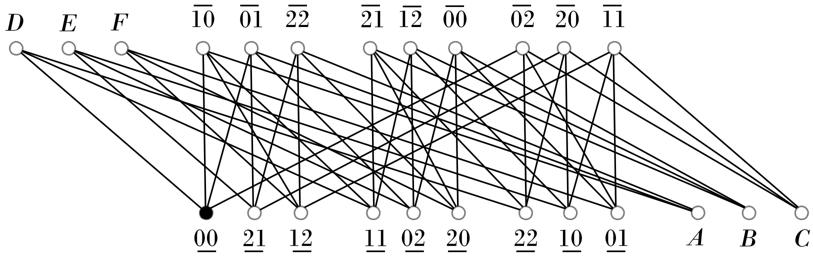

FIGURE 2. A poset representation of $\mathbf{Z}_3^2$

We claim that $\mathbb{P}$ is a poset representation of $\mathbf{Z}_3^2$ with the orbits $\underline{H}$, $\overline{H}$, $\{A, B, C\}$, and $\{D, E, F\}$. As $\mathbf{Z}_3^2$ acts faithfully on P by translations, $\mathbf{Z}_3^2$ embeds into $\mathbf{Aut}(\mathbb{P})$. Note that the orbits of this action on P coincide with the orbits of $\mathbf{Aut}(\mathbb{P})$. Indeed, every element in $\underline{H}$ is covered by exactly four elements, whereas every element in $\{A, B, C\}$ is covered by exactly three elements. So $\{A, B, C\}$ and $\underline{H}$ are orbits of $\mathbf{Aut}(\mathbb{P})$. A dual argument yields that $\{D, E, F\}$ and $\overline{H}$ are also orbits. Since $\underline{H}$ is a 9-element orbit of $\mathbf{Aut}(\mathbb{P})$ and $|H| = |\mathrm{Aut}(\mathbb{P})|/|G_0|$ where $\mathbf{G}_0$ is the stabilizer of $\underline{00}$, to complete the proof of the claim, it suffices to show that G_0 is 1-element.

So let us assume that $\varphi(\underline{00}) = \underline{00}$ where $\varphi \in \mathrm{Aut}(\mathbb{P})$. First observe that the numbers of length two paths from $\underline{00}$ to A, B, C are 2, 0 and 1 respectively. Since φ fixes $\underline{00}$ and preserves the set $\{A,B,C\}$, φ fixes A, B, C, and $\overline{02}$. For φ fixes $\underline{00}$ and $\overline{02}$, φ preserves the set $\{\underline{22},\underline{01}\}$. Since φ fixes $\underline{00}$ and preserves $\{D,E,F\}$, it fixes D as well. So φ preserves the set $\{\underline{11},\underline{22}\}$. Hence $\underline{22}$, $\underline{11}$ and $\underline{01}$ are fixed by φ. Since $\underline{01}$ is fixed and $\{D,E,F\}$ is preserved, F is fixed by φ. For D and F are fixed, E must be fixed.

Let $\mathbb{S}$ be the subposet spanned by the union of the orbits $\underline{H}$ and $\overline{H}$. The set S is obviously preserved by φ. For C is fixed, the set $\{\overline{02},\overline{20},\overline{11}\}$ is preserved by φ. We already saw that $\overline{02}$ and $\underline{22}$ are fixed, hence $\overline{20}$ and $\overline{11}$ must be also fixed by φ. Since $\underline{10}$ is the only common lower cover of $\overline{20}$ and $\overline{11}$ in $\mathbb{S}$, $\underline{10}$ is also fixed by φ.

Now we use the fact that any maximal element p of P must be fixed by φ if p has a fixed lower cover whose all upper covers different from p are fixed. By this fact and its dual, we obtain that $\overline{21},\overline{12},\overline{00}$ and $\underline{21},\underline{12}$ are fixed, since $\underline{22},\underline{10},\underline{01}$ and $\overline{02},\overline{20},\overline{11}$ are fixed by φ. By using the same fact again, the remaining elements $\underline{02},\underline{20},\overline{10},\overline{01},\overline{22}$ are also fixed by φ. Thus φ is the identity on P, and $\mathbf{Z}_3^2$ has a height one poset representation with at most four orbits.

To conclude the proof, we exhibit a poset representation of the quaternion group. Let $\mathbf{Q}$ be the quaternion group defined as usual on $Q = \{-1,1,-i,i,-j,j,-k,k\}$. Let

$A = \{-1,1,-j,j\}$, $B = \{-i,i,-k,k\}$, $C = \{-1,1,-i,i\}$ and $D = \{-j,j,-k,k\}$.

Notice that A and C are normal subgroups of $\mathbf{Q}$ with related cosets B and D, respectively.

We define the sets

$$\underline{Q} = \{-\underline{1},\underline{1},-\underline{i},\underline{i},-\underline{j},\underline{j},-\underline{k},\underline{k}\} \text{ and } \overline{Q} = \{-\overline{1},\overline{1},-\overline{i},\overline{i},-\overline{j},\overline{j},-\overline{k},\overline{k}\}.$$

The sets $\overline{A}$, $\overline{B}$, $\underline{C}$ and $\underline{D}$ are defined similarly. We define a poset $\mathbb{P}$ with base set

$$P = (\underline{Q} \cup \{A,B\}) \cup (\overline{Q} \cup \{C,D\})$$

and with covering relation

$$\{(\underline{g},\overline{h}) : \ g \in Q,\ h \in \{gi,gj,gk\}\} \cup \{A\} \times \overline{A} \cup \{B\} \times \overline{B} \cup \underline{C} \times \{C\} \cup \underline{D} \times \{D\},$$

see Figure 3.

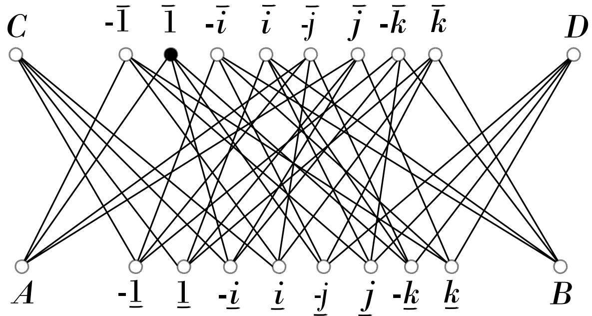

FIGURE 3. A poset representation of the quaternion group

We claim that $\mathbb{P}$ is a poset representation of $\mathbf{Q}$ with the orbits $\underline{Q}$, $\overline{Q}$, $\{A,B\}$, and $\{C,D\}$. The left multiplication of $\mathbf{Q}$ acts faithfully on P, hence $\mathbf{Q}$ embeds into

Aut($\mathbb{P}$). Note that the orbits of this action on P coincide with the orbits of **Aut**($\mathbb{P}$). Indeed, every element in $\underline{Q}$ can be reached from A via a path of length 2, but there is no path of length 2 from A to B. So $\{A,B\}$ is an orbit of **Aut**($\mathbb{P}$). Similarly, $\{C,D\}$ is also an orbit. Therefore, $\underline{Q}$ and $\overline{Q}$ are also orbits of **Aut**($\mathbb{P}$). Since $\overline{Q}$ is an 8-element orbit of **Aut**($\mathbb{P}$), to complete the proof of the claim, it suffices to show that the stabilizer of $\overline{1}$ is 1-element.

So let us assume that $\varphi(\overline{1}) = \overline{1}$ where $\varphi \in \mathrm{Aut}(\mathbb{P})$. Clearly, then φ fixes A and B, and hence φ preserves the sets $\{\overline{1},-\overline{1},\overline{j},-\overline{j}\}$ and $\{\overline{i},-\overline{i},\overline{k},-\overline{k}\}$. For $\overline{1}$ is fixed, φ preserves the set $\{-\underline{i},-\underline{j},-\underline{k}\}$. Since $\overline{1}$ covers a unique element below C and two elements below D, φ fixes C and D as well. Then φ preserves the sets $\{\underline{1},-\underline{1},\underline{i},-\underline{i}\}$ and $\{\underline{j},-\underline{j},\underline{k},-\underline{k}\}$.

Since φ preserves the sets $\{-\underline{i},-\underline{j},-\underline{k}\}$ and $\{\underline{1},-\underline{1},\underline{i},-\underline{i}\}$, φ fixes $-\underline{i}$. Then φ preserves the set $\{\overline{1},\overline{j},-\overline{k}\}$. By using that φ preserves the set $\{\overline{i},-\overline{i},\overline{k},-\overline{k}\}$, φ fixes $-\overline{k}$. Since $\{\overline{1},\overline{j},-\overline{k}\}$ is preserved by φ and $\overline{1}$ and $-\overline{k}$ are fixed by φ, $\overline{j}$ is also fixed by φ.

Since $\overline{j}$ and D are fixed, their unique common lower cover $\underline{k}$ is fixed. Then the unique common upper cover $-\overline{i}$ of $\underline{k}$ and B are fixed. Since $-\overline{k}$ and D are fixed, their unique common lower cover $\underline{j}$ is fixed. Then the unique common upper cover $-\overline{1}$ of $\underline{j}$ and A are fixed. For $\underline{j}$ is fixed, φ preserves $\{-\overline{1},\overline{i},-\overline{k}\}$. So, since $-\overline{1}$ and $-\overline{k}$ are fixed by φ, $\overline{i}$ is also fixed.

So far we proved that all elements above A except $-\overline{j}$ and all elements above B except $\overline{k}$ are fixed by φ. Then, as A and B are fixed, $-\overline{j}$ and $\overline{k}$ must also be fixed. So every maximal element of $\mathbb{P}$ is fixed by φ. Now each minimal element of $\mathbb{P}$ is a unique common lower cover of two maximal elements of $\mathbb{P}$. So we obtain that every minimal element of $\mathbb{P}$ is fixed by φ. Thus φ is the identity on P, and the quaternion group has a height one poset representation with at most four orbits. $\square$

3. Poset representations of finite cyclic groups

Now we turn to the poset representations of the finite cyclic groups. Clearly, an n-cycle is a regular digraph representation of an n-element cyclic group $\mathbf{Z}_n$. The poset defined from an n-cycle in the proof of Proposition 7.3 [2] gives a height two representation of $\mathbf{Z}_n$. The resulting height two posets are sometimes called *subdivided crowns*, see Figure 4. So the following proposition holds.

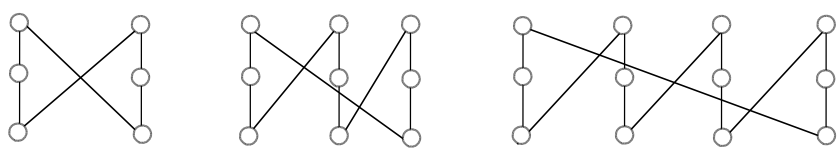

FIGURE 4. Some subdivided crowns

Proposition 3.1. *If $n \geq 2$, then $\mathbf{Z}_n$ is representable by a $3n$-element height two poset with three n-element orbits.*

By the use of Theorem 2.1, starting with an n-cycle, it is possible to construct a height one poset representation of $\mathbf{Z}_n$ with four n-element orbits. In the proof of the next theorem, we construct a more economical height one poset representation of the n-element cyclic group when n is even and $n \geq 4$. We note that this theorem is not used in the further parts of our paper. We also remark that for $n = 4$, the posets constructed in Theorems 3.1 and 3.2 yield 12-element height three and height two representations for $\mathbf{Z}_4$, respectively. It will turn out later that the these poset representations of $\mathbf{Z}_4$ are of minimum size.

Theorem 3.2. *If $n \geq 4$ and n is even, then $\mathbf{Z}_n$ is representable by a $(2n+4)$-element height one poset with two n-element and two 2-element orbits.*

Proof. For any $n \geq 4$, we define the height one poset $\mathbb{T}_n$ as follows. First we take the $2n$-element crown with minimal and maximal elements $\underline{i}$ and $\bar{i}$, where $0 \leq i \leq n-1$, and for any $0 \leq i \leq n-1$, $\underline{i}$ is covered by $\bar{i}$ and $\overline{i+1}$ where $i+k$ is meant modulo n. To obtain $\mathbb{T}_n$ we add four elements denoted by $\underline{a}, \underline{b}, \overline{a}$, and $\overline{b}$ to this crown with covering pairs $\underline{a} < \overline{a}$, $\underline{b} < \overline{b}$, $\underline{a} < \bar{i}$ and $\underline{i} < \overline{a}$ if i is even, and $\underline{b} < \bar{i}$ and $\underline{i} < \overline{b}$ if i is odd for any $0 \leq i \leq n-1$. We claim that $\mathbf{Aut}(\mathbb{T}_n) \cong \mathbf{Z}_n$.

The map that swaps $\underline{a}$ with $\underline{b}$ and $\overline{a}$ with $\overline{b}$ and maps $\underline{i}$ to $\underline{i+1}$ and $\bar{i}$ to $\overline{i+1}$ where $0 \leq i \leq n-1$ is an automorphism of $\mathbb{T}_n$ and acts transitively on the sets of the minimal and the maximal elements of the crown and on the sets $\{\underline{a}, \underline{b}\}$ and $\{\overline{a}, \overline{b}\}$, respectively. If $n = 4$, notice that the distance between every two different minimal (maximal) elements is two in $\mathbb{T}_n$, but the distance between $\underline{a}$ and $\underline{b}$ ($\overline{a}$ and $\overline{b}$) is 4. If $n > 4$, then the number of covers of each of $\underline{a}$ and $\underline{b}$ is $1+\frac{n}{2}$ that differs from 3, the number of covers of any $\underline{i}$ in the crown. Hence each of the sets $\{\underline{0}, \ldots, \underline{n-1}\}$, $\{\overline{0}, \ldots, \overline{n-1}\}$, $\{\underline{a}, \underline{b}\}$ and $\{\overline{a}, \overline{b}\}$ is an orbit of $\mathbf{Aut}(\mathbb{T}_n)$. Clearly, $\mathbf{Z}_n$ embeds into $\mathbf{Aut}(\mathbb{T}_n)$. Therefore, it suffices to prove that $|\mathrm{Aut}(\mathbb{T}_n)| = n$. Then we only have to prove that the stabilizer $\mathbf{G}_0$ of $\underline{0}$ is one-element. Indeed, if $\mathbf{G}_0$ is one-element, by using the fact that the size of the orbit of $\underline{0}$ is $|\mathrm{Aut}(\mathbb{T}_n)|/|G_0|$ on one hand and is n on the other hand, we shall be done.

So let $\varphi \in \mathbf{Aut}(\mathbb{T}_n)$ such that $\varphi(\underline{0}) = \underline{0}$. We want to prove that φ is the identity map on $\mathbb{T}_n$. Since φ fixes $\underline{0}$ and $\{\overline{0}, \ldots, \overline{n-1}\}$ is an orbit in $\mathbf{Aut}(\mathbb{T}_n)$, φ permutes the elements $\overline{0}$ and $\overline{1}$. Also, φ must fix the elements $\overline{a}, \overline{b}, \underline{a}$, and $\underline{b}$. Then by $\underline{a} = \varphi(\underline{a}) \leq \varphi(\overline{0}) \in \{\overline{0}, \overline{1}\}$, $\varphi(\overline{0}) = \overline{0}$. By using the fact that $\varphi(\overline{0}) = \overline{0}$ and applying a dual argument, we obtain that $\varphi(\underline{1}) = \underline{1}$. By continuing the proof in the same fashion, now starting with the condition that $\varphi(\underline{1}) = \underline{1}$, we obtain that $\varphi(\underline{2}) = \underline{2}$, and so on. At the end, we get that φ fixes all of the elements of $\underline{0}, \ldots, \underline{n-1}$ and hence φ is the identity of $\mathbb{T}_n$. This concludes the proof. $\square$

For any $n \geq 4$, we define the $2n$-element height one poset $\mathbb{P}_n$ as follows. The minimal and maximal elements of $\mathbb{P}_n$ are $\underline{i}$ and $\bar{i}$, respectively, where $0 \leq i \leq n-1$, and $\underline{i}$ has exactly three upper covers $\bar{i}$, $\overline{i+1}$, and $\overline{i+3}$ in $\mathbb{P}_n$ where $i+k$ is meant modulo n. It may be surprising that for finite cyclic groups with finitely many exceptions, there are height one poset representations with just two orbits. Here is the main theorem of the section.

Theorem 3.3. *If $n \geq 8$, then $\mathbf{Z}_n$ is representable by a $2n$-element height one poset with two n-element orbits.*

Proof. We shall prove first that for every $n \geq 9$, $\mathbf{Aut}(\mathbb{P}_n)$ is isomorphic to $\mathbf{Z}_n$. Clearly, the natural action of $\mathbf{Z}_n$ on $\mathbb{P}_n$ is contained in the action of $\mathbf{Aut}(\mathbb{P}_n)$ and is transitive on the sets of the minimal and the maximal elements, respectively. Hence $\mathbf{Z}_n$ embeds into $\mathbf{Aut}(\mathbb{P}_n)$, and each of the sets of the minimal and the maximal elements is an n-element orbit of $\mathbf{Aut}(\mathbb{P}_n)$. Therefore, if we prove that $|\mathrm{Aut}(\mathbb{P}_n)| = n$, we are done. Similarly as in the preceding proof, it suffices to prove that the stabilizer of $\underline{0}$ is one-element. So let $\varphi \in \mathbf{Aut}(\mathbb{P}_n)$ such that $\varphi(\underline{0}) = \underline{0}$. We want to prove that φ is the identity map on $\mathbb{P}_n$. The proof can be visualized in Figure 5.

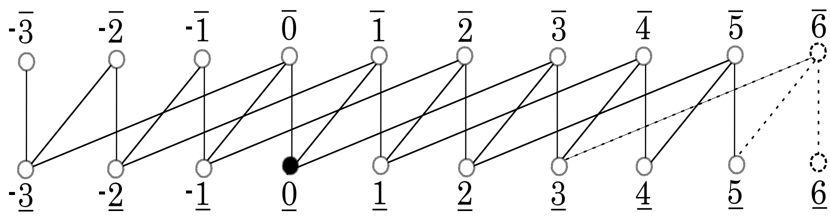

FIGURE 5. A slice of $\mathbb{P}_n$

We call a sequence $\underline{0} < \overline{x} > \underline{y} < \overline{z}$ a 3-*path* where $x \neq z$ and $0 \neq y$. So a 3-path contains 4 different elements, including the starting element $\underline{0}$. We list all of the 3-paths in $\mathbb{P}_n$:

$$\underline{0} < \overline{0} > -\underline{1} < -\overline{1},\quad \underline{0} < \overline{0} > -\underline{1} < \overline{2},\quad \underline{0} < \overline{0} > -\underline{3} < -\overline{3},\quad \underline{0} < \overline{0} > -\underline{3} < -\overline{2},$$
$$\underline{0} < \overline{1} > -\underline{2} < -\overline{2},\quad \underline{0} < \overline{1} > -\underline{2} < -\overline{1},\quad \underline{0} < \overline{1} > \underline{1} < \overline{2},\quad \underline{0} < \overline{1} > \underline{1} < \overline{4},$$
$$\underline{0} < \overline{3} > \underline{2} < \overline{2},\quad \underline{0} < \overline{3} > \underline{2} < \overline{5},\quad \underline{0} < \overline{3} > \underline{3} < \overline{4},\quad \underline{0} < \overline{3} > \underline{3} < \overline{6}.$$

It is obvious that for any 3-path $\underline{0} < \overline{x} > \underline{y} < \overline{z}$, the sequence

$$\underline{0} = \varphi(\underline{0}) < \varphi(\overline{x}) > \varphi(\underline{y}) < \varphi(\overline{z})$$

is a 3-path too. Hence φ permutes the ending elements of the 3-paths. Let

$$E = \{-\overline{3}, -\overline{2}, -\overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{6}\}.$$

Observe that E is the set of ending elements of the 3-paths. Notice that E is a 7-element set if $n > 9$, and E is a 6-element set if $n = 9$. In the latter case, $\overline{6} = -\overline{3}$. We say that a 3-path of the form $\underline{0} < \overline{x} > \underline{y} < \overline{z}$ has *color* x. So every 3-path has a unique color: color 0, color 1, or color 3. We partition the set of the twelve 3-paths into three color classes, see the three lines of our list above. Notice that in E, $\overline{2}$ is the only element that is reached by three 3-paths of different colors. Therefore, $\overline{2}$ is also a fixed point of φ.

An element $\overline{w}$ in E is *lone* iff there is only one 3-path ending at $\overline{w}$. If $n = 9$, then $\overline{5}$ is the only lone element of E, it is the ending element of a 3-path of color 3. Hence the elements of the 3-path leading to $\overline{5}$ are fixed by φ. Thus $\varphi(\overline{3}) = \overline{3}$, $\varphi(\underline{2}) = \underline{2}$, and $\varphi(\overline{5}) = \overline{5}$. Since $\varphi(\underline{0}) = \underline{0}$ implies $\varphi(\overline{3}) = \overline{3}$, $\varphi(\underline{x}) = \underline{x}$ similarly implies $\varphi(\overline{x+3}) = \overline{x+3}$.

Notice that the map ρ that swaps $\underline{x}$ and $-\overline{x}$ is an order reversing bijection of $\mathbb{P}_n$. So $\mathbb{P}_n$ is a self-dual poset and in its dual $\overline{x}$ is covered by $\underline{x}$, $\underline{x-1}$ and $\underline{x-3}$. As φ preserves the dual ordering, we obtain by a dual argument as above that $\varphi(\overline{x}) = \overline{x}$ implies $\varphi(\underline{x-3}) = \underline{x-3}$. Hence by $\varphi(\overline{2}) = \overline{2}$, we obtain that $\varphi(-\underline{1}) = -\underline{1}$. So we proved based only on the assumption $\varphi(\underline{0}) = \underline{0}$ that $\varphi(-\underline{1}) = -\underline{1}$. By the use of $\varphi(-\underline{1}) = -\underline{1}$ we similarly obtain $\varphi(-\underline{2}) = -\underline{2}$, and so on. Eventually, φ fixes all underlined elements in $\mathbb{P}_n$, hence φ must be the identity.

If $n > 9$, then $-\overline{3}$, $\overline{5}$, $\overline{6}$ are the lone elements. The color of the unique 3-paths leading to $\overline{5}$ and $\overline{6}$ is 3. The color of the unique 3-path leading to $-\overline{3}$ is $0 \neq 3$. Hence the elements of the 3-path leading to $-\overline{3}$ are fixed by φ. Therefore, $\varphi(\overline{0}) = \overline{0}$,

$\varphi(-\underline{3}) = -\underline{3}$, and $\varphi(-\overline{3}) = -\overline{3}$. The fixed element $\overline{0}$ has the three lower covers: $\underline{0}$, $-\underline{1}$, and $-\underline{3}$. Since $\underline{0}$ and $-\underline{3}$ are fixed by φ, φ fixes $-\underline{1}$ as well. From here, the proof of case $n > 9$ finishes in the same way as in case $n = 9$.

Now it remains to prove the theorem for $n = 8$. We note that the poset $\mathbb{P}_8$ is the point-line poset of the Möbius-Kantor configuration, and its automorphism group has 48 elements and isomorphic to $\mathbf{GL}(2,3)$, cf. [7]. So we need to come up with another 16-element height one poset to finish the proof of this theorem. We define the height one poset $\mathbb{R}_8$ as follows. The minimal and maximal elements of $\mathbb{R}_8$ are $\underline{i}$ and $\overline{i}$, respectively, where $i \in \{0,\dots,7\}$, and in $\mathbb{R}_8$, for any $0 \le i \le 7$, $\underline{i}$ has exactly four upper covers $\overline{i-1}$, $\overline{i}$, $\overline{i+1}$, and $\overline{i+3}$ where $i+k$ is meant modulo 8, see Figure 6.

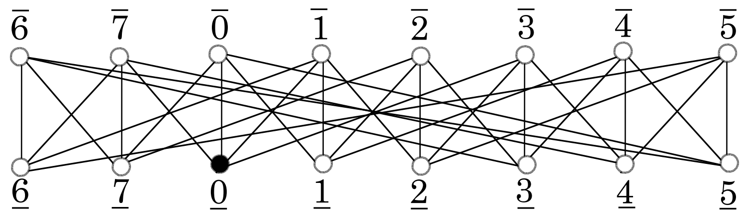

FIGURE 6. $\mathbb{R}_8$

We shall prove that $\mathbf{Aut}(\mathbb{R}_8)$ is isomorphic to $\mathbf{Z}_8$. We follow the structure of the proof for $\mathbb{P}_n$, $n \ge 9$. First observe that the natural action of $\mathbf{Z}_8$ on $\mathbb{R}_8$ yields a subgroup $\mathbf{Aut}(\mathbb{R}_8)$ and is transitive on the sets of the minimal and the maximal elements, respectively. Hence each of these two sets is an 8-element orbit of $\mathbf{Aut}(\mathbb{R}_8)$, and $\mathbf{Z}_8$ embeds into $\mathbf{Aut}(\mathbb{R}_8)$. Therefore, if we prove that $|\mathrm{Aut}(\mathbb{R}_8)| = 8$ we are done. Similarly as earlier, it suffices to prove that the stabilizer of $\underline{0}$ is one-element.

So let $\varphi \in \mathbf{Aut}(\mathbb{R}_8)$ such that $\varphi(\underline{0}) = \underline{0}$. We want to prove that φ is the identity map on $\mathbb{R}_8$. We call a sequence $\underline{w} < \overline{x} > \underline{y}$ a 2-path, where $w \neq y$. So a 2-path contains three different elements. Here is the list of all 2-paths starting with $0 \in R_8$, the list for the 2-paths starting with $w \in R_8$ is similar:

$$\begin{array}{lll} \underline{0} < \overline{7} > \underline{6}, & \underline{0} < \overline{7} > \underline{7}, & \underline{0} < \overline{7} > \underline{4}, \\ \underline{0} < \overline{0} > \underline{7}, & \underline{0} < \overline{0} > \underline{1}, & \underline{0} < \overline{0} > \underline{5}, \\ \underline{0} < \overline{1} > \underline{1}, & \underline{0} < \overline{1} > \underline{2}, & \underline{0} < \overline{1} > \underline{6}, \\ \underline{0} < \overline{3} > \underline{2}, & \underline{0} < \overline{3} > \underline{3}, & \underline{0} < \overline{3} > \underline{4}. \end{array}$$

We call a sequence $\underline{w} < \overline{x} > \underline{y} < \overline{z} > \underline{w}$ a *4-cycle* if $w \neq y$ and $x \neq z$. So a 4-cycle contains four different elements. Note that a 4-cycle is oriented, hence $\underline{w} < \overline{x} > \underline{y} < \overline{z} > \underline{w}$ and $\underline{w} < \overline{z} > \underline{y} < \overline{x} > \underline{w}$ are two different 4-cycles. Every 4-cycle is put together from two 2-paths in one of the previous lists of 2-paths. Now we list all 4-cycles through $0 \in R_8$, the list for the 4-cycles through $w \in R_8$ is similar:

$$\underline{0} < \overline{7} > \underline{6} < \overline{1} > \underline{0}, \quad \underline{0} < \overline{7} > \underline{7} < \overline{0} > \underline{0}, \quad \underline{0} < \overline{7} > \underline{4} < \overline{3} > \underline{0},$$
$$\underline{0} < \overline{0} > \underline{1} < \overline{1} > \underline{0}, \quad \underline{0} < \overline{0} > \underline{7} < \overline{7} > \underline{0},$$
$$\underline{0} < \overline{1} > \underline{2} < \overline{3} > \underline{0}, \quad \underline{0} < \overline{1} > \underline{6} < \overline{7} > \underline{0}, \quad \underline{0} < \overline{1} > \underline{1} < \overline{0} > \underline{0},$$
$$\underline{0} < \overline{3} > \underline{4} < \overline{7} > \underline{0}, \quad \underline{0} < \overline{3} > \underline{2} < \overline{1} > \underline{0}.$$

It is clear that $\mathbf{Aut}(\mathbb{R}_8)$ acts on the edge set of the Hasse diagram of $\mathbb{R}_8$ in the natural way: $\beta(\underline{x}\overline{x+k}) = \beta(\underline{x})\beta(\overline{x+k})$ for $\beta \in\mathbf{Aut}(\mathbb{R}_8)$ and $\underline{x} < \overline{x+k}$. Since the number of 4-cycles starting with an edge $\underline{w}\overline{w+3}$ or with an edge $\underline{w}\overline{w}$ is 2 and the the number of 4-cycles starting with an edge $\underline{w}\overline{w-1}$ or with an edge $\underline{w}\overline{w+1}$ is 3, the edge set

$$F := \{\underline{w}\overline{w+3} :\ w \in \mathbf{Z}_8\} \cup \{\underline{w}\overline{w} : w \in \mathbf{Z}_8\},$$

is a union of orbits of the action of $\mathbf{Aut}(\mathbb{R}_8)$ on the edge set of the Hasse diagram of $\mathbb{R}_8$. These edges form a 16-element crown $\mathbb{C}_F$ in $\mathbb{R}_8$:

$$\underline{0} < \overline{3} > \underline{3} < \overline{6} > \underline{6} < \overline{1} > \underline{1} < \overline{4} > \underline{4} < \overline{7} > \underline{7} < \overline{2} > \underline{2} < \overline{5} > \underline{5} < \overline{0} > \underline{0}.$$

Since $\underline{0}$ is a fixed point of φ and $\mathbf{Aut}(\mathbb{C}_F)$ is isomorphic to $\mathbf{D}_8$, φ is the identity map or the reflection on $\mathbb{C}_F$ that fixes $\underline{0}$ and $\underline{4}$. In the latter case, φ swaps $\overline{4}$ and $\overline{7}$, which contradicts $\underline{0} = \varphi(\underline{0}) < \varphi(\overline{7}) = \overline{4}$ in $\mathbb{R}_8$. Thus φ indeed is the identity map on R_8. □

In the following corollary, we describe the automorphism groups of the posets $\mathbb{P}_n$ for all $n \geq 4$.

Corollary 3.4. $\mathbf{Aut}$*(*$\mathbb{P}_4$*)*$\cong \mathbf{S}_4$*,* $\mathbf{Aut}$*(*$\mathbb{P}_5$*)*$\cong \mathbf{D}_5$*,* $\mathbf{Aut}$*(*$\mathbb{P}_6$*)*$\cong \mathbf{A}_4 \times \mathbf{Z}_2$*,* $\mathbf{Aut}$*(*$\mathbb{P}_7$*)*$\cong\mathbf{PSL}(2,7)$*,* $\mathbf{Aut}$*(*$\mathbb{P}_8$*)*$\cong\mathbf{GL}(2,3)$*, and for each* $n \geq 9$*,* $\mathbf{Aut}$*(*$\mathbb{P}_n$*)*$\cong \mathbf{Z}_n$*.*

Proof. The first two isomorphisms hold, since $\mathbf{Aut}(\mathbb{P}_4)$ coincides with the automorphism group of the disjoint sum of five two element chains and $\mathbf{Aut}(\mathbb{P}_5)$ coincides with the automorphism group of a 10-element crown.

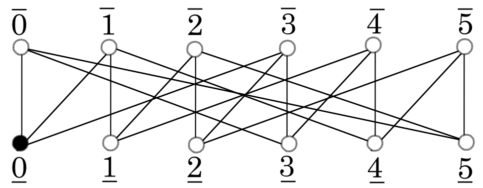

FIGURE 7. Poset $\mathbb{P}_6$

Now we prove the third isomorphism, see Figure 7. First we determine the stabilizer $\mathbf{G}_0$ of $\underline{0}$ in $\mathbf{Aut}(\mathbb{P}_6)$. Let φ be an arbitrary element of $\mathbf{G}_0$. Observe that $\underline{3}$ is the only element that can be reached from $\underline{0}$ by two different paths of length 2 in $\mathbb{P}_6$. So φ fixes $\underline{3}$. Since $\underline{0}$ and $\underline{3}$ are fixed by φ, the sets $\{\overline{0},\overline{1},\overline{3}\}$, $\{\overline{0},\overline{3},\overline{4}\}$ and their intersection $\{\overline{0},\overline{3}\}$ are preserved by φ. So φ also fixes the elements $\overline{1}$ and $\overline{4}$. Then φ preserves the set $\{\underline{1},\underline{4}\}$. Thus φ preserves the subsets $\{\overline{0},\overline{3}\}$, $\{\overline{2},\overline{5}\}$, $\{\underline{1},\underline{4}\}$, $\{\underline{2},\underline{5}\}$ and fixes $\underline{0},\underline{3},\overline{1},\overline{4}$. Then $\mathbf{G}_0$ is the 4-element group generated by $\beta = \{(\underline{1}\ \underline{4})(\overline{2}\ \overline{5})\}$ and $\gamma = \{(\underline{2}\ \underline{5})(\overline{0}\ \overline{3})\}$. So $\mathbf{Aut}(\mathbb{P}_6)$ has 24 elements and is generated by α,β and γ.

We decompose $\mathbf{Aut}(\mathbb{P}_6)$ into a product of its two normal subgroups. Since α^3 commutes with β and γ, α^3 generates a 2-element normal subgroup $\mathbf{H}$ of $\mathbf{Aut}(\mathbb{P}_6)$. The 4-element subgroup $\mathbf{M}$ generated by $\alpha^3\beta$ and $\beta\gamma$ consists of the identity map and all elements of $\mathbf{Aut}(\mathbb{P}_6)$ whose cycle type is $2+2+2+2$, hence it is closed under conjugation, so $\mathbf{M}$ is a normal subgroup of $\mathbf{Aut}(\mathbb{P}_6)$. Therefore the product of the two complexes $\mathbf{M}$ and the 3-element subgroup generated by α^2 is a subgroup $\mathbf{N}$ of $\mathbf{Aut}(\mathbb{P}_6)$. Since $\mathbf{N}$ has index 2, $\mathbf{N}$ is also normal in $\mathbf{Aut}(\mathbb{P}_6)$. So $\mathbf{Aut}(\mathbb{P}_6)$

is the inner product of $\mathbf{H}$ and $\mathbf{N}$. Hence $\mathbf{Aut}(\mathbb{P}_6) \cong \mathbf{H} \times \mathbf{N}$. Since $\mathbf{N}$ is a non-commutative 12-element group, and it has 8 elements of order 3, $\mathbf{N} \cong \mathbf{A}_4$. Thus $\mathbf{Aut}(\mathbb{P}_6) \cong \mathbf{Z}_2 \times \mathbf{A}_4$.

It is well known that $\mathbb{P}_7$ is the point-line poset of the Fano plane and $\mathbf{Aut}(\mathbb{P}_7)$ is isomorphic to the 168-element simple group $\mathbf{PSL}(2,7)$, cf. [7]. As we mentioned in the proof of the preceding theorem, it is also well known that $\mathbf{Aut}(\mathbb{P}_8) \cong \mathbf{GL}(2,3)$. Finally, we also saw in the proof that $\mathbf{Aut}(\mathbb{P}_n) \cong \mathbf{Z}_n$ if $n \geq 9$. □

4. MINIMUM SIZE OF POSET REPRESENTATIONS OF $\mathbf{Z}_n$

In this section, for each n, we determine the minimum size of the poset representations of $\mathbf{Z}_n$. Let f_n denote the minimum size of the posets that represent $\mathbf{Z}_n$. Clearly, $f_1 = 1$ and $f_2 = 2$. First, by the use of Theorem 3.3, we determine the values of f_n when n is a prime power. Let $\mathbf{S}_n$ and $\mathbf{A}_n$ respectively denote the symmetric group and the alternating group on n-elements.

Corollary 4.1. *Let n be a prime power. Then*

$$f_n = 3n \text{ if } 3 \leq n \leq 7, \text{ and } f_n = 2n \text{ if } n \geq 8.$$

Proof. Let $n = p^k$ where p is a prime. Let $\mathbb{P}$ be a poset representation of $\mathbf{Z}_n$ of minimum size. In [4], Barmak proved that in this case $\mathbf{Aut}(\mathbb{P})$ has at least two orbits of size n, if $n \geq 3$. This and Theorem 3.3 imply that $f_n = 2n$ if $n \geq 8$. If $3 \leq n \leq 7$, then by Proposition 3.1, $f_n \leq 3n$. So it suffices to prove the inequality $3n \leq f_n$ for $n \in \{3,4,5,7\}$. The proof is explicitly given for $n = 3$ by Barmak in [4], and is very similar for $n = 4,\ 5,\ 7$ as we see it in what follows.

Let us assume that $n \in \{3,4,5,7\}$. Let $\mathbb{P}$ be a minimum size poset representation of $\mathbf{Z}_n$. Then, as it was proved in [4], there are two n-element orbits in $\mathbb{P}$. First we want to exclude that $|P| = 2n$. To the contrary, let us suppose that $|P| = 2n$. Then a generating element α of $\mathbf{Aut}(\mathbb{P})$ must be a product of two disjoint n-cycles, and so the comparability graph of $\mathbb{P}$ is a regular bipartite graph. To get a contradiction, it suffices to prove that the automorphism group of such a $2n$-element height one poset is not commutative. Let A and B be respectively the set of minimal and the set of maximal elements of $\mathbb{P}$. Clearly, A and B are the two n-element orbits of α. Let $\mathbb{P}'$ be the height one poset whose base set equals P and whose ordering is defined by $i < j$ iff $i \in A$, $j \in B$ and $i \not< j$ in $\mathbb{P}$. Let v be the valency of every vertex in the comparability graph of $\mathbb{P}$. We may assume that $1 \leq v \leq [\frac{n}{2}]$ or $v = n$. Indeed, if $[\frac{n}{2}] < v < n$, then the automorphism groups of $\mathbb{P}$ and $\mathbb{P}'$ are the same, and in the comparability graph of $\mathbb{P}'$ every vertex has valency $n - v$. By using the assumption that $1 \leq v \leq [\frac{n}{2}]$ or $v = n$, $\mathbb{P}$ must be a poset of one of the following types: a disjoint sum of n two-element chains, a $2n$-element crown, the disjoint sum of two 4-element crowns, a linear sum of two n-element antichains, and a 14-element height one poset that has a 3-regular comparability graph and admits an automorphism with cycle decomposition of two 7-cycles. It turns out that the automorphism groups of all of these posets are non-commutative groups in which $\mathbf{Z}_n$ is embedded as a subgroup. The automorphism groups $\mathbf{S}_n$, $\mathbf{D}_n$, $\mathbf{S}_2 \wr \mathbf{S}_2^2$, $\mathbf{S}_n^2$ of the first four types are indeed non-commutative. For the last type when $n = 7$ and $v = 3$, we give a detailed proof of the non-commutativity of $\mathbf{Aut}(\mathbb{P})$ as follows.

Let $\{\underline{0},\dots,\underline{6}\}$ be the orbit of minimal and $\{\overline{0},\dots,\overline{6}\}$ the orbit of maximal elements in $\mathbb{P}$. We may assume that $\mathbf{Aut}(\mathbb{P})$ is generated by the permutation $\alpha = (\underline{0}\dots\underline{6})(\overline{0}\dots\overline{6})$. Let $C = \{j \in Z_7 :\ \underline{0} < \overline{j} \text{ in } \mathbb{P}\}$. Now we know that $|C| = 3$. We may also assume that $0 \in C$. Then for each $0 \le i \le 6$, the covers of $\underline{i}$ are the $\overline{i+j}$ where $j \in C$ in $\mathbb{P}$. So $\mathbb{P}$ is determined by the set C. Up to an isomorphism of $\mathbb{P}$, C coincides with one of the following 3-element sets:

$$\{\overline{6},\overline{0},\overline{1}\},\ \{\overline{5},\overline{0},\overline{2}\},\ \{\overline{4},\overline{0},\overline{3}\},\ \{\overline{0},\overline{1},\overline{3}\}.$$

In the first three cases, the automorphism group of the corresponding poset $\mathbb{P}$ contains the involution given by

$$\underline{i} \mapsto -\underline{i} \text{ and } \overline{i} \mapsto -\overline{i} \text{ for each } 0 \le i \le 6.$$

In the forth case, $\mathbb{P} \cong \mathbb{P}_7$. So by Corollary 3.4, $\mathbf{Aut}(\mathbb{P})$ is isomorphic to $\mathbf{PSL}(2,7)$. Thus, $\mathbf{Aut}(\mathbb{P})$ is a non-commutative group in all of the cases when $n \in \{3,4,5,7\}$, $|P| = 2n$ and $\alpha \in \mathbf{Aut}(\mathbb{P})$.

So $\mathbf{Aut}(\mathbb{P})$ has at least three orbits, two of which have n-elements. Next we prove that $\mathbb{P}$ cannot have just one-element orbits and two n-element orbits. Let us suppose the contrary. Let H denote the set of the 1-element orbits in $\mathbb{P}$, and $\mathbb{P}-H$ the poset obtained by removing the elements of H from $\mathbb{P}$. Then $\mathbf{Aut}(\mathbb{P})$ is clearly isomorphic to the subgroup $\mathbf{K}$ of $\mathbf{Aut}(\mathbb{P}-H)$ that consists of the automorphisms that preserve the two n-element orbits of $\mathbf{Aut}(\mathbb{P})$. Since $\mathbb{P}$ is of minimum size, $\mathbf{Aut}(\mathbb{P}-H)$ has only one orbit. So $\mathbb{P}-H$ is an antichain, and $\mathbf{Aut}(\mathbb{P}-H) \cong \mathbf{S}_{2n}$. Therefore, $\mathbf{K} \cong \mathbf{S}_n^2$ which contradicts $\mathbf{K} \cong \mathbf{Aut}(\mathbb{P}) \cong \mathbf{Z}_n$. Thus for the primes $n \in \{3,5,7\}$, there must be a third orbit of size n in $\mathbb{P}$, and hence $3n \le f_n$.

To conclude the proof for $n = 4$, we have to exclude two cases: $|P| = 10$ and $|P| = 11$ such that in both cases $\mathbb{P}$ has two 4-element orbits and one 2-element orbit. So let us suppose that $|P| = 10$ or $|P| = 11$, and let

$$O_0 = \{\underline{0},\underline{1},\underline{2},\underline{3}\},\ O_1 = \{\overline{0},\overline{1},\overline{2},\overline{3}\},\ O_2 = \{0,1\}$$

be the at least 2-element orbits of $\mathbb{P}$. We denote the corresponding subposets of $\mathbb{P}$ by $\mathbb{O}_0$, $\mathbb{O}_1$ and $\mathbb{O}_2$, respectively. Let α be a generating element of $\mathbf{Aut}(\mathbb{P})$. Clearly, α acts transitively on the orbits O_i, $1 \le i \le 3$.

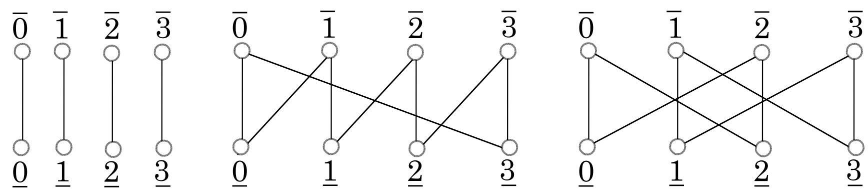

FIGURE 8. Shapes of $\mathbb{O}_0 \cup \mathbb{O}_1$ with vertices of valency at most 2

The comparability graph of $\mathbb{O}_0 \cup \mathbb{O}_1$ must be a bipartite regular graph preserved by α. To determine $\mathbf{Aut}(\mathbb{O}_0 \cup \mathbb{O}_1)$, we may assume as before that the valency of every vertex is at most 2 in this graph or the graph is full bipartite. So by inspecting of the possible shapes of the poset $\mathbb{O}_0 \cup \mathbb{O}_1$, see Figure 8, we may assume that the automorphism group of $\mathbb{O}_0 \cup \mathbb{O}_1$ restricted to O_0 admits the transposition $(\underline{0}\ \underline{2})$. We may analogously assume that the automorphism group of $\mathbb{O}_0 \cup \mathbb{O}_1$ restricted to O_1 admits the transposition $(\overline{0}\ \overline{2})$. Besides the above assumptions, we may also assume that

$$\alpha = (0\ 1)(\underline{0}\ \underline{1}\ \underline{2}\ \underline{3})(\overline{0}\ \overline{1}\ \overline{2}\ \overline{3}).$$

Now observe that the comparability graph of the subposet $\mathbb{O}_0 \cup \mathbb{O}_2$ is either the empty graph, or the full bipartite graph between O_0 and O_2 , or it consists of the edges $a\underline{0}$, $a\underline{2}$ and $b\underline{1}$, $b\underline{3}$ where $\{a,b\} = \{0,1\}$. Clearly, an analogous statement is satisfied by the comparability graph of the subposet $\mathbb{O}_1 \cup \mathbb{O}_2$.

Let $\mathbb{O}$ denote the subposet $\mathbb{O}_0 \cup \mathbb{O}_1 \cup \mathbb{O}_2$ in $\mathbb{P}$. Let us assume that $\mathbb{O}$ is a height one poset. Without loss of generality, we may assume that the elements of O_0 are minimal, the elements of O_2 are maximal in $\mathbb{O}$. Then the restriction of $\mathbf{Aut}(\mathbb{O}_0 \cup \mathbb{O}_1)$ to O_0 contains the transposition $(\underline{0}\ \underline{2})$. Clearly, this transposition extends to $\mathbb{P}$ as an automorphism of $\mathbb{P}$, so $\mathbf{Aut}(\mathbb{P})$ is not commutative, a contradiction.

So $\mathbb{O}$ has to be a height two poset. Up to duality and symmetry we have two cases. We assume in the first case that the elements of O_2 are maximal in $\mathbb{O}$. Without loss of generality we may assume that $\underline{0} < \overline{0}$. If the comparability graph $\mathbb{O}_1 \cup \mathbb{O}_2$ is the full bipartite graph, then there is an automorphism of $\mathbb{O}_0 \cup \mathbb{O}_1$ that equals $(\underline{0}\ \underline{2})$ on O_0 and extends to an automorphism of $\mathbb{P}$. So $\mathbf{Aut}(\mathbb{P})$ is not commutative, a contradiction. This means that up to symmetry, $\mathbb{O}_1 \cup \mathbb{O}_2$ contains only the covering pairs $\overline{0}, \overline{2} < 0$ and $\overline{1}, \overline{3} < 1$. We also know that $\underline{i} < \overline{i}$, $0 \le i \le 3$, in $\mathbb{O}$. Notice that there are no covering pairs of $\mathbb{O}$ in $O_0 \cup O_2$ or the comparability graph of $\mathbb{O}_0 \cup \mathbb{O}_2$ would be full bipartite. In both cases, we take again an automorphism of $\mathbb{O}_0 \cup \mathbb{O}_1$ that is $(\overline{0}\ \overline{2})$ restricted to O_1 and extend it to an automorphism of $\mathbb{P}$. Then again $\mathbf{Aut}(\mathbb{P})$ is not commutative, a contradiction.

Now for the remaining case, $\mathbb{O}$ has height two, O_0 has the minimal and O_1 has the maximal elements of $\mathbb{O}$. By a similar argument as in preceding paragraph we may assume that the comparability graphs of $\mathbb{O}_0 \cup \mathbb{O}_2$ and $\mathbb{O}_2 \cup \mathbb{O}_1$ are not full bipartite. Then up to symmetry, $\mathbb{O}$ has either the covering pairs

$$\underline{0}, \underline{2} < 0 \text{ and } \underline{1}, \underline{3} < 1 \text{ and } 0 < \overline{0}, \overline{2} \text{ and } 1 < \overline{1}, \overline{3}$$

or these covering pairs plus the covering pairs

$$\underline{0} < \overline{1} \text{ and } \underline{1} < \overline{0} \text{ and } \underline{3} < \overline{4} \text{ and } \underline{4} < \overline{3}.$$

In each of the resulting posets the permutation $(\underline{0}\ \underline{2})(\overline{0}\ \overline{2})$ is an automorphism of $\mathbb{P}$. So in this case, $\mathbf{Aut}(\mathbb{P})$ is not commutative as well, a contradiction. $\square$

We remark that by Theorem 3.3 and Corollary 4.1, $\mathbf{Z}_n$ has a $2n$-element poset representation with two n-element orbits whenever $n \le 2$ and $n \ge 8$, and there are no such representations if $n = 3,4,5,7$. For the sake of completeness, we give a proof that there is no such representation for $n = 6$ as well.

Corollary 4.2. *$\mathbf{Z}_n$ is representable by a $2n$-element height one poset with two n-element orbits if and only if $n \le 2$ or $n \ge 8$.*

Proof. By Theorem 3.3 and Corollary 4.1, we only have to prove that there is no 12-element height one poset $\mathbb{P}$ with two 6-element orbits such that $\mathbf{Aut}(\mathbb{P}) \cong \mathbf{Z}_6$. To the contrary, let us suppose that $\mathbb{P}$ is such a poset. Let $\{\underline{0}, \dots, \underline{5}\}$ be the orbit of minimal and $\{\overline{0}, \dots, \overline{5}\}$ the orbit of maximal elements in $\mathbb{P}$. We may assume that $\mathbf{Aut}(\mathbb{P})$ is generated by the permutation $\alpha = (\underline{0} \dots \underline{5})(\overline{0} \dots \overline{5})$.

Let $B := \{j \in Z_6 :\ \underline{0} < \overline{j} \text{ in } \mathbb{P}\}$. Clearly, $B \ne \emptyset$, Z_6. We may also assume that $0 \in B$. Then for each $0 \le i \le 5$, the covers of $\underline{i}$ are the $\overline{i+j}$ where $j \in B$ in $\mathbb{P}$. So $\mathbb{P}$ is determined by the subset B of Z_6. Similarly as in the proof of the preceding corollary, the posets determined by B and its complement $Z_6 \setminus B$ have the same automorphism groups. Hence, it suffices to prove that $\mathbf{Aut}(\mathbb{P}) \ncong \mathbf{Z}_6$ if $1 \le |B| \le 3$.

When $|B| = 1$, then $\mathbf{Aut}(\mathbb{P}) \cong \mathbf{S}_6$. When $|B| = 2$, then $\mathbb{P}$ is a 12-element crown, or a disjoint sum of two 6-element crowns, or a disjoint sum of three 4-element crowns. Then the corresponding automorphism groups are $\mathbf{D}_6$, $\mathbf{S}_3 \wr \mathbf{S}_2$, and $\mathbf{S}_2 \wr \mathbf{S}_3$, respectively. These groups are clearly not isomorphic to $\mathbf{Z}_6$.

In the remaining cases, $|B| = 3$. Up to isomorphism, we only have to look at the three cases $B = \{5,0,1\}$, $B = \{4,0,2\}$, and $B = \{0,1,3\}$. In the first two cases when $B = \{5,0,1\}$ or $B = \{4,0,2\}$, the involution defined by

$$\underline{i} \mapsto -\underline{i} \text{ and } \overline{i} \mapsto -\overline{i} \text{ for each } 0 \leq i \leq 5$$

is an automorphism of $\mathbb{P}$ that is not a power of α. Hence in these cases, $\mathbf{Aut}(\mathbb{P}) \not\cong \mathbf{Z}_6$.

In the third case, $B = \{0,1,3\}$, that is, $\mathbb{P} = \mathbb{P}_6$, and by Corollary 3.4, we know that $\mathbf{Aut}(\mathbb{P}_6) \cong \mathbf{A}_4 \times \mathbf{Z}_2$. Thus $\mathbf{Aut}(\mathbb{P}) \not\cong \mathbf{Z}_6$, which concludes the proof. $\square$

Now we turn to investigating the value of f_n when n is not a prime power. First we give an example of a poset that plays a crucial role in our characterization of f_n for an arbitrary n.

We define the a 20-element height one poset $\mathbb{E}$ as follows. Let $\mathbb{C}_4$ be the 8-element crown whose elements and covering relation are given by

$$\underset{\sim}{i} < \widetilde{i}, \widetilde{j} \text{ where } 0 \leq i \leq 3, \; j = i+1$$

where the addition is meant modulo 4. First we take the disjoint sum of $\mathbb{C}_4$ with the poset $\mathbb{P}_6$, see Figure 9. Then to obtain $\mathbb{E}$, we add to this sum all pairs between the minimal elements of $\mathbb{C}_4$ and the maximal elements of $\mathbb{P}_6$ with the same parity, furthermore, all pairs between the minimal elements of $\mathbb{P}_6$ and the maximal elements of $\mathbb{C}_4$ with the same parity.

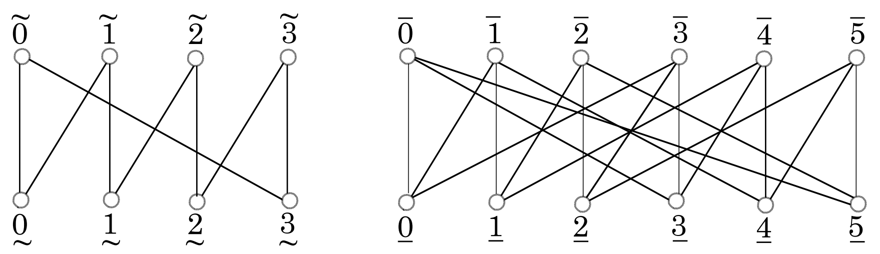

FIGURE 9. The disjoint sum of posets $\mathbb{C}_4$ and $\mathbb{P}_6$

The disjoint sum of two minimum size posets representing $\mathbf{Z}_3$ and $\mathbf{Z}_4$ respectively, is a 21-element poset representation of $\mathbf{Z}_{12}$. By the following proposition, the 20-element $\mathbb{E}$ gives us a more economical poset representation of $\mathbf{Z}_{12}$.

Proposition 4.3. $\mathbf{Aut}(\mathbb{E}) \cong \mathbf{Z}_{12}$.

Proof. Let $\underset{\sim}{C}$ be the set of minimal and $\widetilde{C}$ the set of maximal elements of $\mathbb{C}_4$. Let $\underline{P}$ be the set of minimal and $\overline{P}$ the set of maximal elements of $\mathbb{P}_6$. Let us choose an element of $\underset{\sim}{C}$ and count the elements that are reached from this element by a unique path of length 2 in $\mathbb{E}$. Without loss of generality, we choose $\underset{\sim}{0} \in \underset{\sim}{C}$. Then the ending vertices of these paths from $\underset{\sim}{0}$ are $\underset{\sim}{1}$, $\underset{\sim}{3}$, $\underline{0}$, $\underline{2}$, $\underline{4}$. So for any element of $\underset{\sim}{C}$, the number of elements reached from it by a unique path of length 2 in $\mathbb{E}$ is

5. Similarly, the vertices that are reached from $\underline{0} \in \underline{P}$ by a unique path of length 2 are $\underset{\sim}{0}$, $\underset{\sim}{2}$, $\underline{1}$, $\underline{2}$, $\underline{4}$, $\underline{5}$. So for any element of $\underline{P}$, the number of elements that can be reached from it by a unique path of length 2 in $\mathbb{E}$ is 6. This shows that $\underset{\sim}{C}$ and $\underline{P}$ are preserved by the automorphisms of $\mathbb{E}$. A dual argument proves that both of the sets $\widetilde{C}$ and $\overline{P}$ are also preserved by the automorphisms of $\mathbb{E}$.

Since the permutation

$$(\underset{\sim}{0}\ \underset{\sim}{1}\ \underset{\sim}{2}\ \underset{\sim}{3})(\widetilde{0}\ \widetilde{1}\ \widetilde{2}\ \widetilde{3})(\underline{0}\ \underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{5})(\overline{0}\ \overline{1}\ \overline{2}\ \overline{3}\ \overline{4}\ \overline{5})$$

is an automorphism of $\mathbb{E}$, $\mathbf{Z}_{12}$ embeds into $\mathbf{Aut}(\mathbb{E})$, and $\underset{\sim}{C}$, $\widetilde{C}$, $\underline{P}$, $\overline{P}$ are the orbits of $\mathbf{Aut}(\mathbb{E})$. As $4 = |\underset{\sim}{C}| = |\mathrm{Aut}(\mathbb{E})|/|G_0|$ where $\mathbf{G_0}$ is the stabilizer of $\underset{\sim}{0}$ in $\mathbf{Aut}(\mathbb{E})$, it suffices to prove that $|G_0| = 3$. Let β be an element of G_0. Since β preserves C_4, $\beta|_{C_4}$ is an automorphism of $\mathbb{C}_4$ that fixes $\underset{\sim}{0}$. So $\beta|_{C_4}$ is the identity or the reflection $(\underset{\sim}{1}\ \underset{\sim}{3})(\widetilde{0}\ \widetilde{1})(\widetilde{2}\ \widetilde{3})$ on $\mathbb{C}_4$.

We argue that $\beta|_{C_4}$ is the identity. If not, then the subsets of the even and respectively the odd elements in $\overline{P}$ are preserved by β, and the subsets of the even and respectively the odd elements in $\underline{P}$ are swapped by β. So the subposet induced by the even elements of $\mathbb{P}_6$ is mapped into the subposet induced by the odd elements of $\underline{P}$ and the even elements of $\overline{P}$. This is impossible, since the first subposet is a disjoint sum of three 2-element chains and the second one is a 6-element crown.

So we have that β restricted to C_4 is the identity. Then the subsets of the even and respectively the odd elements in $\underline{P}$ are preserved by β. Similarly, the subsets of the even and respectively the odd elements in $\overline{P}$ are preserved by β. In this way, β permutes the edges of the form $(\underline{i}, \overline{i})$ in $\mathbb{P}_6$. By leaving out these edges from $\mathbb{P}_6$, we obtain a poset that is a disjoint sum of two six-element crowns. Let $\mathbb{A}$ denote one of these crowns induced by $\underline{0}, \underline{2}, \underline{4}, \overline{1}, \overline{3}, \overline{5}$. Let $\mathbb{B}$ denote the other crown, so $\mathbb{B}$ is induced by $\underline{1}, \underline{3}, \underline{5}, \overline{0}, \overline{2}, \overline{4}$. Clearly, β preserves the subsets A and B. Moreover, $\beta|_A$ uniquely determines $\beta|_{P_6}$ (hence β as well) since β permutes the edges of the form $(\underline{i}, \overline{i})$ in $\mathbb{P}_6$. So if $\beta|_A$ was the reflection $(\underline{2}\ \underline{4})(\overline{1}\ \overline{3})$ on $\mathbb{A}$, then $\beta|_B$ would be $(\underline{1}\ \underline{3})(\overline{2}\ \overline{4})$ on $\mathbb{B}$. This is impossible, since the latter permutation is not an automorphism of $\mathbb{B}$. Thus, $\beta|_A$ is one of the three rotations on $\mathbb{A}$. Thus G_0 has indeed 3-elements. □

By the preceding corollary and proposition, we get an upper bound for f_n. We prove also the that the so obtained upper bound is a lower bound as well.

Theorem 4.4. *Let*

$$n = \prod_{q|n} q \geq 2$$

where the q are the full prime power factors of n. Let

$$s_n = \sum_{2=q|n} q + \sum_{3 \leq q \leq 7,\ q|n} 3q + \sum_{8 \leq q|n} 2q.$$

Then $f_n = s_n - 1$ if $n = 12k$ where k is coprime to 6 and $f_n = s_n$ otherwise.

Proof. Since the automorphism group of the disjoint sum of pairwise non-isomorphic connected posets is the product of their automorphism groups, by Corollary 4.1 and Proposition 4.3, we have that $f_n \leq s_n - 1$ if $n = 12k$ where k is coprime to 6 and $f_n \leq s_n$ otherwise.

Let $\mathbb{P}$ be a minimum size poset representation of $\mathbf{Z}_n$. Let α be a generating element of $\mathbf{Aut}(\mathbb{P})$. To prove the other direction of the corollary, we require some claims on the cycle structure of α.

Claim 1. *For any full prime power factor q of n, there is a cycle β in the cycle decomposition of α such that q divides the length of β.*

Proof of Claim 1. Since the order of α is n and equals the least common multiple of the lengths of the cycles of α, we have the claim.

Claim 2. *For any full prime power factor $q \geq 3$ of n, there are at least two cycles of α such that q divides their lengths.*

Proof of Claim 2. Suppose that β is the only cycle of α whose length is divisible by q, and $q = p^m$ where p is a prime. Then $\alpha^{\frac{n}{p}} = \beta^{\frac{n}{p}}$ is a product of p-cycles that only move elements which are also moved by β. So there are no comparabilities between the elements moved by these p-cycles. For each element fixed by $\beta^{\frac{n}{p}}$ and for each p-cycle of $\beta^{\frac{n}{p}}$, the fixed element is greater or smaller than all elements moved by the p-cycle or is incomparable to all of them. This yields that any transposition that switches two elements moved by one of the p-cycles is in $\mathbf{Aut}(\mathbb{P})$. It is a contradiction, since α does not have a power which is a transposition.

We define the *small cycles* of α to be the cycles that have length q where $q = 2$ or q is a full prime power factor of n in the cycle decomposition of α. The *large cycles* are the non-small cycles in the cycle decomposition of α that have even length not divisible by 4 or length divisible by at least a full prime power factor of n. The small and large cycles altogether are called *important* cycles.

Claim 3. *Let q be a full prime power factor of n where $q \in \{3,4,5,7\}$. If there are exactly two cycles of α whose lengths are divisible by q and the lengths of the other cycles of α are coprime to q, then both cycles whose lengths are divisible by q are large.*

Proof of Claim 3. Suppose the contrary. Let β and γ be a q-cycle and a kq-cycle of α, respectively. Then $\alpha^{\frac{n}{q}} = \beta^{\frac{n}{q}}\gamma^{\frac{n}{q}}$, and the elements moved by β and $\beta^{\frac{n}{q}}$ coincide. Observe that α acts transitively on the set W of the $2q$-element subposets of $\mathbb{P}$ induced by the moved elements of β and any one of the q-cycles of $\gamma^{\frac{n}{q}}$. So the posets in W are isomorphic to each other via a restriction of a suitable power α. Let us fix an element $\mathbb{R}$ in W. Since each poset in W is isomorphic to $\mathbb{R}$, the restriction of any automorphism of $\mathbb{R}$ to the set of elements moved by β extends to an automorphism of any poset in W. Let us take such an automorphism extension for each poset in W. Then the union of these extensions is an automorphism μ of the subposet induced by the moved elements of β and γ in $\mathbb{P}$. Clearly, the automorphism μ preserves each subset of P that consists of the elements moved by any of the q-cycles of $\alpha^{\frac{n}{q}}$. Since $\alpha^{\frac{n}{q}}$ fixes the elements apart from the moved elements of β and γ, the automorphism μ extends further onto the whole poset $\mathbb{P}$ by mapping the elements that are fixed by $\alpha^{\frac{n}{q}}$ into themselves. At the beginning of the proof of Corollary 4.1, we saw that for any $q \in \{3,4,5,7\}$, the automorphism group of any $2q$-element poset of height 1 that admits an automorphism with two q-cycles has a non-commutative automorphism group. Hence the automorphism group of $\mathbb{R}$ is non-commutative. Since each automorphism of $\mathbb{R}$ extends to an automorphism of $\mathbb{P}$, $\mathbf{Aut}(\mathbb{P}) \not\cong \mathbf{Z}_n$, a contradiction. Thus β and γ are large.

Claim 4. *Let* 4 *be a full prime power factor of n. If there are exactly three cycles of α of even lengths and the length of one of them is not divisible by 4, then at least two of these cycles are large.*

Proof of Claim 4. To the contrary, let us suppose first that β and γ are two 4-cycles of α. Then $\alpha^{\frac{n}{4}}$ consists of two 4-cycles and some 2-cycles where each of the 2-cycles are located in the antichain of the elements moved by the cycle whose length is divisible by 2 but not 4. Then α acts transitively on the set of subposets determined by the elements moved by β, γ, and any one of the 2-cycles of $\alpha^{\frac{n}{4}}$. So these subposets are isomorphic to each other. Similarly, as in the preceding proof every element of the automorphism group of such a subposet $\mathbb{R}$ extends to an automorphism of $\mathbb{P}$. But at the end of the proof of Corollary 4.1, we saw that the automorphism group of $\mathbb{R}$ is non-commutative, hence $\mathbf{Aut}(\mathbb{P})$ is non-commutative, a contradiction. The proof is completely similar, if β is a 2-cycle and γ is a 4-cycle of α.

Now we give some lower bounds for the cycle lengths of the cycles of α. For the lengths of unimportant cycles, we let our lower bound be 0. For the length of a small cycle of α, our lower bound is q where q is the length of the cycle. Let l be the length of a large cycle of α where $2|l$. If $4 \not| l$, then we take the lower bound $\frac{3}{2}+\frac{3}{2}\sum_{r|l} r$ for l where r runs through the full odd prime power factors of l. If $2 \not| l$ or $4|l$, our lower bound for l is $\frac{3}{2}\sum_{r|l} r$ where r runs through the full prime power factors of l.

In order to see that the lower bounds given in the preceding paragraph are correct for l when l is the length of a large cycle, we prove the inequality

$$\frac{3}{2}\sum_{i=1}^{t} r_i \leq \prod_{i=1}^{t} r_i$$

where the r_i are integers at least 3 and t is at least 2. We use an induction on t. If $t=2$, we may assume that $r_1 \leq r_2$, that is, $\frac{r_1}{r_2} \leq 1$. Then we have $\frac{3}{2}(\frac{r_1}{r_2}+1) \leq 3$. Thus by $3 \leq r_1$, $\frac{3}{2}(\frac{r_1}{r_2}+1) \leq r_1$, that is, $\frac{3}{2}(r_1+r_2) \leq r_1 r_2$. If $t>2$, then by using the induction hypothesis, the following inequalities hold

$$\frac{3}{2}\sum_{i=1}^{t} r_i \leq \frac{3}{2}r_t + \frac{3}{2}\sum_{i=1}^{t-1} r_i \leq \frac{3}{2}r_t + \prod_{i=1}^{t-1} r_i \leq \frac{3}{2}(r_t + \prod_{i=1}^{t-1} r_i) \leq \prod_{i=1}^{t} r_i.$$

If $2|l$, $4 \not| l$ and l has exactly one full odd prime power factor, then this factor is at least 3, and our lower bound for l in the preceding paragraph is correct. If $2|l$, $4 \not| l$ and l has more than one full odd prime power factors, then by the use of the inequality we just proved,

$$\frac{3}{2}+\frac{3}{2}\sum_{r|l} r \leq \frac{3}{2}+\prod_{r|l} r \leq 2\prod_{r|l} r = l$$

where r runs through the full odd prime power factors of l. If $2 \not| l$ or $4|l$, then l has at least 2 full prime power factors. All of these factors are at least 3. So by the inequality we just proved $\frac{3}{2}\sum_{r|l} r \leq \prod_{r|l} r = l$ where r runs through the full prime power factors of l.

Let s denote the sum of lower bounds just introduced for the lengths of the cycles of α. Clearly, $|P|$ is greater than or equal to the sum of the cycle lengths of α. Therefore, $\lceil s \rceil \leq |P|$. In what follows in the proof, we are vying for the inequality $s_n \leq \lceil s \rceil$. We narrow the cases for n until this inequality is unachievable.

In this case, it will turn out that $n = 12k$ where k is coprime to 6 and we still have the inequality $s_n - 1 \leq s$.

So first observe that if $q \geq 8$ is a full prime power factor of n, then by Claim 2, there are at least two cycles of α whose cycle lengths are divisible by q. So in s, q adds up at least two times. If $q \in \{3,5,7\}$ is a full prime power factor of n, then it is possible that q may occur as a full prime power factor of the lengths of only two cycles, but when this happens, by Claim 3, these two cycles are large. So in s, q is counted at least 3 times, no matter if q divides the lengths of two or more cycles of α. If $q = 2$ is a full prime power factor of n, then 2 or $\frac{3}{2}$ appears as a summand in s. So if 4 is not a full prime power factor of n, then we have indeed that $s_n \leq \lceil s \rceil$.

For the remaining cases, we assume that 4 is a full prime power factor of n. Then the only problem occurs when there are exactly two cycles with length divisible by 4 and some cycles with length divisible by 2 but not 4, for otherwise in s, we have the summand 4 at least 3 times and we get that $s_n \leq s$. So from now on, we assume that among the cycles of even length of α there are exactly two cycles whose lengths are divisible by 4.

If α has only two cycles of even length, necessarily divisible by 4, then by Claim 3, these two cycles are large. So 4 adds up 3 times in s, and we are done. If α has exactly three cycles of even length, not all of them divisible by 4, then Claim 4 implies that α has at least two large cycles of even length. Then in s, 4 adds up with a coefficient at least $\frac{5}{2}$, and we have a summand at least $\frac{3}{2}$ for the cycle whose length is divisible by 2 but not 4. So in this case, $s_n \leq \lceil s \rceil$ and we are done. If α has at least five cycles of even length, then s has the summand 4 at least 2 times and the summand $\frac{3}{2}$ at least 3 times, and the lower bound s_n is exceeded.

So the only remaining case occurs, when α has four cycles of even length, two cycles whose lengths are divisible by 4 and two cycles whose lengths are not divisible by 4. If one of the cycles divisible by 4 is large, then 4 and 2 contribute to s with at least 13, that exceeds the lower bound we want. If one of the cycles has length 2, then 4 and 2 contribute to s with at least 11.5, so the required lower bound s_n for $\lceil s \rceil$ is met. Therefore, we may assume that α has exactly two 4-cycles and exactly two large cycles with lengths divisible by 2 but not 4. Notice then that the lower bounds given for the cycle lengths of the latter two cycles are strict, that is, the sum of the lengths of these cycles are larger than 11, except if both of these lengths coincide with 6.

So, we may assume that α has a cycle decomposition of two 4-cycles, two 6-cycles, and some odd cycles. The problem in this case with our lower estimate is that the factors 4 and 2 of the even cycle lengths contribute to s with 11, not by more than 11. If one of the odd cycles is divisible by a power of 3, then there is a small or large odd cycle of α divisible by a power of 3. Hence 3 produces a surplus in s and we exceed the required lower bound s_n for $\lceil s \rceil$ due to this surplus.

Hence we may assume that the lengths of the odd cycles of α are coprime to 3. So $n = 12k$ where k is coprime to 6. As the factors 4 and 2 of the even cycle lengths contribute to s with 11, we have the lower bound $s_n - 1$ for s, and the proof is finished. $\square$

ACKNOWLEDGEMENT

We would like to thank one of the anonymous referees for their detailed comments and suggestions that led to an improved version of our manuscript.

Funding. The research of authors was partially supported by the grants NKFIH-K128042, NKFIH-K138892, and TKP2021-NVA-09 of the Ministry for Innovation and Technology, Hungary.

Data availability statement. Data sharing not applicable - no new data generated.

Conflict of interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author contribution statement. All authors contributed to the study, conception and design. Preparation of the material was performed by Gergő Gyenizse, Péter Hajnal and László Zádori. All authors read and approved the final manuscript.

REFERENCES

[1] W. Arlinghaus, The classification of minimal graphs with given abelian automorphism group. Memoirs Amer. Math. Soc. 57/330 (1985).

[2] L. Babai, Infinite digraphs with given regular automorphism groups. J. Combin. Theory Ser. B 25 (1978), pp. 26-46.

[3] L. Babai, Finite digraphs with given regular automorphism groups. Per. Math. Hungarica 11 (1980), pp. 257-270.

[4] J.A. Barmak, Automorphism groups of finite posets II. arXiv:2008.04997v1 (2020).

[5] J. A. Barmak, A. N. Barreto, Smallest posets with given cyclic automorphism group, arXiv:2301.08701 (Submitted on 20 January 2023).

[6] A. N. Barreto, Sobre los posets más chicos con grupo de automorfismos abeliano dado, Tesis Licenciatura, Universidad de Buenos Aires, Argentina (20 September 2021).

[7] H. S. M. Coxeter, Self-dual configurations and regular graphs, Bull. Amer. Math. Soc. 56/5 (1950), pp. 413-455.

[8] R. L. Meriwether, Smallest graphs with a given cyclic group (1963) unpublished, see Math. Reviews 33 (1967) #2563.

Email address: gergogyenizse@gmail.com

Email address: hajnal@math.u-szeged.hu

Email address: zadori@math.u-szeged.hu

BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, HUNGARY